Mean curvature flow with obstacles: a viscosity approach

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Abstract

We introduce a level-set formulation for the mean curvature flow with obstacles and show existence and uniqueness of a viscosity solution. These results generalize a well known viscosity approach for the mean curvature flow without obstacle by Evans and Spruck and Chen, Giga and Goto in 1991. In addition, we show that this evolution is consistent with the variational scheme introduced by Almeida, Chambolle and Novaga (2012) and we study the long time behavior of our viscosity solutions.

1 Introduction

In this article, we introduce the level set formulation for a generalized motion by mean curvature with obstacles. More precisely, let $M(t) = \partial\Omega(t)$ be a $n-1$ submanifold of $\mathbb{R}^n$, we say that it evolves by mean curvature if for any $x \in M(t)$, the velocity of $M(t)$ at x is given by

$$v(x) = -H\nu(x) \tag{1}$$

where H is the mean curvature of $M(t)$ at x (nonnegative if $\Omega(t)$ is a convex set with boundary) and ν is the normal vector to $M(t)$ pointing to $\Omega(t)^c$.

Motivated by recent works from Almeida, Chambolle and Novaga [1] and Spadaro [16] about a discrete scheme for the mean curvature flow with obstacles, we want to constrain (1) forcing

$$\Omega^-(t) \subset \Omega(t) \subset \Omega^+(t) \tag{2}$$

where $\Omega^\pm$ are two open sets (which can depend of the time variable).

Mean curvature flow has been widely studied in the 30 past years. One can in particular cite [5] for a first paper on this motion, [9] for a geometric study of (1) and [11] and [7] for a level-set formulation and the use of viscosity solutions. In the sequel we follow the last approach.

It is well known (see for example [11]) that if $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function with a nonzero gradient at x_0, the mean curvature of the level set $\{u = u(x_0)\}$ is given by $\operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)(x_0)$. As a result, making this set (and every other level-set of u) evolve by mean curvature leads to the following equation for u:

$$u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right). \tag{3}$$

In the whole paper, we will think of $M(t)$ as the zero-level-set of $u(\cdot, t)$.
To add the constraint to (3), we define $u^\pm(x,t)$ such that

$$\Omega^-(t) \subset \Omega(t) \subset \Omega^+(t) \Leftrightarrow \{u^+ < 0\} \subset \{u < 0\} \subset \{u^- < 0\}$$

and impose

$$\forall x, t, \quad u^-(x,t) \leqslant u(x,t) \leqslant u^+(x,t). \tag{4}$$

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As in [11], [7], we study (3) with constraint (4) using viscosity solutions. We first present a suitable viscosity framework prove a uniqueness and existence result for bounded uniformly continuous initial data and obstacles and Lipschitz forcing term in the spirit of [8]. Then, we link the regularity of the solution to the regularity of the initial data.

We also show that our level-set approach really defines a geometric flow: the α-level set of the solution depends only on the α-level set of the initial data and the obstacles. Nonetheless, as expected, there is no uniqueness: level sets of the solution can develop non empty interiors because of the obstacles. In an upcoming paper with Matteo Novaga [15], we study the MCF with obstacles by a geometrical point of view, proving short time existence, uniqueness and regularity of solutions.

Finally, in Section 4, we compare the approach followed by [16] and [1] to ours. More precisely, we show that the discrete scheme has a limit which is the viscosity solution to (3) with constraint (4). In addition, this variational approach gives monotonicity of the flow and therefore information on the long time behavior of the viscosity solution.

2 Notation

In what follows, we consider the equation (slightly more general than (3), but the latter has to be kept in mind), for $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$

$$\forall t \geqslant 0, x \in \mathbb{R}^n, \quad u_t + F(Du, D^2u) + k|Du| = 0, \tag{5}$$

where $k : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is a forcing term and $F : \mathbb{R}^n \times \mathcal{S}^n \to \mathbb{R}$ satisfies

i) F is continuous in space and time when $p \neq 0$,

ii) F is geometric : $\forall \lambda > 0, \sigma \in \mathbb{R},\ F(\lambda p, \lambda X + \sigma p \otimes p) = \lambda F(p, X)$,

iii) For X and Y symetric matrices with $X \leqslant Y$, $F(p, X) \leqslant F(p, Y)$.

In the following, ∇u, Du and D^2u denote space derivatives only.

We also introduce $\overline{F}$ and $\underline{F}$ which are respectively the upper semicontinuous and lower semi-continuous envelopes of u[1] (see Definition 3).

To play the role of obstacles, we consider u^- and $u^+ : \mathbb{R}^n \to \mathbb{R}$, with $u^- \leqslant u^+$. The function u will be forced to stay between u^- and u^+. Geometrically, the constraint reads $\{u^+ < s\} \subset \{u < s\} \subset \{u^- < s\}$.

To adapt the classical theory of viscosity solutions (we will use the same scheme of proof as in [8]), the key point is to define correctly sub and super solutions of

$$u_t + F(Du, D^2u) = 0 \quad \text{with} \quad u^- \leqslant u \leqslant u^+. \tag{6}$$

This definition for two obstacles has been already given, for instance in [18].

Definition 1. A function $u : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}$ is said to be a (viscosity) subsolution on $[0, T]$ of the motion equation with obstacles u^+, u^- and initial condition g if

- u is upper semicontinous (usc),
- for all $x, t \in \mathbb{R}^n \times [0, T]$, $u^-(x, t) \leqslant u(x, t) \leqslant u^+(x, t)$,

[1] This quantity is useful to make the following results apply for the mean curvature motion, where

$$F(p, X) = -\operatorname{Tr}\left(\left(I - \frac{p \otimes p}{|p|^2}\right) X\right).$$

- for all $x \in \mathbb{R}^n$, $u(x,0) \leqslant g(x)$,
- if φ is a $\mathcal{C}^2$ function of x,t, if $(\hat{x},\hat{t}) \in \mathbb{R}^n \times (0,T]$ is a maximum of $u-\varphi$ and if $u(\hat{x},\hat{t}) > u^-(\hat{x},\hat{t})$, then
$$\varphi_t + \underline{F}(D\varphi, D^2\varphi) + k|D\varphi| \leqslant 0.$$

Similarly, u is said to be a (viscosity) supersolution of the motion equation with obstacles u^+, u^- and initial condition g if

- u is lower semicontinous (lsc),
- for all $x,t \in \mathbb{R}^n \times [0,T]$, $u^-(x,t) \leqslant u(x,t) \leqslant u^+(x,t)$,
- for all $x \in \mathbb{R}^n$, $u(x,0) \geqslant g(x)$,
- if φ is a $\mathcal{C}^2$ function of x,t, if $(\hat{x},\hat{t}) \in \mathbb{R}^n \times (0,T]$ is a minimum of $u-\varphi$ and if $u(\hat{x},\hat{t}) < u^+(\hat{x},\hat{t})$, then
$$\varphi_t + \overline{F}(D\varphi, D^2\varphi) + k|D\varphi| \geqslant 0.$$

Finally, u is said to be a (viscosity) solution of the motion equation with obstacles u^+, u^- if u is both a super and a sub solution.
To simplify, we write

$$u_t + F(Du, D^2u) + k|Du| = 0 \quad \text{on } \{u^- \leqslant u \leqslant u^+\}. \tag{7}$$

A supersolution (resp subsolution) of the motion equation with obstacles u^+, u^- will be called a supersolution (resp. subsolution) of (7).

Looking at the very definition, one can make the

Remark 1. Let u be a subsolution with obstacles $u^- \leqslant u^+$. Then, u is a subsolution with obstacles u^- and v^+ for every $v^+ \geqslant u^+$.

Remark 2. It has to be noticed that using this definition, obstacle can depend on the time variable. Moreover, the contact zone $\{u^+ = u^-\}$ can be nonempty.

We also want to point out that the obstacle problem can be defined using a modified Hamiltonian (see [8], Example 1.7). For instance, let

$$G(x,t,u,Du,D^2u) = \max\left(\min\left(u_t + F(Du,D^2u), u-u^-\right), u-u^+\right). \tag{8}$$

One can easily show that the (usual) viscosity solutions of $G=0$ coincide with our definition above (the only difference is the subsolutions of $G=0$ do not have to satisfy $u \geqslant u^-$, but must remain below u^+). Nonetheless (8) cannot be written

$$u_t + G(x,t,u,Du,D^2u) = 0,$$

which is the usual form for parabolic equations, for which known results (see [8, 13, 7]) could apply. As a result, despite of this convenient formulation, we have to check that the usual results still apply. That is why we decided to use the definition above with a standard Hamiltonian but with (explicit) obstacles.

There is another equivalent definition of such solutions, which can be useful (see [8]).

Definition 2. Let $f : \mathbb{R}^n \times (0,T] \to \mathbb{R}$. We said that $(a,p,X) \in \mathbb{R} \times \mathbb{R}^n \times \mathcal{S}_n(\mathbb{R})$ is a superjet for f in (x_0,t_0) and we denote $(a,p,X) \in \mathcal{J}^{2,+}f(x_0,t_0)$ if, when $x,t \to x_0,t_0$ in $\mathbb{R}^n \times (0,T]$,

$$f(x,t) \leqslant f(x_0,t_0) + a(t-t_0) + \langle p\,,\, x-x_0\rangle + \frac{1}{2}\langle X(x-x_0)\,,\, x-x_0\rangle + o(|t-t_0| + |x-x_0|^2).$$

We likewise say that $(a,p,X) \in \mathbb{R}\times\mathbb{R}^n\times\mathcal{S}_n(\mathbb{R})$ is a subjet for f in (x_0,t_0) and we denote $(a,p,X)\in\mathcal{J}^{2,-}f(x_0,t_0)$ if, for every $x,t\to x_0,t_0$,

$$f(x,t)\geqslant f(x_0,t_0)+a(t-t_0)+\langle p\,,\,x-x_0\rangle+\frac{1}{2}\left\langle X(x-x_0)\,,\,x-x_0\right\rangle+o(|t-t_0|+|x-x_0|^2).$$

Then, u is a subsolution of (7) if it satisfies the three first assumptions of the previous definition and if

$$\forall x,t\in\mathbb{R}^n\times(0,T],\forall(a,p,X)\in\mathcal{J}^{2,+}u(x,t),\quad u(x)>u^-(x)\Rightarrow a+\underline{F}(p,X)+k|p|\leqslant 0.$$

Of course, u is a supersolution of (7) if the three assumptions of the first definition are satisfied and if

$$\forall x,t\in\mathbb{R}^n\times(0,T],\forall(a,p,X)\in\mathcal{J}^{2,-}u(x,t),\quad u(x)<u^+(x)\Rightarrow a+\overline{F}(p,X)+k|p|\geqslant 0.$$

We also use the following notation.

Definition 3. For $f:\mathbb{R}^n\to R$, we denote by f^* the upper semicontinuous envelope of f. More precisely

$$f^*(x)=\limsup_{y\to x}f(y).$$

We define in a similar way the lower semicontinuous envelope of f.

$$f_*(x)=\liminf_{y\to x}f(y).$$

Note that f^* (resp. f_*) is the smaller (resp. larger) semicontinuous function g such that $g\geqslant f$ (resp. $g\leqslant f$).

3 Existence and uniqueness

The aim of this section is to show the

Theorem 1. *We assume that u^- and u^+ are uniformly continuous and bounded and that k is Lipschitz. Then, if $g:\mathbb{R}^n\to\mathbb{R}$ is uniformly continuous and $u^-(x,0)\leqslant g(x)\leqslant u^+(x,0)$, (7) has an unique solution, which is uniformly continuous.*

The structure of the proof is classical when dealing with viscosity solutions. A comparison principle will show uniqueness, and existence will follow by standard methods.

3.1 Uniqueness

We begin by proving a comparison principle, adapted from [8], Theorem 8.2. It has to be noticed that the same result with no obstacles has been proved in [13] (Th. 4.1) in a very general framework. We could adapt this result to the obstacle case but we prefer to present a simpler and self consistent proof based on [8] (nonetheless, we will use some ideas of [13]).

Proposition 1 (Comparison principle)**.** *We assume that u is a subsolution and v a supersolution of (7) on $(0,T)$, and that $u(x,0)\leqslant v(x,0)$. Then, $u\leqslant v$ in $\mathbb{R}^n\times(0,T)$.*

Proof. Throughout the proof, ω will denote a modulus of continuity for u^-, u^+ and g and L a Lipschitz bound on k.

We proceed by contradiction. Since $\tilde{u}=(u-\frac{\eta}{T-t})\vee u^-$ is still a subsolution, but with

$$F(D\tilde{u},D^2\tilde{u})+k|D\tilde{u}|\leqslant -c<0,$$

it is enough to prove the comparison principle with $\tilde{u}$ and then pass to the limit (nonetheless, we still write u). Suppose that there exists $\overline{x}, \overline{t}$ such that $u(\overline{x}, \overline{t}) - v(\overline{x}, \overline{t}) \geqslant 2\delta > 0$. One defines

$$\Phi(x, y, t) = u(x, t) - v(y, t) - \frac{\alpha}{4}|x - y|^4 - \frac{\varepsilon}{2}(|x|^2 + |y|^2).$$

If ε is sufficiently small, $\Phi(\overline{x}, \overline{x}, \overline{t}) \geqslant \delta$. Hence, $M := \max_{x,y,t} \Phi(x, y, t) \geqslant \delta$ (the penalization at infinity reduces searching for the maximum to a compact set). Let $\hat{x}, \hat{y}, \hat{t}$ be a maximum point. Since u and v are bounded, there is C depending only on $\|u\|_\infty$ and $\|v\|_\infty$ such that

$$|\hat{x} - \hat{y}| \leqslant \frac{C}{\alpha^{1/4}}.$$

First, let us show by contradiction that $u(\hat{x}, \hat{t}) > u^-(\hat{x}, \hat{t})$ and $v(\hat{y}, \hat{t}) < u^+(\hat{y}, \hat{t})$. Suppose for example that $u(\hat{x}, \hat{t}) = u^-(\hat{x}, \hat{t})$. Then

$$\begin{aligned} 0 < \delta \leqslant u^-(\hat{x}, \hat{t}) - v(\hat{y}, \hat{t}) &\leqslant u^-(\hat{y}, \hat{t}) + \omega(|\hat{x} - \hat{y}|) - v(\hat{y}, \hat{t}) \\ &\leqslant \omega(|\hat{x} - \hat{y}|) + 0 \leqslant \omega(C\alpha^{-1/4}). \end{aligned}$$

Hence, if α is sufficiently large (independently of ε), $\omega(C\alpha^{-1/4}) \leqslant \delta/3$. Contradiction (this shows moreover that $\hat{t} < T$). Similarly, $v(\hat{y}, \hat{t}) < u^+(\hat{y}, \hat{t})$.

In what follows, α is fixed sufficiently big to satisfy these conclusions.

As

$$M + \alpha|x - y|^4 + \frac{\varepsilon}{2}(|x|^2 + |y|^2) \geqslant u(\hat{x}) - v(\hat{y}) \tag{9}$$

with equality in $\hat{x}, \hat{y}, \hat{t}$, we are able to apply Ishii's lemma [8] which provides (a, b, X, Y) such that $(a, \underbrace{\alpha|\hat{x} - \hat{y}|^2(\hat{x} - \hat{y})}_{:=\hat{p}} - \varepsilon\hat{y}, Y - \varepsilon I) \in \overline{\mathcal{J}}^{2,+} v(\hat{y}, \hat{t})$ and $(b, \alpha|\hat{x} - \hat{y}|^2(\hat{x} - \hat{y}) + \varepsilon\hat{x}, X + \varepsilon I) \in \overline{\mathcal{J}}^{2,-} u(\hat{x}, \hat{t})$.

It provides moreover $a + b = 0$ and

$$-C|\hat{x} - \hat{y}|^2\alpha \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant \alpha C|\hat{x} - \hat{y}|^2 \begin{bmatrix} I & -I \\ -I & I \end{bmatrix},$$

which shows in particular that $X \leqslant Y$ and $\|X\|, \|Y\| \leqslant C_1\alpha|\hat{x} - \hat{y}|^2$.

Since u and v are respectively subsolution and supersolution near $(\hat{x}, \hat{t})$ and $(\hat{y}, \hat{t})$, one has

$$c \leqslant \underbrace{a + b}_{=0} + \underline{F}(\hat{p} - \varepsilon\hat{y}, Y - \varepsilon I) - \overline{F}(\hat{p} + \varepsilon\hat{x}, X + \varepsilon I) + k(\hat{x}, \hat{t})|\hat{p} + \varepsilon\hat{x}| - k(\hat{y}, \hat{t})|\hat{p} - \varepsilon y|.$$

One can write

$$k(\hat{x}, \hat{t})|\hat{p} + \varepsilon\hat{x}| - k(\hat{y}, \hat{t})|\hat{p} - \varepsilon y| \leqslant (k(\hat{x}, \hat{t}) - k(\hat{y}, \hat{t}))|\hat{p} + \varepsilon\hat{x}| + 2|k(\hat{y}, t)|(|\varepsilon\hat{x}| + |\varepsilon\hat{y}|),$$

which gives

$$c \leqslant \underline{F}(\hat{p} + \varepsilon\hat{x}, X + \varepsilon I) - \overline{F}(\hat{p} - \varepsilon y, Y - \varepsilon I) + L(|\hat{x} - \hat{y}|)|\hat{p} + \varepsilon\hat{x}| + N(|\varepsilon\hat{x}| + |\varepsilon\hat{y}|).$$

Then, we want to let ε go to 0.

Since $M \geqslant \delta > 0$, we have

$$\delta + \frac{1}{4}\alpha|x - y|^4 + \frac{\varepsilon}{2}(|x|^2 + |y|^2) \leqslant u(\hat{x}) - v(\hat{y}) \leqslant 2N,$$

which implies that $\varepsilon|\hat{x}|^2$ is bounded, hence $\varepsilon\hat{x} \to 0$ (same for $\varepsilon\hat{y}$), whereas for $i \in \{2, 3, 4\}$, $\alpha|\hat{x} - \hat{y}|^i$ is bounded (so is $\hat{p}$, X and Y). Indeed, α is fixed here. Hence one can assume that $\hat{p} \to p$, $X \to X_0$, $\alpha|\hat{x} - \hat{y}|^4 \to \mu_\alpha$.

We now use a short lemma, which is an easy adaptation of [13], Proposition 4.4 (see also Lemma 2.8 in the preprint of [12], which has a form which is closer to ours) and whose proof is reproduced here for convenience.

Lemma 1. *One has*

$$\lim_{\alpha\to\infty}\lim_{\varepsilon\to 0}\alpha|\hat{x}-\hat{y}|^4=0.$$

Proof. Let $M_h=\sup_{\substack{|x-y|\leqslant h\\ t\in[0,T)}}u(x,t)-v(y,t))$ and (x_h^n,y_h^n,t_h^n) such that $u(x_h^n,t_h^n)-v(y_h^n,t_h^n)\geqslant M_h-\frac{1}{n}$ and $|x_h^n-y_h^n|\leqslant h$. Then,

$$M_h-\frac{1}{n}-\alpha h^4-\frac{\varepsilon}{2}\left(|x_h^n|^2+|y_h^n|^2\right)\leqslant M\leqslant u(\hat{x},\hat{t})-v(\hat{y},\hat{t}).$$

As x_h^n and y_h^n do not depend on ε, one can let it go to zero (considering the liminf of the right term) to get

$$M_h-\frac{1}{n}-\alpha h^4\leqslant\liminf_{\varepsilon\to 0}u(\hat{x},\hat{t})-v(\hat{y},\hat{t}).$$

Let $h\to 0$ (We denote by M' the decreasing limit of M_h). One obtains

$$M'-\frac{1}{n}\leqslant\liminf_{\varepsilon\to 0}(u(\hat{x},\hat{t})-v(\hat{y},\hat{t})).$$

Let α go to infinity :

$$\begin{aligned}M'-\frac{1}{n}&\leqslant\liminf_{\alpha\to\infty}\liminf_{\varepsilon\to 0}(u(\hat{x},\hat{t})-v(\hat{y},\hat{t}))\\&\leqslant\limsup_{\alpha\to\infty}\left(\sup_{\substack{|x-y|\leqslant C\alpha^{-1/4}\\ t\in[0,T)}}(u(x,t)-v(y,t))\right)\\&\leqslant\limsup_{h\to 0}\sup_{|x-y|\leqslant h}(u(x,t)-v(y,t))=M'\end{aligned}$$

hence

$$\lim_{\alpha\to\infty}\lim_{\varepsilon\to 0}u(\hat{x},\hat{t})-v(\hat{y},\hat{t})=M'.$$

We proves similarly that $\lim_{\alpha\to\infty}\lim_{\varepsilon\to 0}M=M'$. As a matter of fact,

$$\lim_{\alpha\to\infty}\lim_{\varepsilon\to 0}\left(\alpha|\hat{x}-\hat{y}|^4+\frac{\varepsilon}{2}(|\hat{x}|^2+|\hat{y}|^2)\right)=0,$$

which proves the lemma. □

One can now choose α such that $\lim_{\varepsilon\to 0}\alpha|\hat{x}-\hat{y}|^4\to\mu_\alpha$ with $\mu_\alpha\leqslant c/2L$ and pass to the liminf in $\varepsilon\to 0$. One gets (using $X\leqslant Y$),

$$\frac{\delta}{2}\leqslant\liminf\left(\underline{F}(\hat{p},X)-\overline{F}(\hat{p},X)\right).$$

To conclude, we distinguish two cases :

- if $p\neq 0$, then $\overline{F}(p,X_0)=\underline{F}(p,X_0)$ and we get the contradiction.
- if $p=0$, we have $\alpha|\hat{x}-\hat{y}|^2(\hat{x}-\hat{y})\underset{\varepsilon\to 0}{\longrightarrow}0$, so $X_0=0$ and $\overline{F}(p,X_0)=\underline{F}(p,X_0)=0$ and we get the contradiction too.

□

3.2 Existence

We will build a solution using Perron's method. Since we know that the supersolutions of (7) remain larger than subsolutions, the solution, if it exists, must be the larger subsolution (or equivalently, the smaller supersolution). Hence we introduce

$$W(x,t) = \sup\{w(x,t),\ w \text{ (usc) subsolution on } [0,T)\}.$$

We show that W is in fact the expected solution to (7).

Let us first state a straightforward but useful proposition.

Proposition 2. *i) Let u be a subsolution of the motion without obstacles which satisfies $u \leqslant u^+$. Then, $u_{ob} := u \vee u^-$ is a subsolution of (7) with obstacles (the same happens for $v \geqslant u^-$ supersolution and $v_{ob} = v \wedge u^+$).*

ii) More generally, if u is a solution with initial conditions and obstacles (u_0, u^-, u^+) and if v^- and v^+ are other obstacles which satisfy $u^- \leqslant v^- \leqslant u^+ \leqslant v^+$, then $u \vee v^-$ is a subsolution of the equation with initial condition $u_0 \vee v^-|_{t+0}$ and obstacles v^- and u^+.

iii) With the same notation as in ii), u is a subsolution of the equation with initial conditions u_0 and obstacles u^-, v^+.

Proof. The proof is quite simple: consider a smooth function φ and some x_0, t_0 such that $\varphi - u \vee u^-$ has a maximum at (x_0, t_0). Then, using the definition of subsolutions, either $u(x_0,t_0) \vee u^-(x_0,t_0) = u^-(x_0,t_0)$ and nothing has to be done, or $u(x_0,t_0) > u^-(x_0,t_0)$. In the second alternative (x_0,t_0) is in fact a maximum of $u - \varphi$. Since u is a viscosity subsolution of the motion, we have $\varphi_t + \underline{F}(D\varphi, D^2\varphi) + k|D\varphi| \leqslant 0$, what was expected.

Let us now show the second part of the proposition. The initial condition $u \vee v^- \leqslant u_0 \vee v^-|_{t=0}$ is satisfied. Once again, we consier φ smooth and (x_0,t_0) such that $u \vee v^- - \varphi$ has a maximum at (x_0,t_0). Then, either $u(x_0,t_0) \vee v^-(x_0,t_0) = v^-(x_0,t_0)$ and nothing has to be checked, or $u(x_0,t_0) > v^-(x_0,t_0)$. The latter implies that $u(x_0,t_0) > u^-(x_0,t_0)$, so $\varphi_t + \underline{F}(D\varphi, D^2\varphi) + k|D\varphi| \leqslant 0$, what was wanted.

The fact that u is a subsolution of the equation with obstacles u^- and v^+ is obvious using the very definition. □

Lemma 2. *Let $\mathcal{F}$ be a family of subsolutions of (7) and define $U := \sup\{u(x), u \in \mathcal{F}\}$. Then, U^* is a subsolution of (7).*

To prove this lemma, we need the

Proposition 3. *Let v be a upper semicontinuous function, $(x,t) \in \mathbb{R}^n \times \mathbb{R}$ and $(a,p,X) \in J^{2,+}v(x,t)$. Assume there exists a sequence (u_n) of usc functions which satisfy*

i) There exists (x_n,t_n) such that $(x_n,t_n,u_n(x_n,t_n)) \to (x,t,v(x,t))$

ii) $(z_n,s_n) \to (z,s)$ in $\mathbb{R}^n \times \mathbb{R}$ implies $\limsup u_n(z_n,s_n) \leqslant v(z,s)$.

Then, there exists $(\hat{x}_n,\hat{t}_n) \in \mathbb{R}^n \times \mathbb{R}$, $(a_n,p_n,X_n) \in \mathcal{J}^{2,+}u_n(\hat{x}_n,\hat{t}_n)$ such that

$$(\hat{x}_n,\hat{t}_n,u_n(\hat{x}_n,\hat{t}_n),a_n,p_n,X_n) \to (x,t,v(x,t),a,p,X).$$

The proof of the proposition and the lemma can be found in [8], Lemma 4.2 (with obvious changes due to the parabolic situation and obstacles).

In our way to prove that W is the solution of (7), we need to show that it is a subsolution of (7). Lemma 2 shows that it satisfies the viscosity equation with obstacles, because every subsolution u

satisfies $u^- \leqslant u \leqslant u^+$ which imples that $u^- \leqslant W \leqslant u^+$ and, taking the enveloppe (and using the continuity of $u^\pm$),

$$u^- = (u^-)^* \leqslant W^* \leqslant (u^+)^* = u^+.$$

Nevertheless, we know nothing on the initial conditions. Indeed even if for all subsolution, one has $u(x,0) \leqslant g(x)$, which implies $W(x,0) \leqslant g(x)$, taking the semicontinuous envelope could break this inequality. We thus need to build some continuous barriers which will force W^* to remain below g at time zero. More precisely, we build a continuous supersolution w^+ which gets the initial data g. Then, by comparison principle, every subsolution u will satisfy $u \leqslant w^+$ and $W \leqslant w^+$. Taking the enveloppe will yield

$$W^* \leqslant (w^+)^* = w^+$$

which will imply

$$W^*(x,0) \leqslant g(x).$$

Similarly, we build a continuous subsolution w^- which also gets the initial data. By the very definition of W, it gives $W(x,0) \geqslant g(x)$.

For technical reasons, we begin building the solution in the case where $k = 0$.

3.2.1 Construction of barriers in the non forcing case

Let us construct w^-. Without a forcing term, we note that

$$h(x,t) = -(|x|^2 + 2nt + \omega(t))$$

is a classical subsolution of (7) but with neither initial conditions nor obstacles. We define

$$\theta_\xi(r) = \inf\{g(y) \mid |y-\xi|^2 + r \leqslant 0\}$$

The function θ_ξ is bounded, non decreasing, continuous and satisfies $\theta_0(0) = g(0)$ and $\theta_\xi(-|x-\xi|^2 - 2nt) \leqslant g(x)$. As the equation is geometric, $\theta_\xi(-|x-\xi|^2 - 2nt)$ is also a classical subsolution. Let us then define

$$\phi(x,t) = \left(\sup_\xi \theta_\xi(-|x-\xi|^2 - 2nt) \vee u^-(x,t) \right)^*.$$

Since $\theta_\xi(-|x-\xi|^2 - 2nt) \leqslant g(x)$ and g is continuous, it is also true for $\psi(x,t)$. In addition, we can check that

$$\phi(x,t) \geqslant \theta_x(-|x-x|^2 - 2nt) = \theta_x(-2nt) \geqslant g(x) - \omega(\sqrt{2nt}). \tag{10}$$

Hence, $\phi(x,0) = g(x)$. Thanks to Lemma 2, ϕ is a subsolution with $\phi(x,0) \leqslant g(x)$. We conclude this proof defining

$$w^-(x,t) = \phi(x,t) - \omega(t) \vee u^-(x,t).$$

It is clear that w^- is a subsolution with obstacles. Indeed, by definition, $w^- \geqslant u^-$. Moreover, $\phi(x,t) - \omega(t) \leqslant g(x) - \omega(t) \leqslant u^+(x,0) - \omega(t) \leqslant u^+(x,t)$. Proposition 2 concludes the proof.

The other barrier w^+ is obtained similarly.

3.2.2 Perron's method

We have already seen that W is a subsolution of (7). We want to show that it is also a supersolution.

Before finishing the proof of existence, let us notice a useful property of the no-forcing-term case.

Remark 3. If $k(x,t) = 0$, then W is ω-uniformly continuous in space. In time, W is uniformly continuous with modulus $\tilde{\omega} : r \mapsto \max(\omega(r), \omega(\sqrt{2nr}))$. Indeed, the proof is contained in the following lemma.

Lemma 3. *Let $u(x,t)$ be a subsolution of* (7) *with no forcing term (and g, u^-, u^+ ω-uniformly continuous in space and time). Then,*

$$u_{z,\delta}(x,t) = (u(x+z,t+\delta) - \omega(|z|) - \tilde{\omega}(|\delta|)) \vee u^-(x,t)$$

is also a subsolution.

Proof. To begin, we notice that $u(x+z,t+\delta) - \omega(|z|) - \tilde{\omega}(|\delta|) \leqslant u^+(x,t)$.
Now, let φ be a smooth function with $\forall x,t,\ u_{z,\delta}(x,t) \leqslant \varphi(x,t)$ with equality at $(\overline{x},\overline{t})$. Then, either $u_{z,\delta}(\overline{x},\overline{t}) = u^-(\overline{x},\overline{t})$, and nothing has to be done, or $u_{z,\delta}(\overline{x},\overline{t}) > u^-(\overline{x},\overline{t})$. In the second alternative, we have

$$u(\overline{x}+z,\overline{t}+\delta) - \omega(|z|) - \tilde{\omega}(\delta) > u^-(\overline{x},\overline{t}) = u^-(\overline{x}+z,\overline{t}+\delta) + (u^-(\overline{x},\overline{t}) - u^-(\overline{x}+z,\overline{t}+\delta))$$

hence

$$u(\overline{x}+z,\overline{t}+\delta) > u^-(\overline{x}+z,\overline{t}+\delta) + \underbrace{(u^-(\overline{x},\overline{t}) - u^-(\overline{x}+z,\overline{t}+\delta) + \omega(|z|) + \tilde{\omega}(|\delta|))}_{\geqslant 0} \geqslant u^-(\overline{x}+z,\overline{t}+\delta).$$

As u is a subsolution at $(\overline{x}+z,\overline{t}+\delta)$ and $u(x+z,t+\delta) \leqslant \varphi(x,t) + \omega(|z|) + \tilde{\omega}(|\delta|)$ with equality at $(\overline{x}+z,\overline{t}+\delta)$, one can write, with $y = x+z, s = t+\delta$,

$$u(y,t) \leqslant \varphi(y-z,t-\delta) + \omega(|z|) + \tilde{\omega}(|\delta|) =: \phi(y,t),$$

equality at $(\overline{y},\overline{t})$, and deduce that $\phi_t + \underline{F}(D\phi(\overline{y},\overline{t}), D^2\phi(\overline{y},\overline{t})) \leqslant 0$. Since $D\phi(\overline{y},\overline{t}) = D\varphi(\overline{x},\overline{t})$ (so are the second derivatives), we get

$$\varphi_t + \underline{F}(D\varphi(\overline{x},\overline{t}), D^2\varphi(\overline{x},\overline{t})) \leqslant 0,$$

what was expected.

Concerning the initial conditions, we have (we use (10))

$$u(x+z,0+\delta) - \omega(|z|) - \tilde{\omega}(\delta) \leqslant w^+(x+z,\delta) - \omega(|z|) - \tilde{\omega}(\delta) \leqslant g(x+z) - \omega(|z|) \leqslant g(x).$$

□

Applying this lemma to W shows $(x,t) \mapsto W(x+z,t+\delta) - \omega(|z|) - \tilde{\omega}(|\delta|) \vee u^-(x+z,t)$ is a subsolution. By definition of W, one can write

$$W(x,t) \geqslant (W(x+z,t+\delta) - \omega(|z|) - \tilde{\omega}(\delta)) \vee u^-(x+z,t) \geqslant W(x+z,t+\delta) - \omega(|z|) - \tilde{\omega}(\delta)$$

which shows exactly that W is uniformly continuous.

We now want to show that W is in fact a supersolution of (7). We need the following lemma which is adapted from [8], Lemma 4.4.

Lemma 4. *Let u be a subsolution of* (7). *If u_* fails to be a solution of $u_t + \overline{F}(Du, D^2u) + k|Du| \geqslant 0$ in some point $(\hat{x},\hat{t})$ (there exists $(a,p,X) \in \mathcal{J}^{2,-}u_*(\hat{x},\hat{t})$ such that $a + \overline{F}(p,X) + k|p| < 0$), then for all sufficiently small κ, there exists a solution u_κ of $u_t + \underline{F}(Du, D^2u) + k|Du| \leqslant 0$ satisfying $u_\kappa(x,t) \geqslant u(x,t)$, $\sup_{\mathbb{R}^n}(u_\kappa - u) > 0$, $u_\kappa(x,t) \leqslant u^+(x,t)$ and such that u and u_κ coincide for all $|x-\hat{x}|, |t-\hat{t}| \geqslant \kappa$.*

Proof. We can suppose that u_* fails to be a supersolution in $(0,1)$ (this implies in particular $u_*(0,1) < u^+(0,1)$). We get $(a,p,X) \in \mathcal{J}^{2,-}u_*(0,1)$ such that $a + \overline{F}(p,X) + k(0,1)|p| < 0$. We introduce

$$u_{\delta,\gamma}(x) = u_*(0,1) + \delta + \langle p\,,\, x\rangle + a(t-1) + \frac{1}{2}\langle Xx\,,\, x\rangle - \gamma(|x|^2 + t - 1).$$

By upper semicontinuity of $\overline{F}$, $u_{\delta,\gamma}$ is a classical subsolution of $u_t + \overline{F}(Du, D^2u) + k|Du| \leqslant 0$ for γ, δ, r sufficiently small.
Since

$$u(x,t) \geqslant u_*(x,t) \geqslant u_*(0,1) + a(t-1) + \langle p\,,\, x\rangle + \frac{1}{2}\langle Xx\,,\, x\rangle + o(|x|^2) + o(|t-1|),$$

choosing $\delta = \gamma\frac{r^2+r}{8}$, we get $u(x,t) > u_{\delta,\gamma}(x,t)$ for $\frac{r}{2} \leqslant |x|, |t-1| \leqslant r$ and r sufficiently small. Moreover, we can reduce r again to have $u_{\delta,\gamma} \leqslant u^+$ on B_r (Choosing r sufficiently small, one has δ sufficiently small and $u_{\delta,\gamma}(0,1) - u_*(0,1) = \delta < u^+(0,1) - u_*(0,1)$. By continuity, one can find an smaller r such that $u_{\delta,\gamma}(x,t) < u^+(x,t)$ for all $\frac{r}{2} \leqslant |x|, |t-1| \leqslant r$.).
Thanks to Lemma 2, the function

$$\tilde{u}(x,t) = \begin{cases} \max(u(x,t), u_{\delta,\gamma}(x,t)) \text{ if } |x, t-1| < r \\ u(x) \text{ otherwise} \end{cases}$$

is a subsolution of (7) (with initial conditions if r is small enough). □

Now, we saw that W is a subsolution of (7) (in particular, $W \leqslant u^+$). If it is not a supersolution at a point $\hat{x}, \hat{t}$, Lemma 4 provides $W_\kappa \geqslant W$ subsolutions of (7) (with initial condition, even if we have to reduce r again, to make t stay far from zero), which is a contradiction with the definition of W.
Finally, W is the expected solution of (7).

3.2.3 With forcing term

1. We assume at this point only that u^-, u^+ and g are K-Lipschitz. Then, thanks to Remark 3, there exists a K-Lipschitz solution ψ of the non forcing term equation. Let us set $w^-(x,t) = (\psi(x,t) + NKt) \vee u^-(x,t)$. It satisfies, as soon as $w^- > u^-$,

   $$u_t - NK + F(Du, D^2u) = 0, \qquad u(x,0) = g(x).$$

   As a consequence, w^- is a continuous subsolution of (7) (with forcing term) satisfying $w^-(x,0) = g(x)$. It is a barrier as in 3.2.1. We build w^+ in a similar way and apply Perron's method to see that W is a solution.

2. Here, u^+, u^- and g are only ω-uniformly continuous. For all $K > 0$, let $g_K = \min_y g(y) + K|x-y|$, $u_K^+ = \max_y u^+(y) - K|x-y|$ and $u_K^- = \min_y g(y) + K|x-y|$. These three new function are K-Lipschitz and converge uniformly to g, u^+ and u^- when $K \to \infty$. Moreover, as g, u^+, u^- are ω-uniformly continuous, so are they.
   Thanks to the previous point, for every K, there exists a solution u_K of (7) with obstacles u_K^+, u_K^- and with initial data g_K, which is (thanks to the following proposition 4, which is admitted for a little time) uniformly continuous with same moduli on $[0,T]$ for every T. One can define, thanks to Ascoli's theorem

   $$u(x,t) = \lim_n u_{K_n}(x,t).$$

   The function u is continuous. We have to check that it is the solution of the motion with obstacles $u^\pm$.

   It is clear that $u^- \leqslant u \leqslant u^+$. Let φ be a smooth function and $(\hat{x}, \hat{t})$ a maximum point of $u - \varphi$ such that $u(\hat{x}, \hat{t}) - u^-(\hat{x}, \hat{t}) =: \eta > 0$. One can assume that the maximum is strict. We then choose ε such that

   $$\forall (x,t) \in B_\varepsilon(\hat{x}, \hat{t}), \quad u(x,t) - u^-(x,t) \geqslant \frac{3\eta}{4}.$$

Let

$$\delta := \min_{B_\varepsilon} |u - \varphi|.$$

It is positive (if necessary, one can reduce ε again). We choose n_0 such that

$$\forall n \geqslant n_0,\ \|u - u_{K_n}\|_{L^\infty(B_\varepsilon)} \leqslant \max\left(\frac{\eta}{4}, \frac{\delta}{2}\right).$$

Then, for every $n \geqslant n_0$, $u_{K_n} - \varphi$ has a maximum (x_n, t_n) on B_ε reached out of u_K^-. It is easy to show that $(x_n, t_n) \to (\hat{x}, \hat{t})$. Since u_K is a viscosity subsolution, one can write, at (x_n, t_n),

$$\varphi_t + \underline{F}(D\varphi, D^2\varphi) + k|D\varphi| \leqslant 0.$$

By smoothness of φ and semicontinuity of $\underline{F}$, we get the same inequality at $(\hat{x}, \hat{t})$.

We prove that u is a supersolution using the same arguments.

Let us conclude this section by an estimation of the solution's regularity, which is essentially [12], Lemma 2.15 (except that the solution here is only uniformly continuous).

Proposition 4. *Let u be the unique solution of (7). Then u is uniformly continuous in space. moreover, one as*

$$\forall x, y, t, \quad |u(x,t) - u(y,t)| \leqslant \omega(e^{Lt}|x - y|).$$

Proof. First, it is well known that one can choose ω to be continuous and nondecreasing. Since u and v are bounded by N, $\omega \wedge 2N$ is a modulus too. In the following, we use this new modulus, sill denoted by ω.

Then, let ρ_n be a $\mathcal{C}^\infty$ nondecreasing function on $[0, \infty[$ such that $0 \leqslant \rho_n - \omega$, for all $r > n + 1$, $\rho_n(r) = 2N + 1$, and for all $r \in [0, n], \rho_n(r) - \omega(r) \leqslant \frac{1}{n}$. Then, let us define

$$\omega_n(r) = \rho_n + \frac{r}{n^2}.$$

It's clear that $\omega_n(r) \underset{n\to\infty}{\longrightarrow} \omega(r)$. Moreover, for a fixed n, $\omega_n'(r)$ is bounded and stay far from zero. In what follows, we work with ω_n.

We will proceed as in Proposition 1. Let $\phi(x, y, t) = \omega_n(e^{Lt}|x - y|)$. We will show by contradiction that $u(x,t) - u(y,t) \leqslant \phi(x, y, t)$. Assume that

$$M := \sup_{(x,y,t)\in\mathbb{R}^n\times\mathbb{R}^n\times[0,T)} u(x,t) - u(y,t) - \phi(x,y,t) > 0.$$

As before, we introduce

$$\tilde{M} = \sup_{x,y,t\leqslant T} u(x,t) - u(y,t) - \phi(x,y,t) - \frac{\alpha}{2}(|x|^2 + |y|^2) - \frac{\gamma}{T - t}.$$

For sufficiently small γ, α, $\tilde{M}$ remains positive and is attained (at $\overline{x}, \overline{y}, \overline{t} < T$). As g is ω-uniformly continuous, $\overline{t} > 0$. Moreover, it is clear that $\overline{x} \neq \overline{y}$.

By assumption, $u^-(\overline{x}, \overline{t}) \leqslant u^-(\overline{y}, \overline{t}) + \omega(|\overline{x} - \overline{y}|) \leqslant u^-(\overline{y}, \overline{t}) + \omega_n(|\overline{x} - \overline{y}|) \leqslant u(\overline{y}, \overline{t}) - \phi(\overline{x}, \overline{y}, \overline{t})$ so $0 \leqslant \tilde{M} < u^(\hat{x}, \hat{t}) - u(\hat{y}, \hat{t}) - \phi(\hat{x}, \hat{y}, \hat{t})$ forces $u(\overline{x}, \overline{t}) > u^-(\overline{x}, \overline{t})$. Similarly, $u(\overline{y}, \overline{t}) < u^+(\overline{y}, \overline{t})$.

Applying Ishii's lemma ([8], Th. 8.3) to $\tilde{u}(x,t) = u(x,t) - \frac{\alpha}{2}|x|^2$ and $\tilde{v}(y,t) = u(y,t) + \frac{\alpha}{2}|y|^2$ where

$$\overline{p} = D_x\phi = \frac{\overline{x} - \overline{y}}{|\overline{x} - \overline{y}|} e^{L\overline{t}} \omega_n'(e^{L\overline{t}}|\overline{x} - \overline{y}|) = -D_y\phi \neq 0,$$

$$Z = D_x^2\phi = \frac{e^{L\bar{t}}}{|\overline{x}-\overline{y}|}\omega_n'(e^{L\bar{t}}|\overline{x}-\overline{y}|)I + \frac{(\overline{x}-\overline{y})\otimes(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^3}e^{Lt}\omega_n'(e^{L\bar{t}}|\overline{x}-\overline{y}|) + \frac{(\overline{x}-\overline{y})\otimes(\overline{x}-\overline{y})}{|\overline{x}-\overline{y}|^2}e^{2L\bar{t}}\omega_n''(e^{L\bar{t}}|\overline{x}-\overline{y}|).$$

and

$$A = D^2\phi = \begin{bmatrix} Z & -Z \\ -Z & Z \end{bmatrix},$$

we get the following. For all β such that $\beta A < I$, there exists $\tau_1, \tau_2 \in \mathbb{R}$, $X, Y \in \mathcal{S}_n$ such that

$$\tau_1 - \tau_2 = \frac{\gamma}{(T-t)^2} + Le^{L\bar{t}}|\overline{x}-\overline{y}|\omega_n'(e^{Lt}|\overline{x}-\overline{y}|),$$

$$(\tau_1, \overline{p}+\alpha\overline{x}, X+\alpha I) \in \overline{\mathcal{J}}^{2,+}u(\overline{x},\bar{t}),$$
$$(\tau_2, \overline{p}-\alpha\overline{y}, Y-\alpha I) \in \overline{\mathcal{J}}^{2,-}u(\overline{y},\bar{t}),$$
$$\frac{-1}{\beta}\begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \leqslant \begin{bmatrix} X & 0 \\ 0 & -Y \end{bmatrix} \leqslant (I-\beta A)^{-1}A.$$

In particular, the last equation provides $X \leqslant Y$.

As u is a subsolution and a supersolution, one has

$$\tau_1 + k(\overline{x},\bar{t})|\overline{p}+\alpha\overline{x}| + \underline{F}(\overline{p}+\alpha\overline{x}, X+\alpha I) \leqslant 0, \tag{11}$$
$$\tau_2 - k(\overline{y},\bar{t})|\overline{p}-\alpha\overline{y}| + \overline{F}(\overline{p}-\alpha\overline{y}, Y-\alpha I) \geqslant 0.$$

$X \leqslant Y$ in the last equation gives

$$-\tau_2 + k(\overline{y},\bar{t})|\overline{p}-\alpha\overline{y}| - \overline{F}(\overline{p}-\alpha\overline{y}, X-\alpha I) \leqslant 0. \tag{12}$$

Adding (12) to (11) leads to

$$\frac{\gamma}{(T-\bar{t})^2} + Le^{L\bar{t}}|\overline{x}-\overline{y}|\omega_n'(e^{L\bar{t}}|\overline{x}-\overline{y}|) - k(\overline{x},\bar{t})|\overline{p}+\alpha\overline{x}| + k(\overline{y},\bar{t})|\overline{p}-\alpha\overline{y}| + \underline{F}(\overline{p}+\alpha\overline{x}, X+\alpha I) - \overline{F}(\overline{p}-\alpha\overline{y}, X-\alpha I) \leqslant 0. \tag{13}$$

Notice that

$$Le^{L\bar{t}}|\overline{x}-\overline{y}|\omega_n'(e^{L\bar{t}}|\overline{x}-\overline{y}|) - k(\overline{x},\bar{t})|\overline{p}| + k(\overline{y},\bar{t})|\overline{p}| \geqslant Le^{L\bar{t}}|\overline{x}-\overline{y}|\omega_n'(e^{L\bar{t}}|\overline{x}-\overline{y}|) - L|\overline{x}-\overline{y}|e^{L\bar{t}}\omega_n'(e^{L\bar{t}}|\overline{x}-\overline{y}|) \geqslant 0. \tag{14}$$

Then, (13) becomes

$$\frac{\gamma}{(T-\bar{t})^2} + (|\overline{p}| - |\overline{p}+\alpha\overline{x}|)\,k(\overline{x},\bar{t}) - (|\overline{p}| - |\overline{p}-\alpha\overline{y}|)\,k(\overline{y},\bar{t}) + \underline{F}(\overline{p}+\alpha\overline{x}, X+\alpha I) - \overline{F}(\overline{p}-\alpha\overline{y}, X-\alpha I) \leqslant 0.$$

Let α go to zero. $\overline{p}$ and X are bounded : one assume they converge and still denote by $\overline{p}, X$ their limit. As $|\overline{p}| \geqslant \frac{1}{n^2}$ (ρ_n is nondecrasing), $\underline{F}(\overline{p}, H) = \overline{F}(\overline{p}, H)$ for all $H \in \mathcal{S}_n$. Moreover, $\alpha\overline{x}, \alpha\overline{y} \to 0$ and k is bounded, hence

$$\frac{\gamma}{(T-\bar{t})^2} \leqslant 0,$$

which is a contradiction. So

$$u(x,t) - u(y,t) \leqslant \omega_n(e^{Lt}|x-y|).$$

It remains to let n go to $+\infty$ to conclude. □

3.3 The motion is geometric

In all this subsection, a solution u of the motion with initial data u_0 and obstacles u^- and u^+ will be denoted by $u = [u_0, u^-, u^+]$. The corresponding equation will be denoted by (u_0, u^-, u^+).
To agree with the geometric motion, we have to check that the zero level-set of the solution depends only on the zero level sets of the initial condition u_0 and of the obstacles u^+ and u^-.

Lemma 5. *Let $u = [u_0, u^-, u^+]$ and $v = [v_0, v^-, v^+]$. We assume that $u_0 \leqslant v_0$, $u^- \leqslant v^-$ and $u^+ \leqslant v^+$. Then, $u \leqslant v$.*

Proof. This proposition is obvious thanks to Remark 1. Indeed, u is a subsolution of $[u_0, u^-, u^+]$ so is a subsolution of $[u_0, u^-, v^+]$ whereas v is a supersolution of $[v_0, v^-, v^+]$, so of $[u_0, u^-, v^+]$. The comparison principle implies

$$u \leqslant v.$$

□

Proposition 5. *Let u be the solution of (5) with obstacles u^+ and u^-, and let ϕ be a continuous nondecreasing function $[-N, N] \to \mathbb{R}$ such that $\{\phi = 0\} = \{0\}$. Then, the solutions*

$$[u_0 \wedge (\phi(u^+) \vee u^-)|_{t=0}, u^-, \phi(u^+) \vee u^-],$$

$$(u_0 \vee (\phi(u^-) \wedge u^+)|_{t=0}, \phi(u^-) \vee u^+, u^+]$$

$$\text{and} \quad [(\phi(u_0) \wedge u^+|_{t=0}) \vee u^-|_{t=0}, u^-, u^+]$$

have the same zero level set as u.

Proof. We will prove that

$$u_\phi = [u_0 \wedge (\phi(u^+) \vee u^-)|_{t=0}, u^-, \phi(u^+) \vee u^-]$$

has the same zero set as u. All the other equalities can be prove with a similar strategy.

We begin the proof assuming $\phi(x) \geqslant x$. Then, $u_\phi = [u_0, u^-, \phi(u^+)]$.
First, let us notice that the classical invariance proves immediately that $\phi(u)$ is the solution $[\phi(u_0), \phi(u^-), \phi(u^+)]$. In addition, thanks to Lemma 5 $u_\phi \geqslant u$ and $u_\phi \leqslant \phi(u)$. As a result, since $\{\phi(u) = 0\} = \{u = 0\}$, we conclude that $\{u = 0\} = \{u_\phi = 0\}$, what was expected.

Assume now that $\phi(x) \leqslant x$. The same arguments shows that $\phi(u) \leqslant u_\phi \leqslant u$, which leads to the same conclusion.

To conclude the proof for a general ϕ, just introduce $f(x) = \min(x, \phi(x))$ and $g(x) = \max(x, \phi(x))$ and notice that since ϕ is nondecreasing, $\phi = f \circ g$. So,

$$\{u = 0\} = \{u_f = 0\} = \{(u_g)_f = 0\} = \{u_{f\circ g} = 0\} = \{u_\phi = 0\}.$$

□

Now, to be able to define a real geometrical evolution, we want a more general independence, which is contained in the following

Theorem 2. *Let $u = [u_0, u^-, u^+]$. Then, $\{u = 0\} = \{v = 0\}$ with $v = [v_0, v^-, v^+]$ under the (only) assumptions that*

$$\{u_0 = 0\} = \{v_0 = 0\}, \quad \{u^- = 0\} = \{v^- = 0\} \quad \text{and} \quad \{u^+ = 0\} = \{v^+ = 0\}.$$

Proof. This proof is based on the independence with no obstacles which is proved in [11], Theorem 5.1. We assume first that $u^- = v^-$ and $u^+ = v^+$. As in [11], we define

$$\forall k \in \mathbb{Z} \setminus \{0\}, \quad E_k = \left\{ x \in \mathbb{R}^n \,\middle|\, u_0 \geqslant \frac{1}{k} \right\}$$

and

$$a_k = \max_{\mathbb{R}^n \setminus E_k} v_0.$$

It is easy to see that

$$\forall k > 0,\ a_1 \geqslant a_2 \geqslant \cdots \to 0 \quad \text{and} \quad a_{-1} \leqslant a_{-2} \leqslant \cdots \to 0.$$

Let us introduce $\phi : [-N, N] \to [-N, N]$, piecewise affine, by

$$\phi(\pm N) = \pm N, \quad \phi\left(\frac{1}{k}\right) = a_k \quad \text{and} \quad \phi(0) = 0.$$

Then, by definition, $\phi(u_0) \geqslant v_0$, $\{\phi = 0\} = \{0\}$ and ϕ is nondecreasing continuous. Thanks to Proposition 5, the solution $u_\phi := [\phi(u_0) \wedge u^+, u^-, u^+]$ has the same zero level-set as u, and is bigger than v by comparison principle. Hence

$$\{v \geqslant 0\} \subset \{u_\phi \geqslant 0\} = \{u \geqslant 0\}.$$

We prove the reverse inclusion switching u_0 and v_0.

Now, we assume that $u_0 = v_0$, $u^- = v^-$ and $u^+ \leqslant v^+$. Then, by comparison principle, $u \leqslant v$. We have just seen that there exists $\phi : [-N, N] \to [-N, N]$ nondecreasing continuous such that $\phi(u^+) \geqslant v^+$ and $\{\phi = 0\} = \{0\}$. Let $u_\phi = [u_0, u^-, \phi(u^+) \vee u^-]$. We saw that u_ϕ has the same zero set as u. In addition, by comparison, $u_\phi \geqslant v$. As a matter of fact,

$$\{u = 0\} = \{v = 0\} = \{u_\phi = 0\}.$$

If we drop the assumption $u^+ \leqslant v^+$, notice that $[u_0, u^-, u^+]$ and $[u_0, u^-, u^+ \wedge v^+]$ have the same zero level-set, so do $[u_0, u^-, v^+]$ and $[u_0, u^-, u^+ \wedge v^+]$. Hence $[u_0, u^-, u^+]$ and $[u_0, u^-, v^+]$ have the same zero level-set.

Of course, changing only u^- leads to the same result.

To show the general case saying that $[u_0, u^-, u^+]$ and $[u_0, u^-, v^+]$ have the same zero level-set, so do $[u_0, u^-, v^+]$ and $[u_0, v^-, v^+]$, and $[u_0, v^-, v^+]$ and $[v_0, v^-, v^+]$, and the first and the last ones. □

3.4 Obstacles create fattening

Although the fattening phenomenon may already occur without any obstacle (see [3] for examples and [2, 4] for a more general discussion), obstacles will easily generate fattening whereas the free evolution is smooth. Consider A a set of three points in $\mathbb{R}^2$ spanning an equilateral triangle and S a circle enclosing it, centered on the triangle's center. Let $u^- = -1$, $u^+ = d(\cdot, A)$, $u_0 = d(\cdot, S)$ (d is the signed distance) and $F(Du, D^2u) = -|\nabla u| \ \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

It is possible to show (see next section) that the level sets $\{u(\cdot, t) \leqslant \alpha\}$ are minimizing hulls, hence are convex. So, the level set $\{u \leqslant 0\}$ contains the equilateral triangle. On the other hand, the level sets $\{u \leqslant -\delta\}$ behave as if there were no obstacles at all (in Proposition 2, one can take $u^+ \equiv 1$ which has the same $-\delta$-set as $d(\cdot, A)$), so they disappear in finite time. As a result, $u = 0$ in the whole triangle, and $\{u = 0\}$ develops non empty interior.

4 Comparison with a variational discrete scheme and long-time behavior

In this section, we study the behavior of the mean curvature flow only[2] with no forcing term in large times. In particular, we show that for relevant initial conditions, the flow has a limit.

In order to get some monotonicity properties of the flow, we will link our approach to a variational discrete flow built in [16] and [1]. Starting from a set E_0 and an obstacle Ω (which corresponds, in our framework, to $E_0 = \{u_0 \leqslant \alpha\}$ and $\Omega = \{u^+ \leqslant \alpha\}$ for every α-level set), these two papers introduce the following minimizing scheme with step h:

$$E_h(t) = T_h^{[t/h]}(E_0)$$

with

$$T_h(E) = \arg\min_{\Omega \subset F} \left[\mathrm{Per}(F) + \frac{1}{h} \int_{F \Delta E} |d_E| \right]. \tag{15}$$

Spadaro introduces the notion of minimizing hull: E is said to be a minimizing hull if $|\partial E| = 0$ (this is not assumed in the definition in [16], but is assumed stating minimizing hull properties) and

$$\mathrm{Per}(E) \leqslant \mathrm{Per}(F), \quad \forall F \supset E \text{ with } F \setminus E \text{ compact.}$$

He shows that if E is a minimizing hull with measure-zero boundary, then for every h, one can define a maximal minimizer in (15), still denoted in what follows by $T_h(E)$ (for every other solution F of (15), one has $F \subset T_h(E)$). Spadaro proves that $T_h(E) \subset E$ and $T_h(E)$ is still a minimizing hull (the measure of the boundary remains zero thanks to the classical regularity of minimizers (see for example Appendix B in [16])). Moreover, if F satisfies the same assumptions and $F \subset E$, then $T_h(F) \subset T_h(E)$.

We end the general properties of this discrete flow by the

Remark 4. Let E be a minimizing hull and $h > \tilde{h}$. Then, $T_h(E) \subset T_{\tilde{h}}(E)$ almost everywhere.

Indeed, Let $F := T_h(E)$ and $\tilde{F} := T_{\tilde{h}}E$. Since E is a minimizing hull, $F, \tilde{F} \subset E$ so $d_E \leqslant 0$ on $F \cup \tilde{F}$. Using the very definition of F and $\tilde{F}$, one can write

$$\mathrm{Per}(F \cap \tilde{F}) + \frac{1}{h} \int_{F \cap \tilde{F}} d_E \geqslant \mathrm{Per}\, F + \frac{1}{h} \int_F d_E$$

$$\mathrm{Per}(F \cup \tilde{F}) + \frac{1}{\tilde{h}} \int_{F \cup \tilde{F}} d_E \geqslant \mathrm{Per}\, \tilde{F} + \frac{1}{\tilde{h}} \int_{\tilde{F}} d_E.$$

Summing, we get

$$\mathrm{Per}(F \cap \tilde{F}) + \mathrm{Per}(F \cup \tilde{F}) + \frac{1}{h} \int_{F \cap \tilde{F}} d_E + \frac{1}{\tilde{h}} \int_{F \cup \tilde{F}} d_E \geqslant \mathrm{Per}\, F + \mathrm{Per}\, \tilde{F} + \frac{1}{h} \int_F d_E + \frac{1}{\tilde{h}} \int_{\tilde{F}} d_E.$$

Since $\mathrm{Per}(F \cap \tilde{F}) + \mathrm{Per}(F \cup \tilde{F}) \leqslant \mathrm{Per}\, F + \mathrm{Per}\, \tilde{F}$, one has

$$\frac{1}{h} \int_{F \cap \tilde{F}} d_E + \frac{1}{\tilde{h}} \int_{F \cup \tilde{F}} d_E \geqslant \frac{1}{h} \int_F d_E + \frac{1}{\tilde{h}} \int_{\tilde{F}} d_E,$$

which means

$$\frac{1}{\tilde{h}} \int_{F \setminus \tilde{F}} d_E \geqslant \frac{1}{h} \int_{F \setminus \tilde{F}} d_E,$$

[2] That means $u_t = |\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right)$.

hence

$$\int_{F\setminus\tilde{F}} d_E\left(\frac{1}{\tilde{h}}-\frac{1}{h}\right)\geqslant 0.$$

Then, since $|\partial E|=0$, $|F\setminus\tilde{F}|=0$.

To pass to the limit in h, we want to control the motion speed. First, we compare the constrained and the free motions.

Proposition 6. *Let E_0 be a minimizing hull containing Ω. Let E_f be the free evolution of E_0 (E_f solves (15) with no constraint) and E_c the regular evolution (E_c solves (15) and is maximal). Then, $E_f\cup\Omega\subset E_c$.*

Proof. Using the definition of E_f and E_c, one can write

$$\mathrm{Per}(E_f\cap E_c)+\int_{E_f\cap E_c}\frac{d_{E_0}}{h}\geqslant \mathrm{Per}(E_f)+\int_{E_f}\frac{d_{E_0}}{h}$$

$$\mathrm{Per}(E_f\cup E_c)+\int_{E_f\cup E_c}\frac{d_{E_0}}{h}\geqslant \mathrm{Per}(E_c)+\int_{E_c}\frac{d_{E_0}}{h}.$$

Summing and using $\mathrm{Per}(E\cap F)+\mathrm{Per}(E\cup F)\leqslant \mathrm{Per}\,E+\mathrm{Per}\,F$, we get

$$\int_{E_c\cap E_f}\frac{d_{E_0}}{h}+\int_{E_c\cup E_f}\frac{d_{E_0}}{h}\geqslant\int_{E_f}\frac{d_{E_0}}{h}+\int_{E_c}\frac{d_{E_0}}{h},$$

which is an equality. We conclude that all the inequalities above are equalities. In particular,

$$\mathrm{Per}(E_f\cup E_c)+\int_{E_f\cup E_c}\frac{d_{E_0}}{h}=\mathrm{Per}(E_c)+\int_{E_c}\frac{d_{E_0}}{h},$$

which shows that $E_f\cup E_c$ is a minimizer of (15). Since E_c is a maximal minimizer, one has $E_f\subset E_c$.

One can also notice that by definition, $\Omega\subset E_c$ so $E_f\cup\Omega\subset E_c$. □

Then, it is easy to see that

- A ball $B_R(x_0)$ is a minimizing hull,
- For $h\leqslant\frac{R^2}{4n}$ The free evolution of $B_R(x_0)$ is $B_r(x_0)$ with $r=\frac{R+\sqrt{R^2-4nh}}{2}$.

Thanks to the monotonicity of the flow starting from minimizing hulls ([16], Lemma 3.5) and the last proposition, one can conclude that the evolution E_h of a minimizing hull E_0 contains the free evolution of every ball inside E_0.

More generally, for every function $u_0:\mathbb{R}^n\to[-1,1]$ whose level sets are minimizing hulls (we always assume this condition in the following) and an obstacle $v:\mathbb{R}^n\to[-1,1]$ with $u_0\geqslant v$, one want to define an evolution $u_h:\mathbb{R}^n\times[0,T[\to[-1,1]$ by posing for all $s\in[-1,1]$, $E_s:=\{u_0\leqslant s\}$ and

$$\{u_h(t)\leqslant s\}=(E_s)_h(t).$$

This is well defined (in particular, $\{u_h(t)\leqslant s\}\subset\{u_h(t)\leqslant s'\}$ if $s\leqslant s'$) thanks to the

Proposition 7. *Let $\Omega^1\subset\Omega^2$ be two obstacles and $E^1\subset E^2$ be two minimizing hulls containing respectively Ω^1 and Ω^2. Then, $E^1_h\subset E^2_h$.*

Proof. Use the definition to write

$$\mathrm{Per}(E_h^1 \cap E_h^2) + \int_{E_h^1 \cap E_h^2} \frac{d_{E^1}}{h} \geqslant \mathrm{Per}(E_h^1) + \int_{E_h^1} \frac{d_{E^1}}{h},$$

$$\mathrm{Per}(E_h^1 \cup E_h^2) + \int_{E_h^1 \cup E_h^2} \frac{d_{E^2}}{h} \geqslant \mathrm{Per}(E_h^2) + \int_{E_h^2} \frac{d_{E^2}}{h},$$

Summing and simplifying, we get

$$\int_{E_h^1 \cap E_h^2} \frac{d_{E^1}}{h} + \int_{E_h^1 \cup E_h^2} \frac{d_{E^2}}{h} \geqslant \int_{E_1^h} \frac{d_{E^1}}{h} + \int_{E_2^h} \frac{d_{E^2}}{h}$$

which can be read

$$\int_{E_h^1 \setminus E_h^2} \frac{d_{E^2}}{h} \geqslant \int_{E_h^1 \setminus E_h^2} \frac{d_{E^1}}{h}.$$

Since $E_1 \subset E_2$, one has $d_{E_2} \leqslant d_{E_1}$ which shows that the last inequality is in fact an equality, showing as above that $E_h^1 \subset E_h^2$. □

One can easily notice the two following points:

- The scheme is invariant by translation (if $\tilde{u}(x) := u(x+z)$ for some $z \in \mathbb{R}^n$, one has $\tilde{u}_h(x) = u_h(x+z)$ where $\tilde{u}_h$ is computed using the obstacle $v(\cdot + z)$).
- Proposition 7 gives the following monotonicity. If $u \leqslant \tilde{u}$ are two functions whose level sets are minimizing hulls, $v \geqslant \tilde{v}$ two obstacle functions, then $u_h \leqslant \tilde{u}_h$.

Now, we want to pass to the limit in h in the construction above. We will use the

Proposition 8. *If u_0 and v are uniformly continuous (with modulus ω), then the family (u_h) is equicontinuous in space (with modulus ω) and time.*

Proof. The arguments are standard and use the translation invariance of the scheme as well as the comparison principle.

- Space continuity. The space continuity is easy to deduce. By continuity and translation invariance, $\tilde{u}_0(x) := u_0(x+z) \leqslant u_0(x) + \omega(|z|)$ and $\tilde{v} = v(\cdot + z) \leqslant v + \omega(|z|)$ so $\tilde{u}_h \leqslant u_h + \omega(|z|)$, which was expected
- Time continuity. Let $(x,t) \in \mathbb{R}^n \times \mathbb{R}^+$. Let $r > 0$. By uniform continuity, on $B_r(x)$, $u_h(t) \leqslant u_h(x,t) + \omega(r)$, which means that $A^r := \{u_h \leqslant u_h(x,t) + \omega(r)\}$ contains $B_r(x_0)$. Thanks to Proposition 6, the time evolution of A^r contains the free evolution of $B_r(x_0)$, as long as the latter exists. That means $u_h(x, t+s) \leqslant u_h(x,t) + \omega(r)$ for $s \leqslant T_r$, extinction time of $B_r(x_0)$. It is easy to see that this time is controlled, for a sufficiently small h, by $\frac{r^2}{\sqrt{16h}}$. We proved that for h small enough, u_h is continuous in time with modulus $\tilde{\omega}(T_r) \leqslant \omega(r)$.

□

Corollary 1. *Up to a subsequence, the collection $(u_h)_h$ has a limit which is uniformly continuous in space and time.*

Let us denote it by u (we will see that this limit does not depend on the subsequence).
We are now able to show the main proposition of this section.

Proposition 9. *The function u is a viscosity solution of* (5).

Proof. We have just seen that u is uniformly continuous in space and time. In addition, $u \geqslant v$ by construction and the initial conditions are satisfied. We only have to check the fourth point of the definition (we only deal with the subsolution thing, the supersolution one can be treated similarly). Let $(x,t) \in \mathbb{R}^n$. Either $u(x,t) = v(x,t)$ and nothing has to be done, or $u(x,t) > v(x,t)$. In this case, one can directly apply [6], Th. 4.6 or, with a setting closer to ours, [17], Th 3.6.1. See also [10]. □

4.1 The limit is locally minimal

For this section, we deal only with mean curvature motion without forcing term. Thanks to Proposition 9, if u_0 has minimizing hull level sets, so does $u(\cdot,t)$. In addition, u is nondecreasing in time (this is true for u_h). As u is uniformly equicontinuous on each compact set, letting t go to $+\infty$ we have a locally uniform convergence to a limit u_∞ which is a viscosity solution of

$$|\nabla u| \operatorname{div}\left(\frac{\nabla u}{|\nabla u|}\right) = 0$$

with obstacles u^+, u^-, thanks to classical theory of viscosity solutions.

Thanks to [14], Theorem 3.10, one has the following result.

Proposition 10. *Let $s \in u_\infty(\mathbb{R}^n)$ such that $\mathcal{H}^{n-1}(\{u = s\}) < \infty$. Moreover, let $\Omega = \{u^+ \geqslant s\} \cup \{u^- \leqslant s\}$. Then, there exists a relatively open set $U \subset u^{-1}(s)$ with $H^{n-8-\alpha}(u^{-1}(s) \setminus U) = 0$ for all $\alpha > 0$, such that $u^{-1}(s) \setminus \Omega$ is an analytic minimal surface in a neighborhood of each point of U. Moreover, it is stable and stationnary in the varifold sense (classically on U).*

Note in particular that non empty interior can occur for only countable many s.

4.2 Comparison with mean convex hull

In [16], E. Spadaro is interested in the long time behavior of the discrete scheme (15) but with a step h which remains fixed. In this short subsection, we compare his limit to ours. The dimension of the ambiant space n is assumed to be less or equal to 7. Here is the theorem he gets:

Theorem 3 (Spadaro, [16])**.** *Let $\Omega \subset \mathbb{R}^n$, $n \leqslant 7$, be a $\mathcal{C}^{1,1}$ closed set and $E_0 \supset \Omega$ a minimizing hull. Then, for a fixed h, the iterative scheme* (15) *converges to some limit E^h_∞. In addition, the E^h_∞ converge monotonically to some E_∞ which satisfies*

- *E_∞ is $\mathcal{C}^{1,1}$,*
- *E_∞ is a minimizing hull,*
- *$\partial E_\infty \setminus \Omega$ is a (smooth) minimal surface.*

In addition, Spadaro uses this construction starting from E_0 with obstacles $\Omega_\varepsilon := \{x \in \mathbb{R}^n \,|\, d(x,\Omega) \leqslant \varepsilon\}$ to build a limit E^ε_∞.

Theorem 4 (Spadaro)**.** *The set*

$$\Omega^{mc} := \bigcap_{\varepsilon > 0} E^\varepsilon_\infty$$

is the mean convex hull of Ω. That means

$$\Omega^{mc} = \bigcap_{\Omega \subset \Theta \in \mathcal{A}} \Theta$$

where $\mathcal{A}$ is the family of $\Theta \in \mathbb{R}^n$ such that for every minimal surface Σ such that $\partial\Sigma \subset \Theta$, we have $\Sigma \subset \Theta$.

Let u_0 be any function with zero level set equal to S, $u^+ = d(\cdot, \Omega)$ and $u^- = -1$. Since Spadaro's work is in low dimension, the open set U in Proposition 10 is the whole $u^{-1}(s)$. Let us assume that $u^{-1}(0)$ does not fatten. Hence, $\partial\{u \leqslant 0\} = \{u = 0\}$ and $\{u = 0\} \setminus \Omega$ is a minimal hypersurface with boundary in Ω. Using the very definition of the global barrier, we deduce that $\{u \leqslant 0\} \subset \Omega^{mc}$.

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