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CONCURRENT HYLAND-ONG GAMES

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ABSTRACT. We build a cartesian closed category, called Cho, based on event structures. It allows an interpretation of higher-order stateful concurrent programs that is refined and precise: on the one hand it is conservative with respect to standard Hyland-Ong games when interpreting purely functional programs as innocent strategies, while on the other hand it is much more expressive. The interpretation of programs constructs compositionally a representation of their execution that exhibits causal dependencies and remembers the points of non-deterministic branching.

The construction is in two stages. First, we build a compact closed category TCG. It is a variant of Rideau and Winskel's category CG, with the difference that games and strategies in TCG are equipped with *symmetry* to express that certain events are essentially the same. This is analogous to the underlying category of AJM games enriching simple games with an equivalence relations on plays. Building on this category, we construct the cartesian closed category Cho as having as objects the standard arenas of Hyland-Ong games, with strategies, represented by certain events structures, playing on games with symmetry obtained as expanded forms of these arenas.

To illustrate and give an operational light on these constructions, we interpret (a close variant of) Idealized Parallel Algol in Cho.

1. INTRODUCTION

In game semantics, *computation* is represented within a two-player game played between the *program* and its *execution environment* – the program is often called *Player* and the execution environment *Opponent*. The two players make *moves* corresponding to computational events: the program calling an external function is a Player move, and this function returning is an Opponent move. Originally motivated by the very foundational quest of understanding higher-order sequentiality [HO00, AJM00a], game semantics developed into a rich subject, with a wide scope spanning logical aspects of computation, through the

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Curry-Howard correspondence, to the conception of new decision algorithms for equivalence or verification problems on higher-order programs.

Game semantics often plays one of the two following roles in the literature.

(1) *A syntax-free, compositional operational semantics.* The strategy interpreting a program is a syntax-free object – in essence it is a representation of the *behaviour* of the program, with no information as to *how* this behaviour is written down in the syntax. In particular, it abstracts cleanly from the bureaucratic aspects of the syntax and reduction of the language under examination. It is by nature *compositional*, because the strategy for a term is calculated by induction on its syntax tree, following the methodology of denotational semantics. In particular, the application of one term to another is interpreted as the composition of the corresponding strategies.

A compositional fine-grained description of the execution of higher-order programs is a useful tool – for instance, it provides methodologies to study problems such as termination or complexity in the abstract, in a syntax-free manner [CH10, Cla15]. Such a representation is also key to further program analysis. It can provide an invariant for compilation [Ghi07, Sch14], or a compositional model construction on which to perform model-checking [AGMO04]. Even in the purely functional case, it was recently proposed by Jones *et al* [BJ16] as a convenient closure-free way to compute the partial evaluation of a term.

Although historically the focus on game semantics has often been on the second role mentioned below, a good part of its recent developments have been indeed as a syntax-free, compositional operational semantics. In this direction, it is to be compared with various similar frameworks. *Normal form bisimulations* [LL07] are close relatives, as recently emphasized by Levy and Staton [LS14]. Recent developments of the *geometry of interaction* also pursue similar methods and objectives [HMH14, LFVY15]. Finally, Hirschowitz and collaborators have provided a very general framework in which one can give syntax-free descriptions of different kinds of programs [HP12, EHS15].

(2) *An observational classification of effects.* Beyond the use of game semantics as an operational semantics, the separation of the *observable behaviour* of a term from its syntax allowed researchers to study computational features of programs in terms of the observations that they permitted. One of the most acclaimed achievements of early game semantics is the identification of conditions on strategies (visibility, innocence, well-bracketing) in the context of Hyland-Ong games, that characterize, not syntactically but *observationally*, the behaviour of programs having access to certain effects. Indeed, *innocent well-bracketed* strategies are essentially purely functional programs [HO00]: relaxing innocence to *visibility* captures the use of ground state [AM96] while removing well-bracketing captures control operators [Lai97]. Finally, removing visibility captures terms that have access to higher-order references [AHM98]. This is known as the *semantic cube* or *Abramsky cube*.

Each combination of these conditions corresponds to a certain programming language, for which the strategies have exactly the same observation power as programs. In many cases the resulting model is *fully abstract*, without the need for a quotient. This allowed researchers in game semantics, starting with Ghica and McCusker [GM03], to give decision procedures for observational equivalence of programs in certain programming languages where the fully abstract games model is algorithmically presented [MW08, CHMO15].

Despite this impressive flexibility, each game semantics model comes with its limitations. The notion of *play*, which is at the heart of any game semantics model, specifies the observational power of the execution environment. Whereas the capabilities of Player can be adjusted to a certain extent via conditions on strategies, this cannot be pushed further

than what is wired into the model construction. For instance in Hyland and Ong’s original model, plays are well-bracketed and visible for both players – it follows that we only observe parts of programs visible to an evaluation context with no access to higher-order state and control operators. Clearly that can be solved; for instance by allowing more general plays as in [AHM98]. Then again, the whole model has to be rethought if one wishes to allow Opponent to be concurrent. The same line of thought led Ghica and Tzevelekos to define *system-level game semantics* [GT12], in an effort to take as few assumptions as possible as to the power of the execution environment. We advocate here another option, namely *causal* models.

Causality. In *causal* game semantics, a program is not represented by an enumeration of all its possible interactions with an Opponent of observational strength wired into the model. Instead, it is represented by an abstract structure displaying information about the *causal choices* behind the program’s actions. On the one hand, this means that the model is more intensional and most likely further away from quotient-free full abstraction results. On the other, the representation makes absolutely no assumption as to the computational features available to the execution environment. This makes the model more modular, and a finer representation: from the causal game semantics of a program, it is always possible to recover – in an effective manner – the set of plays corresponding to an observation by a certain kind of environment. The causal representation has other advantages. For instance, as long advocated by Melliès, it allows us to get rid of artificialities in the standard play-based composition mechanism for innocent strategies, making more explicit the fact that it is relational (this is key to the full completeness results for fragments of linear logic [AM99, Mel05]). Furthermore, the importance of causal representations for programs has been advocated in the past for various purposes, ranging from error diagnostics [BFHJ03] to the study of reversible aspects of computation [CKV13]. Last but not least, causal representations display the evolution of a concurrent system with partial orders rather than interleavings (it is “*truly concurrent*”). Such representations avoid the *state explosion* problem of interleaving-based ones [God96], leading to potential applications to the verification of concurrent systems.

Contributions. Giving causal representations of the execution of programs is not a new problem. Such models have been for various process languages, starting from CCS [Win86], up to (recently) the full π -calculus [EHS15, CKV15]. There seem to exist few developments on truly concurrent semantics of concurrent languages with shared state, with the notable exception of a Petri net semantics for a simple imperative programming language [HW06].

In this paper, we give a general framework in which one can define such truly concurrent models for higher-order concurrent languages, with various synchronization primitives. This has the form of a cartesian closed category of arenas and concurrent strategies, which are certain event structures. The approach is conservative over the category of Hyland-Ong innocent strategies for PCF (and over the more recent work [TO15] in the non-deterministic case), in the sense that a pure term is interpreted as its forest of *P-views*. In this paper, we develop this category in detail, and illustrate it by spelling out the interpretation of Idealized Parallel Algol (IPA), a higher-order concurrent programming language with shared state and semaphores as synchronization primitives. The methodology is that of game semantics, which provides a well-rooted hope that any of the many languages that one can model in game semantics could be given a truly concurrent representation in this framework as well.

Outline. In Section 2, we give some basic ideas behind the formalization of game semantics on top of event structures, and introduce the key issues that we will have to solve in order to push these ideas to a fully-fledged games model. In particular, we will show that we need to move to a setting of event structures *with symmetry*, in order to handle uniformity of strategies with respect to replicated resources. In Section 3, we introduce our basic framework of games with symmetry, called *thin concurrent games*. This yields a category TCG equipped to handle uniformity of strategies. In Section 4, we show that TCG is compact closed. In Section 5, we leverage this compact closed structure to construct our cartesian closed category Cho. Finally in Section 6, we illustrate our framework by describing the interpretation of IPA in Cho.

2. ARENAS, CONCURRENT STRATEGIES, AND UNIFORMITY

This section will stay at a mostly informal level, and is best read with some basic familiarity with Hyland-Ong games. It should be understandable without fluency in event structures or the games based on them, but the reader interested in learning the details of the model should certainly start with [CCRW].

2.1. Preliminaries on Idealized Parallel Algol. Before presenting our game semantics, we fix a syntax (inspired by [GM08]) for Idealized Parallel Algol (IPA). It will not be exactly the same language as in [GM08] – notably, it lacks semaphores. We do not need them because we do not aim to prove any full abstraction result in this paper; and the language is just there to fix notations and for providing examples and illustrations. Indeed, the focus on the paper is on the model construction rather than its applications, which will come later in companion papers.

The **types** of IPA are the following.

$$A, B ::= \mathbf{com} \mid \mathbb{B} \mid \mathbb{N} \mid A \rightarrow B \mid \mathbf{ref}$$

The type **com** is a type of *commands*, which returns no useful value (if it returns at all, it returns **skip**), but may perform read/write operations on the memory. references. The types \mathbb{N} and \mathbb{B} are types for expressions that (if they return) return respectively a natural number or a boolean. Finally, **ref** is the type for integer variables. Note that we consider *active expressions*, *i.e.* the evaluation of a term of type \mathbb{B} or \mathbb{N} can trigger side effects.

Raw **terms** of IPA are described as follows.

$$\begin{aligned} M, N ::= & x \mid MN \mid \lambda x. M \mid Y \mid \\ & \mathbf{tt} \mid \mathbf{ff} \mid \mathbf{if} M N_1 N_2 \mid \\ & n \mid \mathbf{succ} M \mid \mathbf{pred} M \mid \mathbf{iszero} M \mid \\ & \mathbf{skip} \mid M; N \mid M \parallel N \mid \\ & \mathbf{newref} r \mathbf{in} M \mid M := N \mid !M \mid \mathbf{mkvar} \end{aligned}$$

The first three lines describe the syntax of PCF [Plo77]. The fourth line describe *commands* and combinators for them. Finally, the fifth line gives the combinators for manipulating variables. We include the so-called *bad variable* constructor [AM96], but it will only play a very minor role in our development.

These terms are subject to mostly standard typing rules. We give most of them in Figure 1, omitting the standard rules for the λ -calculus, the fixpoint combinator, and constants.

$$\begin{array}{c}
 \frac{\Gamma \vdash M : \mathbf{com} \quad \Gamma \vdash N : \mathbf{com}}{\Gamma \vdash M \parallel N : \mathbf{com}} \quad \frac{\Gamma \vdash M : \mathbf{com} \quad \Gamma \vdash N : \mathbb{X}}{\Gamma \vdash M ; N : \mathbb{X}} \quad \frac{\Gamma \vdash M : \mathbf{ref} \quad \Gamma \vdash N : \mathbb{N}}{\Gamma \vdash M := N : \mathbf{com}} \\
 \\
 \frac{\Gamma, x : \mathbf{ref} \vdash M : \mathbb{X}}{\Gamma \vdash \mathbf{newref} \ x \ \mathbf{in} \ M : \mathbb{X}} \quad \frac{\Gamma \vdash M : \mathbb{B} \quad \Gamma \vdash N_i : \mathbb{X}}{\Gamma \vdash \mathbf{if} \ M \ N_1 \ N_2 : \mathbb{X}} \quad \frac{\Gamma \vdash M : \mathbb{N} \rightarrow \mathbf{com} \quad \Gamma \vdash N : \mathbb{N}}{\Gamma \vdash \mathbf{mkvar} \ M \ N : \mathbf{ref}}
 \end{array}$$

FIGURE 1. Typing rules of IPA

By convention, we use \mathbb{X} to range over ground types: $\mathbf{com}, \mathbb{B}, \mathbb{N}$. By abuse of notation, we will also use \mathbb{N} and \mathbb{B} respectively for the sets of (total) natural numbers and booleans.

Although some of our typing rules seem restricted to output ground types, the full rules can be derived as syntactic sugar. For instance, a version of **if** that eliminates to **ref** can be obtained as:

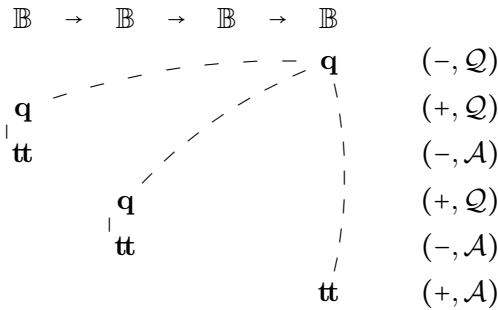
$$\frac{\Gamma \vdash M : \mathbb{B} \quad \Gamma \vdash N_i : \mathbf{ref}}{\Gamma \vdash \mathbf{mkvar} \ (\lambda x. \mathbf{if} \ M \ (N_1 := x) \ (N_2 := x)) \ (\mathbf{if} \ M ! N_1 ! N_2) : \mathbf{ref}}$$

It is an easy verification that the other rules can be generalized similarly.

The language can be equipped with standard (small-step) operational semantics, see [GM08] for details. We omit it here since it will play no role in the technical development.

2.2. Partial orders and conflicts for strategies. We now start introducing our semantics. In the remainder of this section we introduce gradually the main ideas behind our model, relying as much as possible on examples. Our starting point will be the standard Hyland-Ong innocent semantics for PCF, which we will use to motivate concurrent games on event structures. This section will culminate on the issues of replication and uniformity, which will prompt the developments of Section 3.

2.2.1. Dialogue games. *Hyland-Ong games* formalize the intuition that a program is a strategy having a dialogue with its execution environment. A possible dialogue on the type $\mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}$ could be:


 FIGURE 2. A dialogue on $\mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}$

The diagram is read from top to bottom. Each move is either by Player or Opponent, and is either a Question or an Answer. Questions correspond to variable calls, whereas Answers indicate a call terminating. The dashed lines between moves (traditionally called *justification pointers*) convey information about thread indexing; in this example they are redundant but become required at higher types – we will see more on them later.

In natural language, this diagram would read: “*The environment starts the evaluation of a term of type $\mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}$ by interrogating its return type $(-, \mathcal{Q})$. The evaluation requires information on the first argument, so the term triggers its evaluation by playing under the corresponding component of the type $(+, \mathcal{Q})$. The evaluation of the argument terminates with value \mathbf{tt} $(-, \mathcal{A})$. Knowing that its first argument is \mathbf{tt} , the term now needs information on its second argument $(+, \mathcal{Q})$. This argument returns \mathbf{tt} $(-, \mathcal{A})$, and now the term computes and returns \mathbf{tt} at toplevel $(+, \mathcal{A})$.*” The reader should recognize here a description of an execution of **if**.

In Hyland-Ong games, **Sequential innocent strategies** consist of sets of dialogues as above, where Opponent moves are *justified* by the preceding one – such dialogues are known as *P-views*. A strategy for a term of PCF contains several such dialogues, specifying the term entirely. The full strategy for **if** contains in total four maximal P-views, displayed in Figure 3. Such non-empty sets of P-views (satisfying further conditions: determinism and

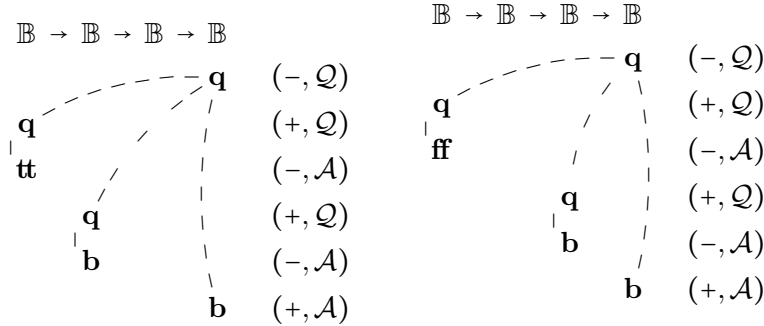


FIGURE 3. Maximal *P*-views for **if**

well-bracketing) are usually called *well-bracketed innocent strategies*. Because of their correspondence with certain normal forms (*PCF Böhm trees* [Cur98]), they are the cornerstone of Hyland-Ong games and of the full abstraction results they allowed.

2.2.2. Partially ordered dialogues. In Hyland-Ong games, every P-view is a total order, meaning that the whole sequential innocent strategy is a tree. In our framework, we question this premise. For instance, informally, the intuitive behaviour of the parallel composition operation $\parallel: \mathbf{com} \rightarrow \mathbf{com} \rightarrow \mathbf{com}$ of IPA is most elegantly represented as in Figure 4.

The diagram of Figure 4 is analogous to the previous ones, but is now partially ordered rather than totally ordered. The relation \rightarrow denotes immediate causality; it was unnecessary before, as it coincided with chronological contiguity. The justification pointers remain – we will see more on their precise nature later. Note that in the standard game semantics for IPA [GM08], this partial order would be only implicit; and given by all the possible linear orderings of the partial order above. Here instead, the partial order will be the first-class object of interest. Strategies, in particular, will be partially ordered.

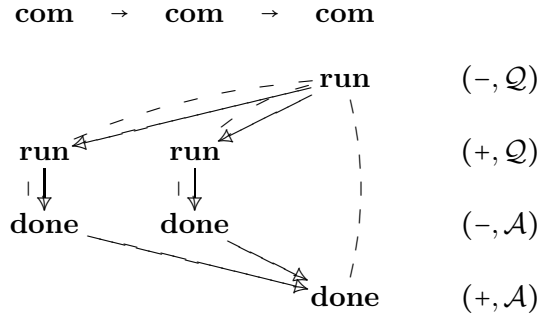


FIGURE 4. A partially ordered dialogue

2.2.3. *Non-determinism.* Concurrent languages are, in general, non-deterministic. Note that we do not mean non-deterministic in the sense that, as above, the execution admits multiple distinct linear orderings. For us, non-determinism means that execution takes irreconciliable routes, even up to permutation of independent events. We illustrate that by the following two examples.

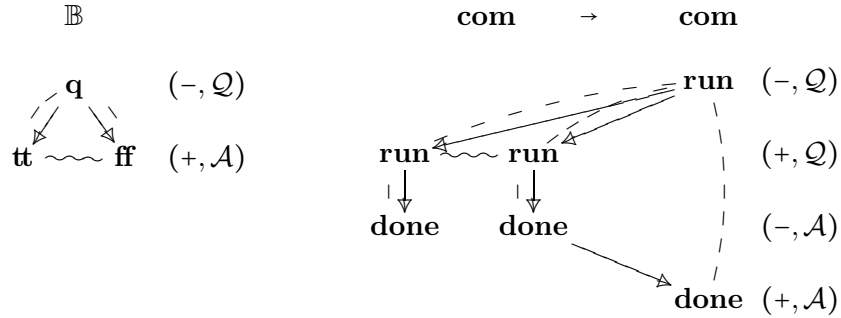


FIGURE 5. Non-deterministic dialogues

In the above two diagrams, the wiggly line \sim indicates *immediate conflict*. Two moves/events related by immediate conflict are incompatible, and can never occur together in an execution. Accordingly, the first example is a representation of our strategy for the non-deterministic boolean, which answers either true or false. The second example illustrates another key aspect of our model: we remember explicitly the point of non-deterministic choice. Here, Player silently throws a coin. If the result is heads, they evaluate the argument, then terminate. However, if the result is tails, they evaluate the argument, then diverge. Typical play-based game semantics would forget the halting branch, which is contained in the other. Instead, our model represents the two branches explicitly.

We now show how to make such diagrams formal.

2.3. Prestrategies on arenas.

2.3.1. *Event structures.* Such a combination of causality and non-determinism is elegantly expressed via Winskel's *event structures*[Win86].

Definition 2.1. An **event structure** (*es* for short) is a tuple $(E, \leq_E, \text{Con}_E)$ where E is a set of *events*, \leq_E is a partial order indicating *causality* and Con_E is a set of finite subsets of E , satisfying:

$$\begin{aligned} \forall e \in E, [e]_E &= \{e' \in E \mid e' \leq_E e\} \text{ is finite} \\ \forall e \in E, \{e\} &\in \text{Con}_E \\ \forall X \in \text{Con}_E, \forall Y \subseteq X, Y &\in \text{Con}_E \\ \forall X \in \text{Con}_E, \forall e \in E, \forall e' \in X, e &\leq_E e' \implies X \cup \{e\} \in \text{Con}_E \end{aligned}$$

The set Con_E of *consistent subsets* specifies which events can occur together in an execution of the system. The *states* of an event structure E , called the (finite) **configurations**, are those finite sets $x \subseteq E$ that are both consistent and **down-closed** (*i.e.* for all $e \in x$, for all $e' \leq e$, then $e' \in x$) – the set of configurations on E is written $\mathcal{C}(E)$, and is partially ordered by inclusion. Configurations with a maximal element are called **prime configurations**, they are those of the form $[e]$ for $e \in E$ (note that we drop the E in $[e]_E$ whenever, as above, this is clear from the context). We will use the notation $[e] = [e] \setminus \{e\}$. We write $x \overset{e}{\dashv} c$ to mean that $e \notin x$ and $x \cup \{e\} \in \mathcal{C}(E)$. Finally, when drawing event structures as above, we do not represent the full partial order \leq but the **immediate causality** generating it, defined as $e \rightarrow e'$ whenever $e < e'$ and for any $e \leq e'' \leq e'$, either $e = e''$ or $e'' = e'$. We will often omit the subscripts in \leq or \rightarrow when they are clear from the context.

In this paper, most of the event structures we consider (such as those in the previous subsection) have a simpler consistency structure.

Definition 2.2. An **event structure with binary conflict** is a triple $(E, \leq_E, \#_E)$, where \leq_E is a partial order and $\#_E$ is an irreflexive symmetric binary relation on E , such that:

$$\begin{aligned} \forall e \in E, [e]_E &\text{ is finite} \\ \forall e_1 \#_E e_2, \forall e_2 \leq_E e'_2, e_1 &\#_E e'_2 \end{aligned}$$

It is easy to check that an event structure with binary conflict is an event structure, with $\text{Con}_E = \{X \in \mathcal{P}_f(E) \mid \forall e_1, e_2 \in X, \neg(e_1 \#_E e_2)\}$. On the other hand, not every event structure can be described via a binary conflict (take *e.g.* three events with any subset of cardinal less than two being consistent). The strategies in the cartesian closed category we aim to build will only have binary conflict, and accordingly in Section 5 we will restrict to event structures with binary conflict. In the meantime, some aspects of the theoretical development are smoother when carried out with arbitrary consistency.

In an event structure with binary conflict, we can trace back conflicts to their original cause. For $e_1 \#_E e_2$ we say that the conflict is **minimal**, written $e \rightsquigarrow_E e'$, iff for all $e'' \leq e$ we have $\neg(e'' \#_E e')$ and for all $e'' \leq e'$ we have $\neg(e \#_E e'')$. As above, we will often drop the subscripts in $\#$ or \rightsquigarrow when they are clear from the context. Following this notation, all the diagrams of the previous subsection can be regarded as representations of event structures with binary conflict (ignoring the dashed lines).

2.3.2. *Games and arenas.* In game semantics, dialogues as in Subsection 2.2 obey the rules of a *game* inherited from the type. In order to define it, let us first recall the following notion from [CCRW].

Definition 2.3. An **event structure with polarities (esp)** is an event structure A along with a **polarity function**

$$\text{pol}_A : A \rightarrow \{-, +\}$$

associating to any event an polarity, that is either $-$ for Opponent or $+$ for Player.

By a **game**, we simply mean an event structure with polarities.

Those games form the objects of the category CG of concurrent games of [CCRW]. However in this paper we are interested in reconstructing a version of Hyland-Ong games, so we will eventually consider restricted games called *arenas*, imported from [HO00]. Since we do not aim to prove full abstraction results in this paper, our arenas will not carry Question/Answer labeling – and our strategies will not be assumed to be well-bracketed.

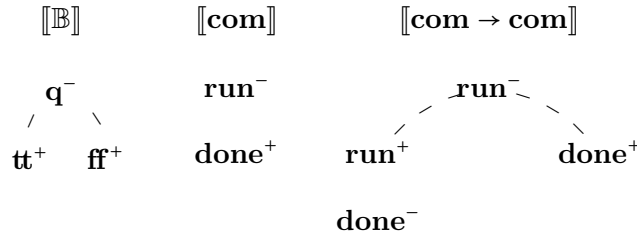
Definition 2.4. An **arena** is a **conflict-free** (all finite sets consistent) esp/game satisfying:

- *Forest.* if $a_1, a_2 \leq a \in A$, then either $a_1 \leq a_2$ or $a_2 \leq a_1$.
- *Alternation.* if $a_1 \rightarrow a_2$, then $\text{pol}(a_1) \neq \text{pol}(a_2)$.

An arena A is **negative** if all its minimal events have negative polarity.

Arenas are close representations of types. Although formulated a bit differently, our arenas are the same as in [HO00] (with the exception of the absence of the Question/Answer labeling).

Example 2.5. Leaving for later the general interpretation of types, we have:



Throughout this paper, we will often omit the semantic brackets on types when this causes no confusion and simply refer to these arenas as \mathbb{B} , \mathbf{com} , *etc.*

By convention, we represent immediate causality in arenas by dashed lines $- - -$ rather than \rightarrow . Events are annotated with their polarity and Question/Answer labeling. We observe in the third example – and it will be a general fact once we give the formal definitions – that each move in the arena $\llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket$ *comes from* a well-defined occurrence of a base type \mathbf{com} in $\mathbf{com} \rightarrow \mathbf{com}$: \mathbf{run}^- and \mathbf{done}^+ come from the output \mathbf{com} , and \mathbf{run}^+ and \mathbf{done}^- come from the input \mathbf{com} . As usual in game semantics, this is used in the representation of dialogues (as in Subsection 2.2): whenever possible, moves are placed under the corresponding base type occurrence.

2.3.3. Prestrategies. The dialogues of Subsection 2.2, and our notion of strategies (called *prestrategies* for now – more conditions are to come), will be event structures *labeled by a game*. In other words, a prestrategy will be an event structure S along with a labeling function $\sigma : S \rightarrow A$ associating to each event in S an image in the game. These labeling functions need to satisfy conditions corresponding to the notion of *map of event structures*.

Definition 2.6. Let E, F be event structures. A **morphism (map) of event structures** $f : E \rightarrow F$ is a function, satisfying:

- *Preservation of configurations.* For all $x \in \mathcal{C}(E)$, $fx \in \mathcal{C}(F)$,
- *Local injectivity.* For all $e, e' \in x \in \mathcal{C}(E)$, if $fe = fe'$ then $e = e'$.

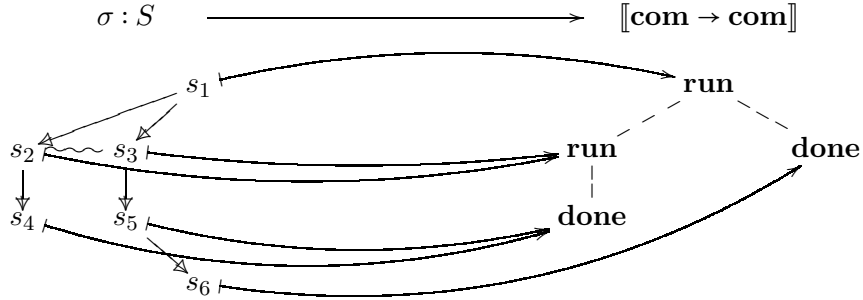
Event structures and maps between them form a category \mathcal{E} .

A **prestrategy** on a game A is a map of event structures $\sigma : S \rightarrow A$.

So a prestrategy $\sigma : S \rightarrow A$ must only reach valid states of A , and behaves *linearly*: in a configuration, each event of the game appears at most once. We note in passing that for non-linear languages, this linearity assumption will be circumvented by creating duplicates of events – more on that later.

If $\sigma : S \rightarrow A$ is a prestrategy, then S automatically inherits from A a polarity function that we write $\text{pol}_S : S \rightarrow \{-, +\}$, leaving the dependency on σ implicit. Of course, it would be equivalent (as done in [RW11]) to require S to be explicitly equipped with polarities, in a way preserved by σ .

2.3.4. Representations of prestrategies. We will only draw prestrategies with binary conflict. When drawing such a $\sigma : S \rightarrow A$ as in Subsection 2.2, we only draw S (more precisely, with immediate causality \rightarrow and immediate conflict \rightsquigarrow), where each event is presented as its image through A , and placed under the corresponding ground type occurrence in the type. We use the dashed lines $- - -$ to represent the relation on S induced by immediate causality on A . For instance, the second diagram of Figure 5 is a representation of the map of event structures below.



As the reader can see, this explicit map notation is a bit cumbersome. Its representation as in Figure 5 conveys the relevant information – the only thing lost is the “name” (s_1, \dots, s_6) of moves in S . More formally, it should be clear to the reader that such a representation displays a finite prestrategy $\sigma : S \rightarrow A$ adequately up to **isomorphism** of prestrategies:

Definition 2.7. Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A$ be prestrategies. A **morphism** from σ to τ is a map of event structures $f : S \rightarrow T$ such that $\tau \circ f = \sigma$.

Accordingly, an **isomorphism** between σ and τ is given by (f, g) , where $f : S \rightarrow T$ and $g : T \rightarrow S$ are maps between σ and τ such that $g \circ f = \text{id}_S$ and $f \circ g = \text{id}_T$. We write $\sigma \cong \tau$ to mean that σ and τ are isomorphic – in that case we might sometimes say that σ and τ are **strongly isomorphic** to emphasize the distinction with *weak isomorphisms*, to be defined in Definition 2.28.

2.4. Compositional structure. In order to obtain such representations of programs compositionally, the standard methodology of denotational semantics suggests to organize them as a category. Rideau and Winskel [RW11] give the basic ingredients for the construction of a (bi)category of games on event structures. We give here the main ideas and definitions, but refer the reader to [CCRW] for a more in-depth discussion with proofs.

We start with the following simple definition.

Definition 2.8. The **simple parallel composition** $E_1 \parallel E_2$ of two event structures E_1 and E_2 has:

- *Events.* The disjoint union $\{1\} \times E_1 \cup \{2\} \times E_2$,
- *Causality.* We have $(i, e) \leq_{E_1 \parallel E_2} (j, e')$ iff $i = j$ and $e \leq_{E_i} e'$,
- *Consistency.* For $X = \{1\} \times X_1 \uplus \{2\} \times X_2$ (often written simply $X_1 \parallel X_2$) a finite subset of $E_1 \parallel E_2$, we have $X \in \text{Con}_{E_1 \parallel E_2}$ iff $X_1 \in \text{Con}_{E_1}$ and $X_2 \in \text{Con}_{E_2}$.

The simple parallel composition of event structures with binary conflict still has binary conflict. If A and B are games, so is $A \parallel B$, with $\text{pol}_{A \parallel B}((1, a)) = \text{pol}_A(a)$ and $\text{pol}_{A \parallel B}((2, b)) = \text{pol}_B(b)$.

In other words, the two event structures are put side by side, without any interaction. If A is a game, then there is also its **dual** A^\perp , defined as having the same events, causality, consistency as A , but reversed polarity: $\text{pol}_{A^\perp}(a) = -\text{pol}_A(a)$. Both operations $- \parallel -$ and $(-)^\perp$ are defined on all games, but preserve arenas.

2.4.1. Morphisms. Given games A and B , a **prestrategy from A to B** is a prestrategy:

$$\sigma : S \rightarrow A^\perp \parallel B$$

We will sometimes write $\sigma : A \dashrightarrow B$ to keep the S anonymous. The basic example of a prestrategy from A to A is the *copycat strategy*.

Definition 2.9. Let A be a game. We define an event structure \mathbb{C}_A as having:

- *Events.* Those of $A^\perp \parallel A$,
- *Causality.* The transitive closure of the relation:

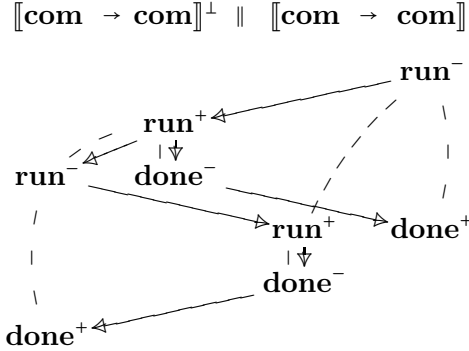
$$\begin{aligned} & \{((1, a), (1, a')) \mid a \leq_{A^\perp} a'\} \cup \{((2, a), (2, a')) \mid a \leq_A a'\} \cup \\ & \{((1, a), (2, a)) \mid \text{pol}_A(a) = +\} \cup \{((2, a), (1, a)) \mid \text{pol}_A(a) = -\} \end{aligned}$$
- *Consistency.* For X a finite subset of \mathbb{C}_A , we have $X \in \text{Con}_{\mathbb{C}_A}$ iff $[X]_{\mathbb{C}_A} \in \text{Con}_{A^\perp \parallel A}$.

In particular, if A is an arena, then \mathbb{C}_A is conflict-free.

The **copycat prestrategy** is the identity function, which is a map of es:

$$\mathfrak{c}_A : \mathbb{C}_A \rightarrow A^\perp \parallel A$$

Example 2.10. The copycat prestrategy from $\llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket$ to itself is:



Note that the partial order above is a tree, whose branches are exactly the P-views of the usual corresponding copycat strategy in Hyland-Ong games.

2.4.2. Interaction. As usual in game semantics, composition is obtained by a two-step process: parallel interaction, plus hiding. The main difficulty in defining the composition of prestrategies is parallel interaction – we first explain how it is done on a closed interaction between $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$. The *interaction* of σ and τ , written $\sigma \wedge \tau : S \wedge T \rightarrow A$, will be a labeled event structure describing the behaviours accepted by both σ and τ .

Its construction is done in several stages. Firstly, its *states* should correspond to certain pairings between matching states of σ and states of τ , *i.e.* pairs $(x, y) \in \mathcal{C}(S) \times \mathcal{C}(T)$ such that $\sigma x = \tau y$. Note that in such a case, the local injectivity assumption on σ and τ induces a bijection $\varphi_{x,y}$ between x and y – in fact matching pairs (x, y) are in one-to-one correspondence with bijections $\varphi : x \simeq y$ between configurations of S and T such that for all $s \in x$, $\tau(\varphi s) = \sigma s$, indicating which events *synchronise* with each other. However, not all such bijections represent valid states of the interaction, as σ and τ might not agree on the *order* in which to play events in x, y . This is addressed by requiring bijections to be *secured*, as below.

Definition 2.11. Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$ be prestrategies. A **secured bijection** between $x \in \mathcal{C}(S)$ and $y \in \mathcal{C}(T)$ is a bijection

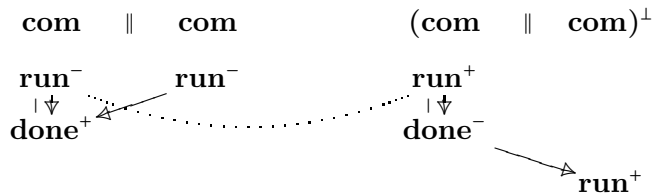
$$\varphi : x \simeq y$$

such that for all $s \in x$ we have $\tau(\varphi s) = \sigma s$, and which is **secured**, in the sense that the reflexive transitive closure of

$$(s, t) \triangleleft (s', t') \Leftrightarrow s <_S s' \vee t <_T t'$$

is a partial order written \leq_φ on (the graph of) φ , making (φ, \leq_φ) a poset. We write $\mathcal{B}_{\sigma, \tau}^{\text{sec}}$ the set of secured bijections between σ and τ .

Example 2.12. In the diagram below are represented two prestrategies σ on $\mathbf{com} \parallel \mathbf{com}$, and τ on $(\mathbf{com} \parallel \mathbf{com})^\perp$.



The dotted line is the only pair in the unique maximal secured bijection in $\mathcal{B}_{\sigma,\tau}^{\text{sec}}$. The maximum configurations of σ and τ are matching, but not in a secured way.

This gives a notion of state of the interaction, but we expected to build a labeled event structure. Hence we wish to present $\mathcal{B}_{\sigma,\tau}^{\text{sec}}$ (up to isomorphism) as the set of configurations of an event structure $S \wedge T$. This is done via the *prime construction*. Say that a secured bijection (φ, \leq_φ) is a **prime** when it has exactly one maximal element (s_φ, t_φ) . In other words, a prime secured bijection is the data of one synchronisation (s_φ, t_φ) , plus a causally valid history for it. We now form:

Definition 2.13. The event structure $S \wedge T$ is obtained as follows:

- *Events.* Prime secured bijections $\varphi \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$.
- *Causal order.* Inclusion of secured bijections.
- *Consistency.* For X a finite subset of $\mathcal{B}_{\sigma,\tau}^{\text{sec}}$ of prime secured bijections, we have $X \in \text{Con}_{S \wedge T}$ iff $\cup X \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$.

There is a map of es $\sigma \wedge \tau : S \wedge T \rightarrow A$, given by $(\sigma \wedge \tau) \varphi = \sigma s_\varphi = \tau t_\varphi$.

In passing, we note that if $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$ have binary conflict (meaning S and T have binary conflict), then so does $\sigma \wedge \tau : S \wedge T \rightarrow A$, with $\neg(\varphi_1 \#_{S \wedge T} \varphi_2)$ iff $\varphi_1 \cup \varphi_2 \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$ – this easily boils down to the lemma below.

Lemma 2.14. Assume σ, τ have binary conflict, and let X be a finite subset of $\mathcal{B}_{\sigma,\tau}^{\text{sec}}$. Then the two following statements are equivalent.

- (1) We have $\cup X \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$,
- (2) For all $\varphi, \psi \in X$, $\neg(\varphi \# \psi)$.

Proof. (1) \Rightarrow (2). Obvious, since $\varphi \cup \psi$ is a down-closed subset of $\cup X$.

(2) \Rightarrow (1). First we note that for $\varphi, \psi \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$, if $\varphi \cup \psi$ is still a bijection between configurations, then it is automatically in $\mathcal{B}_{\sigma,\tau}^{\text{sec}}$ – indeed both φ and ψ are down-closed sets in a common partial order of synchronised events. But then, we note that $\cup X$ is a bijection between configurations. Indeed, if $(c, d) \in \varphi \in X$ and $(c, d') \in \psi \in X$ then $d = d'$ since $\varphi \cup \psi \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$. \square

It will also be useful later to have a concrete understanding of how minimal conflict arises in an interaction; hence we prove the following lemma.

Lemma 2.15. Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$ be prestrategies, and $\varphi \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$. Then if $\varphi : x \simeq y$ extends in $\mathcal{B}_{\sigma,\tau}^{\text{sec}}$ with (s_1, t_1) and (s_2, t_2) but $\varphi \cup \{(s_1, t_1), (s_2, t_2)\} \notin \mathcal{B}_{\sigma,\tau}^{\text{sec}}$. Then, either $x \cup \{s_1, s_2\} \notin \mathcal{C}(S)$ or $y \cup \{t_1, t_2\} \notin \mathcal{C}(T)$.

In particular, if S, T have binary conflict and φ, ψ are events in $S \wedge T$ such that $\varphi \rightsquigarrow \psi$, then $s_\varphi \rightsquigarrow s_\psi$ or $t_\varphi \rightsquigarrow t_\psi$.

Proof. If $x \cup \{s_1, s_2\} \in \mathcal{C}(S)$, $y \cup \{t_1, t_2\} \in \mathcal{C}(T)$, and s_1, s_2 and t_1, t_2 are distinct, then clearly $\varphi \cup \{(s_1, t_1), (s_2, t_2)\} \in \mathcal{B}_{\sigma,\tau}^{\text{sec}}$. However if e.g. $s_1 = s_2$, then $\tau t_1 = \tau t_2$, hence $y \cup \{t_1, t_2\} \notin \mathcal{C}(T)$ by local injectivity – and similarly if $t_1 = t_2$.

The second part of the statement follows easily. \square

In fact, what we have described above is the *pullback* construction in \mathcal{E} . There are maps of event structures:

$$\begin{array}{ccc} \Pi_1 : S \wedge T & \rightarrow & S \\ & \varphi \mapsto & s_\varphi \end{array} \quad \begin{array}{ccc} \Pi_2 : S \wedge T & \rightarrow & T \\ & \varphi \mapsto & t_\varphi \end{array}$$

making the following square commute, and a pullback (from Lemma 2.11 of [CCRW]):

$$\begin{array}{ccc}
 & S \wedge T & \\
 \Pi_1 \swarrow & \downarrow \sphericalangle & \searrow \Pi_2 \\
 S & & T \\
 \sigma \searrow & & \swarrow \tau \\
 & A &
 \end{array}$$

We motivated the pullback by asking for an es whose configurations are secured bijections. And indeed, those are in a very close correspondence.

Proposition 2.16. For any $x \in \mathcal{C}(S \wedge T)$, we have $\varphi_x = \cup x : \Pi_1 x \simeq \Pi_2 x$. Moreover, the assignment:

$$\begin{array}{ccc}
 \mathcal{C}(S \wedge T) & \rightarrow & \mathcal{B}_{\sigma, \tau}^{\text{sec}} \\
 x & \mapsto & \varphi_x
 \end{array}$$

is an order-isomorphism (with both sets ordered by inclusion). Finally, there is a family of order-isomorphisms:

$$\begin{array}{ccc}
 \nu_x & : & x \cong \varphi_x \\
 & & \psi \mapsto (s_\psi, t_\psi)
 \end{array}$$

that is natural in x .

Proof. Direct extension of Lemma 2.9 in [CCRW]. □

This allows us, when reasoning on configurations of a pullback, to manipulate directly secured bijections rather than compatible sets of prime secured bijections. Likewise, when reasoning on events of the pullback in an ambient configuration, we can directly apply ν and reason on synchronised pairs. In the proofs, we will often use this proposition implicitly and transfer silently between the different representations.

Finally, we are in position to define the parallel interaction of two prestrategies $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$. We simply form the pullback:

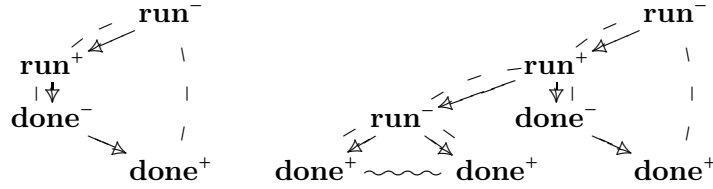
$$\begin{array}{ccc}
 & (\sigma \parallel C) \wedge (A \parallel \tau) & \\
 \Pi_1 \swarrow & \downarrow \sphericalangle & \searrow \Pi_2 \\
 S \parallel C & & A \parallel T \\
 \sigma \parallel C \searrow & & \swarrow A \parallel \tau \\
 & A \parallel B \parallel C &
 \end{array}$$

We write $T \otimes S = (\sigma \parallel C) \wedge (A \parallel \tau)$ for the interaction, and $\tau \otimes \sigma : T \otimes S \rightarrow A \parallel B \parallel C$ for either side of the pullback square. Hence we get the interaction of σ and τ , $\tau \otimes \sigma$, as a labeled event structure.

Example 2.17. Consider the following prestrategies $\sigma : S \rightarrow \llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket$ and $\tau : T \rightarrow \llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket^\perp \parallel \llbracket \mathbf{com} \rrbracket$ (note that to match the definition of interaction above, we can

consider $\sigma : S \rightarrow \mathbb{1}^\perp \parallel \llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket$, where $\mathbb{1}$ is the empty arena).

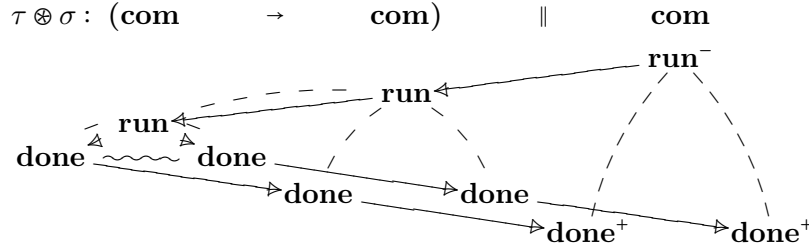
$$\sigma : \llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket \quad \tau : \llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket^\perp \parallel \mathbf{com}$$



We display below a representation of the interaction:

$$\tau \otimes \sigma : T \otimes S \rightarrow (\mathbf{com} \rightarrow \mathbf{com}) \parallel \mathbf{com}$$

We only display polarities for the moves in the right hand side \mathbf{com} . Indeed events on the left hand side (synchronised) part of the interaction have no well-defined polarity, as the two strategies disagree on them.



We leave it to the reader to check that each event in this diagram corresponds uniquely to a configuration in $S \parallel \mathbf{com}$ and a matching configuration in T such that the induced bijection is secured.

2.4.3. *Hiding*. Once we have performed the interaction, it is fairly simple to obtain the composition by ignoring the *synchronised events*, *i.e.* those that map to B . This is an instance of the following *projection* operation.

Definition 2.18. Let E be an event structure, and $V \subseteq E$ a set of events of E . The **projection of E to V** , written $E \downarrow V$, has components:

- *Events.* V .
- *Causality.* The order \leq_E restricted to V .
- *Consistency.* The sets $X \in \text{Con}_E$ such that $X \subseteq V$.

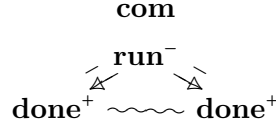
This gives an event structure – it is clear that a hiding of an event structure with binary conflict still has binary conflict. Note as well the *unique witness* property reminiscent of that used in studying the composition of deterministic strategies in standard game semantics: for any $x \in \mathcal{C}(E \downarrow V)$, there exists a *unique* $[x]_E = \{e \in E \mid \exists e' \in x, e \leq_E e'\} \in \mathcal{C}(E)$ such that $[x]_E \cap V = x$, and whose maximal events are those of x .

Finally, we define composition. From $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$, first compute the interaction $\tau \otimes \sigma : T \otimes S \rightarrow A \parallel B \parallel C$. Then, set $V \subseteq T \otimes S$ to comprise all $\varphi \in T \otimes S$ such that $(\tau \otimes \sigma)\varphi \in B$. Writing $T \circ S = T \otimes S \downarrow V$, the composition of σ and τ is:

$$\begin{aligned} \tau \circ \sigma & : T \circ S \rightarrow A^\perp \parallel C \\ \varphi & \mapsto (\tau \otimes \sigma)\varphi \end{aligned}$$

From the fact that interaction and hiding preserve binary conflict, it follows that for $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$, if S, T, A, C have binary conflict, then so does $T \odot S$.

Example 2.19. Consider the interaction of Example 2.17. After hiding, the resulting composition is:



Note that the conflict between the two maximal events, although it was inherited in the interaction, becomes minimal after projection as its original cause has been hidden away.

Composition is associative up to isomorphism [CCRW]. However, copycat is not neutral for composition with respect to prestrategies – it is only the case for *strategies* (see [CCRW] for details):

Definition 2.20. A prestrategy $\sigma : S \rightarrow A$ on a game A is a **strategy** if it is:

- *Receptive.* For all $x \in \mathcal{C}(S)$, if $\sigma x \xrightarrow{a^-}$, then there exists a unique $s \in S$ such that $x \xrightarrow{s}$ and $\sigma s = a$.
- *Courteous.* If $s_1 \rightarrow_S s_2$ and $\text{pol}(s_1) = +$ or $\text{pol}(s_2) = -$, then $\sigma s_1 \rightarrow_A \sigma s_2$.

Putting everything together, we get [CCRW]:

Theorem 2.21. There is a compact closed category CG of games and strategies up to isomorphism.

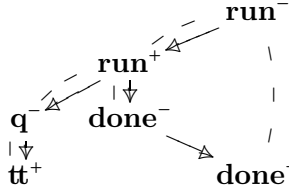
2.5. Interpreting programs and replication. The category CG is a general framework for composing concurrent strategies. We can rely on it to build a model of an affine variant of IPA: that involves restricting to negative arenas, and interpreting function space in IPA via the usual arrow arena construction. We refrain from giving the details here, since we will give them in the non-affine case later on. However, before going on to handling replication, we will give examples in the affine case and try to convey further intuition as to what the model computes. Then we will present the expanded arenas used to handle replication, and we will introduce the issue of uniformity.

2.5.1. Concurrent strategies and view functions. As the reader familiar with Hyland-Ong games may have noticed, our examples before showed how to represent as concurrent strategies, *view functions* rather than expanded strategies – or, in Curien’s terminology [Cur98], *meager* rather than *fat* innocent strategies. And indeed, in our framework it is the case that a pure program will be interpreted directly as its view function, never constructing the full set of plays. For illustration, the interpretation of the affine pure program:

$$\lambda f^{\mathbb{B} \rightarrow \mathbf{com}}. f \mathbf{tt} : (\mathbb{B} \rightarrow \mathbf{com}) \rightarrow \mathbf{com}$$

will be the strategy:

$$(\mathbb{B} \rightarrow \mathbf{com}) \rightarrow \mathbf{com}$$



which the reader can match against the tree of P -views for the corresponding Hyland-Ong innocent strategy. The composition of such strategies is computed directly using pullbacks in \mathcal{E} , never constructing the expanded plays. In other words, we never work with full Hyland-Ong strategies, but always with their causal representations: the view functions.

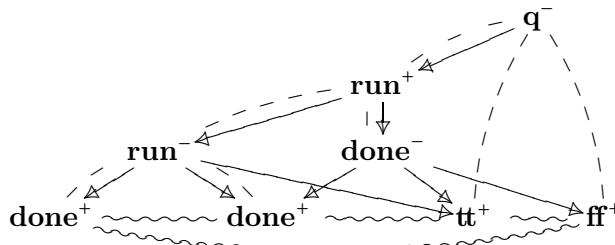
But the usual strategies for stateful programs [AM96] are not generalizations of *meager* innocent strategies, but of *fat* ones: the behaviour of programs must be observed not only on P -views but on general plays. Hence, the reader may wonder if evading them causes us to lose that ability. Fortunately it is not the case, and strategies for stateful programs can be represented causally just as innocent strategies. For instance, consider the following example.

Example 2.22. Consider the following term of IA.

$$\mathbf{newref} \ b \ \mathbf{in} \ \lambda f^{\mathbf{com} \rightarrow \mathbf{com}}. f(b := \mathbf{tt}); !b : (\mathbf{com} \rightarrow \mathbf{com}) \rightarrow \mathbb{B}$$

Following (the affine variant of) the interpretation of Section 6, the corresponding strategy is:

$$(\mathbf{com} \rightarrow \mathbf{com}) \rightarrow \mathbb{B}$$



The \mathbf{done}^+ to the left is duplicated, witnessing the two outcomes of the race in the memory that happens if the argument does not respect the evaluation stack, and concurrently returns \mathbf{done}^- and asks for its argument. The reader familiar with the game semantics for Idealized Algol [AM96] can check that taking the set of (well-bracketed) alternating linear orderings of configurations of this event structure yields the expected set of plays.

2.5.2. *Replication.* But so far, we have only seen *affine* programs and strategies, *i.e.* that call each resource at most once. As it stands, the local injectivity condition in Definition 2.6 forbids us from having two compatible events corresponding to the same move in the game. It is natural to consider dropping it, but then we lose access to the nice structural properties of \mathcal{E} (such as pullbacks). It is unclear to us how one would do about defining composition of strategies in such a setting, let alone proving that it forms a category; in particular if one insists of remembering the point of non-deterministic branching.

Instead, our solution takes inspiration from AJM games [AJM00b] and from the reconstruction of HO games in [HHM07] : we explicitly duplicate moves in arenas. Rather than playing directly on an arena A , our strategies will play in $!A$, a variant of A where all events have been duplicated a countably infinite amount of times, in depth. More formally, we define:

Definition 2.23. Let A be an arena, and $a \in A$. An **indexing function** for a is a function:

$$\alpha : [a] \rightarrow \omega$$

which associates, to a and its dependencies, a **copy index**. From $\alpha : [a] \rightarrow \omega$, we write $\text{lbl } \alpha = a$ for its **label**, and $\text{ind } \alpha = \alpha(\text{lbl } \alpha)$ for the copy index of a .

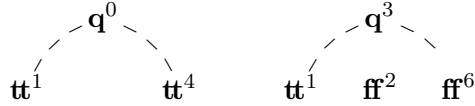
Indexing functions will be the *events* of $!A$. Its full structure will be:

Definition 2.24. From an arena A , we build a new arena $!A$, comprising:

- *Events.* indexing functions $\alpha : [a] \rightarrow \omega$,
- *Causal order.* for $\alpha : [a] \rightarrow \omega$ and $\beta : [b] \rightarrow \omega$, we have $\alpha \leq_{!A} \beta$ iff $a \leq_A b$ and for all $a' \leq_A a$, $\alpha(a') = \beta(a')$.
- *Polarity.* For $\alpha \in !A$, $\text{pol}_{!A}(\alpha) = \text{pol}_A(\text{lbl } \alpha)$.

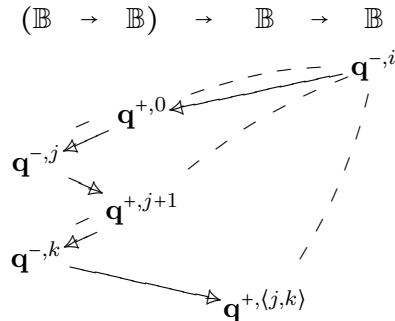
Moves in $!A$ have a rather complex structure. However, note that just as A – which was required to be an arena rather than a general game – $!A$ is a forest. For each $\alpha : [a] \rightarrow \omega$, either a is minimal and then so is α , or there is a unique $a' \rightarrow_A a$ – in which case the restriction of α to $[a']$ gives a unique α' such that $\alpha' \rightarrow_{!A} \alpha$. In other words, α is entirely determined by the data of $\text{lbl } \alpha = a$, $\text{ind } \alpha = \alpha(a)$, and its immediate predecessor α' , called its **justifier** $\text{just}(\alpha)$. Using this decomposition inductively, we can unambiguously draw configurations of $!A$ by annotating each event by its copy index, and its justifier.

Example 2.25. The following is a representation of a configuration of $!\mathbb{B}$:



where, for instance, the two events labeled tt^1 respectively denote $\{\mathbf{q} \mapsto 0, \mathbf{tt} \mapsto 1\}$ and $\{\mathbf{q} \mapsto 3, \mathbf{tt} \mapsto 1\}$.

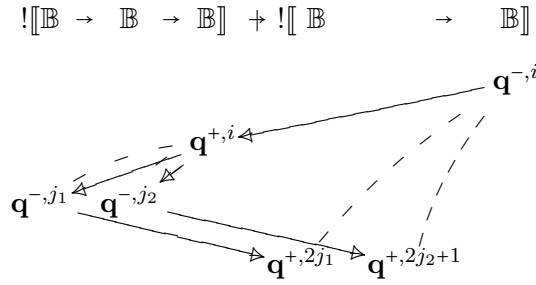
Using the additional space granted by $!A$, we can now represent programs evaluating their arguments multiple times. For instance, a valid strategy $\sigma : S \rightarrow ![(\mathbb{B} \rightarrow \mathbb{B}) \rightarrow \mathbb{B} \rightarrow \mathbb{B}]$ for the term $\lambda f^{\mathbb{B} \rightarrow \mathbb{B}} x^{\mathbb{B}}. f(fx)$ could contain, for $i, j \in \omega$ and injective function $\langle -, - \rangle : \omega^2 \rightarrow \omega$, a configuration:



This diagram exploits the representation introduced just above for configurations of $!A$: each event is specified through its label and copy index. The full strategy σ would comprise such configurations for all $i, j \in \omega$. For each positive event, σ must provide a copy index – this choice must be made globally, in a way that avoids collisions to maintain local injectivity of σ .

2.5.3. Uniformity. Using the composition mechanism introduced before, one may define an interpretation of terms of IPA as concurrent strategies on expanded arenas. However, as observed above, such strategies not only carry information about the events they play and their causal history, but also the data of specific *copy indices* that seem largely irrelevant – *e.g.*, as above, the choice of an injection $\langle -, - \rangle$. In fact, for reasons familiar from AJM games [AJM00b], strategies will not satisfy the laws of cartesian closed categories unless we consider them *up to their specific choice of copy indices*. Let us observe that on an example.

Example 2.26. Consider the term M to be $f : \mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B} \vdash \lambda x^{\mathbb{B}}. f x x : \mathbb{B} \rightarrow \mathbb{B}$. Its interpretation could contain:



Because of the contraction, the Opponent events of indices j_1 and j_2 corresponding to different events in the arena trigger Player events corresponding to the same event in the arena. To ensure local injectivity, we exploit that the functions $2n$ and $2n + 1$ have disjoint codomain.

Likewise, consider two terms, with chosen configurations of their strategies:

$$[[\lambda x^{\mathbb{B}} y^{\mathbb{B}}. x]] : ![[\mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}]] \quad [[\lambda x^{\mathbb{B}} y^{\mathbb{B}}. y]] : ![[\mathbb{B} \rightarrow \mathbb{B} \rightarrow \mathbb{B}]]$$



Then we have:

$$[[M]] \circ [[\lambda x^{\mathbb{B}} y^{\mathbb{B}}. x]] : ![[\mathbb{B} \rightarrow \mathbb{B}]] \quad [[M]] \circ [[\lambda x^{\mathbb{B}} y^{\mathbb{B}}. y]] : ![[\mathbb{B} \rightarrow \mathbb{B}]]$$



But these are required to be the same by the laws of cartesian closed categories, since we have $(\lambda f. M) (\lambda x y. x) =_{\beta} (\lambda f. M) (\lambda x y. y)$. However, they are not isomorphic as strategies on $!A$.

In order to solve this mismatch, we first need to formalize what it means for two configurations of $!A$ to be the same up to the choice of copy indices.

Definition 2.27. Let $x, y \in \mathcal{C}(!A)$. A **reindexing iso** between x and y is an order-isomorphism:

$$\theta : x \cong y$$

which *preserves labels*: for all $\alpha \in x$, $\text{lbl } \alpha = \text{lbl } (\theta \alpha)$.

A reindexing iso $\theta : x \cong y$ is **positive** iff it preserves the copy index of negative events, *i.e.* for all $\alpha^- \in x$, $\text{ind } \alpha = \text{ind } (\theta \alpha)$. **Negative** reindexing isos are defined dually.

Intuitively, two configurations of $\mathcal{C}(!A)$ related by a reindexing iso are distinct specific representations of one *thick subtree* of A in the sense of Boudes [Bou09], *i.e.* a subtree of the arena with duplicated sub-arenas. Two strategies are to be identified iff they are isomorphic, with the commuting triangle to $!A$ being weakened to a commutation *up to reindexing iso* – in fact, it turns out to be simpler to strengthen that to *positive reindexing isos*. Altogether:

Definition 2.28. Let $\sigma : S \rightarrow !A$, $\tau : T \rightarrow !A$ be two strategies. A **weak morphism from σ to τ** is $f : S \rightarrow T$, such that the triangle

$$\begin{array}{ccc} S & \xrightarrow{f} & T \\ \sigma \searrow & & \swarrow \tau \\ & !A & \end{array}$$

commutes *up to positive symmetry*, in the sense that for all $x \in \mathcal{C}(S)$, the set:

$$\{(\sigma s, \tau (f s)) \mid s \in x\}$$

is a positive reindexing iso. If $f : S \rightarrow T$, $g : T \rightarrow S$ are two weak morphisms such that $g \circ f = \text{id}_S$ and $f \circ g = \text{id}_T$, we say that (f, g) is a **weak isomorphism**, and write $\sigma \approx \tau$ to mean that σ and τ are weakly isomorphic.

The two strategies of Example 2.26 are weakly isomorphic. And in fact, a consequence of the developments of this paper is that the natural interpretation of terms $\vdash M : A$ of PCF as strategies on $!A$ hinted at here *is* sound and computationally adequate (it is reasonable to expect the same statement for IPA to be true as well, but it does not follow from the results in this paper). However, proving it bumps into a significant difficulty: without further constraints on strategies, weak isomorphism is *not* a congruence. Indeed, strategies can behave differently depending on Opponent's choice of copy indices. For instance, composing the two weakly isomorphic strategies $\llbracket M \rrbracket \odot \llbracket \lambda x^{\mathbb{B}} y^{\mathbb{B}}. x \rrbracket$ and $\llbracket M \rrbracket \odot \llbracket \lambda x^{\mathbb{B}} y^{\mathbb{B}}. y \rrbracket$ of Example 2.26 with

$$\llbracket \mathbb{B} \rightarrow \mathbb{B} \rrbracket \vdash \llbracket \mathbb{B} \rightarrow \mathbb{B} \rrbracket$$

yields, in the one hand, a strategy that calls its argument, and on the other, one that does not. Clearly, they are not weakly isomorphic. This is because the strategy above is not *uniform*: its behaviour not only depends on Opponent's moves, but also on their copy

index. A useful analogy is that of a program that looks up the address where it is loaded in memory, and uses that information to specify its behaviour.

In AJM games [AJM00b], uniformity is ensured by equipping games with an equivalence relation on plays not unlike our reindexing isomorphisms, and then requiring strategies to satisfy closure properties with respect to it. Here, since our strategies have considerably more structure than sets of plays, the endeavour proved considerably more subtle. One solution, presented in [CCW14], was analogous to the AJM games of [BDER97]: require strategies to be *saturated*, and to play non-deterministically all available copy indices. The next section develops the dual approach, already used in [CCW15], that we call *thin concurrent games*. It is reminiscent of Melliès’ notion of strategies bi-invariant under group actions [Mel03] in the setting of asynchronous games.

3. THIN CONCURRENT GAMES

In order to enforce uniformity, as in [CCW14], we equip event structures with a sort of equivalence relation between configurations, which is *proof relevant* in the sense that two configurations may be related in several different ways. Following [Win07], the resulting structure is called *event structures with symmetry*. In [Win07, CCW14], event structures with symmetry are defined as certain *spans of open maps* – instead, here we mainly use their more concrete presentation as *isomorphism families*: certain sets of bijections, such as the set of reindexing isos between configurations of $!A$.

Isomorphism families plays two important roles in the construction of the framework:

- Adjoined to games, they express “equivalent configurations” of a game, allowing us to switch to a coarser equivalence on strategies (*weak isomorphism*). They give an abstraction of reindexing isos for games not necessarily of the form $!A$.
- Adjoined to strategies, they are used as witnesses of uniformity. Unlike in AJM games, in our development such witnesses are not unique – uniformity is not a property of strategies but a part of their structure.

In this section, we construct a compact closed category TCG of uniform strategies up to weak isomorphism. The category TCG is a generalization of the compact closed category CG of [RW11, CCRW] to deal with uniformity, in the same way that AJM games [AJM00b] extend simple games [Hyl97]. It is the cornerstone of our cartesian closed category Cho.

In Section 3.1 we define isomorphism families and event structures with symmetry, and study their properties. In Section 3.2, we develop a notion of games equipped with isomorphism families, generalizing the situation of $!A$, and introduce a notion of uniform strategies. In Section 3.3, we explain how these notions interact with the composition of strategies developed in [RW11, CCRW] and we generalize the notion of *weak isomorphism* to games not necessarily of the form $!A$. Finally in Section 3.4, we overcome the technical difficulty that weak isomorphism is not a congruence by introducing the notion of *thin symmetries* and proving that the quotient by weak isomorphism yields a category TCG.

3.1. Isomorphism families and symmetry. Isomorphism families extend the partial equivalence relation on plays in AJM games to event structures, in a “proof relevant” way.

Definition 3.1 (Isomorphism families and event structures with symmetry). Let A be an event structure and \tilde{A} be a set of bijections between configurations of A . Then, \tilde{A} is an **isomorphism family** on A if it satisfies:

- (*Groupoid*) The set \tilde{A} contains all identity bijections, and is stable under composition and inverse of bijections.
- (*Restriction*) For every bijection $\theta : x \simeq y \in \tilde{A}$ and $x' \in \mathcal{C}(A)$ such that $x' \subseteq x$, then the restriction $\theta \upharpoonright x'$ of θ to x' is in \tilde{A} . In particular, $\theta x' \in \mathcal{C}(A)$.
- (*Extension*) For every bijection $\theta : x \simeq y \in \tilde{A}$ and every extension $x \subseteq x' \in \mathcal{C}(A)$, there exists a (non-necessarily unique) $y \subseteq y' \in \mathcal{C}(A)$ and an extension $\theta \subseteq \theta'$ such that $\theta' : x' \simeq y' \in \tilde{A}$.

In this case the pair $\mathcal{A} = (A, \tilde{A})$ is called an **event structure with symmetry (ess)**. We will use $\mathcal{S}, \mathcal{T}, \mathcal{A}, \mathcal{B}, \dots$ to range over event structures with symmetry.

An isomorphism family on a game A is an isomorphism family \tilde{A} on the underlying event structure such that all bijections in \tilde{A} preserve polarities.

The definition above does not explicitly mention that the bijections need to be order-isomorphisms. It is actually a consequence of the (Restriction) axiom:

Lemma 3.2. Let \mathcal{A} be an ess and $\theta : x \simeq y \in \tilde{A}$. Then, θ is an order-isomorphism.

Proof. Let $s \leq s' \in x$. Applying the restriction axiom to θ^{-1} and the configuration $[\theta s'] \subseteq y$, it entails that $\theta^{-1}[\theta s']$ is a configuration so it is in particular down-closed. As $s' \in \theta^{-1}[\theta s']$, it follows that $s \in \theta^{-1}[\theta s']$. This directly implies $\theta s \leq \theta s'$ as $\theta s \in [\theta s']$. \square

Since $\theta : x \simeq y \in \tilde{A}$ is an order-isomorphism, we will denote it via $\theta : x \cong y$ to indicate that it preserves and reflects the (implicit, inherited from \leq_A) ordering on x and y . Instead of $\theta : x \simeq y \in \tilde{A}$, we will also often use the more compact notation $\theta : x \cong_{\tilde{A}} y$; and we will refer to θ as a **symmetry** between x and y .

Given a bijection θ , we write $\text{dom } \theta$ and $\text{codom } \theta$ for its domain and codomain respectively. The existence of a symmetry θ between two configurations x and y of A ensures that x and y have isomorphic pasts and bisimilar futures. Another remark is that the axiom (Extension) is equivalent to its one-step counterpart:

Lemma 3.3 (One-step extension). Let A be an event structure and \tilde{A} be a family of bijections. The family \tilde{A} satisfies the (Extension) axiom if and only if for all $\theta : x \cong_{\tilde{A}} y$ and $a \in A$ such that $x \xrightarrow{a} c$, there exists $a' \in A$ such that $\theta \cup \{(a, a')\} \in \tilde{A}$. (In particular $y \xrightarrow{a'} c$)

Proof. Straightforward by induction. \square

3.1.1. *Constructing event structures with symmetry.* We construct some ess of interest. Firstly, expanded arenas and reindexing isos are ess.

Proposition 3.4. Let A be an arena. Recall the expanded arena $!A$ (Definition 2.24) whose events are indexing functions $\alpha : [a] \rightarrow \mathbb{N}$ with $a \in A$. The sets $!\tilde{A}$ of *reindexing isos* (see Definition 2.27), $!\tilde{A}_-$ of *negative reindexing isos*, and $!\tilde{A}_+$ of *positive reindexing isos*, are isomorphism families on $!A$.

Proof. The (Groupoid) axiom is easy to check for these three families. The (Restriction) axiom follows from all the θ being order-isomorphisms.

We check the (Extension) axiom for the first family using Lemma 3.3. Let $\theta : x \cong_{!\tilde{A}} y$ and $\alpha : [a] \rightarrow \mathbb{N}$ an extension of x . Recall from the discussion below Definition 2.24 that events in $!A$ are entirely determined by their label, their *justifier* (immediate causal dependency), and

copy index. Define $\alpha' = \text{just } \alpha$. We set the extension of θ to be (α, β) , where β is set to be the unique event of $!A$ with justifier $\theta(\alpha')$, label a and copy index some fresh k not reached in y yet (or at the very least, not reached by events with label a justified by $\theta(\alpha')$). This yields $\theta \cup \{(\alpha, \beta)\}$ an order-isomorphism between configurations of $!A$, preserving labels.

If $\theta : x \cong_{\tilde{!A}_+} y$ and $\alpha : [a] \rightarrow \mathbb{N}$ is a positive extension of x , the same reasoning applies. If it is a negative extension, then the unique possible extension of θ is (α, β) where β has justifier $\theta(\text{just } \alpha)$ and copy index $\text{ind } \alpha$. Such β cannot be in y already: indeed, its pre-image through θ would be an event with label a , justifier $\text{just } \alpha$ and copy index $\text{ind } \alpha$ – so would be α , absurd since $\alpha \notin x$. The reasoning for $\tilde{!A}_-$ is dual. \square

Event structures with symmetry are also closed under all basic operations on event structures.

Definition 3.5. Let \mathcal{A} and \mathcal{B} be ess. We build their **simple parallel composition** as $(A \parallel B, \tilde{A} \parallel \tilde{B})$ where $\tilde{A} \parallel \tilde{B}$ is the set of bijections of the form $\theta_1 \parallel \theta_2 : x \parallel y \simeq x' \parallel y'$ where $x, x' \in \mathcal{C}(A), y, y' \in \mathcal{C}(B), \theta_1 \in \tilde{A}, \theta_2 \in \tilde{B}$ and $\theta_1 \parallel \theta_2$ is defined as $(i, a) \mapsto (i, \theta_i(a))$.

If \mathcal{A} is a game equipped with symmetry, its **dual** \mathcal{A}^\perp has the same isomorphism family \tilde{A} on the arena A^\perp .

3.1.2. Morphisms. In the setting without symmetry, *morphisms of event structures* played a central role, providing in particular an adequate notion of labeling functions for strategies $\sigma : S \rightarrow A$. In our new setting with symmetry, we will also need to consider a corresponding notion of morphisms.

Definition 3.6. Let \mathcal{A}, \mathcal{B} be event structures with symmetry. A map of event structures $f : \mathcal{A} \rightarrow \mathcal{B}$ **preserves symmetry** iff for all $\theta : x \cong_{\tilde{A}} y$, the bijection $f\theta = \{(fa, fa') \mid (a, a') \in \theta\}$ is in \tilde{B} .

In that case, f is a **map of event structures with symmetry**, written $f : \mathcal{A} \rightarrow \mathcal{B}$.

Event structures with symmetry and their maps form a category written \mathcal{E}_\sim . In \mathcal{E}_\sim , morphisms can be compared *up to symmetry*, abstracting away from the comparison of morphisms *up to the choice of copy indices* of the previous section.

Definition 3.7. Let $f, g : \mathcal{A} \rightarrow \mathcal{B}$ be maps of event structures with symmetry. They are **symmetric** (written $f \sim_{\tilde{B}} g$) when for all $x \in \mathcal{C}(A)$, the bijection $\{(fs, gs) \mid s \in x\}$ is in \tilde{B} .

Note however that the notion of symmetry between maps of ess is not refined enough to express the commutation *up to positive copy indices* of Definition 2.28, as the symmetry in an ess is polarity-agnostic. This will however become possible soon, as our games with symmetry (*thin concurrent games*) will come with *several* isomorphism families, abstracting from $\tilde{!A}, \tilde{!A}_+, \tilde{!A}_-$.

3.2. Games with symmetry and uniform strategies. We now proceed with the reconstruction of the (bi)category of strategies on event structures, in this symmetry-aware setting. As pointed out earlier, the use of symmetry will be two-fold: on the first hand, it will allow us to relax the equivalence on strategies and only compare them *up to symmetry*, and on the other hand it will be used to ensure uniformity.

3.2.1. *Thin concurrent games.* As observed in Definition 2.27, the expanded arena $!A$ has three natural isomorphism families $!\tilde{A}, !\tilde{A}_+, !\tilde{A}_-$. The positive symmetry on $!A$ is used to compare strategies *up to positive copy indices*. By duality, the negative symmetry is needed to compare counter-strategies. The whole symmetry appears naturally when we want to make a strategy interact against a counter-strategy.

In our effort to abstract away from expanded arenas, our notion of game with symmetry will also feature three isomorphism families analogous to $!\tilde{A}, !\tilde{A}_+, !\tilde{A}_-$:

Definition 3.8. A **thin concurrent game (tcg)** is an ess \mathcal{A} (where A is a game) with two additional isomorphism families \tilde{A}_- and \tilde{A}_+ on A such that:

- (a) The families \tilde{A}_+ and \tilde{A}_- are subsets of \tilde{A} ,
- (b) If $\theta \in \tilde{A}_+ \cap \tilde{A}_-$ then θ is an identity bijection,
- (c) If $\theta \in \tilde{A}_-$ and $\theta \sqsubseteq^- \theta' \in \tilde{A}$ then $\theta' \in \tilde{A}_-$,
- (d) If $\theta \in \tilde{A}_+$ and $\theta \sqsubseteq^+ \theta' \in \tilde{A}$ then $\theta' \in \tilde{A}_+$.

where $\theta \sqsubseteq^- \theta'$ (resp. $\theta \sqsubseteq^+ \theta'$) means that $\theta \subseteq \theta'$ such that $\theta' \setminus \theta$ only contains events of negative (resp. positive) polarity. The triple $(\mathcal{A}, \tilde{A}_-, \tilde{A}_+)$ will be often written simply \mathcal{A} to ease the notation.

The key example of a thin concurrent game is given by expanded arenas.

Proposition 3.9. Let A be an arena. Then, $(!A, !\tilde{A}, !\tilde{A}_-, !\tilde{A}_+)$ is a thin concurrent game.

Proof. Straightforward verification. □

In fact, all the thin concurrent games involved in the construction of our cartesian closed category (in Section 5) will be expanded arenas, or isomorphic to expanded arenas. However, the issues of symmetry and uniformity are best addressed in the slightly more abstract setting of thin concurrent games.

Given a tcg \mathcal{A} , we will write \mathcal{A}_- for the event structure with symmetry (A, \tilde{A}_-) and \mathcal{A}_+ for (A, \tilde{A}_+) . We now generalize to tcgs the usual operations on arenas.

Definition 3.10. Given a tcg $(\mathcal{A}, \tilde{A}_-, \tilde{A}_+)$, its **dual** is

$$(\mathcal{A}, \tilde{A}_-, \tilde{A}_+)^{\perp} = (\mathcal{A}^{\perp}, \tilde{A}_+, \tilde{A}_-)$$

Note that the two additional isomorphism families are swapped.

Likewise, the **simple parallel composition** of $(\mathcal{A}, \tilde{A}_-, \tilde{A}_+)$ and $(\mathcal{B}, \tilde{B}_-, \tilde{B}_+)$ is performed componentwise:

$$(\mathcal{A}, \tilde{A}_-, \tilde{A}_+) \parallel (\mathcal{B}, \tilde{B}_-, \tilde{B}_+) = (\mathcal{A} \parallel \mathcal{B}, \tilde{A}_- \parallel \tilde{B}_-, \tilde{A}_+ \parallel \tilde{B}_+)$$

where parallel composition of sets of bijections is defined as in Definition 3.5.

3.2.2. *Properties of tcgs.* The definition of tcgs above has a few interesting consequences, that will be useful later on.

First of all, positive (resp. negative) isomorphism families enjoy a particular property: extensions of the identity bijection by Opponent (resp. Player) moves are still identities (property that we call *thin* in the next section, and which – once added to strategies – will be of crucial importance).

Lemma 3.11. Let \mathcal{A} be a tcg and $x \in \mathcal{C}(A)$.

- If $\text{id}_x \sqsubseteq^+ \theta \in \tilde{A}_-$ then $\theta = \text{id}_y$ for some $x \sqsubseteq y \in \mathcal{C}(A)$.

- If $\text{id}_x \sqsubseteq^- \theta \in \tilde{A}_+$ then $\theta = \text{id}_y$ for some $x \sqsubseteq y \in \mathcal{C}(A)$.

Proof. The two items are dual; we only detail the first. Since $\text{id}_x \in \tilde{A}_+$, it follows from (d) that $\theta \in \tilde{A}_+$, so $\theta \in \tilde{A}_+ \cap \tilde{A}_-$, hence the conclusion follows by (b). \square

We will also use the following fact: bijections in the isomorphism family of a tcg can be uniquely factored into a negative and a positive part.

Lemma 3.12 (Decomposition lemma). Let \mathcal{A} be a tcg. The following function is an order-isomorphism:

$$\begin{aligned} \tilde{A}_- \times_A \tilde{A}_+ &\rightarrow \tilde{A} \\ (\theta^-, \theta^+) &\mapsto \theta^- \circ \theta^+ \end{aligned}$$

where $\tilde{A}_- \times_A \tilde{A}_+ = \{(\theta^-, \theta^+) \in \tilde{A}_- \times \tilde{A}_+ \mid \text{codom } \theta^+ = \text{dom } \theta^-\}$ is ordered by pairwise inclusion and \tilde{A} is ordered by inclusion.

Proof. The map is clearly well defined because \tilde{A}_- and \tilde{A}_+ are included in \tilde{A} .

Injectivity. Assume we have $\theta = \theta_1^- \circ \theta_1^+ = \theta_2^- \circ \theta_2^+ : x \cong_{\tilde{A}} y$. In other words we have the following commutative square:

$$\begin{array}{ccc} & z_1 & \\ \theta_1^+ \cong_{\tilde{A}_+} & \cong_{\tilde{A}_-} & \theta_1^- \\ x & \cong_{\tilde{A}_+} & y \\ \theta_2^+ \cong_{\tilde{A}_+} & \cong_{\tilde{A}_-} & \theta_2^- \\ & z_2 & \end{array}$$

By using groupoid laws we get that $\theta_1^+ \circ (\theta_2^+)^{-1} = (\theta_1^-)^{-1} \circ \theta_2^- : z_1 \cong z_2 \in \tilde{A}_- \cap \tilde{A}_+$ hence it is equal to the identity: $z_1 = z_2$, $\theta_1^+ = \theta_2^+$ and $\theta_1^- = \theta_2^-$.

Surjectivity. By induction on $\theta \in \tilde{A}$ we build a preimage. If θ is empty then (\emptyset, \emptyset) is suitable.

Assume we have the decomposition of $\theta : x \cong_{\tilde{A}} y$ into $x \cong_{\tilde{A}_+}^{+\theta} z \cong_{\tilde{A}_-}^{-\theta} y$ and θ extends to $\theta' : x' \cong y'$ by a pair of fixed polarity, say positive. We use the extension axiom on θ^- to get $\theta^- \sqsubseteq \theta'^- : z' \cong_{\tilde{A}_-} y'$. It follows that $\theta^+ \sqsubseteq (\theta'^-)^{-1} \circ \theta' : x' \cong_{\tilde{A}_+} z'$ is a positive extension of θ^+ so it must belong to \tilde{A}_+ by the properties of tcgs. Hence $\theta' = \theta'^- \circ ((\theta'^-)^{-1} \circ \theta')$ provides the required decomposition.

Monotonicity and monotonicity of the inverse. Clearly, the function is monotonic. We prove that so is its inverse. Assume $\theta_- \circ \theta_+ \sqsubseteq \theta'_- \circ \theta'_+$. We write $\theta_- \circ \theta_+ : x \cong_{\tilde{A}} y$ and $\theta'_- \circ \theta'_+ : x' \cong_{\tilde{A}} y'$; in particular we have $x \sqsubseteq x'$ and $y \sqsubseteq y'$. But then restricting θ'_+ to x yields $\theta''_+ : x \cong_{\tilde{A}_+} z$, and restricting θ'_- to z yields $\theta''_- : z \cong_{\tilde{A}_-} y$, where $\theta''_- \circ \theta''_+$ is the restriction of $\theta'_- \circ \theta'_+$ to x , i.e. $\theta_- \circ \theta_+$. By injectivity (proved above), $\theta_- = \theta''_-$ and $\theta_+ = \theta''_+$. Thus, $\theta_- \sqsubseteq \theta'_-$ and $\theta_+ \sqsubseteq \theta'_+$. \square

As an aside, an interesting consequence of this lemma is that the following commutative diagram:

$$\begin{array}{ccc}
 & (A, \{\text{id}_x \mid x \in \mathcal{C}(A)\}) & \\
 \swarrow & & \searrow \\
 (A, \tilde{A}_+) & & (A, \tilde{A}_-) \\
 \searrow & & \swarrow \\
 & \mathcal{A} &
 \end{array}$$

is both a pullback and a pushout in \mathcal{E}_\sim . In particular, this means that \tilde{A} is uniquely determined from \tilde{A}_- and \tilde{A}_+ .

3.2.3. Uniform strategies. Given a tcg \mathcal{A} , we would like to define when a strategy $\sigma : S \rightarrow \mathcal{A}$ on \mathcal{A} is uniform.

Drawing inspiration from AJM games, a natural condition that comes to mind is to say that any configurations $x, y \in \mathcal{C}(S)$ mapping to symmetric configurations of the game (*i.e.* there exists $\theta : \sigma x \cong_{\tilde{A}} \sigma y$) have similar futures. For instance, if x extends by $s \in S$, then there exists $s' \in S$ such that y extends by s' , and $\theta \cup \{(\sigma s, \sigma s')\} : \sigma x \cup \{\sigma s\} \cong \sigma y \cup \{\sigma s'\}$ is in \tilde{A} . However, in our setting this definition is too naive.

Example 3.13. Consider the strategy σ represented below, extending with copy indices that of Figure 5.

$$\begin{array}{ccc}
 \llbracket \mathbf{com} & \rightarrow & \mathbf{com} \rrbracket \\
 & & \text{---} \mathbf{run}^{-,i} \\
 \mathbf{run}^{+,0} & \sim & \mathbf{run}^{+,1} \\
 \downarrow & & \downarrow \\
 \mathbf{done}^{-,j} & & \mathbf{done}^{-,k} \\
 & & \searrow \\
 & & \mathbf{done}^{+,k}
 \end{array}$$

Such a strategy is definable with the help of an operation $+$ for a non-deterministic choice (definable in IPA), for instance with the term $\lambda x^{\mathbf{com}}.(x; \perp) + x$. There is a reindexing iso between configurations $\{\mathbf{run}^{-,i}, \mathbf{run}^{+,0}, \mathbf{done}^{-,j}\}$ and $\{\mathbf{run}^{-,i}, \mathbf{run}^{+,1}, \mathbf{done}^{-,k}\}$ of the tcg $\llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket$, however only the latter can be extended by a move with label \mathbf{done}^+ .

This example shows that in the presence of explicit non-deterministic branching information, it is not because two configurations have a symmetric image in the game that they should be considered equivalent. To circumvent this problem, we let strategies specify themselves which configurations should be equivalent by equipping the event structure S with a symmetry \tilde{S} . We ask however that if $\theta \in \tilde{S}$, then $\sigma\theta = \{(\sigma s, \sigma s') \mid (s, s') \in \theta\}$ must be in \tilde{A} , hence turning σ into a *map of ess* $\mathcal{S} \rightarrow \mathcal{A}$. Note that in what follows, whenever we speak of a map of ess $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ with \mathcal{A} a tcg, we mean that the symmetries in \tilde{S} are transported to \tilde{A} rather than to one of the sub-symmetries \tilde{A}_+ or \tilde{A}_- .

Nothing so far prevents a strategy to specify that the only equivalent configurations are those which are equal. To force the symmetry on strategies to be non-trivial, we introduce the notion of *~-receptivity*.

3.2.4. *Pre- \sim -strategies.* To make sure that $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ includes enough bijections in $\tilde{\mathcal{S}}$, we ask that it must be receptive to Opponent extensions of the symmetry (without necessarily being *receptive* as in Definition 2.20, or courteous).

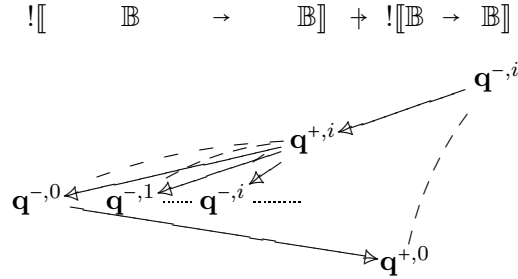
Definition 3.14. If \mathcal{A} is a tcg, a map of ess $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is \sim -**receptive** iff for all $\theta : x_1 \cong_{\tilde{\mathcal{S}}} x_2$, for all $x_1 \xrightarrow{s_1^-} c$ and $\sigma x_2 \xrightarrow{a_2^-} c$ such that $\sigma \theta \cup \{(\sigma s_1, a_2)\} \in \tilde{\mathcal{A}}$, there is a unique s_2 such that $\sigma s_2 = a_2$, and we have $\theta \cup \{(s_1, s_2)\} \in \tilde{\mathcal{S}}$.

A **pre- \sim -strategy** is a \sim -receptive map of ess.

From \sim -receptivity follows the *uniqueness* part of receptivity, but not the *existence*. A pre- \sim -strategy $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ which is additionally receptive is called **strong-receptive**. It is then *receptive at the level of the symmetry*: for all $\theta : x \cong_{\tilde{\mathcal{S}}} y$, for all extension $\sigma \theta \cup \{(a_1^-, a_2^-)\} \in \tilde{\mathcal{A}}$, there are unique $s_1, s_2 \in S$ such that $\sigma s_i = a_i$ and $\theta \cup \{(s_1, s_2)\} \in \tilde{\mathcal{S}}$. In this paper, most of the concrete pre- \sim -strategies we will be interested in will actually be strong-receptive – with the exception of the strategy used in Section 6.3 to handle state.

The following example illustrates how \sim -receptivity ensures uniformity:

Example 3.15. Recall the non-uniform strategy from Section 2:



Assume now there is an isomorphism family $\tilde{\mathcal{S}}$ on this event structure S such that the labelling map $\sigma : S \rightarrow ![[\mathbb{B} \rightarrow \mathbb{B}]^{\perp} \parallel ![[\mathbb{B} \rightarrow \mathbb{B}]]$ is \sim -receptive.

By \sim -receptivity (since the identity on $\{q^{-,0}, q^{+,0}\}$ must be in $\tilde{\mathcal{S}}$), we must have that the bijection $\{q^{-,0}, q^{+,0}, q^{-,1}\} \cong \{q^{-,0}, q^{+,0}, q^{-,i}\}$ is in $\tilde{\mathcal{S}}$. However, only the left hand side part can be extended by $q^{+,0}$, absurd.

In the construction of the category TCG, pre- \sim -strategies will play a role similar to that played by prestrategies in the construction of CG: they support a notion of composition, which will however not yet yield a category. Beyond the motivation of \sim -receptivity as a way to guarantee uniformity, it is also crucial to define composition: indeed, for $\sigma : \mathcal{S} \rightarrow \mathcal{A}^{\perp} \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^{\perp} \parallel \mathcal{C}$ plain maps of ess, one cannot even define their interaction in general, since \mathcal{E}_{\sim} does not have all pullbacks (see Appendix A.1).

Before we go on, let us mention in passing the following lemma, which shows that in checking \sim -receptivity for a map of ess it is enough to look at extensions of identity symmetries.

Lemma 3.16. Let \mathcal{A} be a tcg and $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ be a map of ess. Then, σ is \sim -receptive iff for all $x \in \mathcal{C}(S)$ and $x \xrightarrow{s_1^-} c$, for all $\text{id}_{\sigma x} \cup \{(\sigma s_1, a_2)\} \in \tilde{\mathcal{A}}$, there exists a unique s_2 such that $\sigma s_2 = a_2$, and we have $\text{id}_x \cup \{(s_1, s_2)\} \in \tilde{\mathcal{S}}$.

Proof. only if. Particular case of the definition of \sim -receptivity.

if. Assume $\theta : x_1 \cong_{\tilde{S}} x_2$, $x_1 \xrightarrow{s_1^-} c$ and $\sigma x_2 \xrightarrow{a_2^-} c$ such that $\sigma \theta \cup \{(\sigma s_1, a_2)\} \in \tilde{A}$. By (Extension), there is s'_1 such that $\theta \xrightarrow{(s_1, s'_1)} c$. Since σ is a map of ess, we must have $\sigma \theta \cup \{(\sigma s_1, \sigma s'_1)\} \in \tilde{A}$ as well. By (Groupoid), it follows that $\text{id}_{\sigma x_2} \cup \{(\sigma s'_1, a_2)\} \in \tilde{A}$. By hypothesis, we get a unique s_2 such that $\sigma s_2 = a_2$, satisfying $\text{id}_{x_2} \cup \{(s'_1, s_2)\} \in \tilde{S}$. And finally, by (Groupoid) again, $\theta \cup \{(s_1, s_2)\} \in \tilde{S}$. \square

3.2.5. *Copycat*. As a key example, we show how the copycat strategy presented in Definition 2.9 can be equipped with a symmetry, and made into a pre- \sim -strategy.

Recall that the copycat strategy on game A is a labeled event structure:

$$\mathbb{C}_A : \mathbb{C}_A \rightarrow A^\perp \parallel A$$

where \mathbb{C}_A has the same events as $A^\perp \parallel A$, but additional immediate causal links from negative events on one side to matching positive events on the other side. Consequently, configurations $x \in \mathcal{C}(\mathbb{C}_A)$ decompose as $x = x_1 \parallel x_2 \in \mathcal{C}(A^\perp \parallel A)$.

The following definition is forced by the requirement that the map \mathbb{C}_A should be a map of ess, and that each symmetry should be an order-iso.

Definition 3.17. Let \mathcal{A} be a tcg. Given $x = x_1 \parallel x_2 \in \mathcal{C}(\mathbb{C}_A)$, $y = y_1 \parallel y_2 \in \mathcal{C}(\mathbb{C}_A)$, the set of *symmetries* between x and y (written $\mathbb{C}_{\tilde{\mathcal{A}}}$) comprises any bijection $\theta = \theta_1 \parallel \theta_2$ such that $\theta_1, \theta_2 \in \tilde{A}$, and which is an order-iso (for the order on x, y induced by $\leq_{\mathbb{C}_A}$).

This definition makes a good intuitive sense. However in order to reason on such symmetries, it will be convenient to rely on a more high-level characterisation that does not explicitly require an order-isomorphism. To introduce it, recall first from [CCRW] that configurations $x \in \mathcal{C}(\mathbb{C}_A)$ are exactly those $x_1 \parallel x_2 \in \mathcal{C}(A \parallel A)$ such that (with polarity as in $A \parallel A$):

$$x_2 \supseteq^- x_1 \cap x_2 \subseteq^+ x_1$$

Furthermore, it is observed in [Win13, CCRW] that this relation between x_2 and x_1 is a partial order called the ‘‘Scott order’’, written $x_2 \sqsubseteq_A x_1$. This order is of crucial importance in the construction and study of the bicategory CG.

If \mathcal{A} is a tcg, we now observe the following.

Proposition 3.18. The set $\mathbb{C}_{\tilde{\mathcal{A}}}$ is equivalently defined as comprising the bijections of the form

$$\theta_1 \parallel \theta_2 : x_1 \parallel x_2 \simeq_{\tilde{A}^\perp \parallel \tilde{A}} y_1 \parallel y_2$$

satisfying the further condition that for all $a \in x_1 \cap x_2$, we have $\theta_1(a) = \theta_2(a)$.

In other words, $\mathbb{C}_{\tilde{\mathcal{A}}}$ comprises the bijections $\theta_1 \parallel \theta_2 \in \tilde{A}^\perp \parallel \tilde{A}$ such that $\theta_2 \supseteq^- \theta_1 \cap \theta_2 \subseteq^+ \theta_1$, *i.e.*

$$\theta_2 \sqsubseteq_{\tilde{A}} \theta_1$$

This justifies the notation $\mathbb{C}_{\tilde{\mathcal{A}}}$, as this agrees with the description of configurations of copycat via the Scott order.

Proof. Take $\theta = \theta_1 \parallel \theta_2 : x_1 \parallel x_2 \cong y_1 \parallel y_2$.

If θ is an order-isomorphism, then take $a \in x_1 \cap x_2$. Assume without loss of generality that $\text{pol}_A(a) = +$, so that we have $(1, a) \rightarrow (2, a)$ in \mathbb{C}_A . But then since θ is an order-iso, it preserves immediate causal dependency, therefore $(1, \theta_1 a) \rightarrow (2, \theta_2 a)$. But since these two events are in different components of $A^\perp \parallel A$, this necessarily means that $\theta_1 a = \theta_2 a$ as

required (using *e.g.* the characterisation of immediate causality of copycat in Lemma 3.3 of [CCRW]).

Reciprocally, assume that for all $a \in x_1 \cap x_2$, $\theta_1 a = \theta_2 a$. Using again Lemma 3.3 of [CCRW], it is immediate that θ preserves immediate causal links. The same reasoning applies to θ^{-1} (it is easy to show that the hypothesis is stable under inverse), so it reflects immediate causal links as well; and is an order-iso. \square

We now check the axioms for isomorphism families.

Lemma 3.19. The family $\mathbb{C}_{\tilde{A}}$ satisfies the axioms (Groupoid) and (Restriction) of isomorphism families.

Proof. Immediate from Definition 3.17. \square

We prove the extension axiom separately. Although we will show it holds in tcgs, it fails in general for copycat on event structures with polarity and symmetry without further conditions – the interested reader can find a counter-example in Appendix A.3.

In order to establish it for tcgs, we show that the isomorphism family on tcgs automatically satisfies a technical condition called *race-preservation*.

Lemma 3.20. Let \mathcal{A} be a tcg. Then \tilde{A} is *race-preserving*, in the sense that for any $\theta : x \cong_{\tilde{A}} y$, for any $\theta \sqsubseteq^+ \theta_1 : x_1 \cong_{\tilde{A}} y_1$ and $\theta \sqsubseteq^- \theta_2 : x_2 \cong_{\tilde{A}} y_2$, if x_1 and x_2 are compatible ($x_1 \cup x_2 \in \mathcal{C}(A)$), then so are θ_1 and θ_2 : $\theta_1 \cup \theta_2 \in \tilde{A}$ as well.

Proof. We first prove that \tilde{A}_+ and \tilde{A}_- are race-preserving. Let $\theta : x \cong_{\tilde{A}_+} y$ with a positive extension $\theta_1 : x_1 \cong_{\tilde{A}_+} y_1$ and a negative extension $\theta_2 : x_2 \cong_{\tilde{A}_+} y_2$, with $x_1 \cup x_2 \in \mathcal{C}(A)$.

Using (Extension) of \tilde{A}_+ twice to θ_1 and θ_2 , we get to the following picture:

$$\begin{array}{ccc}
 \theta'_1 : x_1 \cup x_2 \cong_{\tilde{A}_+} y'_1 & & \theta'_2 : x_1 \cup x_2 \cong_{\tilde{A}_+} y'_2 \\
 \downarrow \cup & & \downarrow \cup \\
 \theta_1 : x_1 \cong_{\tilde{A}_+} y_1 & & \theta_2 : x_2 \cong_{\tilde{A}_+} y_2 \\
 & \times \cup & \subset \\
 & \theta : x \cong_{\tilde{A}_+} y &
 \end{array}$$

By the (Groupoid) axiom on \tilde{A}_+ , we have $\text{id}_y \subseteq \theta'_1 \circ \theta'^{-1}_2 : y'_2 \cong_{\tilde{A}_+} y'_1$. By (Restriction), we build $\varphi = \theta'_1 \circ \theta'^{-1}_2 \upharpoonright y_2$. By construction, we have $\text{id}_y \subseteq^- \varphi \in \tilde{A}_+$, so $\varphi = \text{id}_{y_2}$ by Lemma 3.11. It follows that $\theta_2 \subseteq \theta'_1$, hence $\theta'_1 = \theta_1 \cup \theta_2$ as required. A dual reasoning shows that \tilde{A}_- is race-preserving as well.

Now, we deduce the result for \tilde{A} , using the decomposition of Lemma 3.12. Assume $\theta = \theta^- \circ \theta^+$ has extensions $\theta \sqsubseteq^+ \theta_1$ and $\theta \sqsubseteq^- \theta_2$, with decompositions $\theta_1^- \circ \theta_1^+$ and $\theta_2^- \circ \theta_2^+$. By monotonicity of the decomposition, we have $\theta^+ \sqsubseteq^+ \theta_1^+$, $\theta^+ \sqsubseteq^- \theta_2^+$, $\theta^- \sqsubseteq^+ \theta_1^-$ and $\theta^- \sqsubseteq^- \theta_2^-$. By race-preservation of \tilde{A}_+ it follows first that $\theta_1^+ \cup \theta_2^+ \in \tilde{A}_+$, and then by race-preservation of \tilde{A}_- it follows that $\theta_1^- \cup \theta_2^- \in \tilde{A}_-$. Thus $(\theta_1^- \cup \theta_2^-) \circ (\theta_1^+ \cup \theta_2^+) = (\theta_1^- \circ \theta_1^+) \cup (\theta_2^- \circ \theta_2^+) = \theta_1 \cup \theta_2 \in \tilde{A}$. \square

From that, we finally deduce the following.

Proposition 3.21. Let \mathcal{A} be a tcg. Then, writing $\mathbb{C}_{\mathcal{A}} = (\mathbb{C}_{\mathcal{A}}, \mathbb{C}_{\tilde{\mathcal{A}}})$, the map

$$\mathfrak{c}_{\mathcal{A}} : \mathbb{C}_{\mathcal{A}} \rightarrow \mathcal{A}^\perp \parallel \mathcal{A}$$

is a pre- \sim -strategy.

Proof. By Lemma 3.19 we already know that $\mathbb{C}_{\tilde{A}}$ satisfies all axioms for an isomorphism family except for (Extension), which we establish now.

Let $\theta_1 \parallel \theta_2 : x \parallel y \cong_{\mathbb{C}_{\tilde{A}}} x' \parallel y'$. Assume *e.g.* $x \parallel y \xrightarrow{(2,a)} c$. There are two cases:

- If $\text{pol}_A(a) = -$, then by (Extension) for $\tilde{A}^+ \parallel \tilde{A}$ we have $\theta_1 \parallel \theta_2 \subseteq \theta_1 \parallel \theta'_2 \in \tilde{A}^+ \parallel \tilde{A}$ whose domain is $x \parallel y \cup \{a\}$. Its codomain is $x' \parallel y' \cup \{a'\}$. Since $\text{pol}_A(a) = -$, we cannot have $a' \in x'$ – indeed $x' \supseteq^+ x' \cap y' \subseteq^- y'$, so we would have $a' \in y'$ as well, absurd. So we have $x' \cap (y' \cup \{a'\}) = x' \cap y' \subseteq^+ x'$, and $x' \cap (y' \cup \{a'\}) = x' \cap y' \subseteq^- y' \subseteq^- y' \cup \{a'\}$, which establishes that $x' \parallel (y' \cup \{a'\}) \in \mathcal{C}(\mathbb{C}_A)$.

Likewise we have $\theta_1 \cap \theta'_2 = \theta_1 \cap \theta_2$, hence we still have $\theta_1 \cap \theta'_2 \subseteq^+ \theta_1$ but also $\theta_1 \cap \theta'_2 \subseteq^- \theta_2 \subseteq^- \theta'_2$, therefore $\theta_1 \parallel \theta'_2 \in \mathbb{C}_{\tilde{A}}$.

- If $\text{pol}_A(a) = +$ is positive then $a \in x$ as well. Thus, $[a] \subseteq x \cap y$. Therefore, we have $(x \cap y) \cup \{a\} \in \mathcal{C}(A)$, and $(x \cap y) \cup \{a\} \subseteq x$. Define $\theta'_1 = \theta_1 \upharpoonright (x \cap y) \cup \{a\}$. We have:

$$\theta'_1 \supseteq^+ \theta_1 \cap \theta_2 \subseteq^- \theta_2$$

By construction, the domains of θ'_1 (which is $(x \cap y) \cup \{a\}$) and the domain of θ_2 (which is y) are compatible, so by Lemma 3.20, $\theta'_2 = \theta'_1 \cup \theta_2 \in \tilde{A}$, and by construction its domain is $y \cup \{a\}$. To sum up, we have:

$$\theta_1 \supseteq^+ \theta_1 \cap \theta'_2 \subseteq^- \theta'_2$$

Hence $\theta_1 \parallel \theta'_2 \in \mathbb{C}_{\tilde{A}}$ provides the required extension.

We have established that $\mathbb{C}_{\tilde{A}}$ is an isomorphism family. It is obvious that $\mathfrak{c}_A : \mathbb{C}_A \rightarrow \mathcal{A}^+ \parallel \mathcal{A}$ preserves symmetry. It remains finally to show that it is \sim -receptive, for which we apply Lemma 3.16. Assume $x \parallel y \in \mathcal{C}(\mathbb{C}_A)$ can be extended by $(2, a^-)$ in \mathbb{C}_A and by $(2, b^-)$ in $\mathcal{A}^+ \parallel A$ (in which case it is immediate that it is a valid extension in \mathbb{C}_A as well), such that:

$$\text{id}_x \parallel (\text{id}_y \cup \{(a, b)\}) \in \tilde{A}^+ \parallel \tilde{A}$$

We need to check that this is a valid extension in $\mathbb{C}_{\tilde{A}}$ as well. By the characterisation of Proposition 3.18, we only have to check that $\text{id}_x c = (\text{id}_y \cup \{(a, b)\}) c$ for each $c \in x \cap (y \cup \{a\})$, but in fact we must have $c \in x \cap y$. Indeed, we cannot have $a \in x$, as by $x \supseteq^+ x \cap y \subseteq^- y$ and $\text{pol}_A(a) = -$ that would imply $a \in y$ as well, absurd. So the verification is obvious. \square

We have a notion of pre- \sim -strategy that – as we have seen – enforces some notion of uniformity, and includes the copycat strategies. We will now extend to the presence of symmetry the composition operation presented in Section 2.

3.3. Composition of pre- \sim -strategies. In order to define the composition of pre- \sim -strategies, the first step is to define their interaction. As for the plain concurrent games described in Section 2, the interaction of pre- \sim -strategies will be adequately formulated as a *pullback*.

The category \mathcal{E}_{\sim} has no pullbacks in general – the reader can find a proof of this in Appendix A.1. We will start by showing that however, thanks to \sim -receptivity, pullbacks involved in interactions of pre- \sim -strategies do exist.

3.3.1. *Interaction.* As in Section 2, we start by describing a closed interaction between a pre- \sim -strategy $\sigma : \mathcal{S} \rightarrow \mathcal{A}$, and a counter-pre- \sim -strategy $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ (where \mathcal{A} is a tcg). We construct the pullback of their underlying map of event structures $\sigma \wedge \tau : \mathcal{S} \wedge \mathcal{T} \rightarrow \mathcal{A}$. Throughout this section we will reuse the same notations as in Section 2 for operations on strategies.

First, we notice that bijections on configurations of the (plain) pullback induce bijections on their projections:

Lemma 3.22. Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A$ be maps of event structures. Let $\theta : w \simeq z$ be a bijection, where $w, z \in \mathcal{C}(S \wedge T)$. There are (unique) bijections $\theta_S : \Pi_1 w \simeq \Pi_1 z$ and $\theta_T : \Pi_2 w \simeq \Pi_2 z$ satisfying $\Pi_1 \circ \theta = \theta_S \circ \Pi_1$ and $\Pi_2 \circ \theta = \theta_T \circ \Pi_2$. Moreover, the mapping $\theta \mapsto (\theta_S, \theta_T)$ is monotonic *w.r.t.* inclusion.

Proof. We only define θ_S , the definition of θ_T is similar. By local injectivity, Π_1 defines a bijection $w \simeq \Pi_1 w$ and $z \simeq \Pi_1 z$. With this remark, θ_S is simply defined as $\Pi_1 \circ \theta \circ \Pi_1^{-1}$. The equation and uniqueness are by definition, and monotonicity is obvious. \square

Define $\tilde{S} \wedge \tilde{T}$ to contain those bijections $\theta : w \cong z$ such that $\theta_S : \Pi_1 w \cong \Pi_1 z \in \tilde{S}$ and $\theta_T : \Pi_2 w \cong \Pi_2 z \in \tilde{T}$. Bearing in mind the correspondence between configurations of $S \wedge T$ and secured bijections $x \simeq y$, there is an order-isomorphism between those bijections $\theta \in \tilde{S} \wedge \tilde{T}$ and commutative squares between secured bijections $x \simeq y$ and $x' \simeq y'$ (ordered by componentwise union):

$$\begin{array}{ccc} x & \xrightarrow{\sigma} & \sigma x = \tau y \xrightarrow{\tau} y \\ \theta_S \Downarrow & & \Downarrow \theta_T \\ x' & \xrightarrow{\sigma} & \sigma x' = \tau y' \xrightarrow{\tau} y' \end{array}$$

This definition indeed yields a pullback in \mathcal{E}_\sim :

Lemma 3.23. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ be \sim -receptive maps of ess. The set $\tilde{S} \wedge \tilde{T}$ is an isomorphism family on $S \wedge T$ and the ess $(S \wedge T, \tilde{S} \wedge \tilde{T})$ is a pullback in \mathcal{E}_\sim of σ and τ , written $\mathcal{S} \wedge \mathcal{T}$.

Proof. The (Groupoid) and (Restriction) axioms are direct consequences of the corresponding conditions for \tilde{S} and \tilde{T} .

(*Extension*). Let $\theta : w \cong_{\tilde{S} \wedge \tilde{T}} z$. Assume w can be extended by an event $e \in S \wedge T$ to w' . Write $s = \Pi_1 e$ and $t = \Pi_2 e$, and assume eg. σs is positive in A . We then have the following picture:

$$\begin{array}{ccccc} \Pi_1 w' & \simeq & (\sigma \wedge \tau) w' & \simeq & \Pi_2 w' \\ \searrow s & & & & \swarrow t \\ \Pi_1 w & \xrightarrow{\sigma} & (\sigma \wedge \tau) w \xrightarrow{\tau} & \Pi_2 w \\ \theta_S \Downarrow & & \Downarrow \theta_T & & \\ \Pi_1 z & \xrightarrow{\sigma} & (\sigma \wedge \tau) z \xrightarrow{\tau} & \Pi_2 z \end{array}$$

We first use the extension property on θ_S as $\Pi_1 w \xrightarrow{s} \Pi_1 w'$: θ_S extends by (s, s') . Since $\sigma \theta_S = \tau \theta_T$, this means that $\tau \theta_T$ extends by $(\sigma s, \sigma s')$ which is negative in \mathcal{A}^\perp . By \sim -receptivity of τ , it

follows that θ_T extends by (t, t') with $\tau t = \sigma s$ and $\tau t' = \sigma s'$. The picture is now:

$$\begin{array}{ccccc}
 \Pi_1 w' & \simeq & (\sigma \wedge \tau) w' & \simeq & \Pi_2 w' \\
 \downarrow s & & & & \downarrow t \\
 \Pi_1 w & \xrightarrow{\sigma} (\sigma \wedge \tau) w \xrightarrow{\tau} & \Pi_2 w & & \\
 \downarrow \theta_S \wr \wr \tilde{S} & & \downarrow \theta_T \wr \wr \tilde{T} & & \\
 \Pi_1 z & \xrightarrow{\sigma} (\sigma \wedge \tau) z \xrightarrow{\tau} & \Pi_2 w' & & \\
 \downarrow s' & & \downarrow t' & & \\
 z_1 & \simeq & \sigma z_1 = \tau z_2 & \simeq & z_2
 \end{array}$$

The obtained bijection $\varphi : z_1 \simeq z_2$ is secured by construction, so as observed in Definition 2.11 its graph is ordered by \leq_φ compatible with both \leq_S and \leq_T . Therefore restricting φ to the causal history of (s', t') yields $e' = [(s', t')]_\varphi$ a prime secured bijection, *i.e.* an event $e' \in S \wedge T$ such that $z \xrightarrow{e'} z'$. Finally, $\theta \cup \{(e, e')\} \in \tilde{S} \wedge \tilde{T}$ because $\theta_S \cup \{(s, s')\} \in \tilde{S}$ and $\theta_T \cup \{(t, t')\} \in \tilde{T}$.

If σs is negative the reasoning is dual: we use first the extension on \tilde{T} and then \sim -receptivity of σ .

It is a pullback. Clearly the maps $\Pi_1 : S \wedge T \rightarrow S$ and $\Pi_2 : S \wedge T \rightarrow T$ preserve symmetry: they map θ to θ_S and θ_T respectively. We only need to check the universal property. Assume we have two morphisms of ess $\varphi : \mathcal{X} \rightarrow S$ and $\psi : \mathcal{X} \rightarrow T$ such that the square commutes:

$$\begin{array}{ccc}
 & \mathcal{X} & \\
 \varphi \swarrow & & \searrow \psi \\
 S \wedge T & & T \\
 \sigma \swarrow & & \searrow \tau \\
 \tilde{S} & & \mathcal{A}
 \end{array}$$

Because $S \wedge T$ is a pullback in \mathcal{E} there is a map of event structures $\langle \varphi, \psi \rangle : \mathcal{X} \rightarrow S \wedge T$ making the two triangles commute, which is unique in \mathcal{E} . This uniqueness lifts to \mathcal{E}_\sim as the forgetful functor $\mathcal{E}_\sim \rightarrow \mathcal{E}$ is faithful. To conclude we need only to prove that $\langle \varphi, \psi \rangle$ preserves symmetry and is thus a morphism in \mathcal{E}_\sim . Let $\theta : x \simeq_{\tilde{X}} y$. It is transported to a bijection $\langle \varphi, \psi \rangle \theta : \langle \varphi, \psi \rangle x \simeq \langle \varphi, \psi \rangle y$ such that $(\langle \varphi, \psi \rangle \theta)_S = \varphi \theta$ and $(\langle \varphi, \psi \rangle \theta)_T = \psi \theta$, thus $\langle \varphi, \psi \rangle \theta \in \tilde{S} \wedge \tilde{T}$ by definition. \square

3.3.2. Hiding and composition. We now define the composition operation for pre- \sim -strategies.

Given two pre- \sim -strategies $\sigma : S \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : T \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$, we now wish to define their composition. To apply the interaction process above, we need to obtain to dual pre- \sim -strategies; for that we notice that $\sigma \parallel \mathcal{C}^\perp : S \parallel \mathcal{C}^\perp \rightarrow \mathcal{A}^\perp \parallel \mathcal{B} \parallel \mathcal{C}^\perp$ and $\mathcal{A} \parallel \tau : \mathcal{A} \parallel T \rightarrow \mathcal{A} \parallel \mathcal{B}^\perp \parallel \mathcal{C}$ are indeed dual pre- \sim -strategies, so we can form the interaction pullback and the associated map of ess:

$$(\sigma \parallel \mathcal{C}) \wedge (\mathcal{A} \parallel \tau) : (S \parallel \mathcal{C}) \wedge (\mathcal{A} \parallel T) \rightarrow \mathcal{A} \parallel \mathcal{B} \parallel \mathcal{C}$$

written $\tau \otimes \sigma : T \otimes S \rightarrow \mathcal{A} \parallel \mathcal{B} \parallel \mathcal{C}$.

Ignoring symmetry, recall from Section 2 that $\tau \odot \sigma : T \odot S \rightarrow A^\perp \parallel C$ is then obtained using the *projection* (Definition 2.18): given $V = \{p \in S \otimes T \mid (\tau \otimes \sigma)p \notin B\}$ we set $T \odot S = T \otimes S \downarrow V$, and $\tau \odot \sigma$ to be the corresponding restriction of $\tau \otimes \sigma$. We now extend this operation in the presence of symmetry.

Lemma 3.24. Let \mathcal{E} be an ess and $V \subseteq E$ closed under symmetry, in the sense that for all $\theta : x \cong_{\tilde{E}} y$, for all $e \in V \cap x$, we have $\theta e \in V$ as well. Then, defining

$$\tilde{E} \downarrow V = \{\theta : x \simeq y \mid x, y \in \mathcal{C}(E \downarrow V), \exists \theta \subseteq \theta' \in \tilde{E}, \theta' : [x]_E \cong_{\tilde{E}} [y]_E\}$$

we have that $\tilde{E} \downarrow V$ is an isomorphism family, making $\mathcal{E} \downarrow V = (E \downarrow V, \tilde{E} \downarrow V)$ into an event structure with symmetry.

Proof. As usual the axiom (Groupoid) is clear. In this proof we abbreviate $[x]_E$ to $[x]$ for $x \in \mathcal{C}(E \downarrow V)$ for clarity reasons.

(*Restriction*) Let $\theta : x \simeq y \in \tilde{E} \downarrow V$, and $x_0 \in \mathcal{C}(E \downarrow V)$ such that $x_0 \subseteq x$. By definition there is $\theta \subseteq \theta' : [x] \cong_{\tilde{E}} [y]$. We have $[x_0] \subseteq [x]$. Therefore, by (Restriction) on \tilde{E} we have $\theta'_0 \subseteq \theta'$ with $\theta'_0 : [x_0] \cong_{\tilde{E}} y'_0$. Since V is closed under symmetry, $\theta'_0 \cap V^2 : x_0 \simeq y'_0 \cap V$ is still a bijection, which by definition is in $\tilde{E} \downarrow V$. It is clear by construction that $\theta'_0 \cap V^2 \subseteq \theta$.

(*Extension*) Let $\theta : x \simeq y \in \tilde{E} \downarrow V$, and $x \subseteq x_0 \in \mathcal{C}(E \downarrow V)$. By definition there is $\theta \subseteq \theta' : [x] \cong_{\tilde{E}} [y]$. We have $[x] \subseteq [x_0] \in \mathcal{C}(E \downarrow V)$, therefore by (Extension) for \tilde{E} there is $\theta'_0 : [x_0] \cong_{\tilde{E}} y'_0$. Again since V is closed under symmetry, $\theta'_0 \cap V^2 : x_0 \simeq y'_0 \cap V$ is still a bijection. By definition it is in $\tilde{E} \downarrow V$, and by construction it contains θ . \square

Finally, given \sim -receptive $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ (where \mathcal{A}, \mathcal{B} and \mathcal{C} are tcgs), we note that $V = \{p \in S \otimes T \mid (\tau \otimes \sigma)p \notin B\}$ is closed under symmetry; we can therefore apply the lemma above. Accordingly we define $\tilde{T} \odot \tilde{S}$ as $\tilde{T} \otimes \tilde{S} \downarrow V$, *i.e.* as comprising bijections $\theta : x \simeq y$ such that there exists

$$\theta \subseteq \bar{\theta} : [x]_{T \otimes S} \cong_{\tilde{T} \otimes \tilde{S}} [y]_{T \otimes S}$$

This makes $\mathcal{T} \odot \mathcal{S} = (T \odot S, \tilde{T} \odot \tilde{S})$ an event structure with symmetry. As an aside, we mention that in general the choice of $\bar{\theta}$ is *not* unique. However, in the next section, the *thinness* condition on strategies that will (as a side effect) ensure that $\bar{\theta}$ is unique.

Summing up, we state:

Lemma 3.25. If $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ are pre- \sim -strategies, then

$$\tau \odot \sigma : \mathcal{T} \odot \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{C}$$

is a map of ess.

Proof. First, we prove that $\tau \odot \sigma$ preserves symmetry. Let $\theta : x \cong_{\tilde{T} \odot \tilde{S}} y$. By definition, there is $\theta \subseteq \bar{\theta} : [x] \cong_{\tilde{T} \otimes \tilde{S}} [y]$. Then, $(\tau \otimes \sigma)\bar{\theta}$ is some

$$\theta_A \parallel \theta_B \parallel \theta_C : x_A \parallel x_B \parallel x_C \cong_{\tilde{A} \parallel \tilde{B} \parallel \tilde{C}} y_A \parallel y_B \parallel y_C$$

since $(\tau \otimes \sigma)$ preserves symmetry. But then $(\tau \odot \sigma)\theta$ is

$$\theta_A \parallel \theta_C : x_A \parallel x_C \cong_{\tilde{A} \parallel \tilde{C}} y_A \parallel y_C$$

which is a valid symmetry in $\tilde{A} \parallel \tilde{C}$ as required. \square

As said earlier, pre- \sim -strategies give a general notion of strategy with symmetry for which the composition operation is well-defined. However, the composition of pre- \sim -strategies is not in general a pre- \sim -strategy; it might fail \sim -receptivity. In combination with courtesy, \sim -receptivity *is* preserved by composition – but it will be convenient for later (for the interpretation of state in Section 6.3) to be able to compose pre- \sim -strategies which are *not* courteous. So we briefly deviate from our main narrative and provide a sufficient condition for \sim -receptivity to be stable under composition.

Definition 3.26. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ be a pre- \sim -strategy. We say that σ is (A, B) -courteous iff for all $s_1 \rightarrow s_2$ in \mathcal{S} , if $\text{pol}_{\mathcal{S}}(s_2) = -$ (i.e. $\text{pol}_{\mathcal{A}^\perp \parallel \mathcal{B}}(\sigma s_2) = -$), then s_1 and s_2 map to the same A/B component.

We will also say that $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ is **componentwise courteous** to mean that it is (A, B) -courteous, when \mathcal{A} and \mathcal{B} are clear from the context.

So σ is not necessarily courteous, but is not allowed to influence negative moves across components. As announced, we have the following.

Lemma 3.27. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be pre- \sim -strategies, such that σ is (A, B) -courteous and τ is (B, C) -courteous. Then, $\tau \circ \sigma$ is \sim -receptive and (A, C) -courteous.

Proof. As a preliminary to the proof, we note that thanks to (A, B) -courtesy of σ and (B, C) -courtesy of τ , the immediate dependencies of negative $p \in T \odot S$ in $T \otimes S$ *have* to be visible as well, and must map to the same component – this is the key argument of the proof. Indeed assume $p' \rightarrow p$ in $T \otimes S$ with visible p mapping to a negative event in $A^\perp \parallel C$, for instance in C . Then by general properties of the pullback in \mathcal{E} , since p maps to C , its immediate causal dependencies $p' \rightarrow p$ in $T \otimes S$ must be such that $\Pi_2 p' \rightarrow \Pi_2 p$ in $A \parallel T$ (consequence of e.g. Lemma 2.7 of [CCRW] with Proposition 2.16), but since p maps to C those must actually both be in T , and since τ is (B, C) -courteous p' must map to C as well, therefore it is visible.

From that, it is clear that $\tau \circ \sigma$ is (A, C) -courteous. We now show that it is \sim -receptive. We prove it via Lemma 3.16. Take $z \in \mathcal{C}(T \odot S)$, assume z extends via some negative p , say in C . The configuration z has a witness $[z] \in \mathcal{C}(T \otimes S)$, however in general this witness might not extend with p , as it may need to perform some invisible events prior to that. In our case though, the preliminary above shows that this is not possible: the immediate dependencies in $T \otimes S$ of p are visible as well, and hence in $z \subseteq [z]$. Now, if we also have that $(\tau \circ \sigma)z$ extends with c^- with $\text{id}_{(\tau \circ \sigma)z} \cup \{((\tau \circ \sigma)p, c)\} \in \tilde{A} \parallel \tilde{B} \parallel \tilde{C}$, then

$$\Pi_2 [z] \xrightarrow{\Pi_2 p} \text{id}_{\Pi_2 [z]} \cup \{((A \parallel \tau)(\Pi_2 p), c)\} \in \tilde{A} \parallel \tilde{B} \parallel \tilde{C}$$

so using \sim -receptivity of $A \parallel \tau$, we can uniquely lift c to $A \parallel T$, hence to $T \otimes S$ and $T \odot S$, and that lifting is by construction compatible with $\tilde{T} \otimes \tilde{S}$ and $\tilde{T} \odot \tilde{S}$. \square

At this point, we have a notion of (A, B) -courteous pre- \sim -strategy comprising copycat, and a notion of composition which, as will follow from Section 4, is associative up to (strong) isomorphism. In particular, it will also follow from 4 that the pre- \sim -strategies which are additionally *receptive* (so they are *strong-receptive*) and *courteous* (in the sense of Definition 2.20) form a bicategory.

3.3.3. *Weak isomorphisms.* However, the point of introducing symmetry in the first place was to be able to compare strategies *up to symmetry*, e.g. considering equal the two strategies of Example 2.26. To that end, we will restate Definition 2.28 in our current more general setting. But first, we note that the two sub-symmetries of a tcg allow us to compare pre- \sim -strategies *up to positive symmetry*.

Definition 3.28. Let \mathcal{A} be a tcg, $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\sigma' : \mathcal{S} \rightarrow \mathcal{A}$ be maps of ess. We say that σ and σ' are **positively symmetric**, if for all $x \in \mathcal{C}(\mathcal{S})$,

$$\{(\sigma s, \sigma' s) \mid s \in x\} \in \tilde{\mathcal{A}}_+$$

If σ and σ' are positively symmetric, we write $\sigma \sim^+ \sigma'$.

We can now reformulate Definition 2.28 in the setting of tcgs.

Definition 3.29. Let \mathcal{A} be a tcg, $\sigma : \mathcal{S} \rightarrow \mathcal{A}, \tau : \mathcal{T} \rightarrow \mathcal{A}$. A **weak morphism** from \mathcal{S} to \mathcal{T} is a map of ess $f : \mathcal{S} \rightarrow \mathcal{T}$ such that the triangle

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{f} & \mathcal{T} \\ \sigma \searrow & & \swarrow \tau \\ & \mathcal{A} & \end{array}$$

commutes *up to positive symmetry*, i.e. $\tau \circ f \sim^+ \sigma$.

If $f : \mathcal{S} \rightarrow \mathcal{T}$ and $g : \mathcal{T} \rightarrow \mathcal{S}$ are two weak morphisms such that $g \circ f = \text{id}_{\mathcal{S}}$ and $f \circ g = \text{id}_{\mathcal{T}}$, we say that (f, g) is a **weak isomorphism** between σ and τ . We write $\sigma \approx \tau$ to mean that σ and τ are weakly isomorphic.

The definition above, instantiated with the tcg $!A$, agrees with Definition 2.28.

Recall that in the previous section (below Definition 2.28), we motivated the necessity of enforcing *uniformity* by arguing that weak isomorphism should be a congruence, i.e. preserved by operations on strategies and in particular by composition. This is key if we want to build a quotient category of strategies up to weak isomorphism, in which the mismatch of Example 2.26 is solved.

So, is weak isomorphism a congruence? Unfortunately, we are not there yet. The reason is that if a strategy behaves like a strategy σ_1 if Opponent plays a certain move with index k_1 , and behaves like a strategy σ_2 if Opponent plays with an index k_2 , \sim -receptivity along with the axioms of isomorphism families only give us that σ_1 and σ_2 are *bisimilar* (in a sense that we avoid making formal here) but we need actual *morphisms* between σ_1 and σ_2 to establish a weak isomorphism. There is a quite subtle issue – the reader will find in Appendix A.2 a counter-example to the fact that weak isomorphism between pre- \sim -strategies is a congruence.

3.4. **Thin strategies.** To circumvent this issue, we will now add a new condition, *thinness*. For *thin* pre- \sim -strategies, weak isomorphism will be a congruence.

3.4.1. *Definition.* We start by illustrating the difficulty of the problem on an example.

Example 3.30. Consider the two following strategies on $![[\mathbf{com}]]$:

$$\begin{array}{ccc} \sigma_1 : ![\mathbf{com}] & & \sigma_2 : ![\mathbf{com}] \\ \mathbf{run}^{-,i} & & \mathbf{run}^{-,i} \\ \downarrow & & \downarrow \\ \mathbf{done}^{+,0} & & \mathbf{done}^{+,1} \end{array}$$

that only differ by the choice of copy index for **done**. There are obviously \sim -receptive (in fact they are strong-receptive). There is an obvious weak isomorphism $\varphi : \sigma_1 \approx \sigma_2$. Consider now the following strategy τ :

$$\begin{array}{ccc} ![\mathbf{com}] & + & ![\mathbf{com}] \\ & & \mathbf{run}^{-,i} \\ \mathbf{run}^{+, (i,0)} & \leftarrow & \mathbf{run}^{-,i} \\ \downarrow & & \downarrow \\ \mathbf{done}^{-,j} & & \mathbf{done}^{-,j} \\ \downarrow & & \downarrow \\ \mathbf{run}^{+, (i,j+1)} & & \mathbf{run}^{-,i} \\ \downarrow & & \downarrow \\ \mathbf{done}^{-,k} & \rightarrow & \mathbf{done}^{+, (j,k)} \end{array}$$

which represents $x : \mathbf{com} \vdash x; x : \mathbf{com}$.

In order to build a weak isomorphism between the resulting compositions $\tau \odot \sigma_1$ and $\tau \odot \sigma_2$, a reasonable first step is to build a weak isomorphism between the interactions $\tau \otimes \sigma_1$ and $\tau \otimes \sigma_2$. In particular, given a configuration of $T \otimes S_1$, we should be able to build a canonical configuration of $T \otimes S_2$. Consider *e.g.* the following configuration of $T \otimes S_1$.

$$\begin{array}{ccc} ![\mathbf{com}] & + & ![\mathbf{com}] \\ & & \mathbf{run}^{-,i} \\ \mathbf{run}^{(i,0)} & \leftarrow & \mathbf{run}^{-,i} \\ \downarrow & & \downarrow \\ \mathbf{done}^0 & & \mathbf{done}^{-,j} \\ \downarrow & & \downarrow \\ \mathbf{run}^{(i,1)} & & \mathbf{run}^{-,i} \\ \downarrow & & \downarrow \\ \mathbf{done}^0 & \rightarrow & \mathbf{done}^{+, (0,0)} \end{array}$$

where events on the left hand side are drawn without polarity, as they are synchronised between σ_1 and τ . It is easy to extract from this representation configurations $x \in \mathcal{C}(S_1 \parallel ![\mathbf{com}])$ and $y \in \mathcal{C}(T)$ such that

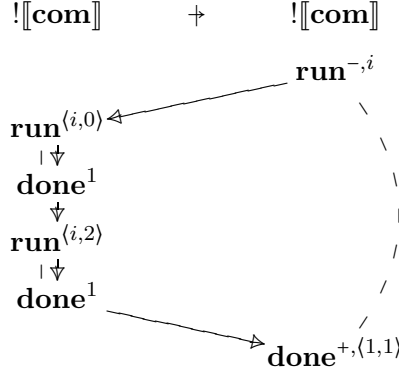
$$(\sigma_1 \parallel ![\mathbf{com}]) x = \tau y$$

and such that the induced bijection is secured.

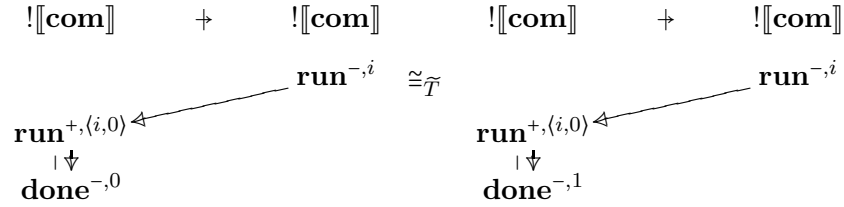
In order to construct a configuration in $T \otimes S_2$, it is natural to try and replace x with $\varphi(x)$ – and that would work out if φ was a *strong* isomorphism. But as it is only a weak isomorphism, we do not have $(\sigma_2 \parallel ![\mathbf{com}]) (\varphi x) = \tau y$, only

$$(\sigma_2 \parallel ![\mathbf{com}]) (\varphi x) \cong_{![\mathbf{com}]] ![\mathbf{com}]} \tau y$$

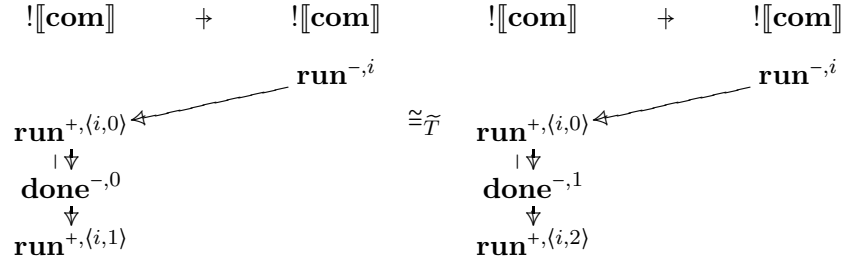
However, we can indeed extract from φx and y a valid configuration of $T \otimes S_2$. For our example, the only possibility is:



It appears that *both* φx and y had to change, in order to find an agreement as to the choice of copy indices. Firstly, by \sim -receptivity, \tilde{T} comprises a bijection:



By (Extension) in \tilde{T} , we know that this bijection can be extended to some:



Likewise, by \sim -receptivity of $\sigma_2 \parallel \llbracket \mathbf{com} \rrbracket$ this extension is lifted to $\widetilde{S}_2 \parallel \llbracket \mathbf{com} \rrbracket$, and we then apply (Extension) on \widetilde{S}_2 . And the process goes on, interactively between σ_2 and τ , until we get $x' \cong_{\widetilde{S} \parallel \llbracket \mathbf{com} \rrbracket} \varphi x$ and $y' \cong_{\tilde{T}} y$ such that $(\sigma_2 \parallel \llbracket \mathbf{com} \rrbracket) x' = \tau y'$ (which in our example, is the configuration of the interaction represented above).

Formalizing this process of using \sim -receptivity on one strategy and extension on the other yields the following lemma:

Lemma 3.31 (Weak bipullback property). Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ be pre- \sim -strategies. Let $x \in \mathcal{C}(\mathcal{S})$ and $y \in \mathcal{C}(\mathcal{T})$ and $\theta : \sigma x \cong_{\tilde{\mathcal{A}}} \tau y$, such that the composite bijection

$$x \stackrel{\sigma}{\simeq} \sigma x \cong_{\tilde{\mathcal{A}}} \tau y \stackrel{\tau}{\simeq} y$$

is secured. Then, there exists $z \in \mathcal{C}(\mathcal{S} \wedge \mathcal{T})$ along with $\theta_S : x \cong_{\tilde{\mathcal{S}}} \Pi_1 z$ and $\theta_T : \Pi_2 z \cong_{\tilde{\mathcal{T}}} y$, such that $\tau \theta_T \circ \sigma \theta_S = \theta$. Moreover, z is unique up to symmetry.

Proof. Uniqueness. Assume we have such (z, θ_S, θ_T) and $(z', \theta'_S, \theta'_T)$. Then it is easy to see that $\theta'_S \circ \theta_S^{-1} : \Pi_1 z \cong_{\tilde{S}} \Pi_1 z'$ and similarly $\theta'_T \circ \theta_T^{-1} : \Pi_2 z \cong_{\tilde{T}} \Pi_2 z'$. Those match on the game \mathcal{A} , so they induce a $z \cong z'$ in $\tilde{S} \wedge \tilde{T}$ as desired.

Existence. We proceed by induction on θ ; the base case is trivial. Assume θ extends by $(\sigma s, \tau t)$ to $\theta' : \sigma x' \cong \tau y'$. For instance, s is positive. We have $\theta_S : x \cong \Pi_1 z$ and x can be extended to x' by s , so by the extension property of the symmetry θ_S extends to $\theta'_S : x' \cong z'_S$. This means that $\tau \theta_T$ can be extended by symmetric negative (for T) events so by \sim -receptivity, θ_T can extend to $\theta'_T : z'_T \cong_{\tilde{T}} y'$, with $\sigma z'_S = \tau z'_T$ by construction. Since the bijection $z'_S \cong z'_T$ is obviously secured, we get $z' \in \mathcal{C}(S \wedge T)$ that satisfies our property. \square

Assume now that for some $\sigma_1 : \mathcal{S}_1 \rightarrow \mathcal{A}, \sigma_2 : \mathcal{S}_2 \rightarrow \mathcal{A}, \tau : \mathcal{T} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$, given a weak isomorphism φ from σ_1 to σ_2 , we wish to build a weak isomorphism $\mathcal{T} \otimes \varphi : \mathcal{T} \otimes \mathcal{S}_1 \rightarrow \mathcal{T} \otimes \mathcal{S}_2$. This would be in particular an isomorphism between the event structures $T \otimes S_1$ and $T \otimes S_2$, and it is proved in *e.g.* [CCRW] that such isomorphisms coincide with order-isomorphisms between $\mathcal{C}(T \otimes S_1)$ and $\mathcal{C}(T \otimes S_2)$. So in order to construct $\mathcal{T} \otimes \varphi$, we need to have, for each $z \in \mathcal{C}(T \otimes S_1)$, a canonical choice of $z' \in \mathcal{C}(T \otimes S_2)$. There is a clear way to obtain z' from z : necessarily $\sigma_2(f(\Pi_1 z)) \cong_{\tilde{\mathcal{A}}} \tau(\Pi_2 z)$, so by Lemma 3.31 we obtain $z' \in \mathcal{C}(T \otimes S_2)$.

This looks promising, however it is not enough – such a z' is only unique *up to symmetry*, so there is a choice to be made. One needs to pick such a z' for *each* $z \in \mathcal{C}(T \otimes S_1)$ such that the overall assignment gives an order-iso $\mathcal{C}(T \otimes S_1) \cong \mathcal{C}(T \otimes S_2)$. In the best case scenario one needs some choice principle to do that, in the worst case scenario it is not possible at all. The reader can find such a case in Appendix A.2, showing that in general, weak isomorphism is *not* a congruence *w.r.t.* composition by pre- \sim -strategies.

The issue here is relying too much on the (Extension) axiom: such extensions are not uniquely determined. In general they cannot be unique. Indeed, when we have $\theta : x \cong_{\tilde{S}} y$ and x can be extended by a negative s , we might have infinitely many choices to extend y with (*e.g.* by choosing the copy index). However, in the proof of Lemma 3.31, we only applied extension for s positive. This suggests to consider only strategies where the extension to positive events is always unique. It turns out this condition is exactly what we want and has a simple characterisation:

Definition 3.32. For $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ a map of ess with \mathcal{A} a tcg, then the following are equivalent:

- (i) For all $\theta : x \cong_{\tilde{S}} y$ such that $x \xrightarrow{s^+} x'$, there is a **unique** $s' \in S$ such that $\theta \cup \{(s, s')\} \in \tilde{S}$.
- (ii) Let x be a configuration of S that can be extended by s^+ . Then, for all $\theta \in \tilde{S}$ whose domain is $x \cup \{s\}$ and which contains id_x , we have that $\theta = \text{id}_{x \cup \{s\}}$.

Then we say \mathcal{S} (and σ) is **thin**. A \sim -**strategy** is a *thin, courteous, receptive pre- \sim -strategy*.

Proof. The condition (ii) is just a special case of (i). We prove (i) assuming (ii). Let $\theta : x \cong_{\tilde{S}} y$ with $x \xrightarrow{s^+} x'$. Assume there are two extensions of θ to x' : $\theta_1 : x' \cong_{\tilde{S}} y'_1$ and $\theta_2 : x' \cong_{\tilde{S}} y'_2$. Since both contain θ , $\theta_2 \circ \theta_1^{-1}$ extends id_y by a positive $(\theta_1(s), \theta_2(s))$. By (ii) we have $\theta_1(s) = \theta_2(s)$, hence $\theta_1 = \theta_2$. \square

We start by proving that our symmetry on copycat is thin:

Lemma 3.33. Let \mathcal{A} be a tcg. The copycat strategy on \mathcal{A} is a \sim -strategy.

Proof. Let $x \parallel y \in \mathcal{C}(\mathbb{C}_{\mathcal{A}})$. Assume $\text{id}_{x \parallel y}$ can be extended by, say, positive $((2, a), (2, a'))$ to $\theta : x \parallel y \cup \{a\} \cong_{\mathbb{C}_{\tilde{\mathcal{A}}}} x \parallel y \cup \{a'\}$. By definition of copycat, we must have $(1, a) \in x$ and

$(1, a') \in x$. We have $(1, a) \rightarrow_{\mathbb{C}_A} (2, a)$ hence $\theta(1, a) \rightarrow \theta(2, a)$ hence $(1, a) \rightarrow (2, a')$. By definition of copycat, this implies $a = a'$ hence $(2, a) = (2, a')$ as desired. \square

It is routine to check that parallel composition of thin symmetries remains thin, so thin pre- \sim -strategies are stable under parallel composition. The rest of this section is dedicated to proving that thinness is stable under composition, and that indeed weak isomorphism is preserved by composition. Putting these together, we will obtain a category of \sim -strategies up to weak isomorphism.

3.4.2. Stability under composition. The subtlety in considering the composition of thin pre- \sim -strategies is that not all symmetries involved are thin. Indeed, remember that the interaction of $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ is defined as the pullback of $\sigma \parallel \mathcal{C}$ and $\mathcal{A} \parallel \tau$. But \mathcal{A} and \mathcal{C} are of course not thin symmetries in general. For instance, the symmetry on $!\llbracket \mathbf{com} \rrbracket$ has a bijection $\{\mathbf{run}^{-,0} \mapsto \mathbf{run}^{-,0}, \mathbf{done}^{+,0} \mapsto \mathbf{done}^{+,1}\}$. However, this bijection does not belong to $!\llbracket \mathbf{com} \rrbracket_-$ which contains only bijections preserving positive indices. As a matter of fact, there is no non-trivial positive symmetry in $!\llbracket \mathbf{com} \rrbracket_-$. This observation can be generalized:

Lemma 3.34. Let \mathcal{A} be a tcg. The map of ess $\text{id} : \mathcal{A}_- \rightarrow \mathcal{A}$ is a \sim -strategy.

Proof. Consequence of Lemma 3.11. \square

One may be tempted to simply replace the pullback $(\mathcal{S} \parallel \mathcal{C}) \wedge (\mathcal{A} \parallel \mathcal{T})$ with $(\mathcal{S} \parallel \mathcal{C}_+) \wedge (\mathcal{A}_- \parallel \mathcal{T})$ in the composition of thin pre- \sim -strategies so that the interaction is reduced to a pullback between thin pre- \sim -strategies. But that would be incorrect: it is very important to compute the interaction against the full symmetries on the game, in order to allow Opponent to declare moves symmetric as they wish. In fact, the pullback of a thin pre- \sim -strategy against a dual thin pre- \sim -strategy has a degenerate symmetry: we start with the empty bijection $\emptyset \simeq \emptyset$ and no player is allowed to add non-trivial pairs (e, e') to the bijection as it would violate thinness. This observation yields:

Lemma 3.35. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ be thin pre- \sim -strategies. The isomorphism family $\tilde{S} \wedge \tilde{T}$ is trivial (reduced to identities).

Proof. We prove by induction that all bijections in $\tilde{S} \wedge \tilde{T}$ are identities. Let $z \in \mathcal{C}(S \wedge T)$ and assume id_z extends by (e, e') to $\theta \in \tilde{S} \wedge \tilde{T}$. Assume for instance $\Pi_2 e$ is positive in T (the other case is similar). By construction $\text{id}_{\Pi_2 z}$ extends to $\theta_T = \Pi_2 \theta \in \tilde{T}$ by positive events $(\Pi_2 e, \Pi_2 e')$, hence $\Pi_2 e = \Pi_2 e'$ and θ_T is the identity because \tilde{T} is thin. By local injectivity of Π_2 it follows that e and e' must be equal, or incompatible extensions of z . But if they are incompatible, by Lemma 2.15 (and Proposition 2.16) it means that $\Pi_1 e$ and $\Pi_1 e'$ are incompatible extensions of $\Pi_1 z$ mapping to the same event in the game, contradicting the \sim -receptivity of σ . Hence $e = e'$ and θ is the identity. \square

Even though there are non-trivial symmetries in the interaction of thin pre- \sim -strategies because the symmetries on \mathcal{A} and \mathcal{C} are not thin, whenever the interaction stays within B the phenomenon above applies, and the symmetry is fixed. As a result, a bijection in the symmetry of the interaction is fully determined by its restriction to visible events:

Lemma 3.36 (Unique witness). Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be thin pre- \sim -strategies. Recall the set of visible events of the interaction:

$$V = \{e \in T \otimes S \mid (\tau \otimes \sigma) e \notin B\}$$

Let $\theta : x \cong_{\tilde{T} \otimes \tilde{S}} y$ and $\theta' : x \cong_{\tilde{T} \otimes \tilde{S}} y'$ such that $\theta \cap V^2 = \theta' \cap V^2$. Then $\theta = \theta'$.

Proof. By hypothesis, we have that $y \cap V = y' \cap V$. Note that $\theta \circ \theta'^{-1} : y' \cong y \in (\tilde{S} \parallel \tilde{C}) \wedge (\tilde{A} \parallel \tilde{T})$ and contains $\text{id}_{y \cap V}$. As a result, $\theta \circ \theta'^{-1}$ actually belongs to $(\tilde{S} \parallel \tilde{C}_+) \wedge (\tilde{A}_- \parallel \tilde{T})$. By Lemma 3.34, this is a pullback of thin pre- \sim -strategies, so $\theta = \theta'$ are both identity bijections by Lemma 3.35. \square

Note that this implies that given a bijection in $\tilde{T} \odot \tilde{S}$ there is exactly one way to extend it to a bijection in $\tilde{T} \otimes \tilde{S}$, property which is not true in general for pre- \sim -strategies. With this remark, we can prove that the composition of thin pre- \sim -strategies is thin.

Lemma 3.37. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be thin pre- \sim -strategies.

Then $\tau \odot \sigma$ is thin.

Proof. Let $z \in \mathcal{C}(T \odot S)$ such that id_z extends by (e, e') to $\theta : x \cong y \in \tilde{T} \odot \tilde{S}$ with witness $\bar{\theta} : [x] \cong_{T \otimes S} [y]$. Write θ_0 for $\bar{\theta} \setminus \{(e, e')\} : x_0 \cong y_0$. By hypothesis, θ_0 behaves like the identity on the visible part of x_0 . Hence, by Lemma 3.36, θ_0 is the identity on x_0 .

Since $\text{id}_{x_0} = \theta_0$ can be extended by (e, e') to $\bar{\theta}$ which is positive in $T \odot S$ we can assume eg. $\Pi_2 e$ and $\Pi_2 e'$ are positive in T . Hence $\Pi_2 \theta_0$ (which is also an identity) extends by positive $(\Pi_2 e, \Pi_2 e')$. Since τ is thin, we have $\Pi_2 e = \Pi_2 e'$ from which $e = e'$ follows (e and e' are positive), as desired. \square

At this point (again, with the proviso that associativity of composition and neutrality of cpycat will be proved in Section 4), we have a category:

- Objects: Thin concurrent games
- Morphisms from \mathcal{A} to \mathcal{B} : \sim -strategies on $\mathcal{A}^\perp \parallel \mathcal{B}$ up to (strong) isomorphism.

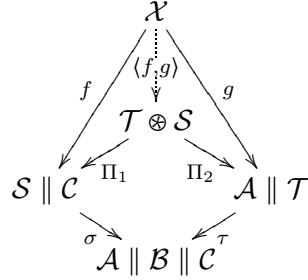
We now proceed to prove that unlike for pre- \sim -strategies, in this category weak isomorphism is a congruence.

3.4.3. The bipullback property. To prove that weak equivalence is a congruence we need to prove that our pullback satisfies a stronger property than Lemma 3.31 that allows us to actually build a map $S \otimes T \rightarrow S' \otimes T$ from a map $S \rightarrow S'$ commuting on the game up to symmetry.

Proposition 3.38 (Bipullback property). Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be thin pre- \sim -strategies.

The ess $\mathcal{T} \otimes \mathcal{S}$ enjoys the following universal property: for all $f : \mathcal{X} \rightarrow \mathcal{S} \parallel \mathcal{C}$ and $g : \mathcal{X} \rightarrow \mathcal{A} \parallel \mathcal{T}$ such that $\tau \circ g \sim_{\tilde{A} \parallel \tilde{B} \parallel \tilde{C}} \sigma \circ f$, there exists $\langle f, g \rangle : \mathcal{X} \rightarrow \mathcal{T} \otimes \mathcal{S}$, unique up to symmetry, such that $\Pi_1 \circ \langle f, g \rangle \sim_{\tilde{S} \parallel \tilde{C}_+} f$ and $\Pi_2 \circ \langle f, g \rangle \sim_{\tilde{A}_- \parallel \tilde{T}}$.

This is summed up by the following diagram (where all squares and triangle commutes up to \sim in the category of event structures with symmetry):



Proof. Uniqueness. Assume there are two such maps $\omega, \omega' : \mathcal{X} \rightarrow \mathcal{T} \otimes \mathcal{S}$. Let $x \in \mathcal{C}(X)$. Then the induced bijection $\Pi_1(\omega x) \cong \Pi_1(\omega' x)$ is in $\tilde{\mathcal{S}} \parallel \tilde{\mathcal{C}}$ as the composition

$$\Pi_1(\omega x) \cong_{\tilde{\mathcal{S}} \parallel \tilde{\mathcal{C}}} f x \cong_{\tilde{\mathcal{S}} \parallel \tilde{\mathcal{C}}} \Pi_1(\omega' x)$$

Similarly $\Pi_2(\omega x) \cong \Pi_2(\omega' x) \in \tilde{\mathcal{A}} \parallel \tilde{\mathcal{T}}$ and those two bijections match on the game, hence $\omega x \cong \omega' x \in \tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}}$ which means $\omega \sim_{\tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}}} \omega'$.

Existence. The main idea is to apply Lemma 3.31 to:

$$\begin{aligned} (\sigma \parallel \mathcal{C}_+) : \mathcal{S} \parallel \mathcal{C}_+ &\rightarrow \mathcal{A}^\perp \parallel \mathcal{B} \parallel \mathcal{C}^\perp \\ (\mathcal{A}_- \parallel \tau) : \mathcal{A}_- \parallel \mathcal{T} &\rightarrow \mathcal{A} \parallel \mathcal{B}^\perp \parallel \mathcal{C} \end{aligned}$$

Those are \sim -strategies hence their pullback has a trivial symmetry by Lemma 3.35. This means that uniqueness of Lemma 3.31 holds on the nose. This pullback has the same events as $\tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}}$, only less symmetry.

Let $x \in \mathcal{C}(X)$. By the above remark applied to $f x \in \mathcal{C}(S \parallel C)$ and $g x \in \mathcal{C}(A \parallel T)$ we get a unique $z \in \mathcal{C}(T \otimes S)$ with $\Pi_1 z \cong_{\tilde{\mathcal{S}} \parallel \tilde{\mathcal{C}}_+} f x$ and $\Pi_2 z \cong_{\tilde{\mathcal{A}}_- \parallel \tilde{\mathcal{T}}} g x$. This construction induces a map $\psi : \mathcal{C}(X) \rightarrow \mathcal{C}(T \otimes S)$ such that $\Pi_1(\psi x) \cong f x$ and $\Pi_2(\psi x) \cong g x$. The map ψ is monotonic and preserves cardinality. We show it preserves compatible unions: let $x, y \in \mathcal{C}(X)$ such that $x \cup y \in \mathcal{C}(X)$. Because ψ is monotonic, we have $\psi(x) \cup \psi(y) \subseteq \psi(x \cup y)$. Moreover, since $\text{card } \psi(x) = \text{card } f(x)$, and $f(x \cup y) = f(x) \cup f(y)$, it follows that $\psi(x \cup y) = \psi(x) \cup \psi(y)$.

Hence, we define $\langle f, g \rangle(s)$ to be the unique element of $\psi[s] \setminus \psi[s]$ (which is a singleton since ψ preserves cardinality). Using preservation of compatible unions, it follows that $\langle f, g \rangle(x) = \psi x \in \mathcal{C}(T \otimes S)$ for all $x \in \mathcal{C}(X)$ and local injectivity follows by construction. Hence $\langle f, g \rangle$ is a map of event structures. It preserves symmetry and satisfies the desired equivalence by construction. \square

3.4.4. *The category TCG.* The bipullback property is the key argument to prove that weak isomorphism is a congruence. However, there is still a missing step: the uniqueness up to symmetry of the bipullback property will only allow us to build maps that are inverse of each other *up to symmetry*. This lemma shows that they are automatically inverse of each other on the nose.

Lemma 3.39 (Weak equivalence and weak isomorphism coincide). Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}$ be thin pre- \sim -strategies such that there exists $f : \mathcal{S} \rightarrow \mathcal{T}$ and $g : \mathcal{T} \rightarrow \mathcal{S}$ such that $\tau \circ f \sim^+ \sigma$, $\sigma \circ g \sim^+ \tau$, $f \circ g \sim \text{id}_{\mathcal{T}}$ and $g \circ f \sim \text{id}_{\mathcal{S}}$.

Then $f \circ g = \text{id}_{\mathcal{T}}$ and $g \circ f = \text{id}_{\mathcal{S}}$ – in particular σ and τ are weakly isomorphic.

Proof. By hypothesis, for all $x \in \mathcal{C}(S)$, the obvious bijection $\theta_x : x \cong g(fx)$ is in \tilde{S} . We show by induction on x it is always the identity hence $g \circ f = \text{id}_S$.

The base case is trivial. Assume the result for $x \in \mathcal{C}(S)$ and suppose x extends by $s \in S$ to x' . If s is positive, then $\theta_x = \text{id}_x$ extends by $(s, \theta_{x'} s)$ which is positive, so since \tilde{S} is thin we have $s = \theta_{x'} s$ as desired.

If s is negative, we know that $\tau \circ f \sim^+ \sigma$ and that $\tau \sim^+ \sigma \circ g$. Hence $\sigma \sim^+ \sigma \circ g \circ f$. As a result, the obvious $\varphi_{x'} : \sigma x' \cong \sigma(g(fx'))$ is in \tilde{A}_+ . By induction hypothesis, we know it is an extension of the identity on σx . Hence $\text{id}_{\sigma x}$ extends in \tilde{A}_+ with $(\sigma s, \sigma(g(fs)))$ with s negative, so $\sigma s = \sigma(g(fs))$ by Lemma 3.11. By \sim -receptivity of σ it follows that $s = g(fs) = \theta_{x'} s$. \square

We are now ready to prove that weak isomorphism is a congruence.

Proposition 3.40. Let $\sigma : S \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$, $\sigma' : S' \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be thin pre- \sim -strategies such that $\sigma \approx \sigma'$, with maps $f : S \rightarrow S'$ and $g : S' \rightarrow S$.

Then, $\tau \circ \sigma \approx \tau \circ \sigma'$.

Proof. First, we notice that there are obvious maps:

$$\begin{array}{ccccc}
 & & \mathcal{T} \otimes \mathcal{S} & & \\
 & \swarrow \Pi_1 & & \searrow \Pi_2 & \\
 \mathcal{S} \parallel \mathcal{C} & & \mathcal{T} \otimes \mathcal{S}' & & \mathcal{A} \parallel \mathcal{T} \\
 \downarrow f \parallel \mathcal{C} & \swarrow \Pi'_1 & & \searrow \Pi'_2 & \parallel \\
 \mathcal{S}' \parallel \mathcal{C} & & & & \mathcal{A} \parallel \mathcal{T} \\
 & \swarrow \sigma' \parallel \mathcal{C} & & \searrow \mathcal{A} \parallel \tau & \\
 & \mathcal{A} \parallel \mathcal{B} \parallel \mathcal{C} & & &
 \end{array}$$

The outer diagram commutes up to symmetry because

$$(\sigma' \parallel \mathcal{C}) \circ (f \parallel \mathcal{C}) \circ \Pi_1 \sim (\sigma \parallel \mathcal{C}) \circ \Pi_1 = (\mathcal{A} \parallel \tau) \circ \Pi_2$$

By Proposition 3.38, there exists a map of ess $\mathcal{T} \otimes f : \mathcal{T} \otimes \mathcal{S} \rightarrow \mathcal{T} \otimes \mathcal{S}'$ (unique up to symmetry), such that $\Pi'_1 \circ (\mathcal{T} \otimes f) \sim_{\tilde{S}' \parallel \tilde{\mathcal{C}}_+} (f \parallel \mathcal{C}) \circ \Pi_1$ and $\Pi'_2 \circ (\mathcal{T} \otimes f) \sim_{\tilde{A}_- \parallel \tilde{\mathcal{T}}} \Pi_2$. A similar argument yields $\mathcal{T} \otimes g : \mathcal{T} \otimes \mathcal{S}' \rightarrow \mathcal{T} \otimes \mathcal{S}$ such that $\Pi_1 \circ (\mathcal{T} \otimes g) \sim_{\tilde{S}' \parallel \tilde{\mathcal{C}}_+} (g \parallel \mathcal{C}) \circ \Pi'_1$ and $\Pi_2 \circ (\mathcal{T} \otimes g) \sim_{\tilde{A}_- \parallel \tilde{\mathcal{T}}} \Pi'_2$. By the uniqueness up to symmetry of the bipullback property, we have that $(\mathcal{T} \otimes f) \circ (\mathcal{T} \otimes g) \sim \text{id}_{\mathcal{S}' \parallel \mathcal{C}}$ and $(\mathcal{T} \otimes g) \circ (\mathcal{T} \otimes f) \sim \text{id}_{\mathcal{S} \parallel \mathcal{C}}$.

Since $\Pi'_1 \circ (\mathcal{T} \otimes f) \sim_{\tilde{S}' \parallel \tilde{\mathcal{C}}_+} (f \parallel \mathcal{C}) \circ \Pi_1$, we also have

$$\begin{aligned}
 (\sigma' \parallel \mathcal{C}) \circ \Pi'_1 \circ (\mathcal{T} \otimes f) &\sim_{\tilde{A} \parallel \tilde{B} \parallel \tilde{\mathcal{C}}_+} (\sigma' \parallel \mathcal{C}) \circ (f \parallel \mathcal{C}) \circ \Pi_1 \\
 &\sim_{\tilde{A} \parallel \tilde{B} \parallel \tilde{\mathcal{C}}_+} (\sigma \parallel \mathcal{C}) \circ \Pi_1
 \end{aligned}$$

In other words, $(\tau \otimes \sigma') \circ (\mathcal{T} \otimes f) \sim_{\tilde{A} \parallel \tilde{B} \parallel \tilde{\mathcal{C}}_+} \tau \otimes \sigma$. But likewise, exploiting $\Pi'_2 \circ (\mathcal{T} \otimes f) \sim_{\tilde{A}_- \parallel \tilde{\mathcal{T}}} \Pi_2$, we deduce $(\tau \otimes \sigma') \circ (\mathcal{T} \otimes f) \sim_{\tilde{A}_- \parallel \tilde{B} \parallel \tilde{\mathcal{C}}} \tau \otimes \sigma$. By definition of symmetric maps, together this implies that:

$$(\tau \otimes \sigma') \circ (\mathcal{T} \otimes f) \sim_{\tilde{A}_- \parallel \tilde{B} \parallel \tilde{\mathcal{C}}_+} \tau \otimes \sigma$$

This implies in particular that $\mathcal{T} \otimes f$ preserves visible events. Hence we can restrict to $\mathcal{T} \circ f : \mathcal{T} \circ \mathcal{S} \rightarrow \mathcal{T} \circ \mathcal{S}'$, which, by restriction of the above, satisfies:

$$(\tau \circ \sigma') \circ (\mathcal{T} \circ f) \sim_{(\overline{A^\perp \parallel C})_+} \tau \circ \sigma$$

so $\mathcal{T} \circ f$ is a weak morphism from $\tau \circ \sigma$ to $\tau \circ \sigma'$. By the same reasoning, we define $\mathcal{T} \circ g$ as a weak morphism from $\tau \circ \sigma'$ to $\tau \circ \sigma$. Since $\mathcal{T} \otimes f$ and $\mathcal{T} \otimes g$ are inverses up to symmetry, so

are $\mathcal{T} \odot f$ and $\mathcal{T} \odot g$. But they are both weak morphisms, so by Lemma 3.39 they are inverses of each other on the nose, so they form a weak isomorphism. Therefore, $\tau \odot \sigma \approx \tau \odot \sigma'$. \square

Again, assuming the developments of the next section, this allows us to conclude:

Corollary 3.41. The following is a category written TCG:

- *Objects:* Thin concurrent games
- *Morphisms from \mathcal{A} to \mathcal{B} :* \sim -strategies on $\mathcal{A}^\perp \parallel \mathcal{B}$ up to weak isomorphism.

We will sometimes write $\sigma : \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{B}$ to mean that σ is a \sim -strategy from \mathcal{A} to \mathcal{B} , keeping the \mathcal{S} anonymous.

4. A COMPACT CLOSED CATEGORY

In the previous section, we have defined notions of games and strategies equipped with symmetry, in a tentative to express the symmetry induced by copy indices in games of the form $!A$ and to ensure that strategies behave uniformly *w.r.t.* this symmetry. We proved in particular that composition of strategies could be extended to these \sim -strategies, that respect symmetry. Finally, we proved that thanks to the uniformity constraints, weak isomorphism is a congruence for compositions with \sim -strategies.

In this section, we perform a more complete investigation of the categorical structure of \sim -strategies in concurrent games with symmetry. In particular, we show that just as for our plain concurrent games of [CCRW], concurrent games with symmetry and \sim -strategies between them form a *compact closed category*. This categorical structure will be key in constructing the cartesian closed category of Concurrent Hyland-Ong games in the next section.

In an attempt to avoid duplication of work *w.r.t.* [CCRW], we will show that many properties of concurrent games with symmetry and \sim -strategies can be *deduced* from the corresponding properties in [CCRW]. This will be done by exploiting the representation of event structures with symmetry as *spans of event structures* in \mathcal{E} .

4.1. Event structures with symmetry as spans. In [Win07], event structures with symmetry are described differently than in the present paper. They are defined as spans of event structures

$$\begin{array}{ccc} & \widehat{E} & \\ l_E \swarrow & & \searrow r_E \\ E & & E \end{array}$$

satisfying some further properties: l_E, r_E are jointly monic, they are *open maps* [Win07], and they satisfy the diagrams of (categorical) equivalence relations. The detail of these conditions will not be useful here; however we will use that the category \mathcal{E}_\sim can be embedded fully and faithfully in $\text{Span}(\mathcal{E})$, defined as below.

Definition 4.1. Let \mathbb{C} be a (small) category. As objects, the category $\text{Span}(\mathbb{C})$ has spans in \mathbb{C} whose both legs map to the same object of \mathbb{C} . As morphisms, $\text{Span}(\mathbb{C})$ has pairs (f, \tilde{f})

yielding commuting diagrams:

$$\begin{array}{ccccc} A & \xleftarrow{l_A} & \widehat{A} & \xrightarrow{r_A} & A \\ \downarrow f & & \downarrow \widehat{f} & & \downarrow f \\ B & \xleftarrow{l_B} & \widehat{B} & \xrightarrow{r_B} & B \end{array}$$

From isomorphism families to spans. In [Win07], it is proved that event structures with symmetry can be equivalently defined through such spans, plus further conditions. Here we will not need this equivalence, but only the fact that event structures with symmetry can be *represented* as spans.

Let $\mathcal{E} = (E, \widetilde{E})$ be an event structure with symmetry. Similarly to the definition of events in an interaction pullback, the events of \widetilde{E} will be certain prime bijections. Recall that any $\theta \in \widetilde{E}$ is an order-isomorphism. Hence, it is trivially a secured bijection, and its graph is ordered. So, as in Definition 2.13, we can form an event structure from its *primes*, *i.e.* those symmetries whose graph has a top element. For such a prime θ , write (e_θ^l, e_θ^r) for its top element.

Definition 4.2. From the isomorphism family \widetilde{E} , we build the event structure $\text{Pr}(\widetilde{E})$, with:

- *Events.* Prime symmetries in \widetilde{E} ,
- *Causality.* Inclusion,
- *Consistency.* For X a finite set of prime symmetries, we set $X \in \text{Con}_{\text{Pr}(\widetilde{E})}$ iff $\cup X \in \widetilde{E}$.

Moreover, there are projections maps of event structures

$$\begin{array}{ccc} l_E : \text{Pr}(\widetilde{E}) & \rightarrow & E \\ \theta & \mapsto & e_\theta^l \end{array} \qquad \begin{array}{ccc} r_E : \text{Pr}(\widetilde{E}) & \rightarrow & E \\ \theta & \mapsto & e_\theta^r \end{array}$$

forming a span $E \xleftarrow{l_E} \text{Pr}(\widetilde{E}) \xrightarrow{r_E} E$ as above.

The axioms of event structures are direct to check, along with the fact that l_E, r_E are maps of event structures. Note also that this definition extends in the presence of polarity in a straightforward way.

Just like for interactions (Proposition 2.16), configurations of $\text{Pr}(\widetilde{E})$ are in direct correspondence with symmetries.

Proposition 4.3. For any $x \in \mathcal{C}(\text{Pr}(\widetilde{E}))$, writing $y_l = l_E x$ and $y_r = r_E x$, we have $\theta_x = \cup x : y_l \cong y_r \in \widetilde{E}$. Moreover, the assignment:

$$\begin{array}{ccc} \mathcal{C}(\text{Pr}(\widetilde{E})) & \rightarrow & \widetilde{E} \\ x & \mapsto & \theta_x \end{array}$$

is an order-isomorphism (with both sets ordered by inclusion). Finally, there is a family of order-isomorphisms:

$$\begin{array}{ccc} \nu_x : x & \cong & \theta_x \\ \theta & \mapsto & (e_\theta^l, e_\theta^r) \end{array}$$

that is natural in x .

Proof. Straightforward adaptation of Proposition 2.16. □

In order to transport thin concurrent games to spans, we will make use of the following key proposition.

Proposition 4.4. There is a full and faithful strong monoidal functor (where the monoidal structure of $\text{Span}(\mathcal{E})$ is obtained by component-wise action of that of \mathcal{E}).

$$\text{Span} : \mathcal{E}_{\sim} \rightarrow \text{Span}(\mathcal{E})$$

This extends trivially in the presence of polarities.

Proof. Details can be found in [Win07]. \square

4.2. Spanning games and strategies. In order to inherit properties of TCG from CG, we will use the functor above. For that to be useful, we show that the construction of the compact closed category CG can be replicated with *spans* rather than mere games.

A **span-game** is a span $\mathbb{A} = A \xleftarrow{l_A} \widehat{A} \xrightarrow{r_A} A$, where \widehat{A}, A are esps. A **span-strategy** on \mathbb{A} is a morphism $(\sigma, \widehat{\sigma}) : \mathbb{S} \rightarrow \mathbb{A}$ in $\text{Span}(\mathcal{EP})$ (where \mathcal{EP} is the category of event structures with polarities, with maps preserving polarities).

$$\begin{array}{ccccc} S & \xleftarrow{l_S} & \widehat{S} & \xrightarrow{r_S} & S \\ \downarrow \sigma & & \downarrow \widehat{\sigma} & & \downarrow \sigma \\ A & \xleftarrow{l_A} & \widehat{A} & \xrightarrow{r_A} & A \end{array}$$

where $\sigma, \widehat{\sigma}$ are strategies in CG, *i.e.* are receptive and courteous.

Constructions on games can be generalized to span-games. The **dual** of a span-game $A \xleftarrow{l_A} \widehat{A} \xrightarrow{r_A} A$ is $A^\perp \xleftarrow{l_A^\perp} \widehat{A}^\perp \xrightarrow{r_A^\perp} A^\perp$, making use of the functor $(-)^\perp : \mathcal{EP} \rightarrow \mathcal{EP}$. Likewise, **simple parallel composition** is extended to spans following the functor $\parallel : \mathcal{EP}^2 \rightarrow \mathcal{EP}$. As before we define span-strategies **from \mathbb{A} to \mathbb{B}** as span-strategies $(\sigma, \widehat{\sigma}) : \mathbb{S} \rightarrow \mathbb{A}^\perp \parallel \mathbb{B}$, also written $(\sigma, \widehat{\sigma}) : \mathbb{A} \dashrightarrow \mathbb{B}$.

We define the copycat span-strategy:

Lemma 4.5. The esp construction $A \mapsto \mathbb{C}_A$ extends to a functor $\mathbb{C} : \mathcal{EP} \rightarrow \mathcal{EP}$.

Proof. From $f : A \rightarrow B$, we form $f^\perp \parallel f : A^\perp \parallel A \rightarrow B^\perp \parallel B$. It is direct that we have also $\mathbb{C}_f = f^\perp \parallel f : \mathbb{C}_A \rightarrow \mathbb{C}_B$. \square

Using the above, we construct the copycat span-strategy as the diagram:

$$\begin{array}{ccccc} \mathbb{C}_A & \xleftarrow{\mathbb{C}_{l_A}} & \mathbb{C}_{\widehat{A}} & \xrightarrow{\mathbb{C}_{r_A}} & \mathbb{C}_A \\ \downarrow \mathbb{C}_A & & \downarrow \mathbb{C}_{\widehat{A}} & & \downarrow \mathbb{C}_A \\ A^\perp \parallel A & \xleftarrow{l_A^\perp \parallel l_A} & \widehat{A}^\perp \parallel \widehat{A} & \xrightarrow{r_A^\perp \parallel r_A} & A^\perp \parallel A \end{array}$$

Besides copycat, other constructions on strategies act in a functorial way. In particular, in order to extend strategy composition to span-strategies, we recall ([CCRW], Lemma 4.4) the following lemma.

Lemma 4.6. Consider two commuting diagrams between strategies:

$$\begin{array}{ccc} S_1 & \xrightarrow{f} & S_2 \\ \sigma_1 \downarrow & & \downarrow \sigma_2 \\ A_1^\perp \parallel B_1 & \xrightarrow{h_1^\perp \parallel h_2} & A_2^\perp \parallel B_2 \end{array} \quad \begin{array}{ccc} T_1 & \xrightarrow{g} & T_2 \\ \tau_1 \downarrow & & \downarrow \tau_2 \\ B_1^\perp \parallel C_1 & \xrightarrow{h_2^\perp \parallel h_3} & B_2^\perp \parallel C_2 \end{array}$$

Then, the following diagram commutes.

$$\begin{array}{ccc} T_1 \odot S_1 & \xrightarrow{g \odot f} & T_2 \odot S_2 \\ \tau_1 \odot \sigma_1 \downarrow & & \downarrow \tau_2 \odot \sigma_2 \\ A_1^\perp \parallel C_1 & \xrightarrow{h_1^\perp \parallel h_3} & A_2^\perp \parallel C_2 \end{array}$$

(recall that $g \odot f$ is the restriction of $g \otimes f : T_1 \otimes S_1 \rightarrow T_2 \otimes S_2$ to visible events, which in turn is obtained by universal property of the interaction pullback)

Thus, from two span-strategies

$$\begin{array}{ccc} S & \xleftarrow{l_S} & \widehat{S} & \xrightarrow{r_S} & S \\ \sigma \downarrow & & \downarrow \widehat{\sigma} & & \downarrow \sigma \\ A^\perp \parallel B & \xleftarrow{l_A \parallel l_B} & \widehat{A}^\perp \parallel \widehat{B} & \xrightarrow{r_A \parallel r_B} & A^\perp \parallel B \end{array} \quad \begin{array}{ccc} T & \xleftarrow{l_T} & \widehat{T} & \xrightarrow{r_T} & T \\ \tau \downarrow & & \downarrow \widehat{\tau} & & \downarrow \tau \\ B^\perp \parallel C & \xleftarrow{l_B^\perp \parallel l_C} & \widehat{B}^\perp \parallel \widehat{C} & \xrightarrow{r_C^\perp \parallel r_C} & B^\perp \parallel C \end{array}$$

we obtain componentwise a new span-strategy:

$$\begin{array}{ccc} T \odot S & \xleftarrow{l_T \odot l_S} & \widehat{T} \odot \widehat{S} & \xrightarrow{r_T \odot r_S} & T \odot S \\ \tau \odot \sigma \downarrow & & \downarrow \widehat{\tau} \odot \widehat{\sigma} & & \downarrow \tau \odot \sigma \\ A^\perp \parallel C & \xleftarrow{l_A^\perp \parallel l_C} & \widehat{A}^\perp \parallel \widehat{C} & \xrightarrow{r_A^\perp \parallel r_C} & A^\perp \parallel C \end{array}$$

Together, we expect composition of span-strategies and the copycat span-strategy to form a category. In fact they will form a *bicategory*, whose 2-cells will be **morphisms of span-strategies**: commuting diagrams as below.

$$\begin{array}{ccccc} & & S_2 & \xleftarrow{l_{S_2}} & \widehat{S}_2 & \xrightarrow{r_{S_2}} & S_2 \\ & & \nearrow f & & \nearrow \widehat{f} & & \nearrow f \\ S_1 & \xleftarrow{l_{S_1}} & \widehat{S}_1 & \xrightarrow{r_{S_1}} & S_1 & & \\ & & \searrow \sigma_1 & & \searrow \widehat{\sigma}_1 & & \searrow \sigma_1 \\ & & A & \xleftarrow{l_A} & \widehat{A} & \xrightarrow{r_A} & A \\ & & \searrow \sigma_2 & & \searrow \widehat{\sigma}_2 & & \searrow \sigma_2 \end{array}$$

By the construction of the associator $\alpha_{\sigma, \tau, \rho} : \rho \odot (\tau \odot \sigma) \simeq (\rho \odot \tau) \odot \sigma$ in [CCRW] through a universal property, it follows easily that it extends to an isomorphism of span-strategies. Likewise by Lemma 4.12 of [CCRW], so do the unitors $\lambda_\sigma : \mathbb{C}_B \odot \sigma \simeq \sigma$ and $\rho_\sigma : \sigma \odot \mathbb{C}_A \simeq \sigma$. All naturality and coherence laws are inherited from [CCRW], so it follows:

Theorem 4.7. There is a bicategory SpanCG having span-games as objects, span-strategies as morphisms and, as 2-cells, morphisms of span-strategies.

Besides its bicategorical structure, SpanCG inherits from CG its compact closed structure. From two span-games \mathbb{A} and \mathbb{B} , their tensor $\mathbb{A} \otimes \mathbb{B}$ is simply defined as $\mathbb{A} \parallel \mathbb{B}$. Likewise, the action of \otimes on span-strategies is obtained componentwise from its action on strategies [CCRW].

In [CCRW], we showed how to *lift* any courteous receptive map $f : A \rightarrow B$ in \mathcal{EP} to a strategy $\bar{f} : A \rightarrow B$. This lifting process was instrumental in defining the compact closed structure of CG. This process is easily extended to span-games. For a morphism (f, \hat{f}) from span \mathbb{A} to span \mathbb{B} such that f and \hat{f} are receptive and courteous, its **lifting** is the span-strategy $(\bar{f}, \bar{\hat{f}})$ obtained by composition in $\text{Span}(\mathcal{E})$:

$$\begin{array}{ccccc}
 \mathbb{C}_A & \xleftarrow{c_{l_A}} & \mathbb{C}_{\hat{A}} & \xrightarrow{c_{r_A}} & \mathbb{C}_A \\
 \downarrow c_A & & \downarrow c_{\hat{A}} & & \downarrow c_A \\
 A^\perp \parallel A & \xleftarrow{l_A^\perp \parallel l_A} & \hat{A}^\perp \parallel \hat{A} & \xrightarrow{r_A^\perp \parallel r_A} & A^\perp \parallel A \\
 \downarrow A^\perp \parallel f & & \downarrow \hat{A}^\perp \parallel \hat{f} & & \downarrow A^\perp \parallel f \\
 A^\perp \parallel B & \xleftarrow{l_B^\perp \parallel l_B} & \hat{A}^\perp \parallel \hat{B} & \xrightarrow{r_B^\perp \parallel r_B} & A^\perp \parallel B
 \end{array}$$

In other words, it is defined as the span morphism with components \bar{f} and $\bar{\hat{f}}$, where $\bar{}$ denotes the lifting operation of [CCRW].

Following [CCRW], using lifting, structural morphisms of the obvious symmetric monoidal structure for $\text{Span}(\mathcal{EP})$ (with \parallel as tensor) can be lifted to SpanCG. From Lemma 4.12 of [CCRW] and the construction of lifting above, these structural morphisms in SpanCG are componentwise those of CG. It follows that all equations for a compact closed category hold up to isomorphism of span-strategies.

4.3. Embedding TCG in SpanCG. From the constructions above, we will transfer to TCG the compact closed structure of SpanCG. In fact, we will prove that all equations of the compact closed structure hold up to *strong isomorphism*, rather than just weak isomorphism – but obviously a strong isomorphism is in particular a weak isomorphism.

4.3.1. Copycat and composition. In order to embed TCG into SpanCG, we show that the functor $\text{Span} : \mathcal{EP}_\sim \rightarrow \text{Span}(\mathcal{EP})$ (where \mathcal{EP}_\sim is the category of event structures with polarities and symmetry and structure-preserving maps) also preserves all our main constructions on strategies: copycat and composition (and lifting).

Lemma 4.8. Let \mathcal{A} be a tcg. Write $\hat{\mathcal{A}} = \text{Pr}(\tilde{\mathcal{A}})$. Then, there is an iso making the following diagram commute

$$\begin{array}{ccc}
 & \mathbb{C}_A & \\
 l_{\mathbb{C}_A} \nearrow & & \nwarrow c_{l_A} \\
 \text{Pr}(\mathbb{C}_{\hat{\mathcal{A}}}) & \simeq & \mathbb{C}_{\hat{\mathcal{A}}} \\
 r_{\mathbb{C}_A} \searrow & & \swarrow c_{r_A} \\
 & \mathbb{C}_A &
 \end{array}$$

and which also preserves the projections to the span-game

$$A^\perp \parallel A \xleftarrow{l_A} \hat{A}^\perp \parallel \hat{A} \xrightarrow{r_A} A^\perp \parallel A$$

In particular, this yields an isomorphism of span-strategies.

Proof. Thanks to Lemma 2.12 of [CCRW], it suffices to check that $\Pr(\mathbb{C}_{\tilde{A}})$ and $\mathbb{C}_{\tilde{A}}$ have isomorphic domains of configurations. Using Lemma 2.14 of [CCRW], the required commutations can be checked pointwise on configurations.

By Proposition 4.3, configurations of $\Pr(\mathbb{C}_{\tilde{A}})$ canonically correspond to symmetries in $\mathbb{C}_{\tilde{A}}$. By their definition (Definition 3.17), they are those of the form:

$$\theta_1 \parallel \theta_2 : x \parallel y \cong x' \parallel y'$$

where $x \parallel y, x' \parallel y' \in \mathcal{C}(\mathbb{C}_A)$, and where $\theta_2 \sqsubseteq_{\tilde{A}^\perp \parallel \tilde{A}} \theta_1$. But $\theta_1, \theta_2 \in \tilde{A}^\perp, \tilde{A}$, which, by Proposition 4.3 again, correspond canonically to configurations $z_1, z_2 \in \mathcal{C}(\tilde{A}^\perp), \mathcal{C}(\tilde{A})$ such that $z_2 \sqsubseteq_{\tilde{A}} z_1$ – in other words, by Lemma 3.10 of [CCRW], to configurations $z_2 \parallel z_1 \in \mathcal{C}(\mathbb{C}_{\tilde{A}})$.

All commutations are immediate to check. \square

From the lemma above, we know that copycat on tcgs is defined in a way compatible with the copycat span-strategy in the compact closed category of span-games. We now prove a similar lemma for composition.

Lemma 4.9. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be thin pre- \sim -strategies. Write $\tilde{\mathcal{S}}$ for $\Pr(\tilde{\mathcal{S}})$ and $\tilde{\mathcal{T}}$ for $\Pr(\tilde{\mathcal{T}})$. There is an isomorphism of esps making the following diagram commute

$$\begin{array}{ccc} & T \otimes S & \\ l_{T \otimes S} \nearrow & & \nwarrow l_{T \otimes S} \\ \Pr(\tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}}) & \cong & \tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}} \\ & & \\ r_{T \otimes S} \searrow & & \swarrow r_{T \otimes S} \\ & T \otimes S & \end{array}$$

and which also preserve the projections to the underlying span-game. In particular, this yields an isomorphism of span-strategies.

Proof. We do it for interactions. The property for composition will follow, as the isomorphism will (by consequence of preserving the projections to games) preserve visible events. Again, we rely on Lemma 2.12 of [CCRW] and build an order-isomorphism between the underlying domains and configurations.

As above, configurations of $\Pr(\tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}})$ correspond canonically to symmetries in $\tilde{\mathcal{T}} \otimes \tilde{\mathcal{S}}$. By definition (above Lemma 3.23), those correspond to commuting squares between composite secured bijections $x \simeq y$ and $x' \simeq y'$

$$\begin{array}{ccc} x & \xrightarrow{\sigma} & \sigma x & = & \tau y & \xrightarrow{\tau} & y \\ \theta_A \wr \parallel_{\tilde{\mathcal{S}}} & & & & & & \theta_T \wr \parallel_{\tilde{\mathcal{T}}} \\ x' & \xrightarrow{\sigma} & \sigma x' & = & \tau y' & \xrightarrow{\tau} & y' \end{array}$$

In particular, this gives a bijection between pairs $(s, s') \in \theta_S$ and $(t, t') \in \theta_T$. This bijection is secured, since the upper and lower sides of the diagram are secured by hypothesis and θ_S and θ_T are order-isos. By Proposition 4.3, θ_S and θ_T canonically represent configurations $z_S \in \mathcal{C}(\tilde{\mathcal{S}})$ and $z_T \in \mathcal{C}(\tilde{\mathcal{T}})$ – so overall, diagrams as above canonically correspond to secured composite bijections between z_S and z_T , as required. By construction these correspondences commute with the projections to $T \otimes S$ and to the underlying span-game. \square

From this, we can use Theorem 4.7 to transport the bicategorical structure of SpanCG to TCG, and in particular show that TCG is a category up to isomorphism – recall that along with Proposition 3.40, it will follow that it is also a category w.r.t. *weak* isomorphism. Beyond \sim -strategies, it also follows that the composition of the component-wise courteous pre- \sim -strategies of Definition 3.26 is associative.

We now inherit from SpanCG its compact closed categorical structure.

4.3.2. Lifting and compact closure. First of all, we note that just like \mathcal{EP} and $\text{Span}(\mathcal{EP})$, the category \mathcal{EP}_{\sim} has a symmetric monoidal structure. We have, for instance, a natural isomorphism

$$\lambda_{\mathcal{A}} : 1 \parallel \mathcal{A} \rightarrow \mathcal{A}$$

Just as above, the structural \sim -strategies involved in the symmetric monoidal structure of TCG will be lifted from \mathcal{EP}_{\sim} . Detailing a bit more:

Definition 4.10. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be a strong-receptive, courteous map of \mathcal{EP}_{\sim} . Then its **lifting** is the \sim -strategy

$$\overline{f} = (\mathcal{A}^{\perp} \parallel f) \circ \sigma : \mathbb{C}_{\mathcal{A}} \rightarrow \mathcal{A}^{\perp} \parallel \mathcal{B}$$

which is a morphism from \mathcal{A} to \mathcal{B} in TCG (in particular, it is thin).

All required verifications are immediate. Applying this to $\lambda_{\mathcal{A}}$ yields, for instance:

$$\overline{\lambda_{\mathcal{A}}} : 1 \otimes \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{A}$$

Moreover, $\text{Span}(\overline{\lambda_{\mathcal{A}}}) = \overline{\text{Span}(\lambda_{\mathcal{A}})}$, and structural \sim -strategies of TCG are sent to those of SpanCG. The functor Span being full, it follows that all isomorphisms involved in the compact closed structure of SpanCG are imported in TCG. As remarked above they are in particular weak isomorphisms, so we have

Theorem 4.11. The category TCG is compact closed.

Proof. For completeness, we list here all structural morphisms for the symmetric monoidal structure of \mathcal{EP}_{\sim} .

$$\begin{aligned} \rho_{\mathcal{A}} & : & \mathcal{A} \parallel 1 & \rightarrow & \mathcal{A} \\ \lambda_{\mathcal{A}} & : & 1 \parallel \mathcal{A} & \rightarrow & \mathcal{A} \\ s_{\mathcal{A},\mathcal{B}} & : & \mathcal{A} \parallel \mathcal{B} & \rightarrow & \mathcal{B} \parallel \mathcal{A} \\ \alpha_{\mathcal{A},\mathcal{B},\mathcal{C}} & : & (\mathcal{A} \parallel \mathcal{B}) \parallel \mathcal{C} & \rightarrow & \mathcal{A} \parallel (\mathcal{B} \parallel \mathcal{C}) \end{aligned}$$

These isomorphisms are then lifted to \sim -strategies.

$$\begin{aligned} \overline{\rho_{\mathcal{A}}} & : & \mathcal{A} \otimes 1 & \xrightarrow{\text{TCG}} & \mathcal{A} \\ \overline{\lambda_{\mathcal{A}}} & : & 1 \otimes \mathcal{A} & \xrightarrow{\text{TCG}} & \mathcal{A} \\ \overline{s_{\mathcal{A},\mathcal{B}}} & : & \mathcal{A} \otimes \mathcal{B} & \xrightarrow{\text{TCG}} & \mathcal{B} \otimes \mathcal{A} \\ \overline{\alpha_{\mathcal{A},\mathcal{B},\mathcal{C}}} & : & (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} & \xrightarrow{\text{TCG}} & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \end{aligned}$$

As explained above, all coherence and naturality laws follow from the fact that the relevant constructions for games based on \mathcal{EP}_{\sim} are mapped to those based on $\text{Span}(\mathcal{EP})$, and from the compact closed structure of SpanCG.

For completeness, we also mention that there are (copycat) \sim -strategies

$$\eta_{\mathcal{A}} : 1 \xrightarrow{\text{TCG}} \mathcal{A}^{\perp} \otimes \mathcal{A} \quad \epsilon_{\mathcal{A}} : \mathcal{A} \otimes \mathcal{A}^{\perp} \xrightarrow{\text{TCG}} 1$$

satisfying the necessary equations up to isomorphism of \sim -strategies. \square

Finally, we will import from [CCRW] the *lifting lemma*, which we will use later.

Lemma 4.12. Let $f : \mathcal{B} \rightarrow \mathcal{C}$ be a strong-receptive courteous map of essps, and $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ be a \sim -strategy. Then, the following strategies are isomorphic:

$$\begin{aligned} \overline{f} \circ \sigma & : \mathbb{C}_{\mathcal{B}} \circ \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{C} \\ (\mathcal{A}^\perp \parallel f) \circ \sigma & : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{C} \end{aligned}$$

Proof. By Lemma 4.6 and Lemma 4.12 of [CCRW], the lifting lemma of CG (Lemma 5.4 of [CCRW]) lifts to span-strategies. By Lemma 4.8 and functoriality of $\text{Span}(-)$, we have that $\text{Span}(\overline{f}) \cong \overline{\text{Span}(f)}$. The result follows by Lemma 4.9. \square

Note that for $f : \mathcal{B}^\perp \rightarrow \mathcal{A}^\perp$ a strong-receptive courteous map of essps we have the dual lifting $\overline{f}^\perp = (f \parallel \mathcal{B}) \circ \mathbb{C}_{\mathcal{B}} : \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{B}$; and, by duality, the symmetric lemma to the above holds: for $\sigma : \mathcal{B} \xrightarrow{\text{TCG}} \mathcal{C}$, $\sigma \circ \overline{f}^\perp \cong (f \parallel \mathcal{C}) \circ \sigma$. Finally we note:

Lemma 4.13. Let $f : \mathcal{A} \rightarrow \mathcal{B}$ be an isomorphism of essps – so both f and f^{-1} are strong-receptive courteous. Then, $\overline{f} \cong \overline{f^{-1}}^\perp$.

Proof. Straightforward. \square

5. CONCURRENT HYLAND-ONG GAMES

We have constructed a compact closed category TCG, which is equipped to deal with the problem evoked at the end of Section 2. Using it, we can revisit (more formally) the interpretation sketched in Subsection 2.5. Exploiting the developments of the previous section, and in particular the fact that weak isomorphism is a congruence, it will follow that the two terms of Example 2.26 cannot be distinguished by any strategy in the model. Indeed, we will get a cartesian closed category supporting the interpretation of IPA.

We will first construct the category Cho and show that it is cartesian, then prove that it is also closed. Finally, we will prove that it supports the interpretation of a fixpoint combinator.

From now on, all event structures are assumed to have binary conflict. All the operations we will consider on them (simple parallel composition, composition, interaction, *etc.*) have been established throughout the development to preserve that property.

5.1. The cartesian category Cho. We now construct the category Cho proper, of *Concurrent Hyland-Ong games*; and prove that it is *cartesian*. The **objects** of Cho will be negative arenas, as in Definition 2.4 – with the further restriction that arenas should have a *countable* set of events, assumed from now on. The **morphisms** from arena A to arena B will be certain \sim -strategies from $!A$ to $!B$ (up to weak isomorphism):

$$\sigma : \mathcal{S} \rightarrow (!A)^\perp \parallel (!B)$$

Just as in standard Hyland-Ong games, we will have to restrict the set of strategies that we consider in order to satisfy the laws of a cartesian category. We will now inspect the different requirements of a cartesian category, and introduce the additional conditions on strategies when they are required.

5.1.1. *Terminal object and negativity.* First of all, a cartesian category has a terminal object. In our case, this will be the **empty arena** $\mathbb{1}$, defined as having an empty set of events – note that $!\mathbb{1}$ also has an empty set of events. However, as it is, $\mathbb{1}$ is not a terminal object. For each negative arena A , it is easy to see that the unique labelling function

$$\mathbf{e}_A : \mathbb{1} \rightarrow (!A)^\perp \parallel (!\mathbb{1})$$

is a \sim -strategy. Crucially, it is receptive since, by negativity of A , the minimal events of $(!A)^\perp$ are all positive. However, \mathbf{e}_A might not be the *unique* \sim -strategy from $!A$ to $\mathbb{1}$, as illustrated below.

Example 5.1. The following diagram represents a \sim -strategy from $!\mathbf{com}$ to $\mathbb{1}$.

$$\begin{array}{c} (!\mathbf{com})^\perp \parallel (!\mathbb{1}) \\ \mathbf{run}^{+,0} \end{array}$$

The answer to this issue is clear: we need to require morphisms in \mathbf{Cho} to be negative, just as arenas. A \sim -strategy $\sigma : \mathcal{S} \rightarrow (!A)^\perp \parallel (!B)$ is **negative** whenever the underlying event structure S is negative, *i.e.* its minimal events all have negative polarity – note that this definition makes sense in general without symmetry, for a prestrategy $\sigma : S \rightarrow A$.

We then easily have:

Proposition 5.2. For any negative arena A , the empty \sim -strategy:

$$\mathbf{e}_A : \mathbb{1} \rightarrow (!A)^\perp \parallel (!\mathbb{1})$$

is the unique negative \sim -strategy from $!A$ to $!\mathbb{1}$.

In more generality, the only negative prestrategy $\sigma : S \rightarrow A^\perp \parallel \mathbb{1}$ for a negative game A is the empty prestrategy.

Proof. Immediate, as in a negative prestrategy $\sigma : S \rightarrow A^\perp \parallel \mathbb{1}$, any hypothetical minimal events in S have nowhere to map to. \square

Thus, in order to get a category of \sim -strategies with a terminal object, we will require that all \sim -strategies are negative. Clearly, copycat – along with all \sim -strategies obtained by lifting – is negative. Moreover, negative \sim -strategies are stable under composition. Since negativity makes sense without symmetry, we state and prove that in slightly greater generality.

Lemma 5.3. Let $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$ be negative prestrategies (with A, B, C negative). Then, $\tau \odot \sigma$ is still negative.

Proof. First, we notice that any map of event structures preserves minimal events. Indeed let $f : A \rightarrow B$ be a map of event structures and $a \in A$ be a minimal event of A . This means $[a] = \{a\}$ and $f[a] = \{fa\}$ is a configuration of B . Since $f[a]$ is down-closed, this implies that fa is minimal in B .

Hence, minimal events of $T \otimes S$ are projected to minimal events of $S \parallel C$ and $A \parallel T$. Take $e \in T \otimes S$ a minimal event. If $(\tau \otimes \sigma)e$ is in A , then $\Pi_1 e$ is a minimal event of S projected to a (necessarily positive) minimal event of A – absurd because σ is negative. Likewise, if $(\tau \otimes \sigma)e$ is in B , this contradicts the negativity of τ . So minimal events of $T \otimes S$ are visible and are in C .

Now, take any minimal event $e \in T \otimes S$. Since minimal events of $T \otimes S$ are visible, e is also minimal in $T \otimes S$. By the previous remark, $(\tau \otimes \sigma)e$ is in C and is minimal. It is also negative because C is negative. \square

Therefore, the category having arenas as objects and as morphisms from A to B , *negative* \sim -strategies $\sigma : \mathcal{S} \rightarrow (!A)^\perp \parallel (!B)$ up to weak isomorphism, has a terminal object $\mathbb{1}$. We now investigate the existence of products.

5.1.2. *Binary products and single-threadedness.* For two arenas A and B , their **product** $A \times B$ is defined as the parallel composition $A \parallel B$, which is still a negative arena.

Projections. Note that there is an injection map of event structures with symmetry:

$$\begin{array}{ccc} i_A : & !A & \rightarrow & !(A \times B) \\ & (\alpha : [a] \rightarrow \omega) & \mapsto & \left(\begin{array}{ccc} \alpha' : & [(1, a)] & \rightarrow & \omega \\ & (1, a') & \mapsto & \alpha(a') \end{array} \right) \end{array}$$

Likewise, there is $i_B : !B \rightarrow !(A \times B)$. Using those, we define the **projections**

$$\varpi_A : \mathbb{C}_{!A} \rightarrow !(A \times B)^\perp \parallel !A \quad \varpi_B : \mathbb{C}_{!B} \rightarrow !(A \times B)^\perp \parallel !B$$

by lifting the injections, *i.e.* $\varpi_A = \overline{i_A}^\perp$ and $\varpi_B = \overline{i_B}^\perp$ (see Definition 4.10).

Pairing. Now, for two negative \sim -strategies $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !A^\perp \parallel !C$, we wish to define their pairing $\langle \sigma, \tau \rangle$, a \sim -strategy from $!A$ to $!(B \times C)$. This \sim -strategy will simply be obtained by relabeling the parallel composition of \mathcal{S} and \mathcal{T} . In simple cases, it suffices to take the co-pairing:

$$\begin{aligned} \langle \sigma, \tau \rangle & : \mathcal{S} \parallel \mathcal{T} \rightarrow !A^\perp \parallel !(B \times C) \\ & = [(!A^\perp \parallel i_B) \circ \sigma, (!A^\perp \parallel i_C) \circ \tau] \end{aligned}$$

However, this is not always well-defined as a \sim -strategy. Indeed, it might fail local injectivity if some events in S and T have the same image in $!A^\perp$. As a first step towards the general construction of pairing, let us prove that this gives a well-defined \sim -strategy when the images of σ and τ are disjoint.

Lemma 5.4. If negative \sim -strategies $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !A^\perp \parallel !C$ have *disjoint codomain* on $!A^\perp$, then $\langle \sigma, \tau \rangle$ as above is a negative \sim -strategy.

Proof. First, we prove that it is a \sim -strategy. By construction, $\langle \sigma, \tau \rangle$ preserves configurations. Local injectivity follows from local injectivity of σ , τ , and the hypothesis that they have disjoint codomains; so it is a map of event structures. Preservation of symmetry follows from the fact that for $\theta_1 : x_1 \cong y_1, \theta_2 : x_2 \cong y_2 \in \widetilde{!A}$ with $x_1 \cap x_2 = y_1 \cap y_2 = \emptyset$, we still have $\theta_1 \cup \theta_2 \in \widetilde{!A}$, which follows from definition of $\widetilde{!A}$. Courtesy and thinness are obvious by construction.

Strong-receptivity needs further attention. Take $\theta : x_S \parallel x_T \cong y_S \parallel y_T \in \widetilde{\mathcal{S} \parallel \mathcal{T}}$, write $\theta = \theta_S \parallel \theta_T$. Its projection to the game is

$$\langle \sigma, \tau \rangle \theta = ((!A^\perp \parallel i_B) \circ \sigma \theta_S) \uplus ((!A^\perp \parallel i_C) \circ \tau \theta_T)$$

which, as argued above, is a valid symmetry in $G = !A^\perp \parallel !(B \times C)$. Assume it extends by a pair (c_1^-, c_2^-) . Since dependency in the game is forest-shaped, there are unique $d_1 \rightarrow_G c_1^-$ and $d_2 \rightarrow c_2^-$, and since symmetries are order-preserving, we have $(d_1, d_2) \in \langle \sigma, \tau \rangle \theta$. But that means that it must be either in $((!A^\perp \parallel i_B) \circ \sigma) \theta_S$, or in $((!A^\perp \parallel i_C) \circ \tau) \theta_T$. We can then apply strong-receptivity of σ , τ , and the injection maps, to produce the extension to $\theta = \theta_S \parallel \theta_T$. \square

Now, we prove that this simple pairing behaves well *w.r.t.* projections.

Proposition 5.5. Assume negative \sim -strategies $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !A^\perp \parallel !C$ as in the previous lemma. Then, we have isomorphisms:

$$\varpi_B \odot \langle\langle \sigma, \tau \rangle\rangle \cong \sigma \varpi_C \odot \langle\langle \sigma, \tau \rangle\rangle \cong \tau$$

Proof. Let us prove the first. More precisely, we prove that the interactions $\varpi_B \otimes \langle\langle \sigma, \tau \rangle\rangle$ and $\mathfrak{c}_{!B} \otimes \sigma$ are isomorphic. This will entail by restriction an isomorphism between the corresponding compositions, and the latter is isomorphic to σ by neutrality of copycat for composition.

We establish the isomorphism between $\varpi_B \otimes \langle\langle \sigma, \tau \rangle\rangle$ and $\mathfrak{c}_{!B} \otimes \sigma$ first for the plain event structures – by Lemma 2.12 of [CCRW] it suffices to prove that they have an isomorphic domain of configurations.

Using Proposition 2.16, we know that configurations of the event structure for the former interaction correspond to secured bijections

$$(x_S \parallel x_T) \parallel x_B \simeq y_A \parallel (y_B^1 \parallel y_B^2)$$

where $x_S \in \mathcal{C}(S)$, $x_T \in \mathcal{C}(T)$, $\langle\langle \sigma, \tau \rangle\rangle(x_S \parallel x_T) = y_A \parallel (i_B y_B^1)$, and $y_B^1 \parallel y_B^2 \in \mathcal{C}(\mathfrak{C}_{!B})$, and where the bijection is the unique such that image of events through the labelings $\langle\langle \sigma, \tau \rangle\rangle \parallel !B$ and $!A \parallel \varpi_B$ match. In particular, $\langle\langle \sigma, \tau \rangle\rangle(x_S \parallel x_T)$ does not reach $!C$. But any minimal events of x_T are negative by negativity of τ , and hence must be in $!C$ (since A is negative). Therefore, x_T is empty. Getting rid of x_T yields a secured bijection corresponding to a configuration of the event structure of $\mathfrak{c}_{!B} \otimes \sigma$. This association is bijective, and yields the required isomorphism between domains of configurations. By construction, it is clear that this isomorphism preserves symmetry. \square

So, we know how to construct a pairing behaving well with projections, when the paired strategies happen to have a disjoint codomain. However, for arbitrary $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !A^\perp \parallel !C$, there might in general be collisions: events $s \in S$ and $t \in T$ such that $\sigma s = \tau t$. In such a case, the co-pairing as above fails local injectivity, and therefore does not correspond to a strategy. Fortunately, we can *relabel* moves of \mathcal{S} and \mathcal{T} , not changing their weak isomorphism class, to ensure that there are no such collisions. For that, we note that there are maps of event structures with symmetry

$$\iota_e : !A^\perp \rightarrow !A^\perp \quad \iota_o : !A^\perp \rightarrow !A^\perp$$

such that $\iota_e \sim^+ \iota_o \sim^+ \text{id}_{!A^\perp}$, but such that ι_e and ι_o have disjoint codomain. For definiteness, say that ι_e sends (necessarily positive) minimal events with copy index i to the same events with copy index $2i$, and preserves the copy index of other events. Likewise, ι_o could follow the injection $i \mapsto 2i + 1$. These maps preserve the index of negative events, so that $\iota_e \sim^+ \iota_o \sim^+ \text{id}_{!A^\perp}$.

Given arbitrary $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !A^\perp \parallel !C$, we define:

$$\sigma_e = (\iota_e \parallel !B) \circ \sigma \quad \tau_o = (\iota_o \parallel !C) \circ \tau$$

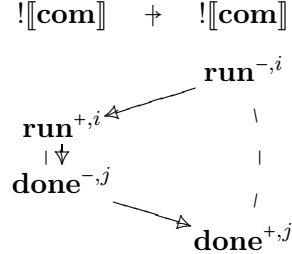
From $\iota_e \sim^+ \iota_o \sim^+ \text{id}_{!A^\perp}$ it is obvious that $\sigma \approx \sigma_e$ and $\tau \approx \tau_o$, but σ_e and τ_o now have disjoint codomains: σ_e (resp. τ_o) only reaches indexing functions in $!A$ whose index for minimal events is even (resp. odd). Therefore, using Proposition 5.5, we define:

$$\langle\langle \sigma, \tau \rangle\rangle = \langle\langle \sigma_e, \tau_o \rangle\rangle$$

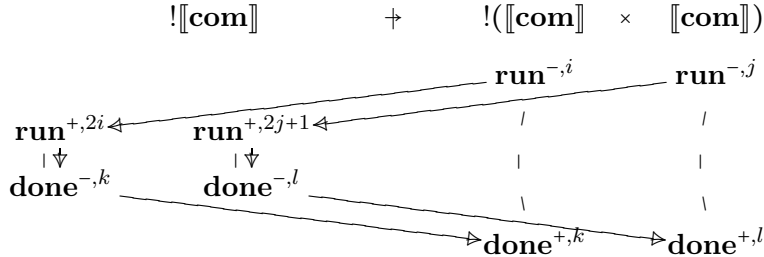
The **pairing** of arbitrary negative \sim -strategies $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !A^\perp \parallel !C$ is defined as $\langle\langle \sigma, \tau \rangle\rangle$. We have, as required, $\varpi_B \odot \langle\langle \sigma, \tau \rangle\rangle = \varpi_B \odot \langle\langle \sigma_e, \tau_o \rangle\rangle \cong \sigma_e \approx \sigma$, and for

the same reason $\varpi_C \odot \langle \sigma, \tau \rangle \approx \tau$. It is an immediate verification that $\langle -, - \rangle$ preserves weak isomorphism, so it will still make sense as an operation on the quotient category.

Example 5.6. Consider the copycat strategy $\mathfrak{C}_{![[\mathbf{com}]]}$ on $![[\mathbf{com}]]$.



Following the definition above, the construction of $\langle \mathfrak{C}_{![[\mathbf{com}]]}, \mathfrak{C}_{![[\mathbf{com}]]} \rangle$ produces the \sim -strategy illustrated below.



As prescribed by the construction, the positive moves on the left hand side had to be relabeled to avoid the collision in the case where $i = j$.

Note that the representation above only displays the event part of the \sim -strategy $\langle \mathfrak{C}_{![[\mathbf{com}]]}, \mathfrak{C}_{![[\mathbf{com}]]} \rangle$, but its construction also equips it with a symmetry ensuring its uniformity.

Surjective pairing. In order to obtain a product, we also need to prove surjective pairing, that is, that for all $\sigma : S \rightarrow !A^+ \parallel !(B \times C)$, we have:

$$\sigma \approx \langle \varpi_B \odot \sigma, \varpi_C \odot \sigma \rangle$$

However, as it stands, this is in general not the case.

Example 5.7. In Figure 6, we display on the left hand side two \sim -strategies $\sigma_1, \sigma_2 : !\mathbb{1} \xrightarrow{\text{TCG}} !([[\mathbf{com}]] \times [[\mathbf{com}]])$, and on the right hand side the corresponding distinct \sim -strategies obtained by projection and pairing.

We observe that surjective pairing fails for these strategies, as behaviours that span both components get erased through composition with the projections.

The analysis of this phenomenon is the same as in standard Hyland-Ong games [Har99]: there, the condition of *single-threadedness* ensures that strategies treat independently events hereditarily caused by distinct minimal events. The definition is independent from symmetry, so we state it first in more generality.

Definition 5.8. Let $\sigma : S \rightarrow A$ be a prestrategy. We say that σ is **single-threaded** if it satisfies the following two conditions.

- (1) For any $s \in S$, $[s]$ has exactly one minimal event written $\text{init}(s)$.

$$\begin{array}{ccc}
 \sigma_1 & : & !(\llbracket \mathbf{com} \rrbracket \times \llbracket \mathbf{com} \rrbracket) \quad \langle \varpi_l \odot \sigma_1, \varpi_r \odot \sigma_1 \rangle \quad : \quad !(\llbracket \mathbf{com} \rrbracket \times \llbracket \mathbf{com} \rrbracket) \\
 & & \begin{array}{ccc}
 \mathbf{run}^{-,i} & \mathbf{run}^{-,j} & \\
 \downarrow & \swarrow & \\
 \mathbf{done}^{+,j} & &
 \end{array} \\
 \\
 \sigma_2 & : & !(\llbracket \mathbf{com} \rrbracket \times \llbracket \mathbf{com} \rrbracket) \quad \langle \varpi_l \odot \sigma_2, \varpi_r \odot \sigma_2 \rangle \quad : \quad !(\llbracket \mathbf{com} \rrbracket \times \llbracket \mathbf{com} \rrbracket) \\
 & & \begin{array}{ccc}
 \mathbf{run}^{-,i} & \mathbf{run}^{-,j} & \\
 \downarrow & \downarrow & \\
 \mathbf{done}^{+,0} & \sim \mathbf{done}^{+,0} & \\
 & & \begin{array}{ccc}
 \mathbf{run}^{-,i} & \mathbf{run}^{-,j} & \\
 \downarrow & \downarrow & \\
 \mathbf{done}^{+,0} & \mathbf{done}^{+,0} &
 \end{array}
 \end{array}
 \end{array}$$

FIGURE 6. Failures to surjective pairing

(2) Whenever $s_1 \# s_2$ in S , $\text{init}(s_1) = \text{init}(s_2)$.

Single-threaded \sim -strategies always satisfy surjective pairing.

Proposition 5.9. Let $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !(B \times C)$ be a single-threaded \sim -strategy. Then, we have:

$$\sigma \approx \langle \varpi_B \odot \sigma, \varpi_C \odot \sigma \rangle$$

Proof. First of all, we define two subsets of S as follows:

$$\begin{aligned}
 S_B &= \{s \in S \mid \sigma(\text{init}(s)) \in B\} \\
 S_C &= \{s \in S \mid \sigma(\text{init}(s)) \in C\}
 \end{aligned}$$

(we abuse notations slightly with $\in B, \in C$).

By single-threadedness, S_B and S_C are disjoint and down-closed, with no immediate conflict spanning both components – in other words, $S = S_B \uplus S_C$. They are obviously still event structures. It is direct to check that the restrictions of σ (along with a simple relabeling to $!B/!C$)

$$\sigma_B : S_B \rightarrow !A^\perp \parallel !B \quad \sigma_C : S_C \rightarrow !A^\perp \parallel !C$$

are receptive and courteous, *i.e.* are strategies.

This decomposition also works at the level of symmetries. Any $\theta \in \widetilde{S}$ preserves S_B and S_C . Indeed if $(s_B, s_C) \in \theta$, then $(\text{init}(s_B), \text{init}(s_C)) \in \theta$ as well: absurd, since one maps to $!B$ and the other to $!C$. It follows that $\theta = \theta_B \uplus \theta_C$ where θ_B and θ_C are bijections between configurations of S_B and S_C respectively. The set of restrictions to S_B (resp. S_C) of symmetries in \widetilde{S} yields a set of bijections between configurations of S_B (resp. S_C), which is easily checked to satisfy the conditions for an isomorphism family \widetilde{S}_B (resp. \widetilde{S}_C). The labeling functions σ_B and σ_C preserve symmetry. Strong-receptivity and thinness follow directly from those for σ , so σ_B and σ_C are \sim -strategies.

By construction, σ_B and σ_C have disjoint codomain; so we can form their pairing $\langle\langle \sigma_B, \sigma_C \rangle\rangle$ without relabeling. Then, the obvious bijection $S = S_B \uplus S_C \cong S_B \parallel S_C$ is an isomorphism of event structures, preserves symmetry, and preserves labeling so as to yield an isomorphism of \sim -strategies:

$$\sigma \cong \langle\langle \sigma_B, \sigma_C \rangle\rangle$$

By Proposition 5.5, it follows that $\varpi_B \odot \sigma \cong \sigma_B$ and $\varpi_C \odot \sigma \cong \sigma_C$. But clearly, $\langle\langle \sigma_B, \sigma_C \rangle\rangle \approx \langle\sigma_B, \sigma_C\rangle$, and $\langle -, - \rangle$ preserves weak isomorphism, so we have surjective pairing. \square

So, single-threadedness ensures surjective pairing. It is clear that copycat \sim -strategies – and lifted \sim -strategies in general – on (expanded) arenas are single-threaded, since $\mathbb{C}_{!A}$ has the shape of a conflict-free forest. In order to get a cartesian category, the last thing to check is that single-threaded strategies are stable under composition.

Single-threadedness and its stability under composition is independent from symmetry, so we state it and prove it below in greater generality.

Proposition 5.10. Let $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : \mathcal{T} \rightarrow B^\perp \parallel C$ be negative single-threaded prestrategies. Then, $\tau \odot \sigma$ is single-threaded.

Proof. We first prove that the interaction $T \otimes S$ satisfies the single-threadedness conditions. More precisely, we prove by induction on φ that for any secured bijection $\varphi : x_S \parallel x_C \simeq y_A \parallel y_T$ representing (via Proposition 2.16) a configuration of $T \otimes S$, then

$$\varphi = \bigsqcup_{1 \leq i \leq n} \varphi_i$$

where each φ_i is a secured bijection with a unique minimal event. Indeed, assume $\varphi \xrightarrow{(c,d)} \varphi'$ where φ' fails this condition. Necessarily, either $c = (1, s^+)$ or $d = (2, t^+)$, *w.l.o.g.* assume the first. Then, the immediate predecessors of (c, d) in $\leq_{\varphi'}$ must be $((1, s_1^-), d_1), \dots, ((1, s_p^-), d_p)$ (using Lemma 2.7 of [CCRW]), with $s_i \rightarrow_S s$. By hypothesis, there are $1 \leq i, j \leq p$ and distinct $1 \leq k \neq l \leq n$ such that $((1, s_i^-), d_i) \in \varphi_k$ and $((1, s_j^-), d_j) \in \varphi_l$. But φ_k (resp. φ_l) must contain an event synchronized with $\text{init}(s_i)$ (resp. $\text{init}(s_j)$). Since σ is single-threaded and $s_i, s_j \in [s]$ we have $\text{init}(s_i) = \text{init}(s_j)$, which contradicts $\varphi_l \cap \varphi_k = \emptyset$.

Now, we go on to prove single-threadedness.

(1) Prime secured bijections have no non-trivial decomposition as above, therefore they have a unique minimal event. This is true in particular for the *visible* prime secured bijections. Condition (1) of single-threadedness follows then from the fact used in the proof of Lemma 5.3 that a minimal event in the interaction of negative strategies is always visible.

(2) Finally, assume there is a minimal conflict $\varphi \sim \psi$ in $T \otimes S$ between visible prime secured bijections. This means that there are non-necessarily visible prime secured bijections $\varphi' \subseteq [\varphi]_{T \otimes S}$, $\psi' \subseteq [\psi]_{T \otimes S}$, such that $\varphi' \sim \psi'$ in $T \otimes S$. Writing φ'' (resp. ψ'') for φ' (resp. ψ') without its top event, minimality of $\varphi' \sim \psi'$ means that $\varphi'' \cup \psi''$ is a valid secured bijection. Therefore, it decomposes:

$$\varphi'' \cup \psi'' = \bigsqcup_{1 \leq i \leq n} \varphi_i$$

With each φ_i a secured bijection having exactly one minimal event. If $n = 1$, we are done since as remarked the unique minimal event is necessarily visible. Otherwise, there are at least two φ_i, φ_j with distinct minimal events.

Then, using Lemma 2.15, $\varphi' \sim \psi'$ implies that their top elements have the form $((1, s_{\varphi'}), d_{\varphi'})$ and $((1, s_{\psi'}), d_{\psi'})$ with $s_{\varphi'} \sim_S s_{\psi'}$, or $(c_{\varphi'}, (2, t_{\varphi'}))$ and $(c_{\psi'}, (2, t_{\psi'}))$ with $t_{\varphi'} \sim_T t_{\psi'}$, *w.l.o.g.* say the first. By receptivity and courtesy of σ , we have $\text{pol}(s_{\varphi'}) = \text{pol}(s_{\psi'}) = +$. Since $n \geq 2$, there are $(c_1, d_1) \rightarrow_{\varphi} ((1, s_{\varphi'}^+), d_{\varphi'})$ with $(c_1, d_1) \in \varphi_i$ and $(c_2, d_2) \rightarrow_{\psi} ((1, s_{\psi'}^+), d_{\psi'})$ with $(c_2, d_2) \in \varphi_j$. By Lemma 2.7 of [CCRW], as an immediate dependency of $((1, s_{\varphi'}^+), d_{\varphi'})$, we have $(c_1, d_1) = ((1, s_1^-), d_1)$ with $s_1 \rightarrow_S s_{\varphi'}$ (similarly, $(c_2, d_2) = ((1, s_2^-), d_2)$ with $s_2 \rightarrow_S s_{\psi'}$). But by single-threadedness of σ , $\text{init}(s_{\varphi'}) = \text{init}(s_{\psi'})$, so there should be an event synchronized with $\text{init}(s_1) = \text{init}(s_2)$ both in φ_i and φ_j , absurd. \square

At this point, we have finished constructing our basic category of Concurrent Hyland-Ong games. Let us call Cho the category having: as objects, negative arenas; and as morphisms from A to B , negative single-threaded \sim -strategies $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$, up to weak isomorphism. We will also sometimes write $\sigma : A \xrightarrow{\text{Cho}} B$ to mean that σ is a morphism from A to B in Cho, keeping \mathcal{S} anonymous.

From all the developments above, we get:

Proposition 5.11. The category Cho has finite products.

In particular, it follows as usual that \times is a bifunctor $\text{Cho}^2 \rightarrow \text{Cho}$, by setting $\sigma_1 \times \sigma_2 = \langle \sigma_1 \circ \varpi_{A_1}, \sigma_2 \circ \varpi_{A_2} \rangle$, for $\sigma_1 : A_1 \xrightarrow{\text{Cho}} B_1$ and $\sigma_2 : A_2 \xrightarrow{\text{Cho}} B_2$.

When constructing the cartesian closed structure, we will leverage the compact closed structure of the underlying category TCG. Therefore, it is useful to connect the cartesian structure of Cho with the monoidal structure of TCG. For that, we note that there is an obvious isomorphism of essps:

$$m_{A,B} : !(A \times B) \cong !A \otimes !B$$

Using Definition 4.10, it follows that there is an isomorphism $\overline{m_{A,B}}$ in TCG between $!(A \times B)$ and $!A \otimes !B$. This isomorphism allows us to connect better the cartesian structure of Cho with the monoidal structure of TCG.

Lemma 5.12. Let $\sigma_1 : A_1 \xrightarrow{\text{Cho}} B_1$ and $\sigma_2 : A_2 \xrightarrow{\text{Cho}} B_2$. Then,

$$\sigma_1 \times \sigma_2 \approx \overline{m_{B_1, B_2}}^{-1} \circ (\sigma_1 \otimes \sigma_2) \circ \overline{m_{A_1, A_2}}$$

Proof. Write $\sigma_1 : \mathcal{S}_1 \rightarrow !A_1^\perp \parallel !B_1$ and $\sigma_2 : \mathcal{S}_2 \rightarrow !A_2^\perp \parallel !B_2$. By definition,

$$\sigma_1 \times \sigma_2 = \langle \sigma_1 \circ \varpi_{A_1}, \sigma_2 \circ \varpi_{A_2} \rangle$$

By Lemma 4.12, $\sigma_1 \circ \varpi_{A_1} \cong (i_{A_1^\perp} \parallel !B_1) \circ \sigma_1$ and $\sigma_2 \circ \varpi_{A_2} \cong (i_{A_2^\perp} \parallel !B_2) \circ \sigma_2$. These maps have disjoint codomain, so up to weak isomorphism their pairing is

$$\langle (i_{A_1^\perp} \parallel !B_1) \circ \sigma_1, (i_{A_2^\perp} \parallel !B_2) \circ \sigma_2 \rangle : \mathcal{S}_1 \parallel \mathcal{S}_2 \rightarrow !(A_1 \times A_2)^\perp \parallel !(B_1 \times B_2)$$

Likewise by Lemma 4.12, $\overline{m_{A_2, B_2}}^{-1} \circ (\sigma_1 \otimes \sigma_2) \circ \overline{m_{A_1, B_1}}$ has (up to isomorphism) essp $\mathcal{S}_1 \parallel \mathcal{S}_2$ and labeling function the obvious relabeling of $\sigma_1 \otimes \sigma_2$ by m_{A_1, A_2} and m_{B_1, B_2} . It is a simple verification that these two coincide. \square

5.2. Cartesian closure. We finish the construction of our cartesian closed category by describing the cartesian closure. We have constructed Cho as a subcategory of TCG – which, as a compact closed category, is symmetric monoidal closed. We wish to leverage this closed structure of TCG in order to transfer it to Cho.

5.2.1. Arrow arena. For two thin concurrent games \mathcal{A} and \mathcal{B} in TCG, the corresponding exponential object (following the compact closed structure) is obtained as $\mathcal{A}^\perp \parallel \mathcal{B}$. In Cho, where objects are arenas, this hints at defining the exponential object of A, B as $A^\perp \parallel B$. Indeed, it is easy to check that $!(A^\perp \parallel B) \cong !A^\perp \parallel !B$, so this matches the closed structure of TCG. However, objects in Cho are required to be *negative* arenas, and $A^\perp \parallel B$ is no longer negative. Therefore, we are brought to introduce a negative variant of $A^\perp \parallel B$, that would be an object of Cho. The natural choice, familiar from Hyland-Ong games, is to make events in A depend on minimal events of B . It would be incorrect to make events of

A depend on *all* minimal events of B , so we will instead create as many copies of A as they are minimal events in B . Writing $\min(B)$ for the set of minimal events of B , we define:

Definition 5.13. Let A, B be two negative arenas. Their **arrow** is $A \Rightarrow B$, with the following components.

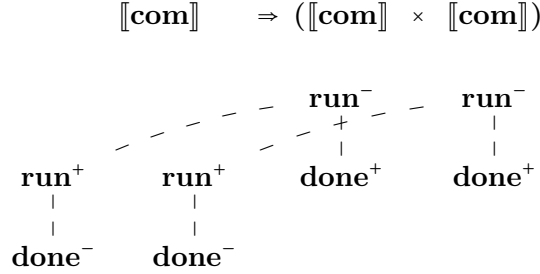
- *Events, and polarity.* Those of:

$$(\|_{b \in \min(B)} A^\perp) \parallel B$$

- *Causality.* As follows:

$$\leq_{(\|_{b \in \min(B)} A^\perp) \parallel B} \uplus \{((2, b), (1, (b, a))) \mid b \in \min(B) \ \& \ a \in A\}$$

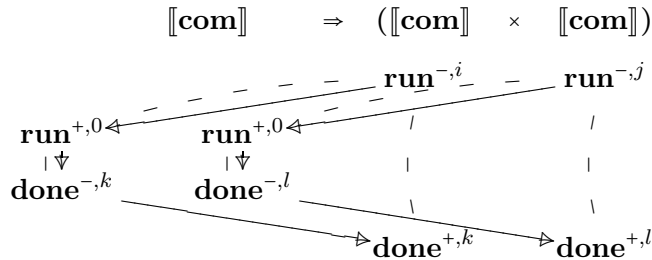
Example 5.14. Following this definition, the reader can check that $\llbracket \mathbf{com} \rrbracket \Rightarrow \llbracket \mathbf{com} \rrbracket$ is the arena presented as $\llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket$ in Example 2.5. As $\llbracket \mathbf{com} \rrbracket$ has only one minimal event, there is no duplication of the left hand side. However, the arena $\llbracket \mathbf{com} \rrbracket \Rightarrow (\llbracket \mathbf{com} \rrbracket \times \llbracket \mathbf{com} \rrbracket)$ is displayed below.



This is exactly the arena construction of [HO00], where arenas are forests.

5.2.2. *Cartesian closed structure.* Our proof of cartesian closure will leverage the compact closed structure of TCG. More precisely, we will show that there is a bijection (up to weak isomorphism) between negative single-threaded \sim -strategies playing respectively on $!A^\perp \parallel !(B \Rightarrow C)$ and $!A^\perp \parallel !(B^\perp \parallel !C)$. This bijection will leave the internal event structure of strategies unchanged, and will only operate through relabeling.

First, we describe the action of the bijection from $!A^\perp \parallel !(B \Rightarrow C)$ to $!A^\perp \parallel !(B^\perp \parallel !C)$. Let us first explain it on an example. Consider a \sim -strategy represented as below – which is, in essence, a curried version of the contraction on $\llbracket \mathbf{com} \rrbracket$ of Example 5.6.



Note that the positive moves on the left hand side have copy index 0, whereas in Example 5.6 they were carefully chosen so as to avoid collisions. This is because the current arena has more causal links. The two positive moves on the left are already made distinct by their justification pointers, so there is no need to distinguish them further via their copy indices. As this example illustrates, we cannot simply relabel this \sim -strategy to $\llbracket \mathbf{com} \rrbracket^\perp \parallel$

($\llbracket \mathbf{com} \rrbracket \times \llbracket \mathbf{com} \rrbracket$) without changing copy indices, as that would result in a collision, *i.e.* a failure of local injectivity of the labeling function.

Therefore, we use countability of the arena in order to do a collision-free relabeling.

Lemma 5.15. There is a strong-receptive, courteous map of event structures with symmetry:

$$\chi_{A,B} : !(A \Rightarrow B) \rightarrow !A^\perp \parallel !B$$

which, additionally, preserves the copy index of negative events.

Proof. For events $b \in B$ we use $\#b$ for the natural number associated to b by the countability of B . As in Section 2.5, we use $\langle - \rangle : \omega^3 \rightarrow \omega$ for any injective function; the collision with the pairing operation should not generate any confusion.

We set:

$$\begin{aligned} \chi_{A,B} : \quad & !(A \Rightarrow B) & \rightarrow & !A^\perp \parallel !B \\ & (\alpha : [(1, (b, a))] \rightarrow \omega) & \mapsto & (1, \alpha') \\ & (\beta : [(2, b)] \rightarrow \omega) & \mapsto & (2, \beta') \end{aligned}$$

where:

$$\begin{aligned} \alpha' : [a] & \rightarrow \omega \\ a' & \mapsto \langle \#b, \alpha((2, b)), \alpha((1, (b, a'))) \rangle \quad (\text{if } a' \in \min(A)) \\ a' & \mapsto \alpha((1, (b, a'))) \quad (\text{otherwise}) \end{aligned}$$

and:

$$\begin{aligned} \beta' : [b] & \rightarrow \omega \\ b' & \mapsto \beta((2, b')) \end{aligned}$$

With this definition $\chi_{A,B}$ preserves symmetry, is strong-receptive (it does not change the copy indices of negative events, since minimal events of A^\perp are positive) and courteous (it only breaks immediate causal links from minimal events of B to minimal events of A^\perp , so from negative to positive). \square

This allows us, from $\sigma : \mathcal{S} \rightarrow !C^\perp \parallel !(A \Rightarrow B)$, to define its relabeling:

$$\begin{aligned} \Phi(\sigma) & : \mathcal{S} \rightarrow !C^\perp \parallel (!A^\perp \parallel !B) \\ & = (!C^\perp \parallel \chi_{A,B}) \circ \sigma \end{aligned}$$

For well-chosen hashing function $\#$ and injection $\langle - \rangle$, this relabeling applied to the curried contraction above yields exactly the \sim -strategy of Example 5.6.

Before going on to the other direction, we note a further property of this relabeling.

Lemma 5.16. Let $\sigma : \mathcal{S} \rightarrow !C^\perp \parallel !(A \Rightarrow B)$ be a negative single-threaded \sim -strategy. Take $s_1, s_2 \in \mathcal{S}$ such that σs_1 has the form $(2, \beta)$ with $\text{lbl} \beta = (2, b)$ ($b \in \min(B)$), and $\sigma s_2 = (2, \alpha)$ with $\text{lbl} \alpha = (1, (b', a))$. Then, $b = b'$ iff $s_1 = \text{init}(s_2)$.

Proof. Straightforward consequence of single-threadedness. \square

Lemma 5.17. Let $\sigma_1, \sigma_2 : \mathcal{S} \rightarrow !C^\perp \parallel !(A \Rightarrow B)$ be two negative single-threaded \sim -strategies sharing the same internal ess. Then, $\sigma_1 \sim^+ \sigma_2$ iff $\Phi(\sigma_1) \sim^+ \Phi(\sigma_2)$.

Proof. if. Assume $\Phi(\sigma_1) \sim^+ \Phi(\sigma_2)$. Take $x \in \mathcal{C}(S)$, and form $\theta = \{(\sigma_1 s, \sigma_2 s) \mid s \in x\}$. We wish to prove that θ is a valid symmetry on $!C^\perp \parallel !(A \Rightarrow B)$. Firstly, we remark that the following diagram of bijections commutes.

$$\begin{array}{ccc}
 & x & \\
 \sigma_1 \swarrow & & \searrow \sigma_2 \\
 \sigma_1 x & \xrightarrow{\theta} & \sigma_2 x \\
 \downarrow !C^\perp \parallel \chi_{A,B} & & \downarrow !C^\perp \parallel \chi_{A,B} \\
 \Phi(\sigma_1) x & \xrightarrow{\theta'_{!C^\perp} \parallel (\theta'_{!A^\perp} \parallel \theta'_{!B})} & \Phi(\sigma_2) x
 \end{array}$$

It follows that θ decomposes as $\theta_{!C^\perp} \parallel \theta_{!(A \Rightarrow B)}$ with $\theta_{!C^\perp} \in \widetilde{!C^\perp}$, and we are left to prove that $\theta_{!(A \Rightarrow B)} \in \widetilde{!(A \Rightarrow B)}$. By construction it is a bijection, so we need to prove that it preserves and reflects causality, that it preserves labels, and that it preserves indices of negative events – which is clear, as they are preserved throughout this diagram.

We prove that it preserves immediate causality. The only nontrivial case concerns immediate causal links not preserved by $\chi_{A,B}$, *i.e.* those of the form:

$$\sigma_1 s_1 = (2, \{(2, b) \mapsto n\}) \rightarrow (2, \{(2, b) \mapsto n, (1, (b, a)) \mapsto p\}) = \sigma_1 s_2$$

But then, by Lemma 5.16, we have $s_1 = \text{init}(s_2)$. Since labels are preserved by $\theta'_{!A^\perp}$ and $\theta'_{!B}$, and using Lemma 5.16 again, we still have $\theta(\sigma_2 s_1) \rightarrow \theta(\sigma_2 s_2)$. The argument also applies to the θ^{-1} , which therefore is an order-isomorphism.

Preservation of labels also follows directly from Lemma 5.16. Finally, θ is a positive symmetry as all bijections involved preserve the copy index of negative events.

only if. By preservation of symmetry for $\chi_{A,B}$, and the fact that it preserves the copy index of negative events. \square

Relabeling from $!C^\perp \parallel (!A^\perp \parallel !B)$ to $!C^\perp \parallel !(A \Rightarrow B)$ is slightly more subtle: indeed, we go from a game having one copy of A to one having as many as there are minimal moves in B . Thus, choosing the label for events formerly mapping to A requires us to choose a copy of A corresponding to some minimal event in B . Here condition (1) of single-threadedness is crucial: each move s mapped to A has a unique minimal dependency $\text{init}(s)$, which must be mapped to a minimal event of B , and hence specifies the copy of A that s should be sent to. More formally, we prove the following.

Lemma 5.18. For any single-threaded \sim -strategy $\sigma : \mathcal{S} \rightarrow !C \parallel (!A \parallel !B)$, there is $\sigma' : \mathcal{S} \rightarrow !C \parallel !(A \Rightarrow B)$, unique up to positive symmetry, such that

$$\sigma \sim^+ (!C \parallel \chi_{A,B}) \circ \sigma'$$

Proof. We define $\sigma' : \mathcal{S} \rightarrow !C \parallel !(A \Rightarrow B)$. For $s \in \mathcal{S}$, then if $\sigma(s) = (1, \gamma)$ we set $\sigma'(s) = (1, \gamma)$ still.

If $\sigma(s) = (2, (2, \beta))$ with $\beta : [b] \rightarrow \omega$, then we set $\sigma'(s) = (2, \beta')$ with

$$\begin{array}{lcl}
 \beta' & : & [(2, b)] \rightarrow \omega \\
 & & (2, b') \mapsto \beta(b')
 \end{array}$$

If $\sigma(s) = (2, (1, \alpha))$ with $\alpha : [a] \rightarrow \omega$, then by condition (1) of single-threadedness it has a unique minimal dependency $\text{init}(s) \leq s$. By hypothesis, $\sigma(\text{init}(s))$ has the form $(2, (2, \beta))$ with $\beta = \{b \mapsto n\}$. Therefore we set:

$$\begin{aligned} \alpha' & : [(1, (b, a))] && \rightarrow \omega \\ & (1, (b, a')) && \mapsto \alpha(a') \\ & (2, b) && \mapsto n \end{aligned}$$

and we define $\sigma'(s) = (2, \alpha')$.

It is routine to check that this map is strong-receptive and courteous, and that its composition with $!C \parallel \chi_{A,B}$ is positively symmetric to σ . It follows from Lemma 5.17 that it preserves symmetry, and that it is unique up to positive symmetry. \square

From that, we deduce the following.

Proposition 5.19. There is a bijection Φ up to weak isomorphism, preserving and reflecting weak isomorphism, between:

- Negative, single-threaded \sim -strategies $\sigma : \mathcal{S} \rightarrow !C^\perp \parallel !(A \Rightarrow B)$,
- Negative, single-threaded \sim -strategies $\sigma' : \mathcal{S} \rightarrow !C^\perp \parallel (!A^\perp \parallel !B)$.

Moreover this bijection is compatible with pre-composition: for all $\tau : \mathcal{T} \rightarrow !D^\perp \parallel !C$, we have:

$$\Phi(\sigma) \odot \tau \simeq \Phi(\sigma \odot \tau)$$

Proof. On the one hand $\Phi(\sigma)$ is obtained as $(!C^\perp \parallel \chi_{A,B}) \circ \sigma$, while $\Phi^{-1}(\sigma')$ is obtained by the unique factorisation of Lemma 5.18. The bijection up to weak isomorphism follows from Lemma 5.18 as well.

We now prove stability under composition. By definition, we have $\Phi(\sigma) = (!C^\perp \parallel \chi_{A,B}) \circ \sigma$. But by Lemma 4.12 this is the same (up to isomorphism) as $\overline{\chi_{A,B}} \odot \sigma$, so the action of Φ can be obtained by post-composition via a lifted map. Stability under composition follows immediately by associativity of composition. \square

And finally, we deduce:

Theorem 5.20. The category Cho is cartesian closed.

Proof. We already know that it is cartesian. Throughout this proof, in the construction of the components of the cartesian closed structure, we ignore the associativity and unity isomorphisms from the compact closed structure of TCG – those can be easily and uniquely recovered from the context.

For any two arenas A, B , we first define the *evaluation* \sim -strategy:

$$\begin{aligned} \text{ev}_{A,B} & : A \times (A \Rightarrow B) \xrightarrow{\text{Cho}} B \\ & = (\epsilon_{!A} \otimes !B) \odot (!A \otimes \Phi(\mathbb{c}_{!(A \Rightarrow B)})) \odot \overline{m_{A, A \Rightarrow B}} \end{aligned}$$

Likewise, for any $\sigma : A \times C \xrightarrow{\text{Cho}} B$, we define its *curryfication* as:

$$\begin{aligned} \Lambda(\sigma) & : C \xrightarrow{\text{Cho}} (A \Rightarrow B) \\ & = \Phi^{-1}(!A^\perp \otimes (\sigma \odot \overline{m_{A,C}^{-1}})) \odot (\eta_{!A} \otimes !C) \end{aligned}$$

It is then a straightforward equational reasoning to prove the two equations [LS88], for $\sigma : A \times C \xrightarrow{\text{Cho}} B$ and $\tau : C \xrightarrow{\text{Cho}} (A \Rightarrow B)$,

$$\begin{aligned} (\beta) \quad & \text{ev}_{A,B} \odot (A \times \Lambda(\sigma)) \approx \sigma \\ (\eta) \quad & \Lambda(\text{ev}_{A,B} \odot (A \times \sigma)) \approx \sigma \end{aligned}$$

using mainly Proposition 5.19 and the laws of the compact closed structure of TCG, in combination with Lemma 5.12 to relate the cartesian structure of Cho and the monoidal structure of TCG – all the structural isomorphisms involved in the definition cancel each other. \square

5.3. Recursion. As the final technical part of this paper, we prove that Cho supports the interpretation of a fixpoint combinator.

Usually in game semantics, the interpretation of the fixpoint combinator Y is obtained by showing that the category of games and strategies is enriched over a category of sufficiently complete partial orders. Here however it will not be the case: indeed, just as in AJM games [AJM00b], our cartesian closed category is a quotient (its morphisms being weak isomorphism classes). It is not clear that the natural ordering on weak isomorphism classes is complete. However, this is not a big issue: although weak isomorphism classes of \sim -strategies might not form a complete partial order, concrete \sim -strategies do. Therefore, when solving recursive strategy equations, we will make sure to work with concrete \sim -strategies rather than weak isomorphism classes.

Our first step will be to order \sim -strategies.

Definition 5.21. Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$, $\tau : \mathcal{T} \rightarrow \mathcal{A}$ be two \sim -strategies on a tcg \mathcal{A} . We write $\sigma \trianglelefteq \tau$ iff $S \subseteq T$, the inclusion map $\mathcal{S} \hookrightarrow \mathcal{T}$ is a map of essps, with all data in \mathcal{S} coinciding with the restriction of that in \mathcal{T} , and such that for all $s \in S$, $\sigma s = \tau s$

The \sim -strategies on \mathcal{A} ordered by \trianglelefteq form a directed complete partial order (dcpo). It is not *pointed* though – it does not have a least element. Indeed, a \trianglelefteq -minimal \sim -strategy must still satisfy receptivity, and hence comprise events for minimal negative events of \mathcal{A} . However, the *name* in \mathcal{S} given to those is arbitrary, so there is one \trianglelefteq -minimal \sim -strategy on \mathcal{A} for each renaming of the minimal negative events of \mathcal{A} . For each \mathcal{A} we distinguish one \trianglelefteq -minimal \sim -strategy

$$\perp_{\mathcal{A}} : \min^-(\mathcal{A}) \rightarrow \mathcal{A}$$

that has as events the negative minimal events of \mathcal{A} with induced symmetry, and as labeling function the identity. Not every \sim -strategy is above $\perp_{\mathcal{A}}$. However, for every \sim -strategy $\sigma : \mathcal{S} \rightarrow \mathcal{A}$, we pick one $\sigma \cong \sigma^\dagger$ such that $\perp_{\mathcal{A}} \trianglelefteq \sigma^\dagger$ obtained by renaming the minimal negative events of \mathcal{S} . We write $\mathcal{D}_{\mathcal{A}}$ for the pointed dcpo of \sim -strategies above $\perp_{\mathcal{A}}$.

Lemma 5.22. For any tcg \mathcal{A} , $\mathcal{D}_{\mathcal{A}}$ is a pointed dcpo with $\perp_{\mathcal{A}}$ as minimal element.

Proof. If $\Gamma = \{\gamma : \mathcal{S}_\gamma \rightarrow \mathcal{A}\} \subseteq \mathcal{D}_{\mathcal{A}}$ is a directed subset of $\mathcal{D}_{\mathcal{A}}$, we form

$$\vee \Gamma = \cup \gamma : \bigcup_{\gamma \in \Gamma} \mathcal{S}_\gamma \rightarrow \mathcal{A}$$

with all components defined as componentwise union.

It is direct that this defines a \sim -strategy, which is the least upper bound of Γ . \square

Additionally, we note that if all \sim -strategies in a directed set Γ are negative or single-threaded, so is $\vee\Gamma$. We now note that all the operations we defined on \sim -strategies in this section are continuous for \preceq .

Lemma 5.23. Composition, tensor, pairing, curryfication and the $(-)^{\dagger}$ operation defined above are continuous for \preceq .

Proof. Straightforward. □

From the above, we deduce the following.

Corollary 5.24. For any arena A there is a fixpoint combinator $\mathcal{Y}_A : (A \Rightarrow A) \xrightarrow{\text{Cho}} A$, *i.e.* a single-threaded \sim -strategy such that:

$$\mathcal{Y}_A \approx ev_{A,A} \odot \langle \mathcal{Y}_A, \mathfrak{C}_{!(A \Rightarrow A)} \rangle$$

Proof. First, we define the following operation, using the combinators on Cho.

$$\begin{aligned} F & : \mathcal{D}_{!(A \Rightarrow A)^+ \| A} \rightarrow \mathcal{D}_{!(A \Rightarrow A)^+ \| A} \\ & \sigma \mapsto (ev_{A,A} \odot \langle \sigma, \mathfrak{C}_{!(A \Rightarrow A)} \rangle)^{\dagger} \end{aligned}$$

By Lemma 5.23 it is continuous, and from the outermost dagger it has indeed value in $\mathcal{D}_{!(A \Rightarrow A)^+ \| A}$. Thus, we can take its least fixpoint $\mathcal{Y}_A \in \mathcal{D}_{!(A \Rightarrow A)^+ \| A}$. The weak isomorphism in the statement actually follows as an *equality*. □

6. INTERPRETATION OF IPA

In this section, we illustrate our model further by defining the interpretation of IPA, displaying the interpretation of programs of interest, and proving a few properties along the way.

We insist here that our purpose is *not* to prove full abstraction, nor to prove deep properties of the interpretation. We feel indeed that given the length of the paper, the specifics of such an endeavour are best left for later. Furthermore, it is our impression that it serves the purpose of this paper better (introducing and developing Concurrent Hyland-Ong games) to give the reader an understanding of what the model computes, what it can and cannot do, rather than delve into additional technical developments.

Throughout all this section, by *strategy* we mean \sim -*strategy*. However, the symmetry will be kept implicit.

6.1. Sequential innocent part. In this subsection, we focus on the interpretation of the (sequential) innocent part of IPA, *i.e.* essentially PCF, plus the combinators for commands. In other words, this contains all of IPA except for variables and parallel composition of commands.

$$\begin{array}{cccc}
\llbracket \mathbf{tt} \rrbracket = !\llbracket \mathbb{B} \rrbracket & \llbracket \mathbf{ff} \rrbracket = !\llbracket \mathbb{B} \rrbracket & \llbracket n \rrbracket = !\llbracket \mathbb{N} \rrbracket & \llbracket \mathbf{skip} \rrbracket = !\llbracket \mathbf{com} \rrbracket \\
\mathbf{q}^{-,i} & \mathbf{q}^{-,i} & \mathbf{q}^{-,i} & \mathbf{run}^{-,i} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
\mathbf{tt}^{+,0} & \mathbf{ff}^{+,0} & n^{+,0} & \mathbf{done}^{+,0}
\end{array}$$

FIGURE 7. Interpretation of constants of IPA

Interpretation of types. The arenas for the types **com** and \mathbb{B} were given in Example 2.5. The interpretation for \mathbb{N} is a countably infinite variant of the interpretation of \mathbb{B} :

$$\begin{array}{c}
\llbracket \mathbb{N} \rrbracket \\
\mathbf{q}^{-} \\
/ \quad \backslash \\
0^+ \quad 1^+ \quad \dots \quad n^+ \quad \dots
\end{array}$$

The interpretation function extends to all types (not containing **ref**), with $\llbracket A \rightarrow B \rrbracket = \llbracket A \rrbracket \Rightarrow \llbracket B \rrbracket$.

Interpretation of terms. The interpretation follows the standard lines of the interpretation of the λ -calculus in a cartesian closed category. A *context* $\Gamma = x_1 : A_1, \dots, x_n : A_n$ is interpreted as the product $\prod_{1 \leq i \leq n} \llbracket A_i \rrbracket$ (which is just the parallel composition of the A_i s). A *typing sequent* $\Gamma \vdash M : A$ is interpreted as a Cho-morphism:

$$\llbracket \Gamma \vdash M : A \rrbracket : \llbracket \Gamma \rrbracket \xrightarrow{\text{Cho}} \llbracket A \rrbracket$$

For the λ -calculus combinators – variables, application, abstraction –, the interpretation is standard (and we do not detail it). For the fixpoint combinators, we use the combinator \mathcal{Y} of Section 5.3. The interpretation of constants is displayed in Figure 7. Note that we only display representations, treating multiple copies of Opponent moves symbolically. The reader should be able to expand them unambiguously to the full event structures, and to detail their isomorphism families. Note also that we give these interpretations over the empty context – they can easily be relabeled to any context Γ .

Likewise, the interpretation of function symbols is given in Figure 8. We have only one figure for a unary function $\text{op} : \mathbb{X} \rightarrow \mathbb{X}$, which covers (up to obvious relabeling) the cases of **succ** : $\mathbb{N} \rightarrow \mathbb{N}$, **pred** : $\mathbb{N} \rightarrow \mathbb{N}$ and **iszero** : $\mathbb{N} \rightarrow \mathbb{B}$. The interpretation of sequents involving those follows as usual, with *e.g.* the following composition in Cho:

$$\llbracket \Gamma \vdash \mathbf{if} M N_1 N_2 : \mathbb{X} \rrbracket = \llbracket \mathbf{if} \rrbracket \odot \langle \llbracket M \rrbracket, \llbracket N_1 \rrbracket, \llbracket N_2 \rrbracket \rangle : \llbracket \Gamma \rrbracket \xrightarrow{\text{Cho}} \llbracket \mathbb{X} \rrbracket$$

At this point, we have finished defining the interpretation function of all well-typed terms of the sequential innocent fragment of IPA. Using the cartesian closed structure and the equations for the fixpoint combinator, it would be straightforward at this point to prove soundness and adequacy of the interpretation. We refrain from detailing this – rather standard – proof.

The paper already contains some examples of the interpretation of terms of the fragment of IPA currently under study, most notably in Section 2 – where for some, copy indices need to be adequately adjoined. In general, the interpretation of such terms yields rather simple

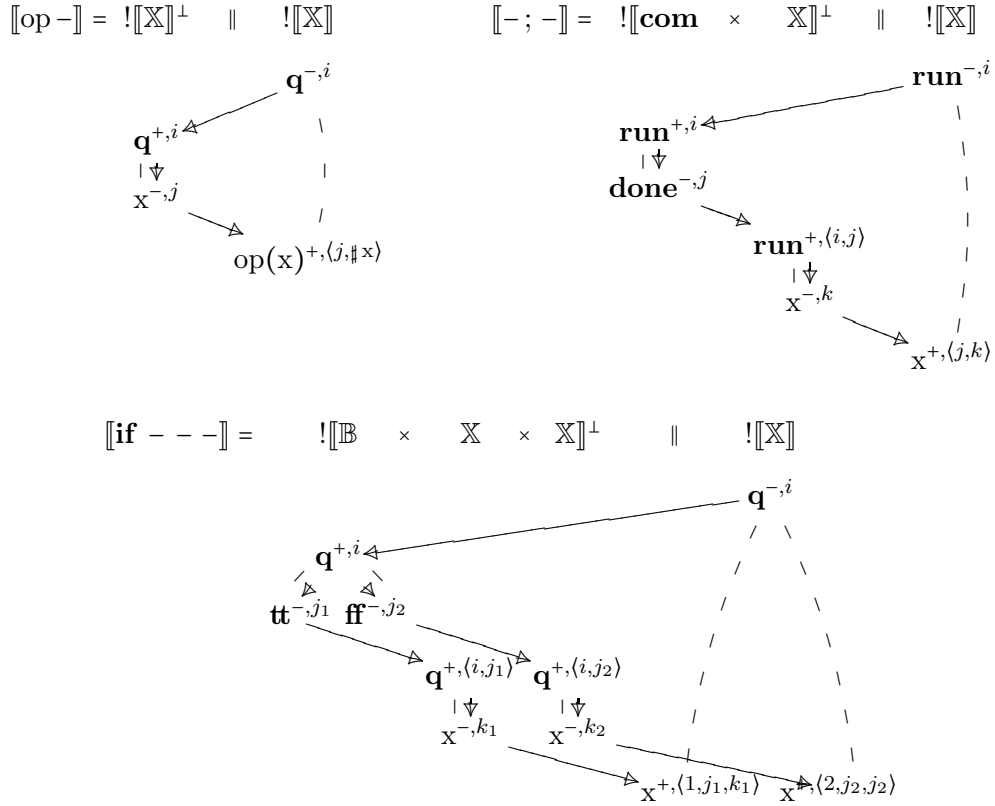


FIGURE 8. Interpretation of sequential function symbols of IPA

event structures, whose causal order is forest-shaped and which have no conflict. Modulo copy indices, and as it was noted in Section 2, these forests exactly coincide with the *view functions* of standard Hyland-Ong games: their branches are exactly the P -views. Hence, our interpretation gives a way to compute the composition of innocent strategies while staying within a causal representation corresponding to view functions, never resorting to expanded plays.

Non-determinism. Although the fragment of the language currently under study is deterministic, we find it interesting to study some examples given by its extension with a non-deterministic primitive. Therefore, we add to the language a new constant $\mathbf{coin} : \mathbb{B}$ which returns a random boolean. Its interpretation is (an obvious extension with copy indices of) the strategy on the left hand side of Figure 5. For $\Gamma \vdash M, N : A$, we define as syntactic sugar a non-deterministic sum $\Gamma \vdash M + N : A$ as $\Gamma \vdash \mathbf{if} \ \mathbf{coin} \ M \ N : A$.

We give in Figure 9 representations of the interpretation of some well-chosen terms. Copy indices are not exactly as given by the interpretation function (though they are up to weak isomorphism): they have been relabeled for convenience of presentation.

As Figure 9a illustrates, the model represents non-determinism in a *non-idempotent way*: redundant non-deterministic choices are kept separate by the interpretation. In

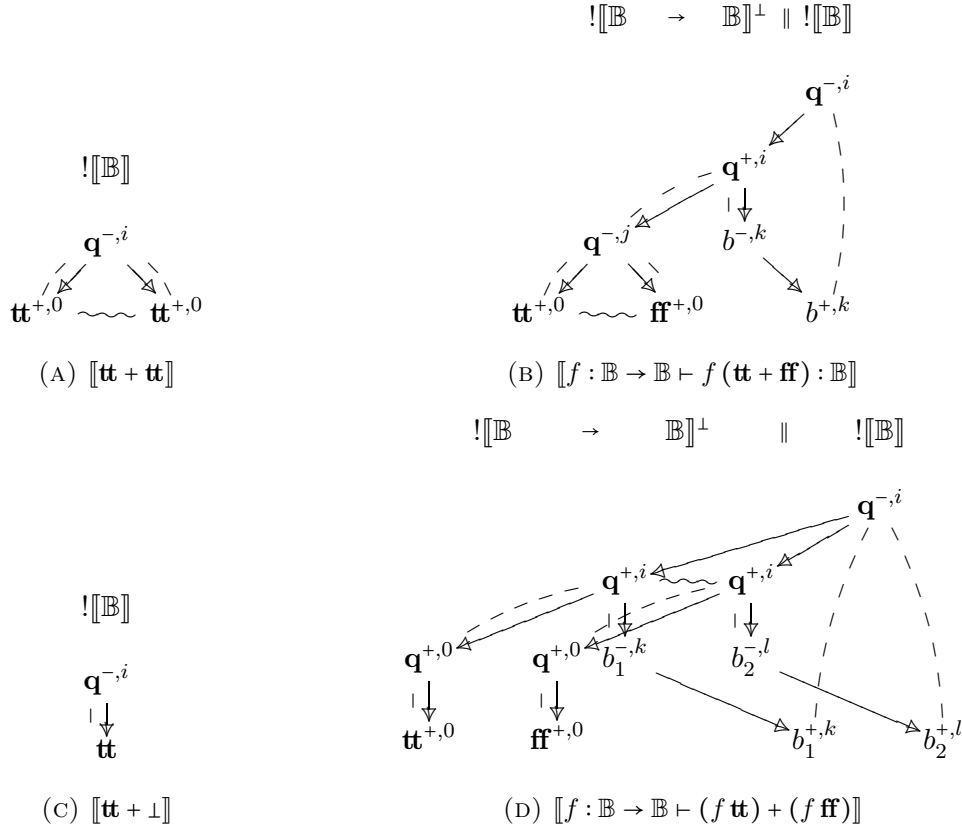


FIGURE 9. Interpretation of some non-deterministic terms

Figure 9c, \perp (which is syntactic sugar for $\mathcal{Y}(\lambda x^{\mathbb{B}}.x)$) is interpreted as the empty strategy. The interpretation of $\mathbf{tt} + \perp$ illustrates that, despite displaying explicitly the point of non-deterministic branching, the hiding step of the interpretation removes some diverging branches of the interaction. Figures 9b and 9d display two strategies which have the same branches (P-views), but differ in their branching points. This gives an interpretation of non-deterministic sequential programs that is similar to Tsukada and Ong's recent presheaf-based model [TO15], although our composition mechanism is very different. It is fairly easy to capture exactly their category as a subcategory of Cho, whose morphisms are *sequential innocent* [CCW14, CCW15] but not deterministic.

6.2. Concurrent innocent part. Now, we go on to show how our model represents concurrent primitives. The only concurrent primitive of IPA is parallel composition, whose

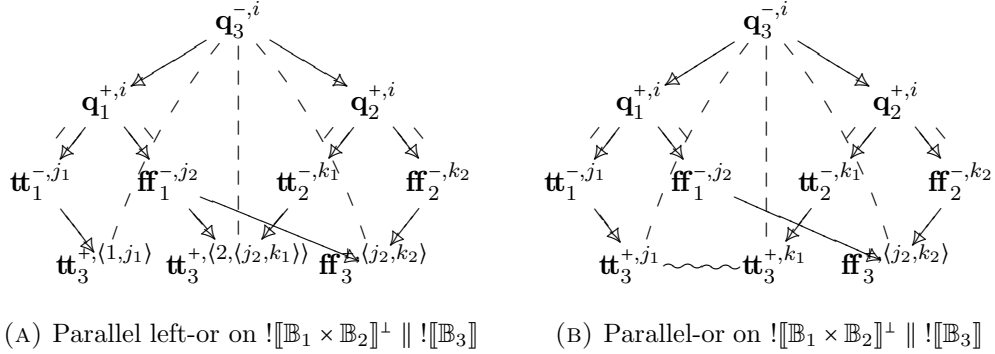
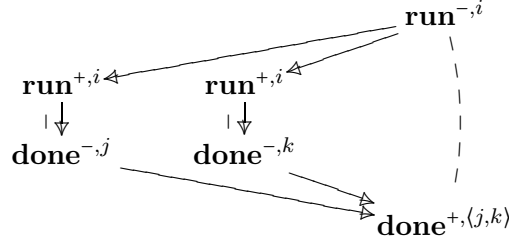


FIGURE 10. Two concurrent innocent strategies

interpretation relies on the following strategy

$$[[\parallel]] = ![\mathbf{com} \times \mathbf{com}]^\perp \parallel [\mathbf{com}]$$



Using this strategy we can define $[[\Gamma \vdash M \parallel N : \mathbf{com}]] = [[\parallel]] \odot \langle [[M], [N]] \rangle$.

This strategy is no longer a forest, but rather a directed acyclic graph. We also note that this is a *deterministic* strategy: there is no conflict in its event structure. As we shall see later, without any non-deterministic primitives, it is only in the presence of shared state that non-deterministic strategies will arise. In fact, a major advantage of our approach to modeling concurrent languages is that, not being based on interleavings, we represent the execution of such non-interfering terms deterministically.

In [CCW15], we exploit this property: we give a concurrent notion of innocence where strategies are directed acyclic graphs rather than forests, and using this notion we give an intensionally fully abstract model of a variant of PCF where independent computations are performed in parallel. The detailed construction is out of the scope of this particular paper, but let us illustrate it with two examples that are both concurrent innocent.

Figure 10 contains representations of two strategies, intended to be concurrent innocent (we associate moves to the corresponding sub-type using indices rather than location). In Figure 10a, we have a strategy for a parallel implementation of the left or, that is strict in its left argument. Indeed, although the strategy starts evaluating both its arguments in parallel, it can only return at toplevel if its first argument has returned. However, this is not true anymore for the strategy of Figure 10b. There, it suffices that *one* argument returns \mathbf{tt} for the overall computation to return \mathbf{tt} – indeed, this strategy computes the well-known *parallel-or* function [Pl077].

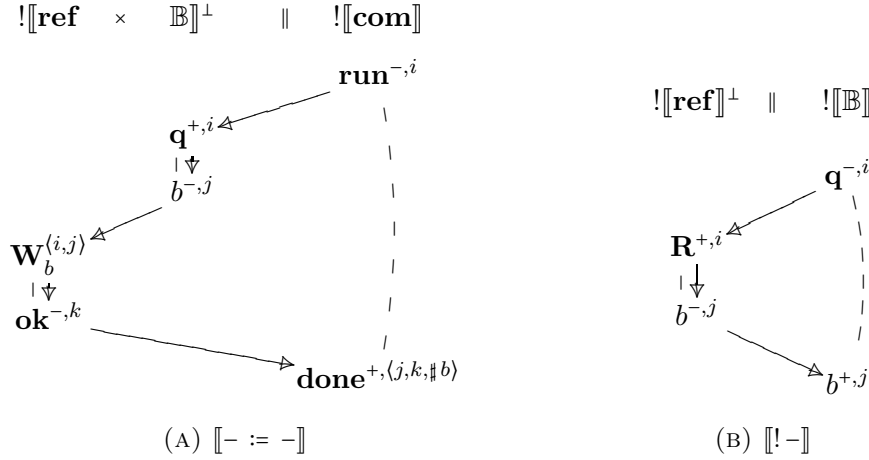


FIGURE 11. Strategies for assignment and dereferenciation

6.3. Stateful part. Finally, we finish the interpretation of IPA and describe how to interpret the primitives of IPA dealing with manipulations of state. For the simplicity of presentation, we set references in IPA to only store booleans; however the method applies just as well to integers.

A variable can be interacted with in two ways: via reading and writing. As usual in game semantics, we follow this idea for the interpretation of variables, and take $\llbracket \mathbf{ref} \rrbracket$ to be a product arena comprising actions for reading the reference or writing on the reference. More precisely, we define:

$$\llbracket \mathbf{ref} \rrbracket = \begin{array}{c} \mathbf{R}^- \quad \mathbf{W}_{\mathbf{tt}}^- \quad \mathbf{W}_{\mathbf{ff}}^- \\ \swarrow \quad \searrow \quad \swarrow \quad \searrow \\ \mathbf{tt}^+ \quad \mathbf{ff}^+ \quad \mathbf{ok}^+ \quad \mathbf{ok}^+ \end{array}$$

We now describe the interpretation of term constructors for the manipulation of state. As usual, assignment and dereferenciation are simply interpreted as (sequential innocent) strategies that interact with the memory cell. We give in Figure 11 the strategies used in the interpretation of those. Using those, we can define:

$$\begin{aligned} \llbracket \Gamma \vdash M := N : \mathbf{com} \rrbracket &= \llbracket - := - \rrbracket \odot \langle \llbracket M \rrbracket, \llbracket N \rrbracket \rangle \\ \llbracket \Gamma \vdash !M : \mathbb{B} \rrbracket &= \llbracket ! - \rrbracket \odot \llbracket M \rrbracket \end{aligned}$$

Before giving the interpretation of genuine references, we mention that the interpretation of \mathbf{mkvar} exploits as usual the isomorphism between $\llbracket \mathbf{ref} \rrbracket$ and $\llbracket \mathbb{B} \rrbracket \times \llbracket \mathbf{com} \rrbracket^2$ [AM96].

New reference. As usual, the more tricky part is the interpretation of the \mathbf{newref} construct. Indeed, while the strategies for assignment and dereferenciation only interact with the interface of the variable in an innocent way, \mathbf{newref} really has to provide an implementation for the memory.

If $\Gamma, x : \mathbf{ref} \vdash M : A$ depends on a reference x , its interpretation plays on (up to iso) $!\llbracket \mathbf{ref} \rrbracket^\perp \parallel !\llbracket \Gamma \rrbracket^\perp \parallel !\llbracket A \rrbracket$. Naively (we will see that this is a slight simplification), all we have to do is to build a strategy cell $:\llbracket \mathbf{ref} \rrbracket$, and compose $\llbracket M \rrbracket$ with it to obtain $\llbracket \mathbf{newref} \ x \ \mathbf{in} \ M \rrbracket$.

To define cell, we keep in mind the operational behaviour of a memory cell. In our (sequentially consistent) understanding of memory in a concurrent setting, although reads

and writes are called concurrently, they are performed in some sequential order by the central memory. Thus the behaviour of a boolean memory cell is best described as the prefix language of the infinite traces:

$$\begin{aligned} \text{Cell}_{\text{tt}} &::= \mathbf{R}^- \cdot \mathbf{tt}^+ \cdot \text{Cell}_{\text{tt}} \mid \mathbf{W}_{\text{tt}}^- \cdot \mathbf{ok}^+ \cdot \text{Cell}_{\text{tt}} \mid \mathbf{W}_{\text{ff}}^- \cdot \mathbf{ok}^+ \cdot \text{Cell}_{\text{ff}} \\ \text{Cell}_{\text{ff}} &::= \mathbf{R}^- \cdot \mathbf{ff}^+ \cdot \text{Cell}_{\text{ff}} \mid \mathbf{W}_{\text{tt}}^- \cdot \mathbf{ok}^+ \cdot \text{Cell}_{\text{tt}} \mid \mathbf{W}_{\text{ff}}^- \cdot \mathbf{ok}^+ \cdot \text{Cell}_{\text{ff}} \end{aligned}$$

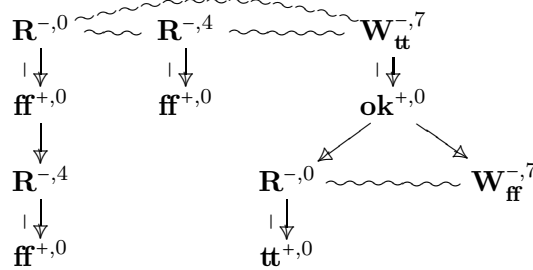
This language is ordered by prefix, so that Cell_{ff} is a forest. Setting all incomparable words to conflict with each other, we get an event structure whose events are words, and configurations are prefix-closed sets of prefixes of a word – so in one-to-one correspondence with words. This event structure, with the obvious labeling function, can be regarded as a prestrategy on $!\llbracket \mathbf{ref} \rrbracket$ (not on $!\llbracket \mathbf{ref} \rrbracket$). But in order to fit in our framework, we need to equip it with copy indices (and symmetry). This calls for extra bookkeeping, as we need to make sure that the same copy index is not used twice in the same branch. We define

$$\begin{aligned} \text{Cell}_{\text{tt}}^{I_{\mathbf{R}}, I_{\text{tt}}, I_{\text{ff}}} &::= \mathbf{R}^{-,i} \cdot \mathbf{tt}^{+,0} \cdot \text{Cell}_{\text{tt}}^{I_{\mathbf{R}} \cup \{i\}, I_{\text{tt}}, I_{\text{ff}}} & (i \notin I_{\mathbf{R}}) \\ &\mid \mathbf{W}_{\text{tt}}^{-,i} \cdot \mathbf{ok}^{+,0} \cdot \text{Cell}_{\text{tt}}^{I_{\mathbf{R}}, I_{\text{tt}} \cup \{i\}, I_{\text{ff}}} & (i \notin I_{\text{tt}}) \\ &\mid \mathbf{W}_{\text{ff}}^{-,i} \cdot \mathbf{ok}^{+,0} \cdot \text{Cell}_{\text{ff}}^{I_{\mathbf{R}}, I_{\text{tt}}, I_{\text{ff}} \cup \{i\}} & (i \notin I_{\text{ff}}) \end{aligned}$$

and similarly for $\text{Cell}_{\text{ff}}^{I_{\mathbf{R}}, I_{\text{tt}}, I_{\text{ff}}}$. Then we define the event structure Cell via $\text{Cell}_{\text{ff}}^{\emptyset, \emptyset, \emptyset}$, as we did above. It has an isomorphism family, that relates any two words differing only on their copy indices. Moreover the names of the events denote the labeling function to $!\llbracket \mathbf{ref} \rrbracket$ (with all positive moves pointing – that is, being immediately dependent in the game – to the previous move). Overall, we get a map of essp:

$$\text{cell} : \text{Cell} \rightarrow !\llbracket \mathbf{ref} \rrbracket$$

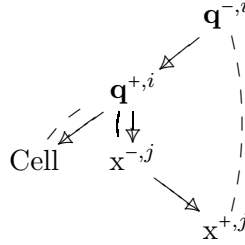
Example 6.1. The following diagram represents a sub-event structure of Cell .



We have constructed $\text{cell} : \text{Cell} \rightarrow !\llbracket \mathbf{ref} \rrbracket$ a map of essps. However, cell is not a valid \sim -strategy in Cho : it neither receptive (after playing $\mathbf{R}^{-,4}$ above one cannot play $\mathbf{R}^{-,4}$, although it is compatible in the game) nor courteous (we have $\mathbf{ff}^{+,0} \rightarrow \mathbf{R}^{-,4}$ which does not hold in $!\llbracket \mathbf{ref} \rrbracket$). However, cell is a *thin pre- \sim -strategy*, and as such can be composed with $\llbracket M \rrbracket : !\llbracket \mathbf{ref} \rrbracket^\perp \parallel !\llbracket A \rrbracket$ to obtain $\llbracket M \rrbracket \odot \text{cell}$ – and it turns out that $\llbracket M \rrbracket \odot \text{cell}$ is *always* a valid \sim -strategy.

However, that is still not quite what we want. The intended semantics for **newref** x in M is that each of its evaluations spawns a new, independent memory cell, whereas the operation above would have it spawned once and for all and shared over all copies of M . In other words, $\llbracket M \rrbracket \odot \text{cell}$ is a valid \sim -strategy indeed, but it might not be *single-threaded*. So finally, we build another pre- \sim -strategy displayed in Figure 12, where Cell means a copy of the pre- \sim -strategy above, with minimal events pointing as indicated.

$$![\mathbf{ref} \rightarrow \mathbb{X}]^\perp \parallel ![\mathbb{X}]$$

FIGURE 12. The pre- \sim -strategy newcell

Finally, from $\Gamma, x : \mathbf{ref} \vdash M : \mathbb{X}$, we define:

$$\llbracket \mathbf{newref} \ r \ \mathbf{in} \ M \rrbracket = \mathbf{newcell} \odot \Lambda(\llbracket M \rrbracket) : ![\Gamma] \xrightarrow{\text{TCG}} ![\mathbb{X}]$$

Then, despite $\mathbf{newcell}$ being a pre- \sim -strategy rather than a \sim -strategy, we have:

Proposition 6.2. For any $\Gamma, x : \mathbf{ref} \vdash M : \mathbb{X}$, the thin pre- \sim -strategy:

$$\llbracket \mathbf{newref} \ r \ \mathbf{in} \ M \rrbracket = \mathbf{newcell} \odot \Lambda(\llbracket M \rrbracket)$$

is a single-threaded \sim -strategy.

Proof. The composition is well-defined (as a map of essps) since both compounds are \sim -receptive. Moreover, both compounds are also *componentwise courteous* (see Definition 3.26), so by Lemma 3.27 the composition $\mathbf{newcell} \odot \Lambda(\llbracket M \rrbracket)$ is a componentwise courteous pre- \sim -strategy. It is also thin, negative and single-threaded as these properties are stable under composition (respectively Lemmas 3.37, 5.3 and Proposition 5.10).

It remains to check that it is receptive and courteous. But that does not involve symmetry at all; and by the results of [RW11, CCRW] it suffices to check that

$$\mathfrak{c}_{![\mathbb{X}]} \odot (\mathbf{newcell} \odot \Lambda(\llbracket M \rrbracket)) \odot \mathfrak{c}_{![\Gamma]} \cong \mathbf{newcell} \odot \Lambda(\llbracket M \rrbracket)$$

but that follows from the fact that composition of componentwise courteous pre- \sim -strategies is associative (and indeed, the embedding of Section 4 shows that it maps to the composition of spans of prestrategies), that $\Lambda(\llbracket M \rrbracket)$ is a strategy, and the easy verification that $\mathfrak{c}_{![\mathbb{X}]} \odot \mathbf{newcell} \cong \mathbf{newcell}$. \square

This concludes the definition of the interpretation of IPA in Cho. As said before, we do not aim in this paper to prove properties of this interpretation, such as soundness or adequacy – this is instead postponed for a further paper. However, we will now illustrate this interpretation by providing some examples.

6.4. Some examples.

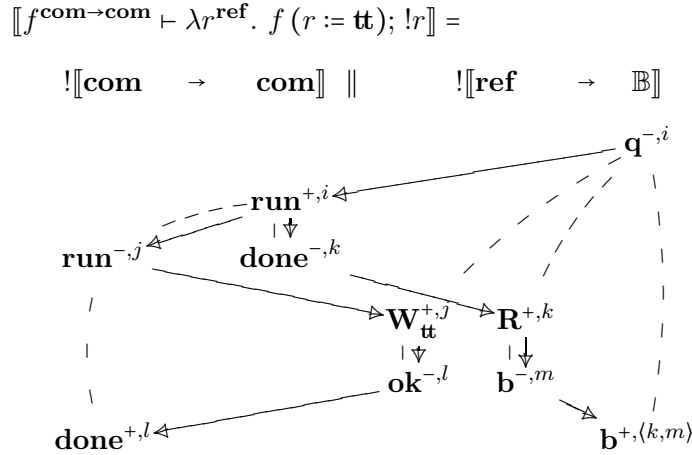
First example: strictness test. As a first example, we detail the interpretation of the term of Example 2.22. Recall that it was:

$$\mathbf{newref} \ b \ \mathbf{in} \ \lambda f^{\mathbf{com} \rightarrow \mathbf{com}}. f(b := \mathbf{tt}); !b : (\mathbf{com} \rightarrow \mathbf{com}) \rightarrow \mathbb{B}$$

As the constructor **new** is only defined on terms of ground type, this is just syntactic sugar for $\lambda f^{\mathbf{com} \rightarrow \mathbf{com}}. \mathbf{newref} \ b \in f(b := \mathbf{tt}); !b : (\mathbf{com} \rightarrow \mathbf{com}) \rightarrow \mathbb{B}$. In order to define its interpretation, the first step is to define:

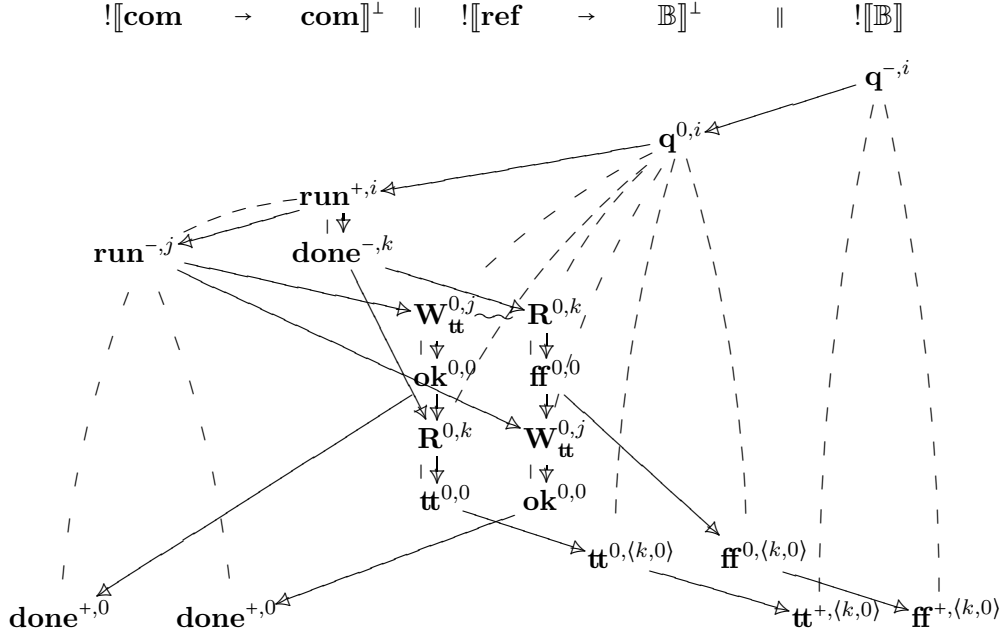
$$\llbracket f : \mathbf{com} \rightarrow \mathbf{com}, b : \mathbf{ref} \vdash f(b := \mathbf{tt}); !b : \mathbb{B} \rrbracket : \llbracket \mathbf{com} \rightarrow \mathbf{com} \rrbracket \times \llbracket \mathbf{ref} \rrbracket \xrightarrow{\text{Cho}} \llbracket \mathbb{B} \rrbracket$$

This is covered by the definitions above, using the cartesian closed structure and the strategies of Figure 11 for assignment and dereferencing. Computing this yields the strategy represented below (again, the copy indices given by the actual interpretation function differ, but this is irrelevant up to weak isomorphism).



Again, this deterministic event structure is forest-shaped and its branches are versions with explicit copy indices of the P-views of the corresponding innocent strategy in Hyland-Ong games.

Now, we compose it with newcell. We represent below the event structure resulting from their interaction. Events of the hidden/synchronised part of the interaction no longer have a well-defined polarity, hence we set it to 0.



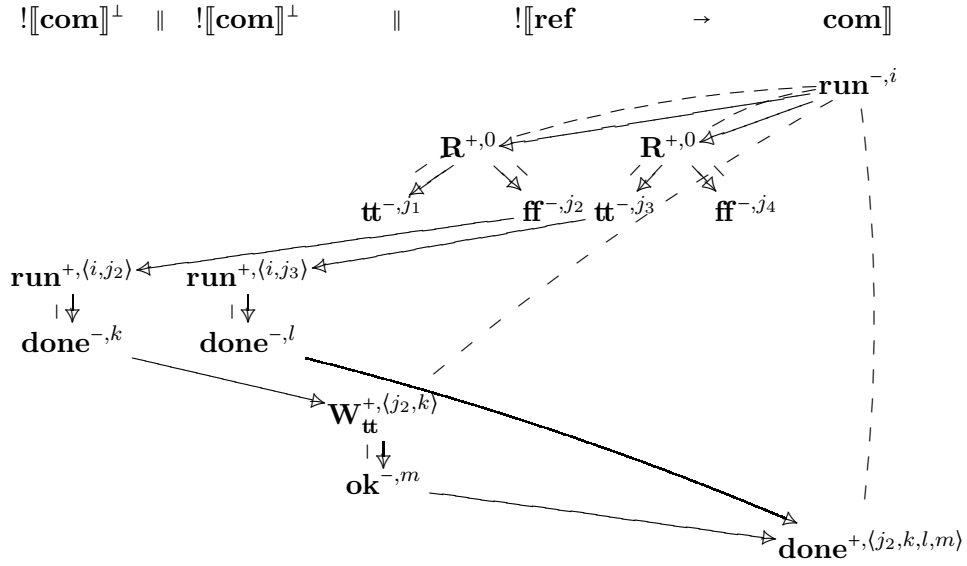
After hiding, the minimal conflict between the first two events in cell is inherited by the final positive events. The reader can check that hiding yields (up to the copy indices) the event structure of Example 2.22.

The reader familiar with Abramsky and McCusker’s model for IA will see that taking the plays – *i.e.* alternating well-bracketed linear orderings of configurations, without copy indices – of the resulting strategy yields the expected sequential strategy. But our model says more, *e.g.* it also specifies the behaviour of the strategy if Opponent decides to *both* ask its argument *and* return in parallel.

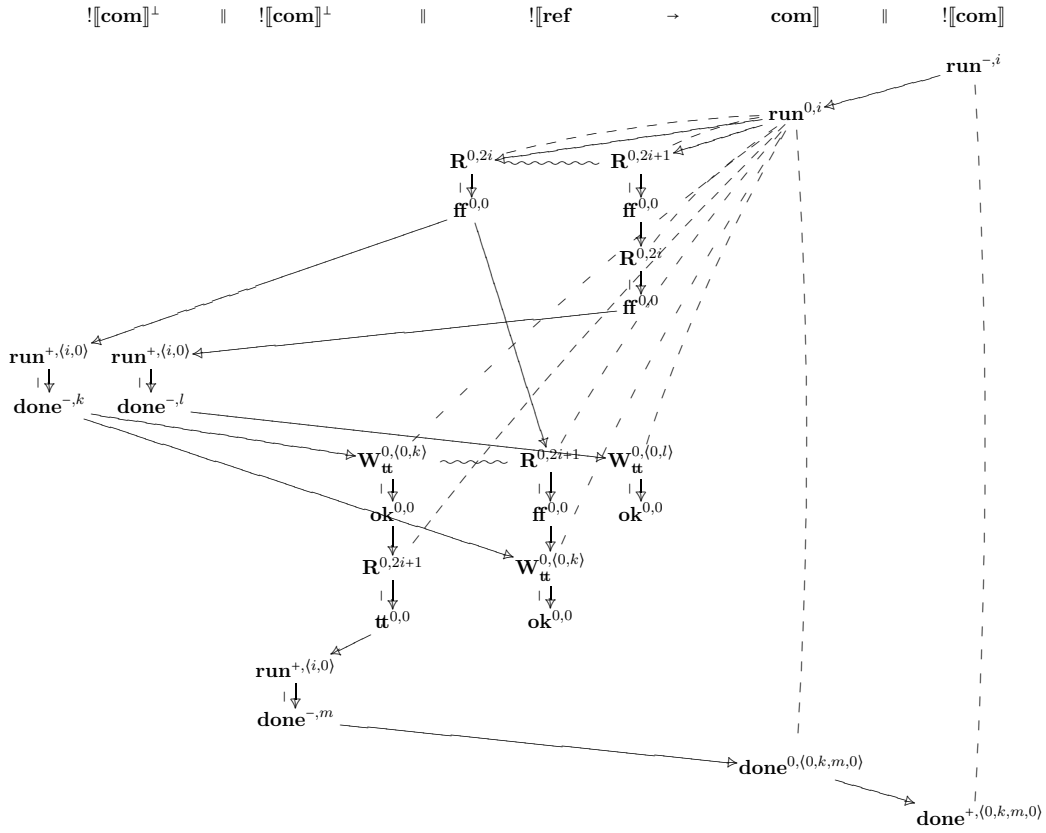
Second example: synchronization through state. Now, we compute the interpretation of the following term of IPA.

$$\begin{aligned}
 &x : \mathbf{com}, y : \mathbf{com} \vdash \mathbf{newref} \ r \ \mathbf{in} \\
 &\quad \mathbf{if} \ (!r) \ \perp \ (x; r := \mathbf{tt}) \ \parallel \\
 &\quad \mathbf{if} \ (!r) \ y \ \perp \\
 &: \mathbf{ref} \rightarrow \mathbf{com}
 \end{aligned}$$

As before, we first compute the interpretation of the variant of this term where the variable has been abstracted away, obtaining the following strategy.



We now compute a part of the interaction with newcell, pictured below.



which, after hiding, yields a strategy with sub-event structures such as:

$$\llbracket \lambda x^{\text{com}}, y^{\text{com}}. \text{newref } r \text{ in } (\text{if } (!r) \perp (x; r := \mathbf{tt})) \parallel (\text{if } (!r) y \perp) \rrbracket =$$

$$\begin{array}{c} \llbracket \mathbf{com} \rrbracket^\perp \quad \parallel \quad \llbracket \mathbf{com} \rrbracket^\perp \quad \parallel \quad \llbracket \mathbf{com} \rrbracket \\ \text{run}^{+, \langle i, 0 \rangle} \quad \text{run}^{+, \langle i, 0 \rangle} \quad \text{run}^{-, i} \\ \downarrow \quad \downarrow \quad \downarrow \\ \text{done}^{-, l} \quad \text{done}^{-, k} \quad \text{done}^{-, m} \\ \text{done}^{+, \langle 0, k, m, 0 \rangle} \end{array}$$

Here, there are several observations to make.

Firstly, note that the copy index of the call to y does not depend on k . This might seem surprising, since it seems that if Opponent plays two occurrences of $\text{done}^{-, k}$, once with $k = 0$ and once with $k = 1$, Player will play the subsequent $\text{run}^{+, \langle i, 0 \rangle}$ twice, breaking local injectivity.

In fact, this is because the symbolic notation used here is a bit misleading, and suggests a uniformity in the behaviour of strategies which is not accurate for strategies that are not innocent. In reality, the interaction presented here is incomplete and does not cover the case where Opponent plays several occurrences of $\text{done}^{-, k}$ (that could happen if, for instance, Opponent has access to call/cc). Indeed, this would trigger new $\mathbf{W}_{\mathbf{tt}}^{0, k}$ events, which would be sequentialized in some order by the memory. The $\mathbf{R}^{0, 2i+1}$ would happen at some point during that sequentialization; each of these possible occurrences of $\mathbf{R}^{0, 2i+1}$ would lead to a call to y – so there would be multiple non-deterministic calls to y .

Secondly, we note that this term, in the Ghica-Murawski model of IPA [GM08], would be interpreted by the same strategy than that for sequential composition. Unlike their model, we keep some information about non-deterministic branching; meaning that we do remember here that the term has a chance to diverge. In the interpretation presented in this paper, we do not remember *all* the information about divergences though. If one was to simplify the term above to $\text{newref } r \text{ in } \lambda x^{\text{com}}, y^{\text{com}}. x; r := \mathbf{tt} \parallel \text{if } !r y \perp$, the branch where the read arrives too early *w.r.t.* the write would be hidden away by composition. The sole purpose of the superfluous read in our example above is to create a race in memory before x , spawning two non-deterministic copies of the execution of x . In one of them the computation is doomed, as the second thread is stuck in a loop.

7. CONCLUSIONS

In this paper, we have given the detailed development leading to our cartesian closed category Cho of *Concurrent Hyland-Ong* games, a setting that we illustrated with an interpretation of IPA. The cartesian closed category Cho conservatively extends standard Hyland-Ong

games, in the sense that in our setting purely function programs are interpreted as (copy-index aware versions of) their tree of P-views – but our setting also supports stateful, non-deterministic, or concurrent languages, or any combination thereof.

The cornerstone of our construction is a compact closed category TCG of *thin concurrent games*, which extends Rideau and Winskel’s category CG of games and strategies as event structures [RW11, CCRW]. Note the interest of TCG is not restricted to the construction of Cho. It supports games that are much more general than those obtained from arenas. The future will tell how this mathematical space is best exploited.

In an extended abstract [CCW15], we relied on the framework presented here to give a full abstraction result for a parallel interpretation of PCF – due to the length of the paper, we believe that the details of the fully abstract model and of the proof are best kept for a separate paper. In another extended abstract [CC16], we investigate the connections between (an affine version of) this interpretation of IPA with that of Ghica and Murawski. There is a lot of ongoing research on this game semantics framework. Research directions include extensions probabilities, applications to the semantics of proofs, or to the semantics of non-interferent programming languages.

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APPENDIX A. COUNTEREXAMPLES

A.1. Absence of pullbacks in \mathcal{E}_\sim . The category of event structures with symmetry does not have pullbacks in general. For that we first note that if a diagram has a pullback in \mathcal{E}_\sim , then, forgetting symmetry, it is also a pullback in \mathcal{E} . The reason for that is the following proposition.

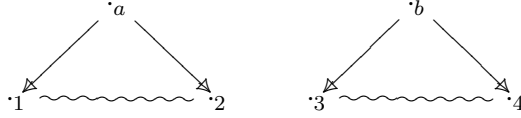
Proposition A.1. The forgetful functor $\mathcal{E}_\sim \rightarrow \mathcal{E}$ which to any event structure with symmetry $\mathcal{A} = (A, \widetilde{A})$ associates A , has a left adjoint.

Proof. The right adjoint associates, to any event structure A , the event structure with symmetry (A, refl_A) , where

$$\text{refl}_A = \{ \{ (a, a) \mid a \in x \} \mid x \in \mathcal{C}(A) \}$$

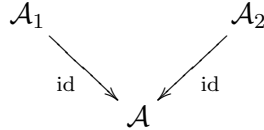
is the minimal symmetry on A . It is straightforward that this indeed defines an adjunction. \square

Hence the symmetry-forgetting functor is a right adjoint, and as such preserves pullbacks. Now, in order to prove that \mathcal{E}_\sim does not have pullbacks, we are going to construct a diagram in \mathcal{E}_\sim whose pullback in \mathcal{E} has no possible isomorphism family. Indeed, consider A the following event structure:



Write \mathcal{A} for A equipped with the maximal isomorphism family: all order-isomorphisms are in the family. Write \mathcal{A}_1 for the sub-event with symmetry where \cdot_1 can only be sent to itself and to \cdot_3 ; and \cdot_2 can only be sent to itself and to \cdot_4 . Similarly, write \mathcal{A}_2 for that where \cdot_1 can only be sent to \cdot_4 and \cdot_2 to \cdot_3 .

Now we have the following diagram:



Assume this diagram has a pullback $(\mathcal{A}_3, \Pi_1, \Pi_2)$. By Proposition A.1 its underlying event structure is A and the projection maps are both identities on objects. The isomorphism $\{(\cdot_a, \cdot_b)\} : \{\cdot_a\} \cong_{\widetilde{A}_3} \{\cdot_b\}$ must be in \widetilde{A}_3 as it is in both \widetilde{A}_1 and \widetilde{A}_2 . However, its left hand side $\{\cdot_a\}$ can be extended with \cdot_1 , so by the extension property we must have

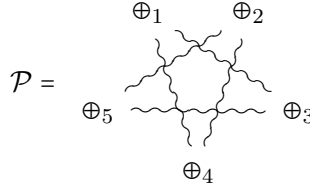
$$\{(\cdot_a, \cdot_b), (\cdot_1, \cdot_i)\} : \{\cdot_a, \cdot_1\} \cong_{\widetilde{A}_3} \{\cdot_b, \cdot_i\}$$

with $i \in \{3, 4\}$. But by construction such an isomorphism cannot be in both \widetilde{A}_1 and \widetilde{A}_2 , absurd.

A.2. Weak isomorphism is not a congruence. This key observation is one of the key facts guiding the design of the saturated games with symmetry of [CCW14] and the thin concurrent games of Section 3.

It first came as a surprise that weak isomorphism between pre- \sim -strategies is not preserved by composition with other pre- \sim -strategies. Indeed, the extension property of symmetries ensures that two symmetric configurations have “bisimilar futures”. For two weakly isomorphic pre- \sim -strategies σ_1, σ_2 on game \mathcal{A} , and another pre- \sim -strategy τ on game $\mathcal{A}^\perp \parallel \mathcal{B}$, from \sim -receptivity of τ and the weak isomorphism of σ_1 and σ_2 one certainly expects $\tau \odot \sigma_1$ and $\tau \odot \sigma_2$ to behave similarly. And they do indeed behave similarly, but in a way less strict than that expressed by weak isomorphisms.

To be more precise, first consider the event structure with polarities and symmetry \mathcal{P} (the “pentagram”):



Its isomorphism family is the maximal one, *i.e.* all bijections between configurations are in the family. In \mathcal{P} , two events will eventually be played. It does not matter which ones, since they are all symmetric – the only thing that matters is the multiplicity.

We will consider \mathcal{P} as a pre- \sim -strategy on a game \mathcal{B} with the same events as \mathcal{P} ($\{\oplus_1, \oplus_2, \oplus_3, \oplus_4, \oplus_5\}$), the maximal isomorphism family, and without conflict. We write α_1 for the obvious labeling function

$$\begin{aligned} \alpha_1 &: \mathcal{P} \rightarrow \mathcal{B} \\ \oplus_i &\mapsto \oplus_i \end{aligned}$$

which indeed informs a pre- \sim -strategy on \mathcal{B} .

We will also be interested in another pre- \sim -strategy:

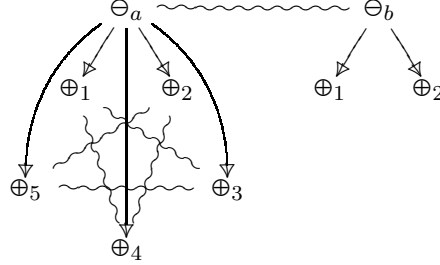
$$\alpha_2 : \mathcal{S} \rightarrow \mathcal{B}$$

where \mathcal{S} has events $\{\oplus_1, \oplus_2\}$ and again maximal isomorphism family. The map α_2 sends \oplus_i in \mathcal{S} to \oplus_i in \mathcal{B} . The pre- \sim -strategies α_1 and α_2 behave similarly, since both will eventually play two events; and we do not care which ones since all possible choices are symmetric in the game. Despite that, α_1 and α_2 are *not* weakly isomorphic. In fact, *there is no map from α_1 to α_2* : such a map would require us to build a map of event structures from \mathcal{P} to \mathcal{S} , but the reader can check that this would induce a 2-coloring of \mathcal{P} , which is not bipartite.

We will now obtain α_1 and α_2 respectively as compositions $\tau \odot \sigma_1$ and $\tau \odot \sigma_2$, for weakly isomorphic σ_1 and σ_2 . We introduce the game

$$\mathcal{A} = \oplus_a \sim \oplus_b$$

with again the maximal isomorphism family. The pre- \sim -strategy $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ selects α_1 or α_2 depending on Opponent's choice in \mathcal{A} . Its events are represented below.



Its isomorphism family is, again, the maximal one: all order-isomorphisms between configurations are valid symmetries. One can check that this satisfies indeed the axioms for an isomorphism family; crucially the extension axiom uses the fact that \mathcal{P} and \mathcal{S} are bisimilar (and that the symmetry on \mathcal{B} is the maximal one).

Finally, consider σ_1, σ_2 on \mathcal{A} , with σ_1 playing only \ominus_a and σ_2 playing only \ominus_b . They are clearly weakly isomorphic, since $\{(\ominus_a, \ominus_b)\}$ is in $\tilde{\mathcal{A}}$. But by construction we have $\tau \circ \sigma_1 \cong \alpha_1$ and $\tau \circ \sigma_2 \cong \alpha_2$, which as we observed are not weakly isomorphic.

Note that the games \mathcal{A} and \mathcal{B} are both tcgs; but crucially τ is not *thin* (Definition 3.32). Indeed, for instance, the symmetry $\{(\ominus_a, \ominus_b)\}$ extends to both $\{(\ominus_a, \ominus_b), (\oplus_1, \oplus_1)\}$ and $\{(\ominus_a, \ominus_b), (\oplus_1, \oplus_2)\}$, which is forbidden by Definition 3.32. For thin pre- \sim -strategies, positive extensions of the symmetry must be canonically chosen, making it impossible that composite strategies as above are bisimilar but not weakly isomorphic.

A.3. Failure of extension for copycat on general games. In the main text, we give (Definition 3.17) a candidate for the isomorphism family on copycat $\mathbb{C}_{\tilde{\mathcal{A}}}$ for any event structure with polarities and symmetry \mathcal{A} . The valid symmetries on $\mathbb{C}_{\mathcal{A}}$ are simply those order-isomorphisms between configurations of $\mathbb{C}_{\mathcal{A}}$ which map to valid symmetries on $\mathcal{A}^\perp \parallel \mathcal{A}$.

Crucially, we proved in Proposition 3.21 that this satisfies the extension property of isomorphism families if the game is a tcg. This boiled down to Lemma 3.20, which shows that tcgs are *race-preserving*: races in the isomorphism family always originate to races in the game. As this phenomenon played an important role in the design of the theory, we find it useful to include here an example demonstrating the fact that without this race-preservation property (so if the games are plain event structures with polarities and symmetry, rather than tcgs), the extension property fails for the isomorphism family on copycat.

Consider an event structure with polarities $A = \{a^-, b^+\}$. From it we form $!_2 A$ with events/causality/polarities/conflict that of $A \parallel A$ (we write $a^{-,i}$ for $(i, a)^-$), and isomorphism families the set of bijections between configurations included in the two maximal ones:

$$\begin{aligned} & \{(a^{-,1}, a^{-,1}), (b^{+,1}, b^{+,1}), (a^{-,2}, a^{-,2}), (b^{+,2}, b^{+,2})\} \\ & \{(a^{-,1}, a^{-,2}), (b^{+,1}, b^{+,2}), (a^{-,2}, a^{-,1}), (b^{+,2}, b^{+,1})\} \end{aligned}$$

So maximal symmetries either globally preserve the copy indices, or globally swap them. It is not possible for a symmetry to, *e.g.* send $a^{-,1}$ to $a^{-,1}$ and $b^{+,1}$ to $b^{+,2}$. So $!_2$ does not act like the $!$ operation of Section 5 – instead, it is a binary version of the bang of [CCW14], imported from AJM games [AJM00b].

From Definition 3.17, the event structure $\mathbb{C}_{!_2A}$ is equipped with a candidate isomorphism family $\mathbb{C}_{!_2A}$. We now show that this however fails the extension axiom of isomorphism family. From the definition, the diagram below represents a valid symmetry in $\mathbb{C}_{!_2A}$.

$$\begin{array}{ccccccc} !_2A^\perp & \parallel & !_2A & \cong_{\mathbb{C}_{!_2A}} & !_2A^\perp & \parallel & !_2A \\ b^{-,1} & \cdots & a^{-,1} & \cdots & b^{-,1} & \cdots & a^{-,2} \end{array}$$

The issue will come from the fact that the symmetry follows irreconcilable courses in the left and the right components of $\mathbb{C}_{!_2}$: in the left component it preserves copy indices, whereas in the right component it swaps them. So it the left hand side of this symmetry extends as depicted below

$$\begin{array}{ccccccc} !_2A^\perp & \parallel & !_2A & \cong_{\mathbb{C}_{!_2A}} & !_2A^\perp & \parallel & !_2A \\ b^{-,1} & \cdots & a^{-,1} & \cdots & b^{-,1} & \cdots & a^{-,2} \\ & \searrow & & & & & \\ & & b^{+,1} & & & & \end{array}$$

the only matching extension on the right hand side is with $b^{+,1}$ as well ($b^{+,2}$ is not possible as it would require to play $b^{-,2}$ first), but $(b^{+,1}, b^{+,1})$ is not a valid extension of the symmetry above, as for that it would need to swap the copy index instead of preserving it.