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► **To cite this version:**

Simon Castellan, Pierre Clairambault, Glynn Winskel. Concurrent Hyland-Ong games. 2014. hal-01068769v1

HAL Id: hal-01068769

<https://hal.science/hal-01068769v1>

Submitted on 26 Sep 2014 (v1), last revised 20 Dec 2018 (v4)

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Concurrent Hyland-Ong games

Simon Castellan, Pierre Clairambault, Glynn Winskel

September 26, 2014

Abstract

In this technical report, we build a cartesian closed category of non-deterministic concurrent strategies playing on arenas. We show that this CCC admits as a sub-CCC the standard category of arenas and Hyland-Ong innocent strategies. Our strategies, have much more possible behaviours than standard Hyland-Ong innocent strategies – however the purpose of this technical report is to define our CCC, and we leave for later its use for semantics of programming languages.

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1 Introduction

In [RW11], Rideau and Winskel have introduced a compositional framework for games based on event structures. The purpose of this note is to build an extension of Hyland-Ong arena games where strategies, instead of trees, are event structures. With this, we hope to be able to use the flexibility of the Hyland-Ong approach within a more expressive metalanguage, and through this get finer models for complex effectful programming languages, with concurrency in mind.

The purpose of this note is to describe the cartesian closed category in which our future developments will take place. In Section 2, we start with some technical preliminaries on event structures, stable families, and symmetry. In Section 3, we build the compact closed category equipped with the adequate notions of symmetry. In Section 4, we use it to build our ccc that we call concurrent Hyland-Ong games, and we show that its subcategory of deterministic sequential strategies coincides with the standard ccc of arena games and innocent strategies.

2 Preliminaries

2.1 Event structures and maps

Let us start with some elementary definitions on event structures. Here we only give some basic definitions, notations and properties, but we will skip proofs. This is intended as a reference only, a more comprehensive introduction to event structures and stable families can be found *e.g.* in [Win11].

2.1.1 The categories \mathcal{ES} and \mathcal{ESP}

Definition 2.1. An *event structure* (es for short) comprises (E, \leq, Con) , consisting of a set E , of *events* which are partially ordered by \leq , the **causal dependency relation**, and a nonempty **consistency relation** Con consisting of finite subsets of E , which satisfy

$$\begin{aligned} \{e' \mid e' \leq e\} &\text{ is finite for all } e \in E, \\ \{e\} &\in \text{Con for all } e \in E, \\ Y \subseteq X \in \text{Con} &\implies Y \in \text{Con}, \text{ and} \\ X \in \text{Con} \ \& \ e \leq e' \in X &\implies X \cup \{e\} \in \text{Con}. \end{aligned}$$

Event structures support the following notion of state.

Definition 2.2. The (finite) **configurations**, $\mathcal{C}(E)$, of an event structure E consist of those finite subsets $x \subseteq E$ which are

Consistent: $x \in \text{Con}$, and

Down-closed: $\forall e, e'. e' \leq e \in x \implies e' \in x$.

Notations. We recall some basic notations on event structures. If $e \in E$, its **prime configuration** is $[e] = \{e' \in E \mid e' \leq e\}$. If $e, e' \in E$ are such that $e < e'$ and no other event lies in between, we write $e \rightarrow e'$. If $x \in \mathcal{C}(E)$ and $e \in E$ is such that $e \notin x$ and $x \cup \{e\} \in \mathcal{C}(E)$, we write $x \dashv e$, or just $x \dashv^x \cup \{e\}$.

Moreover, if two configurations x, y of E are such that $x \cup y$ is consistent then we write that x and y are *compatible*. (It is equivalent to asking that $x \cup y$ is a configuration of E .)

We now introduce total maps between event structures.

Definition 2.3. A **total map** $f : (E, \leq_E, \text{Con}_E) \rightarrow (F, \leq_F, \text{Con}_F)$ between event structures (written $f : E \rightarrow F$ for conciseness) is a function $f : E \rightarrow F$ such that:

- (1) For all $x \in \mathcal{C}(E)$, $f x \in \mathcal{C}(F)$,
- (2) For all $e, e' \in x \in \mathcal{C}(E)$, $f e = f e' \implies e = e'$.

We write \mathcal{ES} for the category of event structures and total maps between them. An **event structure with polarities** (esp for short) is an es E equipped with a **polarity function** $\text{pol}_E : E \rightarrow \{-, +\}$ specifying for each event $e \in E$, whether it is positive/Player ($\text{pol}(e) = +$) or negative/Opponent ($\text{pol}(e) = -$). The names negative/Opponent and positive/Player will be used interchangeably in this paper. Maps of esp are supposed to preserve polarities, and form a category \mathcal{ESP} .

Notations. If E is an esp, when introducing events of E we will sometimes label them with polarities, as in e^-, e^+ . In such a case the labeling is not part of the identifier and can be thought of as typing information – the identifier e can later be used without the annotation. Typically we might say "let $e^- \in E$ " to abbreviate "let $e \in E$ such that $\text{pol}(e) = -$ ".

2.1.2 Two constructions on event structures

We introduce two constructions on event structures, that will be useful later on. We spell them out for plain event structures, but they obviously also make sense in the presence of polarities.

Restriction. Let E be an event structure, and $R \subseteq E$. The *restriction* of E to R comprises events whose causal dependencies are all in R .

Definition 2.4. *There is an event structure $E \upharpoonright R$, the **restriction** of E to R , having:*

- Events:

$$\{e \in E \mid [e] \subseteq R\}$$

- Causality, consistency: *same as in E .*

For (E, \mathcal{F}_E) a stable family one can also form its restriction to R by restricting \mathcal{F}_E to subsequence of R . We can then check that $\mathcal{C}(E \upharpoonright R) = \mathcal{C}(E) \upharpoonright R$, and for all (E, \mathcal{F}_E) stable family we have $\Pr((E, \mathcal{F}_E) \upharpoonright R) = \Pr((E, \mathcal{F}_E)) \upharpoonright \max^{-1}R$.

Projection. Let E be an event structure, and $V \subseteq E$. The *projection* of E to V hides the causal dependencies that are not in V .

Definition 2.5. *For E an event structure and $V \subseteq E$ there is an event structure $E \downarrow V$, the **projection** of E to V , having:*

- Events: *Those of V ,*
- Causality, consistency: *Induced by E .*

2.1.3 Some lemmas on event structures

The two following technical lemmas will be useful in our development.

Lemma 2.6. *Let $f : E \rightarrow F$ be a map of event structures, and let $x, y \in \mathcal{C}(E)$ be compatible configurations such that $x, y \subseteq z$ and $f(x) \subseteq f(y)$, then $x \subseteq y$.*

Proof. Suppose this is not the case, then there is $e \in x$ such that $e \notin y$. But still $f(e) \in f(y)$ so there is $e' \in y$ such that $f(e) = f(e')$. But since $x, y \subseteq z \in \mathcal{C}(E)$, by local injectivity we have $e = e'$, absurd. \square

Lemma 2.7 (Mapification). *Suppose there is a total function $p : \mathcal{C}(R) \rightarrow \mathcal{C}(S)$ preserving \subseteq and preserving bounded unions in the sense that for all $x, y \in \mathcal{C}(R)$ such that x and y are compatible, then $p(x \cup y) = p(x) \cup p(y)$. Then there is a unique total map of event structure $\hat{p} : R \rightarrow S$ such that for all $x \in \mathcal{C}(R)$, $\hat{p}(x) = p(x)$.*

Proof. For an event $r \in R$, we necessarily have $p([e]) \dashv\vdash p([e])$, let us write $p([e]) = p([e]) \cup \{e'\}$. We set $\hat{p}(e) = e'$. Now, suppose $x \xrightarrow{e} y$ in $\mathcal{C}(R)$. Necessarily,

$$\begin{aligned} p(y) &= p(x \cup [e]) \\ &= p(x) \cup p([e]) \\ &= p(x) \cup p([e]) \cup \{\hat{p}(e)\} \\ &= p(x) \cup \{\hat{p}(e)\} \end{aligned}$$

from which it follows by immediate induction that for any configuration $x \in \mathcal{C}(R)$, $\hat{p}(x) = p(x)$. Finally, \hat{p} is a map of event structures since it evidently preserves configurations, and is locally injective since it preserves $\dashv\vdash$. \square

2.2 Stable families, and structure in \mathcal{ES}

In this subsection, we aim to construct some structure in \mathcal{ES} : namely, products and pullbacks. Those are notably difficult to describe combinatorially directly on event structures. Instead, it is more convenient to first describe the category of *stable families*, of which \mathcal{ES} can be seen as a full category. Products (and pullbacks) are more direct to construct in stable families, and are then easily transported to \mathcal{ES} .

2.2.1 Stable families

Definition 2.8. Let E be a set of events equipped with a non-empty $\mathcal{F}_E \subseteq_f E$ (a set of finite subsets) set of configurations. For $X \subseteq \mathcal{F}_E$ we say that X is **compatible**, written $X \uparrow$, iff there is $x \in \mathcal{F}_E$ such that $\cup X \subseteq x$. We define a **stable family** as such a pair (E, \mathcal{F}_E) satisfying the additional axioms:

- (1) **Completeness:** For $X \subseteq \mathcal{F}_E$, if $X \uparrow$ then $\cup X \in \mathcal{F}_E$ as well.
- (2) **Stability:** For $X \subseteq \mathcal{F}_E$, if $X \uparrow$ then $\cap X \in \mathcal{F}_E$ as well.
- (3) **Coincidence-freeness:** For $e, e' \in x \in \mathcal{F}$ with $e \neq e'$, then there is $y \in \mathcal{F}_E$ with $y \subseteq x$ and $e \in y \Leftrightarrow e' \notin y$.

A map $f : (E, \mathcal{F}_E) \rightarrow (F, \mathcal{F}_F)$ of stable families is a function $f : E \rightarrow F$ preserving configurations, and locally injective in the sense that for all $e, e' \in x \in \mathcal{F}_E$, $f(e) = f(e') \implies e = e'$. This forms a category \mathcal{SF} of stable families and maps between them.

When speaking about a stable family (E, \mathcal{F}_E) we will sometimes leave E implicit and refer to it as \mathcal{F}_E . If E is an event structure, $(E, \mathcal{C}(E))$ is a stable family. Obviously the operation $\mathcal{C}(-)$ gives a functor:

$$\mathcal{C}(-) : \mathcal{ES} \rightarrow \mathcal{SF}$$

which is full and faithful.

We now define the product of stable families.

Definition 2.9. For stable families (A, \mathcal{F}_A) and (B, \mathcal{F}_B) their product is $(A \times B, \mathcal{F}_{A \times B})$, where $x \subseteq A \times B$ is in $\mathcal{F}_{A \times B}$ iff it satisfies:

- (1) Projections: $\pi_1 x \in \mathcal{F}_A$ and $\pi_2 x \in \mathcal{F}_B$
- (2) Injectivity of projections: If $e, e' \in x$ with $\pi_i e = \pi_i e'$, then $e = e'$.
- (3) Coincidence-freeness: If $e, e' \in x$ with $e \neq e'$, then there exists $y \subseteq x$ with $\pi_1 y \in \mathcal{F}_A$ and $\pi_2 y \in \mathcal{F}_B$ as well, and $e \in y \Leftrightarrow e' \notin y$.

This definition yields a stable family, and actually gives a product in $S\mathcal{F}$. This product can be transported to event structures as follows.

2.2.2 Products in event structures

We now construct a right adjoint to $\mathcal{C}(-)$ which, to any stable family, associates a canonical event structure representing it.

Primes. Let (E, \mathcal{F}_E) be a stable family, and $x \in \mathcal{F}_E$. Then there is a partial order induced on x by, for $e, e' \in x$:

$$e \leq_x e' \Leftrightarrow \forall y \subseteq x, y \in \mathcal{F}_E \ \& \ e' \in y \implies e \in y$$

We also have the corresponding notion of immediate dependency $e \rightarrow_x e'$.

Then for $e \in x \in \mathcal{F}_E$, we can define the **prime**:

$$[e]_x = \{e' \in x \mid e' \leq_x e\}$$

It is then easy to show that $[e]_x \in \mathcal{F}_E$.

Definition 2.10. For a stable family (E, \mathcal{F}_E) we define an event structure $\text{Pr}((E, \mathcal{F}_E))$ having:

- Events: primes $[e]_x$.
- Causality: inclusion.
- Consistency: For X a finite set of primes, $X \in \text{Con}_{\text{Pr}((E, \mathcal{F}_E))}$ iff $\cup X \in \mathcal{F}_E$.

Then one can check that $\text{Pr}(-)$ extends to a functor from stable families to event structures, which is right adjoint to $\mathcal{C}(-)$. For event structures A and B we can now define:

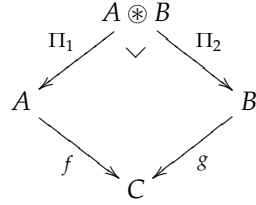
$$A \times B = \text{Pr}(\mathcal{C}(A) \times \mathcal{C}(B))$$

This defines a product of A and B in the category of event structures.

2.2.3 Pullbacks in \mathcal{ES}

We have seen how one can define products in \mathcal{ES} . Take now $f, g : E \rightarrow F$. Defining $R \subseteq E$ as $e \in R$ iff $f(e) = g(e)$, one can check that $E \upharpoonright R$ is an equalizer of f and g . Together with the product introduced above, this permits to construct pullbacks in \mathcal{ES} .

Let $f : A \rightarrow C$ and $g : B \rightarrow C$. Form their pullback by expanding the definition above.



Precisely, we first define $R \subseteq A \times B$ as the set of events $c \in A \times B$ such that $f(\Pi_1 c) = g(\Pi_2 c)$. Then we set:

$$A \otimes B = \text{Pr}(\mathcal{C}(A) \times \mathcal{C}(B) \upharpoonright R)$$

and thereby obtain the pullback of f and g . Unfolding the definitions, we can see that events of $A \otimes B$ have the form:

$$[(a, b)]_x$$

where $x \in \mathcal{C}(A) \times \mathcal{C}(B) \upharpoonright R$ and $(a, b) \in x$, and for all $(a', b') \in x$ we have $f a' = g b'$ (so in particular $f a = g b$).

2.2.4 Technical lemmas on the pullback

The pullback will be of key importance in our development. We will make use of the following technical lemmas, for a pullback $A \otimes B$ of $f : A \rightarrow C$ and $g : B \rightarrow C$.

Lemma 2.11 (Characterization of immediate causality). *Take $f : A \rightarrow C$ and $g : B \rightarrow C$, and take their pullback $A \otimes B$. Suppose moreover that in the pullback, we have:*

$$[(a, b)]_x \rightarrow [(a', b')]_x$$

Then necessarily, $a \rightarrow a'$ or $b \rightarrow b'$.

Proof. Suppose first we have $a \leq a'$ or $b \leq b'$, say *w.l.o.g.* $a < a'$. Suppose, looking for a contradiction, that this dependency is not immediate: there is a'' such that $a < a'' < a'$. But then $[(a', b')]_x$ must contain some (a'', b'') . Then we have:

$$[(a, b)]_x \subset [(a'', b'')]_x \subset [(a', b')]_x$$

Contradicting the immediate causality $[(a, b)]_x \rightarrow [(a', b')]_x$, so $a \rightarrow a'$.

Suppose we have neither $a \leq a'$ nor $b \leq b'$. But then, setting $y = [(a', b')]_x$ we have $y \setminus (a, b) \in \mathcal{C}(A \otimes B)$. To prove that, the only non-trivial condition to check is that its projections are still configurations. Suppose by contradiction that $\pi_1(y \setminus (a, b)) \notin \mathcal{C}(A)$. This means that there is $a'' \in y$ such that $a \leq a''$, so $y \setminus (a, b)$ is not down-closed anymore. But then there is $(a'', b'') \in y$ with:

$$(a, b) \leq_x (a'', b'')$$

But $(a'', b'') \leq_x (a', b')$ as well by definition of y , so that contradicts the hypothesis $[(a, b)]_x \rightarrow [(a', b')]_x$. \square

Lemma 2.12 (Characterization of immediate conflict). *Take $f : A \rightarrow C$ and $g : B \rightarrow C$, and take their pullback $A \otimes B$. Suppose moreover that in the pullback, we have*

$x \in \mathcal{C}(A \otimes B)$ with $x \xrightarrow{[(a,b)]_y} x$, $x \xrightarrow{[(a',b')]_{y'}} x$ but $x \cup \{[(a,b)]_y, [(a',b')]_{y'}\} \notin \mathcal{C}(A \otimes B)$. Then either $\Pi_1 x \cup \{a, a'\} \notin \mathcal{C}(A)$, or $\Pi_2 x \cup \{b, b'\} \notin \mathcal{C}(B)$.

Proof. If we had both $\Pi_1 x \cup \{a, a'\} \in \mathcal{C}(A)$ and $\Pi_2 x \cup \{b, b'\} \in \mathcal{C}(B)$, then $x \cup \{[(a,b)]_y, [(a',b')]_{y'}\}$ would have its two projections respectively in $\mathcal{C}(A)$ and $\mathcal{C}(B)$. However it obviously satisfies the other conditions of configurations of the product, and is in the restriction involved in the definition of the pullback; therefore it is a valid configuration of $A \otimes B$, absurd. \square

Very often, we will reason at the level of configurations of the pullback rather than events. Such configurations can be presented a bit more simply:

Proposition 2.13. *Configurations of $A \otimes B$ uniquely correspond to the composition bijections between $x \in \mathcal{C}(A)$ and $y \in \mathcal{C}(B)$:*

$$x \xrightarrow{f} f(x) = g(y) \xrightarrow{g} y$$

that are secured, in the sense that the transitive relation generated by $(a, b) \leq (a', b')$ iff $a \leq a'$ or $b \leq b'$ is a partial order – in other words there is no causal loop.

Proof. Simple verification. \square

2.3 Symmetry

The basic theory of event structures with symmetry is introduced and developed in [Win07]. In this preliminary section, we only recall some basic definitions, then state and/or prove some lemmas used in our development.

2.3.1 Event structures with symmetry

Definition 2.14 (Open maps). *A map $f : A \rightarrow B$ between event structures is open whenever it is rigid (preserves causal dependency) and for each configuration x of A and y configuration of B such that $fx \subseteq y$ then x can be extended to a configuration x' of A such that $fx' = y$.*

Definition 2.15 (Event structure with symmetry). *An event structure with symmetry (ess for short) \mathcal{A} is a tuple (A, \tilde{A}, l_A, r_A) where A and \tilde{A} are event structures, and l_A, r_A form a span:*

$$A \xleftarrow{l_A} \tilde{A} \xrightarrow{r_A} A$$

such that l_A, r_A are jointly monic (i.e. their pairing $\langle l_A, r_A \rangle$ is monic) and satisfy diagrams presenting categorically equivalence relations, see e.g. [Win07]. Moreover, l_A and r_A are supposed to be open maps.

We do not recall the diagrams expressing the fact that l_A, r_A form an equivalence since they are standard, and actually not used much in our development.

We use the notation $\mathcal{A}, \mathcal{B}, \mathcal{C}$ to range over event structures with symmetry whose underlying event structures will be denoted A, B, C, \dots and symmetries $\tilde{A}, \tilde{B}, \tilde{C}, \dots$. An alternative definition of event structures with symmetry can be given in terms of isomorphism families:

Definition 2.16 (Isomorphism families). *An **isomorphism family** on event structure A is a set \mathbb{S}_A of bijections between configurations of A containing the identities $\text{id}_x : x \cong x$, closed under composition and inverse, and satisfying two more properties:*

- *Whenever $\theta : x \cong y \in \mathbb{S}_A$ and x' a sub-configuration of x then $\theta|_{x'}$ the restriction of θ to x' must belong to \mathbb{S}_A*
- *Whenever $\theta : x \cong y \in \mathbb{S}_A$ and $x \subseteq x' \in \mathcal{C}(A)$ then there exists $\theta' \supseteq \theta$ in \mathbb{S}_A such that the domain of θ' is x' .*

Note that because of restriction, any bijection which belongs to \mathbb{S}_A is also an order-isomorphism, i.e. it preserves and reflects causal order.

Isomorphism families can be regarded as a more concrete presentation of symmetries on event structures. Any symmetry (\tilde{A}, l_A, r_A) on event structure A generates an isomorphism family by setting:

$$\mathbb{S}_A = \{ \{ (l_A \tilde{a}, r_A \tilde{a}) \mid \tilde{a} \in x \} \mid x \in \mathcal{C}(\tilde{A}) \}$$

Likewise \mathbb{S}_A is a stable family on the set of events $A \times A$, therefore one can redefine the symmetry *via* the primes construction, as $\text{Pr}(\mathbb{S}_A)$. One can check that the event structure defined in this way along with the obvious projections is a symmetry isomorphic to the original. This connection is worked out in all details in [Win07]. In our development, we will mostly work with the isomorphism families and leave the symmetry implicit.

If \mathcal{A} is an event structure with symmetry, we will write:

$$x \overset{\theta}{\cong}_{\mathcal{A}} y$$

if $\theta : x \cong y$ is in \mathbb{S}_A .

Definition 2.17 (Maps of event structures with symmetry). A map $f : \mathcal{A} \rightarrow \mathcal{B}$ of event structures with symmetry is a map $f : A \rightarrow B$ of event structures such that for each isomorphism $x \cong_{\mathcal{A}}^{\theta} y$, the bijection induced by f :

$$f\theta = \{(f(a_1), f(a_2)) \mid (a_1, a_2) \in \theta\}$$

is in \mathbb{S}_B .

Or equivalently, there exists a (necessarily unique) map $\tilde{f} : \tilde{A} \rightarrow \tilde{B}$ such that $l_B \circ \tilde{f} = f \circ l_A$, and $r_B \circ \tilde{f} = f \circ r_A$.

There is a category \mathcal{ESS} of event structures with symmetry and maps preserving symmetry, and a category \mathcal{ESSP} in the presence of polarities, which the isomorphisms and maps should preserve.

Symmetry allow us to identify maps that play symmetric events:

Definition 2.18 (Symmetric maps). Let $f, g : \mathcal{A} \rightarrow \mathcal{B}$ be two parallel maps of event structures with symmetry. We say that f and g are symmetric (notation: $f \sim g$) when for any configuration x in A , the induced bijection:

$$\theta = \{(f(a), g(a)) \mid a \in x\}$$

is in \mathbb{S}_B .

Or equivalently, there is a (necessarily unique) $h : A \rightarrow B$ such that $l_B \circ h = f$ and $r_B \circ h = g$.

This relation \sim is preserved by composition, which makes \mathcal{ESS} and \mathcal{ESSP} enriched over equivalence relations, or (rather degenerate) strict 2-categories.

2.3.2 Higher symmetry

If \mathcal{A} is an event structure with symmetry, then its symmetry \tilde{A} is itself canonically equipped with a symmetry. Although it can be described via a universal property (pseudopullbacks – see [CCW14]), here we define it instead through its isomorphism family.

As we have seen above, configurations of \tilde{A} correspond to isomorphisms $\theta \in \mathbb{S}_A$. By abuse of notations, we will often silently go from one to the other. In particular, configurations on \tilde{A} should correspond to certain bijections between isomorphisms $\theta, \theta' \in \mathbb{S}_A$ (seen as subsets of $A \times A$). We ask the isomorphism family of \tilde{A} to comprise bijections between θ, θ' corresponding to commuting squares:

$$\begin{array}{ccc} x & \cong_{\mathcal{A}}^{\phi} & x' \\ \theta \parallel_{\mathcal{R}_A} & & \theta' \parallel_{\mathcal{R}_A} \\ y & \cong_{\mathcal{A}}^{\phi'} & y' \end{array}$$

According to this characterisation, $l_{\tilde{A}}$ sends this square to its left side and $r_{\tilde{A}}$ sends it to its right side. However, there are now additional maps $u_{\tilde{A}}$ sending it to its upper side and $d_{\tilde{A}}$ sending it to its lower side. These additional projections satisfy the diagram:

$$\begin{array}{ccc}
& \tilde{\tilde{S}} & \\
l_{\tilde{S}} \swarrow & & \searrow r_{\tilde{S}} \\
\tilde{S} & \downarrow u_{\tilde{S}} & \tilde{S} \\
l_S \downarrow & & \downarrow r_S \\
S & & S
\end{array}
\quad
\begin{array}{ccc}
& \tilde{\tilde{S}} & \\
l_{\tilde{S}} \swarrow & & \searrow r_{\tilde{S}} \\
\tilde{S} & \downarrow d_{\tilde{S}} & \tilde{S} \\
r_S \downarrow & & \downarrow r_S \\
S & & S
\end{array}$$

We write $\tilde{\tilde{A}}$ for \tilde{A} equipped with its symmetry $\tilde{\tilde{A}}$. We can now automatically lift some properties from maps $f : \mathcal{A} \rightarrow \mathcal{B}$ to $\tilde{f} : \tilde{\tilde{A}} \rightarrow \tilde{\tilde{B}}$:

Lemma 2.19. *We have the following lifting properties.*

1. If $f : \mathcal{A} \rightarrow \mathcal{B}$ is a morphism in $\mathcal{ESS}/\mathcal{ESSP}$, then $\tilde{f} : \tilde{\tilde{A}} \rightarrow \tilde{\tilde{B}}$ preserves symmetry as well.
2. If $f \sim g$, then $\tilde{f} \sim \tilde{g}$.

Proof. (1) We need to check that $\tilde{f} : \tilde{\tilde{A}} \rightarrow \tilde{\tilde{B}}$ preserves symmetry. Take an isomorphism in $\tilde{\tilde{A}}$, corresponding to a square:

$$\begin{array}{ccc}
x & \xrightarrow[\cong_{\mathcal{A}}]{\phi_1} & x' \\
\theta \parallel_{\mathcal{R}_{\mathcal{A}}} & & \theta' \parallel_{\mathcal{R}_{\mathcal{A}}} \\
y & \xrightarrow[\cong_{\mathcal{A}}]{\phi_2} & y'
\end{array}$$

The component-wise image of this isomorphism family by \tilde{f} corresponds to the commuting square:

$$\begin{array}{ccc}
fx & \xrightarrow[\cong_{\mathcal{B}}]{f\phi_1} & fx' \\
f\theta \parallel_{\mathcal{R}_{\mathcal{B}}} & & f\theta' \parallel_{\mathcal{R}_{\mathcal{B}}} \\
fy & \xrightarrow[\cong_{\mathcal{B}}]{f\phi_2} & fy'
\end{array}$$

which corresponds to a configuration of the isomorphism family of $\tilde{\tilde{B}}$ by definition.

(2) Suppose $f \sim g$, this means that for any $x \in \mathcal{C}(A)$, the isomorphism $\theta_x = \{(fa, ga) \mid a \in x\}$ is in the isomorphism family of x . Then it is clear that for any isomorphism $x \cong_{\phi} y$ in the isomorphism family of A , the following square commutes:

$$\begin{array}{ccc}
fx & \cong_{\theta_x} & gx \\
\parallel_{f\phi} & & \parallel_{g\phi} \\
fy & \cong_{\theta_y} & gy
\end{array}$$

Indeed take $(a, b) \in \phi$. Then we have $(fa, fb) \in f\phi$, $(ga, gb) \in g\phi$, $(fa, ga) \in \theta_x$ and $(fb, gb) \in \theta_y$. But this square corresponds to an isomorphism between $f\phi$ and $g\phi$ in $\tilde{\tilde{B}}$, so we do have $\tilde{f} \sim \tilde{g}$. □

2.3.3 Projections with symmetry

In this small subsection, we introduce a few definitions and lemmas regarding the generalisation of projection in the presence of symmetry.

Definition 2.20. *Let \mathcal{A} be an event structure with symmetry. A subset $V \subseteq A$ is closed under symmetry iff, setting*

$$\tilde{V} = \{\tilde{a} \in \tilde{A} \mid l_A(\tilde{a}) \in V\}$$

we have $\sigma_A(\tilde{V}) = \tilde{V}$ (where $\sigma_A : \tilde{A} \rightarrow \tilde{A}$ is the map expressing that as an equivalence relation, the symmetry is symmetric).

Proposition 2.21. *Let \mathcal{A} be an event structure with symmetry, let V be a subset of the events of A closed under symmetry. Then the triple $(\tilde{A} \downarrow \tilde{V}, l_A \upharpoonright \tilde{V}, r_A \upharpoonright \tilde{V})$ (where \upharpoonright denotes function restriction) is a symmetry on $A \downarrow V$.*

*The event structure with symmetry $(A \downarrow V, \tilde{A} \downarrow \tilde{V})$ is called the **projection** of A on V .*

Proof. A subset V is closed under symmetry if and only if it is the domain of a partial map of event structures preserving symmetry. According to this, this proposition is a restatement of the partial-total factorisation result for event structures with symmetry [CCW14, Win07]. \square

3 Thin concurrent games with symmetry

In this section, we introduce the basic compact closed category that serves as a base for our forthcoming concurrent generalisation of Hyland-Ong games. The basic idea is to construct a compact closed (\sim -bi-)category analogous to [RW11] in the presence of symmetry.

In [CCW14], we constructed such a compact closed category of concurrent games based on event structures with symmetry. What we describe here is very close to [CCW14], but does differ in a crucial point. In the basic compact closed category of [CCW14] strategies are *saturated*, in the sense that whenever they are in position to play a particular event they must also be prepared to play (non-deterministically) all events symmetric to it. Saturation enables a greater generality (it is required in particular to define an exponential on non-polarized games). However, for the vast majority of the applications that we have in mind to program semantics, that generality does not seem to be needed. So instead of directly building on [CCW14], we start by developing a non-saturated variant of it.

As the reader will see, developing such a non-saturated variant is far from being a formality. In fact, it is arguably the most significant technical contribution of this paper – it is definitely the part whose development took the most time and effort. Its key advantage is that the symmetry-aware constructions on games and strategies involved are all compatible with [RW11] – meaning that although the *soundness* of the basic operations on games and strategies

heavily involves symmetry, their *definition* is independent from symmetry. So, ignoring the symmetry layer, one immediately gets the games of [RW11]. This is in contrast with [CCW14] where composition and copycat are changed to a symmetry-aware version. Whereas in [CCW14] the key difficulty was to find symmetry-friendly notions of copycat and composition, here we find conditions that make the copycat and composition of [RW11] symmetry-friendly.

Beyond its technical advantages, we believe that this approach yields more economical representations of the execution of programs.

3.1 Thin concurrent games with symmetry

In this first subsection, we define the notion of games that we are going to work with.

Definition 3.1. *An essp \mathcal{A} is thin if for all $x \cong_{\mathcal{A}}^{\theta} y$, if θ has positive extensions to valid isomorphisms $x_1 \cong_{\mathcal{A}}^{\theta_1} y_1$ and $x_2 \cong_{\mathcal{A}}^{\theta_2} y_2$ with $\theta \subseteq^+ \theta_1, \theta \subseteq^+ \theta_2$, if $x_1 \cup x_2 \in \mathcal{C}(A)$ then $\theta_1 \cup \theta_2$ is a valid isomorphism as well.*

Or equivalently, l_A and r_A reflects positive compatibility.

In order to state the definition below, we need the notion of sub-symmetry. A **sub-symmetry** of an essp \mathcal{A} is a sub-event structure \tilde{A}' of \tilde{A} that is still a symmetry for A . We say that it is a **receptive sub-symmetry** if for any $x \in \mathcal{C}(\tilde{A}')$, if $x \subseteq^- y \in \mathcal{C}(\tilde{A})$, then $y \in \mathcal{C}(\tilde{A}')$ as well. In other words a sub-symmetry is receptive when the embedding map $\text{id}_A : (A, \tilde{A}') \rightarrow (A, \tilde{A})$ is strong-receptive (in the sense of [CCW14], also recalled later).

Definition 3.2 (Thin Concurrent game). *A **thin concurrent game** (tcg for short) is an essp such that:*

- It is **race-preserving**, i.e. *the projection $l_A : \tilde{A} \rightarrow A$ preserves races: if $x \in \mathcal{C}(\tilde{A})$ with $x \subseteq^- a_1^-, x \subseteq^- a_2^+$ but $x \cup \{a_1, a_2\} \notin \mathcal{C}(\tilde{A})$, then $l_A x \cup \{l_A a_1, l_A a_2\} \notin \mathcal{C}(A)$ either. It automatically follows that r_A preserves races as well.*
- *\mathcal{A} has \tilde{A}_- and \tilde{A}_+ , thin receptive sub-symmetries of \mathcal{A}^\perp and \mathcal{A} respectively.*

The fact that l_A preserves races will be essential for copycat to have a symmetry. Along with the sub-symmetry condition, they will ensure that composition preserves equivalence between strategies.

The set of concurrent games with symmetry is closed under parallel composition and dual $(-)^{\perp}$, in the obvious way.

Definition 3.3. *A **pre- \sim -strategy** $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is just a map of essps.*

*A **\sim -strategy** $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is a pre- \sim -strategy such that:*

- (1) $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is **courteous**¹: *if $s_1, s_2 \in \mathcal{S}$ such that $s_1 \rightarrow s_2$ and $\text{pol}(s_1) = +$ or $\text{pol}(s_2) = -$, then $\sigma s_1 \rightarrow \sigma s_2$.*

¹Called *innocent* in [RW11] and [CCW14] – we switched to courteous here (following [MM07]) to avoid collision with Hyland-Ong innocence.

(2) $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is **strong-receptive**: if

$$x \cong_{\mathcal{S}}^{\theta} y$$

with $\sigma \theta \dashv (a_1, a_2)$, then there is a unique $\theta \dashv (s_1, s_2)$ such that $\sigma s_1 = a_1$ and $\sigma s_2 = a_2$. Equivalently, $\tilde{\sigma}$ is receptive in the sense of [RW11]. Together with courtesy, this means that $\tilde{\sigma}$ is a strategy in the sense of [RW11].

(3) $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is **thin**, in the sense that \mathcal{S} is thin.

Recall from [RW11] that a map of *essp* $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ (without symmetry) is referred to as a **pre-strategy**, and a **strategy** if it satisfies (1) and (2). We will use this terminology later on in the text when the map has to be considered independently of symmetry.

Note that it automatically follows that $\tilde{\sigma} : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{A}}$ is itself a \sim -strategy:

Lemma 3.4. *Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ be a \sim -strategy, then so is $\tilde{\sigma} : \tilde{\mathcal{S}} \rightarrow \tilde{\mathcal{A}}$.*

Proof. Courtesy is a consequence of the rigidity of l and r . Thus we only need to prove that $\tilde{\sigma}$ is strong-receptive, i.e. that $\tilde{\sigma}$ is receptive, and that $\tilde{\mathcal{S}}$ is thin. Suppose we have a configuration in $\tilde{\mathcal{S}}$, represented by a square:

$$\begin{array}{ccc} x_1 & \cong_{\mathcal{S}}^{\phi} & x_2 \\ \theta \parallel_{\mathcal{S}} & & \theta \parallel_{\mathcal{S}} \\ y_1 & \cong_{\mathcal{S}}^{\phi'} & y_2 \end{array}$$

Suppose that the image by $\tilde{\sigma}$ of this configuration extends in the following way:

$$\begin{array}{ccc} \sigma x_1 \cup \{a_1\} & \cong_{\mathcal{A}}^{\sigma \phi \cup \{(a_1, a_2)\}} & \sigma x_2 \cup \{a_2\} \\ \sigma \theta \cup \{(a_1, a'_1)\} \parallel_{\mathcal{A}} & & \parallel_{\mathcal{A}} \sigma \theta' \cup \{(a_2, a'_2)\} \\ \sigma y_1 \cup \{a'_1\} & \cong_{\mathcal{A}}^{\sigma \phi' \cup \{(a'_1, a'_2)\}} & \sigma y_2 \cup \{a'_2\} \end{array}$$

where all a_1, a_2, a'_1, a'_2 have negative polarity. By receptivity of σ there are unique s_1, s_2, s'_1, s'_2 mapping respectively to a_1, a_2, a'_1, a'_2 by σ . By receptivity of $\tilde{\sigma}$, we necessarily have that $\theta \dashv_{(s_1, s'_1)} \dashv (a_1, a'_1)$, $\theta' \dashv_{(s_2, s'_2)} \dashv (a_2, a'_2)$, $\phi \dashv_{(s_1, s_2)} \dashv (a_1, a_2)$ and $\phi' \dashv_{(s'_1, s'_2)} \dashv (a'_1, a'_2)$, so that the following commutative diagram corresponds to an isomorphism in $\tilde{\mathcal{A}}$.

$$\begin{array}{ccc} x_1 \cup \{s_1\} & \cong_{\mathcal{S}}^{\phi \cup \{(s_1, s_2)\}} & x_2 \cup \{s_2\} \\ \theta \cup \{(s_1, s'_1)\} \parallel_{\mathcal{S}} & & \parallel_{\mathcal{S}} \theta' \cup \{(s_2, s'_2)\} \\ y_1 \cup \{s'_1\} & \cong_{\mathcal{S}}^{\phi' \cup \{(s'_1, s'_2)\}} & y_2 \cup \{s'_2\} \end{array}$$

uniqueness is a trivial consequence of the uniqueness for receptivity of σ .

Let us now show that \tilde{S} is thin. Take a configuration of \tilde{S} , regarded as a commuting square:

$$\begin{array}{ccc} x_1 & \xrightarrow{\phi} & x_2 \\ \theta \parallel \wr_S & & \theta \parallel \wr_S \\ y_1 & \xrightarrow{\phi'} & y_2 \end{array}$$

Suppose it extends positively to $\phi_1, \phi'_1, \theta_1, \theta'_1$, and alternatively to $\phi_2, \phi'_2, \theta_2, \theta'_2$ with compatible left projections. Since S is thin, it follows that $\phi_1 \cup \phi_2, \phi'_1 \cup \phi'_2, \theta_1 \cup \theta_2$ are compatible. By composition, it also follows that $\theta'_1 \cup \theta'_2$ is compatible as well. Putting these together, the two extensions of the commuting square are compatible as well, so the left projection reflects positive compatibility. \square

3.2 Weak equivalence of \sim -strategies

As notion of equivalence between \sim -strategies we introduce *weak equivalence*, that adequately weakens isomorphism of strategies in the presence of symmetry. In particular, it allows us to consider equivalent \sim -strategies that might play the same events, but with a different choice of copy indices.

Definition 3.5 (Weak equivalence). *Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A$ be \sim -strategies on a tcg A . A **weak equivalence** between σ and τ is given by two morphisms of essps $f : S \rightarrow T$ and $g : T \rightarrow S$ such that $f \circ g \sim \text{id}_T$, $g \circ f \sim \text{id}_S$, $\tau \circ f \sim \sigma$ and $\sigma \circ g \sim \tau$.*

A central difficulty in our endeavour will be to show that weak equivalence is preserved under composition. Readers familiar with AJM games may note that this difficulty already exists there, where one has to show that equivalence between strategies is preserved under composition. Here, the richer structure of our strategies makes this fact more technical. This is where our *thin* requirement on essps comes in.

In particular, preservation of weak equivalence under composition will use the fact that the isomorphism family of thin essps satisfy the coherent positive extension property below.

Definition 3.6. *Let A be an essp. We say that A has a **coherent extension** iff for each $x \subseteq^+_{\theta} y$, for each $x \subseteq^+ x' \in \mathcal{C}(A)$ there is $y \subseteq^+ y' \in \mathcal{C}(A)$ and $x' \subseteq^+_{\text{ext}(\theta, x')} y'$ such that $\theta \subseteq^+ \text{ext}(\theta, x')$. Moreover the function ext (the coherent extension) should satisfy the two following properties:*

- (monotonicity) *If $x_1 \cong_{\theta} y_1$ and $x_1 \subseteq^+ x_2 \subseteq^+ x_3$, then*

$$\text{ext}(\text{ext}(\theta, x_2), x_3) = \text{ext}(\theta, x_3)$$

- (stability) If $x_1 \cong_{\theta_1} y_1$, $\theta_1 \subseteq^- \theta_2$ (writing $x_2 \cong_{\theta_2} y_2$) with $x_1 \subseteq^+ x'_1$ such that $\{\text{ext}(\theta_1, x'_1), \theta_2\} \uparrow$, then

$$\text{ext}(\theta_1, x'_1) \subseteq \text{ext}(\theta_2, x_2 \cup x'_1)$$

This condition is implied by the thin condition:

Lemma 3.7. *If $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is a \sim -strategy, then \mathcal{S} has a coherent extension.*

Proof. Note first that for any $x \cong_{\theta}^{\theta} y$ and $x \subseteq^+ x'$, then by the extension property of isomorphism families there is $y \subseteq^+ y'$ and $\theta \subseteq \theta'$ such that $x' \cong_{\theta'}^{\theta'} y'$.

But θ' is actually uniquely defined since if there was another $x' \cong_{\theta''}^{\theta''} y''$ it would yield a positive conflict. Since σ is a thin strategy, this positive conflict should be preserved by $l_{\mathcal{S}}$ but that is obviously not the case, therefore $\theta' = \theta''$. Using that, the monotonicity condition is obvious.

For stability, take $x_1 \cong_{\theta_1} y_1$, $\theta_1 \subseteq^- \theta_2$ (writing $x_2 \cong_{\theta_2} y_2$) with $x_1 \subseteq^+ x'_1$ such that $\{\text{ext}(\theta_1, x'_1), \theta_2\} \uparrow$. By restriction, there is $\theta \subseteq \text{ext}(\theta_2, x_2 \cup x'_1)$ such that $l_A(\theta) = x'_1$. By restriction again it restricts again to $\theta' \subseteq \theta$ with $l(\theta') = x_1$. But then θ' and $\text{ext}(\theta_1, x'_1)$ are two positive extensions of θ_1 whose left projection is x'_1 . If their right projection was different then that would be a positive conflict in $\tilde{\mathcal{S}}$, immediately contradicting preservation of positive conflict. Therefore θ' and $\text{ext}(\theta_1, x'_1)$ must be equal. \square

Lemma 3.8. *Suppose $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ is a \sim -strategy. Then, the coherent extension of $\tilde{\mathcal{S}}$ coincides with that of \mathcal{S} in the sense that for any isomorphism $\theta_1 \cong_{\tilde{\mathcal{S}}}^{\phi} \theta_2$ such that $\theta_1 \subseteq^+ \theta'_1$, we have:*

$$\begin{aligned} u_{\tilde{\mathcal{S}}}(\text{ext}(\phi, \theta'_1)) &= \text{ext}(u_{\tilde{\mathcal{S}}}(\phi), l_{\tilde{\mathcal{A}}}(\theta'_1)) \\ d_{\tilde{\mathcal{S}}}(\text{ext}(\phi, \theta'_1)) &= \text{ext}(d_{\tilde{\mathcal{S}}}(\phi), r_{\tilde{\mathcal{A}}}(\theta'_1)) \end{aligned}$$

Proof. First, we describe the coherent extension of $\tilde{\mathcal{S}}$. Take a configuration of $\tilde{\mathcal{S}}$, described as a commuting square of isomorphisms:

$$\begin{array}{ccc} x_1 & \xrightarrow{\psi} & x_2 \\ \theta_1 \downarrow \mathcal{R}_{\mathcal{S}} & & \theta_2 \downarrow \mathcal{R}_{\mathcal{S}} \\ y_1 & \xrightarrow{\psi'} & y_2 \end{array}$$

Suppose also that θ_1 extends positively to $x'_1 \cong_{\theta'_1} y'_1$. Then, the unique positive

extension is:

$$\begin{array}{ccccccc}
 & & x'_1 & \cong_{\text{ext}(\psi, x'_1)} & x'_2 & & \\
 & & \cup & & \cup & & \\
 x'_1 & \supseteq & x_1 & \cong_{\psi} & x_2 & \subseteq & x'_2 \\
 \wr_{\theta'_1} & & \wr_{\theta_1} & & \wr_{\theta_2} & & \wr_{\theta'_2} \\
 y'_1 & \supseteq & y_1 & \cong_{\psi'} & y_2 & \subseteq & y'_2 \\
 & & \cap & & \cap & & \\
 & & y'_1 & \cong_{\text{ext}(\psi', y'_1)} & y'_2 & &
 \end{array}$$

where $\theta'_2 = \text{ext}(\psi', y'_1) \circ (\theta'_1)^{-1} \circ \text{ext}(\psi, x'_1)^{-1}$. The outer commuting square gives us an extension $\text{ext}(\phi, \theta'_1)$ of ϕ , which satisfies the required equations. \square

3.3 Copycat

We now proceed to show how copycat adapts to our framework. The definition of copycat without symmetry (as given *e.g.* in [RW11, Win11]), gives us:

$$\alpha_A : \mathbb{C}_A \rightarrow A^\perp \parallel A$$

We need to equip it with a symmetry.

We start by showing that \mathbb{C} can be extended to a functor:

$$\mathbb{C} : \mathcal{ESP} \rightarrow \mathcal{ESP}$$

Indeed, one can check that for $f : A \rightarrow B$ a map of esps, the map defined as:

$$\begin{array}{lcl}
 \mathbb{C}_f : \mathbb{C}_A & \rightarrow & \mathbb{C}_B \\
 (1, a) & \mapsto & (1, f a) \\
 (2, a) & \mapsto & (2, f a)
 \end{array}$$

It is still a map of esps, and by definition it is immediate that the construction is indeed functorial. This allows us to formulate the tentative definition of a symmetry on copycat:

$$\begin{array}{lcl}
 \widetilde{\mathbb{C}}_A & = & \mathbb{C}_{\tilde{A}} \\
 l_{\mathbb{C}_A} & = & \mathbb{C}_{l_A} \\
 r_{\mathbb{C}_A} & = & \mathbb{C}_{r_A}
 \end{array}$$

We need to check that $(\widetilde{\mathbb{C}}_{\tilde{A}}, \mathbb{C}_{l_A}, \mathbb{C}_{r_A})$ indeed is a symmetry on \mathbb{C}_A . First we show that it is an equivalence relation, which relies on the following lemma:

Lemma 3.9. *The functor \mathbb{C} preserves pullbacks.*

Proof. First, it is easy to check that the functors $(-)^{\perp} : \mathcal{ESP} \rightarrow \mathcal{ESP}$ and $\| : \mathcal{ESP}^2 \rightarrow \mathcal{ESP}$ preserve pullbacks. So if we have a pullback square in \mathcal{ESP} :

$$\begin{array}{ccc} & P & \\ \Pi_1 \swarrow & \downarrow \sphericalangle & \searrow \Pi_2 \\ A & & B \\ f \searrow & & \swarrow g \\ & C & \end{array}$$

then the following is a pullback square as well:

$$\begin{array}{ccc} & P^{\perp} \parallel P & \\ \Pi_1^{\perp} \parallel \Pi_1 \swarrow & \downarrow \sphericalangle & \searrow \Pi_2^{\perp} \parallel \Pi_2 \\ A^{\perp} \parallel A & & B^{\perp} \parallel B \\ f^{\perp} \parallel f \searrow & & \swarrow g^{\perp} \parallel g \\ & C^{\perp} \parallel C & \end{array}$$

We want to transport the universal property of the above pullback to the following commuting square:

$$\begin{array}{ccc} & \mathbb{C}_P & \\ \mathbb{C}_{\Pi_1} \swarrow & \downarrow \sphericalangle & \searrow \mathbb{C}_{\Pi_2} \\ \mathbb{C}_A & & \mathbb{C}_B \\ \mathbb{C}_f \searrow & & \swarrow \mathbb{C}_g \\ & \mathbb{C}_C & \end{array}$$

To do that, take $h_1 : X \rightarrow \mathbb{C}_A$ and $h_2 : X \rightarrow \mathbb{C}_B$ making the square commute, and form $h = \langle \alpha_A \circ h_1, \alpha_B \circ h_2 \rangle$ using the universal property of the above pullback square. To any $x \in \mathcal{C}(X)$, h associates a configuration $h(x) \in \mathcal{C}(P^{\perp} \parallel P)$. By Proposition 2.13, it corresponds to a secured bijection:

$$\theta : y \cong (f^{\perp} \parallel f)(y) = (g^{\perp} \parallel g)(z) \cong z$$

with $y \in \mathcal{C}(A^{\perp} \parallel A)$ and $z \in \mathcal{C}(B^{\perp} \parallel B)$. By definition of $f^{\perp} \parallel f$ and $g^{\perp} \parallel g$ and writing $y = y_1 + y_2$ and $z = z_1 + z_2$, this decomposes into two secured bijections:

$$\begin{array}{l} \theta_1 : y_1 \cong f(y_1) = g(z_1) \cong z_1 \\ \theta_2 : y_2 \cong f(y_2) = g(z_2) \cong z_2 \end{array}$$

But we also know that $h_1(x) \in \mathbb{C}_A$ and $h_2(x) \in \mathbb{C}_B$, therefore $y_1 \supseteq^+ y_1 \cap y_2 \subseteq^- y_2$ and $z_1 \supseteq^+ z_1 \cap z_2 \subseteq^- z_2$. By definition of θ_1 and θ_2 we must have $\theta_1 \supseteq^+ \theta_1 \cap \theta_2 \subseteq^- \theta_2$ as well, so $\theta_1 \sqsubseteq \theta_2$. From that, we conclude that $h(x) \in \mathbb{C}_P$ as needed, and the square above is a pullback. \square

From that, it is immediate to deduce that $(\mathbb{C}_{\tilde{A}}, \mathbb{C}_{l_A}, \mathbb{C}_{r_A})$ is an equivalence relation from the fact that (\tilde{A}, l_A, r_A) is one. We then need to check that \mathbb{C}_{l_A} and \mathbb{C}_{r_A} are open maps, which relies on the following lemma.

Lemma 3.10. *Let \mathcal{A} be a tcg, then \mathbb{C}_{l_A} and \mathbb{C}_{r_A} are open.*

Proof. Let us detail it for \mathbb{C}_{l_A} . First, we note that it is rigid. Indeed, immediate causal links in $\mathbb{C}_{\tilde{A}}$ have one of the forms:

$$\begin{aligned} (i, \tilde{a}) &\rightarrow (i, \tilde{a}') \\ (i, \tilde{a}) &\rightarrow (3-i, \tilde{a}) \end{aligned}$$

In the first case, we necessarily also have $\tilde{a} \rightarrow \tilde{a}'$ in \tilde{A} . Since l_A is rigid, we have $l_A \tilde{a} \rightarrow l_A \tilde{a}'$ as well. Therefore $(i, l_A \tilde{a}) \rightarrow (i, l_A \tilde{a}')$ in \mathbb{C}_A , but that is by definition $\mathbb{C}_{l_A}(i, l_A \tilde{a}) \rightarrow \mathbb{C}_{l_A}(i, l_A \tilde{a}')$. In the second case we have $\mathbb{C}_{l_A}(i, \tilde{a}) = (i, l_A \tilde{a}) \rightarrow (3-i, l_A \tilde{a}) = \mathbb{C}_{l_A}(3-i, \tilde{a})$ as well by definition of copycat.

We now prove that \mathbb{C}_{l_A} has the configuration extension property. A configuration $x \in \mathcal{C}(\mathbb{C}_{\tilde{A}})$ corresponds to a diagram [RW11, Win11]:

$$\begin{array}{ccccc} x_1 & \supseteq^+ & x_1 \cap x_2 & \subseteq^- & x_2 \\ \parallel_{\theta_1} & & \parallel_{\theta_1 \cap \theta_2} & & \parallel_{\theta_2} \\ y_1 & \supseteq^+ & y_1 \cap y_2 & \subseteq^- & y_2 \end{array}$$

where the upper side corresponds to $l_A(x) \in \mathcal{C}(\mathbb{C}_A)$ and the lower side corresponds to $r_A(x) \in \mathcal{C}(\mathbb{C}_A)$. Suppose now that we have $l_A(x) \xrightarrow{a} \subset$. There are three possible cases.

Suppose first that the upper side of the diagram above extends to:

$$x_1 \supseteq^+ x_1 \cap x_2 \subseteq^- x_2 \cup \{a\}$$

with $pol(a) = -$. By the extension property of isomorphism families, there is an extension $\theta_2 \xrightarrow{(a, a')} \subset$, yielding an obvious extension of the diagram above, hence an extension of x . If the upper side extends to:

$$x_1 \cup \{a\} \supseteq^+ x_1 \cap x_2 \subseteq^- x_2$$

with $pol(a) = +$, then the situation is symmetric and is dealt with similarly.

If the upper side extends to:

$$x_1 \supseteq^+ x_1 \cap x_2 \subseteq^- x_2 \cup \{a\}$$

with $pol(a) = +$. Then, the situation is a bit more subtle. Indeed in this case, we necessarily have $a \in x_1$, so in the domain of θ_1 . Therefore there is $(a, a') \in \theta_1$ with $a' \in y_1$. So any extension θ_2' of θ_2 to a must satisfy $\theta_2'(a) = a'$. This invites us to simply define $\theta_2' = \theta_2 \cup \{(a, a')\}$, however we have to check that this is allowed. Suppose this is not. Then, we note that $(x_1 \cap x_2) \cup \{a\} \in \mathcal{C}(A)$.

Indeed $a \in x_1$ and $x_2 \xrightarrow{a} \subset$, so the dependencies of a are in $x_1 \cap x_2$. So we can restrict θ_1 on the left to it, necessarily yielding the bijection $(\theta_1 \cap \theta_2) \cup \{(a, a')\}$

which is therefore a valid isomorphism of A . So we have two valid extensions of $\theta_1 \cap \theta_2$:

$$(\theta_1 \cap \theta_2) \cup \{(a, a')\} \supseteq^+ \theta_1 \cap \theta_2 \subseteq^- \theta_2$$

If they are compatible, then $\theta \cup \{(a, a')\}$ is a valid isomorphism, providing the required extension of x . Otherwise this is a race. By hypothesis, l_A preserves races, so we have another race:

$$(x_1 \cap x_2) \cup \{a\} \supseteq^+ x_1 \cap x_2 \subseteq^- x_2$$

Therefore, $x_2 \cup \{a\} \notin \mathcal{C}(A)$, absurd. So, $l_{\mathbb{C}_A}$ is open. The same reasoning holds for $r_{\mathbb{C}_A}$. \square

With the two lemmas above, we have proved that if \mathcal{A} is a tcg, then \mathbb{C}_A can be equipped with a symmetry $(\mathbb{C}_{\tilde{A}}, \mathbb{C}_{l_A}, \mathbb{C}_{r_A})$ making \mathbb{C}_A an essp. By definition, we also have $\alpha_{\tilde{A}} : \widetilde{\mathbb{C}_A} \rightarrow \tilde{A}^\perp \parallel \tilde{A}$ a strategy. In order to get a \sim -strategy, the last thing to check is that \mathbb{C}_{l_A} reflects positive compatibility. But for any esp A positive extensions of \mathbb{C}_A are always compatible, so this is trivial. Therefore we have:

Proposition 3.11. *The triple $(\mathbb{C}_{\tilde{A}}, \mathbb{C}_{l_A}, \mathbb{C}_{r_A})$ is a symmetry and makes \mathbb{C}_A an essp, written \mathbb{C}_A . Moreover,*

$$\alpha_{\mathcal{A}} : \mathbb{C}_A \rightarrow \mathcal{A}^\perp \parallel \mathcal{A}$$

is a \sim -strategy.

Here it is crucial that \mathcal{A} is race-preserving, otherwise Lemma 3.10 fails; this was initially one of the motivations for the saturated development of [CCW14].

Let us conclude this sub-section with a few technical lemmas on strategies and \sim -strategies.

Lemma 3.12. *Let $\sigma : S \rightarrow A$ be a strategy, then for any $x \in \mathcal{C}(S)$, if $x \dashv\vdash s_1$ and $x \dashv\vdash s_2$ with $\text{pol}(s_1) = -$ or $\text{pol}(s_2) = -$, and $\sigma x \cup \{\sigma s_1, \sigma s_2\} \in \mathcal{C}(A)$, then $x \cup \{s_1, s_2\} \in \mathcal{C}(S)$ as well. In particular, σ preserves races.*

Proof. Elementary verification, using receptivity and courtesy. \square

Lemma 3.13. *Let $\sigma : S \rightarrow \mathcal{A}$ be a strong-receptive courteous pre- \sim -strategy on a tcg \mathcal{A} , then S is race-preserving as well.*

Proof. Suppose $x \stackrel{\theta}{\cong}_S y$ with two valid extensions $\theta \subseteq^+ \theta_1$ and $\theta \subseteq^- \theta_2$, such that $l_S \theta_1$ and $l_S \theta_2$ are compatible. Then $\sigma(l_S \theta_1) = l_A(\sigma \theta_1)$ and $\sigma(l_S \theta_2) = l_A(\sigma \theta_2)$ are compatible as well. Since l_A preserves races, $\sigma \theta_1$ and $\sigma \theta_2$ are compatible. So $\sigma \theta_1 \subseteq^- \sigma \theta_1 \cup \sigma \theta_2$. It follows by strong-receptivity that there is $\theta_1 \subseteq^- \theta_3$ such that $\sigma \theta_3 = \sigma \theta_1 \cup \sigma \theta_2$. Finally, by Lemma 3.13 we have $\theta_3 = \theta_1 \cup \theta_2$ as well. \square

3.4 Interaction of \sim -strategies

We now give the technical background preliminary to the definition of the composition of \sim -strategies.

3.4.1 Defining pullbacks

As a first step towards defining the composition of \sim -strategies, we now show how to compute their interaction, formulated as a pullback in \mathcal{ESS} . The key remark here is that despite the general phenomenon that the category of event structures with symmetry does not have pullbacks (see [Win07]), the pullbacks involved in the composition of \sim -strategies (really, of strong-receptive pre- \sim -strategies) do exist. Moreover, if $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ are strong-receptive pre- \sim -strategies, then their pullback in \mathcal{ESS} is obtained as the pullback of σ and τ regarded as plain maps of esps, equipped with the symmetry obtained by taking the pullback of $\tilde{\sigma}$ and $\tilde{\tau}$ regarded also as plain maps of esps, as we show in the following lemma.

Lemma 3.14. *Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ be courteous strong-receptive pre- \sim -strategies. Then, forgetting polarities, the pullback*

$$\begin{array}{ccc}
 & \mathcal{S} \otimes \mathcal{T} & \\
 \swarrow & \downarrow & \searrow \\
 \mathcal{S} & & \mathcal{T} \\
 \searrow & & \swarrow \\
 & \mathcal{A} &
 \end{array}$$

σ τ

exists in \mathcal{ESS} and has symmetry; its base event structure is the pullback $\mathcal{S} \otimes \mathcal{T}$ in \mathcal{ES} and its symmetry is the pullback $\tilde{\mathcal{S}} \otimes \tilde{\mathcal{T}}$ in \mathcal{ES} , with the obvious projections.

Proof. We first take the pullback in \mathcal{ES} :

$$\begin{array}{ccc}
 & \mathcal{S} \otimes \mathcal{T} & \\
 \Pi_1 \swarrow & \downarrow & \searrow \Pi_2 \\
 \mathcal{S} & & \mathcal{T} \\
 \searrow & & \swarrow \\
 & \mathcal{A} &
 \end{array}$$

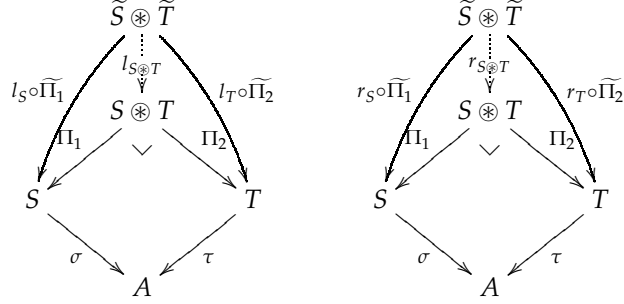
σ τ

in event structures and maps, which we know always exists. Then we take the pullback:

$$\begin{array}{ccc}
 & \tilde{\mathcal{S}} \otimes \tilde{\mathcal{T}} & \\
 \tilde{\Pi}_1 \swarrow & \downarrow & \searrow \tilde{\Pi}_2 \\
 \tilde{\mathcal{S}} & & \tilde{\mathcal{T}} \\
 \searrow & & \swarrow \\
 & \tilde{\mathcal{A}} &
 \end{array}$$

$\tilde{\sigma}$ $\tilde{\tau}$

in plain event structures and maps, which always exists as well. Exploiting the universal property of the first pullback, we get maps:



That $(\tilde{S} \otimes \tilde{T}, l_{S \otimes T}, r_{S \otimes T})$ form an equivalence relation on $S \otimes T$ follows from a simple diagram chasing, and so does the fact that $l_{S \otimes T}$ and $r_{S \otimes T}$ are jointly monic. The key property left is that they are open.

Let us prove rigidity first. Recall that:

$$\tilde{S} \otimes \tilde{T} = \text{Pr}(\mathcal{C}(\tilde{S}) \times \mathcal{C}(\tilde{T}) \upharpoonright R)$$

where $R = \{(\tilde{s}, \tilde{t}) \mid \tilde{\sigma}(\tilde{s}) = \tilde{\tau}(\tilde{t})\}$. Then, suppose that

$$[(\tilde{s}, \tilde{t})]_{\tilde{x}} \rightarrow [(\tilde{s}', \tilde{t}')]_{\tilde{x}}$$

in $\tilde{S} \otimes \tilde{T}$. This means that we also have:

$$(\tilde{s}, \tilde{t}) \rightarrow_{\tilde{x}} (\tilde{s}', \tilde{t}')$$

in $\mathcal{C}(\tilde{S}) \times \mathcal{C}(\tilde{T})$ by definition of Pr and restriction. It follows by lemma 2.11 that either $\tilde{s} \rightarrow \tilde{s}'$ in \tilde{S} or $\tilde{t} \rightarrow \tilde{t}'$ in \tilde{T} ; the cases being symmetric, let us suppose *w.l.o.g.* that $\tilde{s} \rightarrow \tilde{s}'$. Since l_S is open thus rigid, writing $s_1 = l_S(\tilde{s})$ and $s'_1 = l_S(\tilde{s}')$ we have that $s_1 \rightarrow s'_1$ in S . It follows that necessarily, $(s_1, t_1) \rightarrow_{x_1} (s'_1, t'_1)$, where $t_1 = l_T(\tilde{t})$, $t'_1 = l_T(\tilde{t}')$, and:

$$x_1 = (l_S \times l_T) \tilde{x}$$

(obviously $x_1 \in \mathcal{C}(S) \times \mathcal{C}(T)$). By definition of Pr and restriction, this implies that $[(s_1, t_1)]_{x_1} \rightarrow [(s'_1, t'_1)]_{x_1}$ in $\text{Pr}(\mathcal{C}(S) \times \mathcal{C}(T) \upharpoonright R')$ where $R' = \{(s, t) \mid \sigma(s) = \tau(t)\}$. But $[(s_1, t_1)]_{x_1} = l_{S \otimes T}([(s, \tilde{t})]_{\tilde{x}})$ and $[(s'_1, t'_1)]_{x_1} = l_{S \otimes T}([(s', \tilde{t}')]_{\tilde{x}})$, therefore $l_{S \otimes T}$ preserves immediate causality. The same argument applies for $r_{S \otimes T}$ as well, therefore $l_{S \otimes T}$ and $r_{S \otimes T}$ are rigid.

It remains to prove that $l_{S \otimes T}, r_{S \otimes T}$ have the configuration extension property; let us prove it *w.l.o.g.* for $l_{S \otimes T}$. Let $z \in \mathcal{C}(\tilde{S} \otimes \tilde{T})$. By Proposition 2.13, this corresponds to a secured bijection:

$$\phi : \theta_1 \cong \tilde{\sigma} \theta_1 = \tilde{\tau} \theta_2 \cong \theta_2$$

where $x_l \stackrel{\theta_1}{\cong_S} x_r$ and $y_l \stackrel{\theta_2}{\cong_T} y_r$. Then, $l_{S \otimes T}$ maps this secured bijection to:

$$\phi_l : x_l \cong \sigma x_l = \tau y_l \cong y_l$$

where $\phi_l \circ l_S = l_T \circ \phi$. Suppose now that ϕ_l extends to:

$$\phi_l \cup \{(s_l, t_l)\} : x_l \cup \{s_l\} \cong (\sigma x_l) \cup \sigma s = (\tau y_l) \cup \tau t_l \cong y_l \cup \{t_l\}$$

The events s_l and t_l must have complementary polarities in S and T , so one of them is negative – suppose w.l.o.g. that it is s_l . That means that t_l is positive. By the extension property of isomorphism families (or alternatively, since l_T is open), there is $t_r \in T$ such that $y_l \cup \{t_l\} \stackrel{\theta_2 \cup \{(t_l, t_r)\}}{\cong_T} y_r \cup \{t_r\}$. So by receptivity of $\tilde{\sigma}$, there is a corresponding extension of $x_l \stackrel{\theta_1}{\cong_S} x_r$ to $x_l \cup \{s_l\} \stackrel{\theta_1}{\cong_S} x_r \cup \{s_r\}$ such that $\sigma s_l = \tau t_l$ and $\sigma s_r = \tau t_r$. Moreover

$$\theta_1 \cup \{(s_l, s_r)\} \cong \tilde{\sigma} \theta_1 \cup \{(\sigma s_l, \sigma s_r)\} = \tilde{\tau} \theta_2 \cup \{(\tau t_l, \tau t_r)\} \cong \theta_2 \cup \{(t_l, t_r)\}$$

which is still a secured bijection by construction, so we have proved the configuration extension property. \square

With this definition, composition of \sim -strategies can be defined as in [RW11] through the pullback construction followed by projection. We leave the details for later because we still have one key difficulty to overcome: the fact that weak equivalence is preserved by composition.

3.4.2 Pullbacks as bipullbacks

We now get to the bigger obstacle to the definition of thin concurrent games.

In [RW11], preservation of isomorphism by composition follows from the universal property of the pullback. However, here for \sim -strategies $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}$, we have defined their weak equivalence through the existence of certain maps $f : \mathcal{S} \rightarrow \mathcal{T}$ and $g : \mathcal{T} \rightarrow \mathcal{S}$ with respect to which the projections to the game commutes *up to symmetry*. But nothing, in the universal property of the pullback, ensures that it is compatible with this notion of equivalence.

For that, we prove that the pullback of \sim -strategies also satisfies the universal property up to symmetry of *bipullbacks* – which corresponds to a pullback in the quotient category.

Definition 3.15. *A bipullback of two maps $f : \mathcal{A} \rightarrow \mathcal{C}$ and $g : \mathcal{B} \rightarrow \mathcal{C}$ of ess is an object \mathcal{P} and two maps $\Pi_1 : \mathcal{P} \rightarrow \mathcal{A}$ and $\Pi_2 : \mathcal{P} \rightarrow \mathcal{B}$ such that $f \circ \Pi_1 \sim g \circ \Pi_2$ and for all maps $x : \mathcal{X} \rightarrow \mathcal{A}$ and $y : \mathcal{X} \rightarrow \mathcal{B}$ such that $f \circ x \sim g \circ y$, there exists a map of ess $h : \mathcal{X} \rightarrow \mathcal{P}$ unique up to symmetry such that $\Pi_1 \circ h \sim f$ and $\Pi_2 \circ h \sim g$.*

Proposition 3.16. *Suppose $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ are strong-receptive courteous pre- \sim -strategies with receptive thin sub-symmetries. Then, forgetting polarities, the pullback of σ and τ in ESS is also a bipullback.*

Proof. Suppose we have $f : \mathcal{R} \rightarrow \mathcal{S}$ and $g : \mathcal{R} \rightarrow \mathcal{T}$ such that the square commutes up to equivalence, i.e. for any $z \in \mathcal{C}(R)$ there is an isomorphism

$$\sigma f z \stackrel{\phi_z}{\cong}_A \tau g z.$$

Suppose we have a chain:

$$\emptyset = z_0 \text{---} z_1 \text{---} z_2 \text{---} \dots \text{---} z_n$$

of configurations of R . For each $i \leq n$, we will build a diagram of the shape:

$$\begin{array}{ccccc} f(z_i) & \xrightarrow{\theta_i^S} & x_i & & y_i & \xrightarrow{\theta_i^T} & g(z_i) \\ \sigma \downarrow & & \swarrow \sigma & & \nwarrow \tau & & \downarrow \tau \\ \sigma(f(z_i)) & \xrightarrow{\sigma\theta_i^S} & v_i & & v_i & \xrightarrow{\tau\theta_i^T} & \tau(g(z_i)) \end{array}$$

such that $\tau\theta_i^T \circ \sigma\theta_i^S = \phi_{z_i}$, and where θ_i^S and θ_i^T are respectively in the receptive thin sub-symmetries \tilde{S}_+ and \tilde{T}_+ . This is done by induction on i .

For $i = 0$, this diagram is built by setting all components to \emptyset . Suppose now the diagram constructed for i , and take $z_i \xrightarrow{r} z_{i+1}$. Necessarily, either $\text{pol}(f(r)) = +$ or $\text{pol}(g(r)) = +$. Suppose *w.l.o.g.* that it is the former, the other case will be symmetric. We write $f(r) = s_1$, and we set:

$$\theta_{i+1}^S = \text{ext}(\theta_i^S, f(z_i) \cup \{s_1\})$$

We write $\theta_{i+1}^S = \theta_i^S \cup \{(s_1, s_2)\}$. Likewise, we write that $\phi_{z_i} \xrightarrow{(\sigma s_1, a)} \phi_{z_{i+1}}$. Then by hypothesis, we have:

$$\tau\theta_i^T \xrightarrow{(\sigma s_2, a)} \text{---} \text{---}$$

Moreover, $\text{pol}_{A^\perp}(\sigma t) = \text{pol}_{A^\perp}(a) = -$, therefore by strong-receptivity there are unique $t_1, t_2 \in T$ such that $\tau t_1 = \sigma s_2$ and $\tau t_2 = a$, and:

$$\theta_i^T \xrightarrow{(t_1, t_2)} \text{---} \text{---}$$

We set $\theta_{i+1}^T = \theta_i^T \cup \{(t_1, t_2)\}$, yielding a valid extension of the diagram for $i + 1$. By induction, the diagram is well-defined for any $n \in \mathbb{N}$.

Now, we want to show that the diagram obtained for n does not depend on the particular chain $z_0 \text{---} \dots \text{---} z_n$ but only on z_i . This is done by showing that this construction is invariant under permutations of independent events

in the chain (z_i) . Suppose $z_i \xrightarrow{r} z_{i+1}$ and $z_{i+1} \xrightarrow{r'} z_{i+2}$ where r and r' are concurrent. It follows that there is another chain:

$$\emptyset = z'_0 \text{---} z'_1 \text{---} \dots \text{---} z'_n$$

defined by $z'_j = z_j$ for all $j \leq i$, $z'_{i+1} = z_i \cup \{r'\}$, and $z'_j = z_j$ for all $j \geq i + 2$.

We perform the construction described above on (z_i) and (z'_i) , and compare the results. The components of the diagrams obtained by applying the construction to $(z_j)_{0 \leq j \leq n}$ will be called $x_j, y_j, v_j, \theta_j^S, \theta_j^T$ and the components of the diagrams obtained by applying the construction to $(z'_j)_{0 \leq j \leq n}$ will be called $x'_j, y'_j, v'_j, \theta_j^S, \theta_j^T$. By construction, it is obvious that these diagrams coincide for $j \leq i$. We will now compare them for $i+2$.

There are two independent cases: permutation of two events yielding events in A of the same polarity, and permutation of two events yielding events in A of opposite polarity. We start by the former. Without loss of generality, suppose $pol(\sigma(fr)) = pol(\sigma(fr')) = +$ and write $fr = s$ and $fr' = s'$. Since r and r' are concurrent, it follows that s and s' are concurrent as well. We have:

$$\begin{aligned}\theta_{i+2}^S &= \text{ext}(\text{ext}(\theta_i^S, f(z_i) \cup \{s\}), f(z_i) \cup \{s, s'\}) \\ \theta'_{i+2}^S &= \text{ext}(\text{ext}(\theta_i^S, f(z_i) \cup \{s'\}), f(z_i) \cup \{s, s'\})\end{aligned}$$

but we know that $\theta_i^S = \theta_i^S$, so they are both equal to $\text{ext}(\theta_i^S, f(z_i) \cup \{s, s'\})$ by monotonicity. By uniqueness of receptivity, we also have that $\theta_{i+2}^T = \theta'_{i+2}^T$, so the two diagrams coincide at $i+2$. Note that it also follows from this reasoning (and we will use it later) that:

$$\begin{aligned}\theta_{i+2}^S &= \theta_{i+1}^S \cup \theta'_{i+1}^S \\ \theta_{i+2}^T &= \theta_{i+1}^T \cup \theta'_{i+1}^T\end{aligned}$$

Clearly $\theta_{i+1}^S = \text{ext}(\theta_i^S, f(z_i) \cup \{s\}) \subseteq \theta_{i+2}^S$ and the same holds for θ'_{i+1}^S therefore we have the inclusion $\theta_{i+1}^S \cup \theta'_{i+1}^S \subseteq \theta_{i+2}^S$, the equality follows because these two isomorphisms have the same domain and codomain. The second equality follows by uniqueness of receptivity.

Now suppose that r and r' yield events in A of opposite polarity. Suppose *w.l.o.g.* that $pol(\sigma(fr)) = +$ and $pol(\sigma(fr')) = -$. We write $f(r) = s_1$ and $g(r) = t_2$, $f(r') = s'_1$ and $g(r') = t'_2$. We display the diagram obtained for z_{i+1} :

$$\begin{array}{ccccc} f(z_i) \cup \{s_1\} & \xrightarrow{\theta_{i+1}^S} & x_i \cup \{s_2\} & & y_i \cup \{t_1\} & \xrightarrow{\theta_{i+1}^T} & g(z_i) \cup \{t_2\} \\ \sigma \downarrow & & \swarrow \sigma & & \nwarrow \tau & & \tau \downarrow \\ \sigma(f(z_i)) \cup \{\sigma s_1\} & & v_i \cup \{\sigma s_2\} & & \tau(g(z_i)) \cup \{\tau t_2\} \end{array}$$

$\sigma \theta_{i+1}^S \cong_A$ $\tau \theta_{i+1}^T \cong_A$

Where s_2 and θ_{i+1}^S are obtained from s_1 by coherent extension of θ_i^S , τt_1 is fixed to be σs_2 from which t_1 follows uniquely by receptivity, and $\theta_{i+1}^T = \theta_i^T \cup \{(t_1, t_2)\}$ is valid by strong-receptivity.

Likewise, the diagram for z'_{i+1} is:

$$\begin{array}{ccccc}
f(z_i) \cup \{s'_1\} & \xrightarrow{\theta'^S_{i+1} \cong_S} & x_i \cup \{s'_2\} & & y_i \cup \{t'_1\} & \xrightarrow{\theta'^T_{i+1} \cong_T} & g(z_i) \cup \{t'_2\} \\
\sigma \downarrow & & \swarrow \sigma & & \nwarrow \tau & & \downarrow \tau \\
\sigma(f(z_i)) \cup \{\sigma s'_1\} & & v_i \cup \{\sigma s'_2\} & & \tau(g(z_i)) \cup \{\tau t'_2\} & & \\
& \cong_{\mathcal{A}}^{\sigma \theta'^S_{i+1}} & & \cong_{\mathcal{A}}^{\tau \theta'^T_{i+1}} & & &
\end{array}$$

We now show that the diagram obtained for z_{i+2} is the (component-wise) union of those two.

We know by the diagram obtained for z'_{i+1} that $\text{ext}((\theta_i^T)^{-1}, g(z_i) \cup \{t'_2\}) = (\theta_i^T)^{-1} \cup \{(t'_2, t'_1)\}$. Moreover, we know that $g(z_i) \cup \{t_2, t'_2\} \in \mathcal{C}(S)$ since r and r' are concurrent. Moreover $\text{pol}(t_2) \neq \text{pol}(t'_2)$, so since by Lemma 3.13 (using that \mathcal{A} , as a tcg, is race-preserving) \mathcal{T} is race-preserving it follows that $(\theta_i^T)^{-1} \cup \{(t_2, t_1), (t'_2, t'_1)\}$ is in the isomorphism family of T . Therefore by the stability axiom of coherent extensions, it follows that

$$\text{ext}((\theta_i^T)^{-1}, g(z_i) \cup \{t'_2\}) \subseteq \text{ext}((\theta_i^T)^{-1} \cup \{(t_2, t_1)\}, g(z_i) \cup \{t_2, t'_2\})$$

or, in other words:

$$(\theta_i^T)^{-1} \cup \{(t'_2, t'_1)\} \subseteq \text{ext}((\theta_{i+1}^T)^{-1}, g(z_i) \cup \{t_2, t'_2\})$$

which implies $(t'_2, t'_1) \in \text{ext}((\theta_{i+1}^T)^{-1}, g(z_i) \cup \{t_2, t'_2\})$. We also have $(t_2, t_1) \in \text{ext}((\theta_{i+1}^T)^{-1}, g(z_i) \cup \{t_2, t'_2\})$ since $(t_2, t_1) \in (\theta_{i+1}^T)^{-1}$, so it immediately follows that $\theta_{i+2}^T = \theta_{i+1}^T \cup \theta'_{i+1}^T$. The reasoning for $\theta_{i+2}^S = \theta_{i+1}^S \cup \theta'_{i+1}^S$ is symmetric.

We have proved that the construction of this diagram is invariant under permutation of concurrent events, so the diagram for a chain $z_0 \dashv\!\!\!\dashv \dots \dashv\!\!\!\dashv z_n$ only depends on z_n and is independent of the particular chain used to reach it. So we have a construction which to any configuration $z \in \mathcal{C}(R)$ associates a diagram:

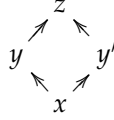
$$\begin{array}{ccccc}
f(z) & \xrightarrow{\cong_{\theta_z^S}} & x_z & & y_z & \xrightarrow{\cong_{\theta_z^T}} & g(z) \\
\sigma \downarrow & & \swarrow \sigma & & \nwarrow \tau & & \downarrow \tau \\
\sigma(f(z)) & \cong_{\sigma \theta_z^S} & v_z & \cong_{\tau \theta_z^T} & & & \tau(g(z))
\end{array}$$

Now, we define the following functions:

$$\begin{array}{ll}
f' : \mathcal{C}(R) \rightarrow \mathcal{C}(S) & g' : \mathcal{C}(R) \rightarrow \mathcal{C}(T) \\
z \mapsto x_z & z \mapsto y_z \\
\\
h'_1 : \mathcal{C}(R) \rightarrow \mathcal{C}(\tilde{S}) & h'_2 : \mathcal{C}(R) \rightarrow \mathcal{C}(\tilde{T}) \\
z \mapsto \theta_z^S & z \mapsto \theta_z^T
\end{array}$$

By definition, it is obvious that these functions preserve $\dashv\!\!\!\dashv$. Additionally they preserve bounded union. Indeed, take any of the functions $f : \mathcal{C}(R) \rightarrow \mathcal{C}(E)$

defined above. We have seen during the construction of the functions above that for any diagram:



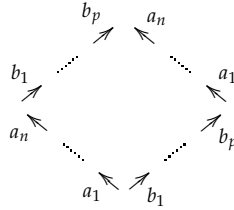
in $\mathcal{C}(R)$, we have $f(z) = f(y) \cup f(y')$. Now suppose $x, y \in \mathcal{C}(R)$ are compatible. Pick a chain

$$\emptyset = x_0 \xrightarrow{a_1} \dots \xrightarrow{a_n} x_n = x$$

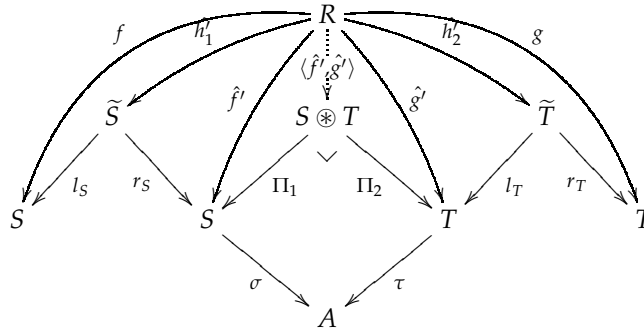
and similarly, a chain:

$$\emptyset = y_0 \xrightarrow{b_1} \dots \xrightarrow{b_p} y_p = y$$

We have the following lattice of sub-configurations of $x \cup y$ whose order preserve the chains $(x_i)_{0 \leq i \leq n}$ and $(y_i)_{0 \leq i \leq p}$:

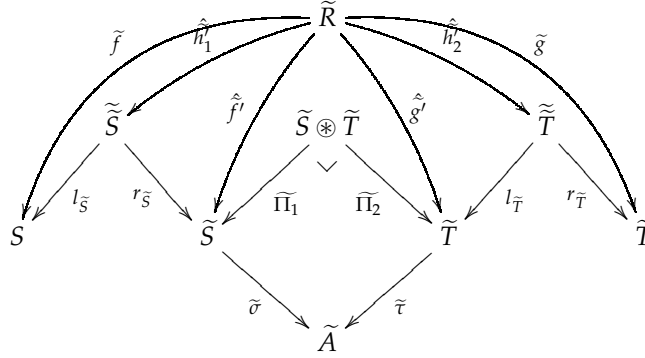


By induction on this diagram using the union property mentioned above, we have that $f(x \cup y) = f(x) \cup f(y)$. Therefore by Lemma 2.7, there are unique maps of event structures $\hat{f}' : R \rightarrow S$, $\hat{g}' : R \rightarrow T$, $\hat{h}'_1 : R \rightarrow \tilde{S}$ and $\hat{h}'_2 : R \rightarrow \tilde{T}$. By the uniqueness property of the map construction, the hat construction preserves commuting diagrams. It follows that the following diagram (in \mathcal{ES}) commutes:



which exactly means that $\langle \hat{f}', \hat{g}' \rangle$ is the required mediating arrow making the two triangles of the pullback diagram commute up to symmetry. Of course for

now $h = \langle \hat{f}', \hat{g}' \rangle$ is just a map of event structures and not of event structures with symmetry, so we need to define $\tilde{h} : \tilde{R} \rightarrow \tilde{S} \otimes \tilde{T}$. Note that by Lemmas 2.19 and 3.8, all the components lift to symmetries, so the same construction can be carried out on the diagram below, where all components have been replaced with their symmetry.



The same process as before can be applied on this diagram, yielding $\langle \hat{\tilde{f}}', \hat{\tilde{g}}' \rangle : \tilde{R} \rightarrow \tilde{S} \otimes \tilde{T}$.

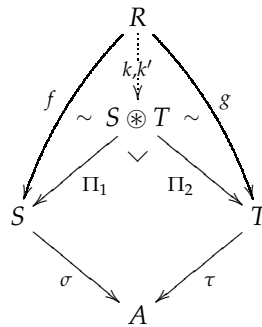
Furthermore, this diagram projects to the previous one in two distinct ways: (1) by mapping each \tilde{E} to E via l_E , and each $\tilde{\tilde{E}}$ to \tilde{E} via $u_{\tilde{E}}$, and (2) by mapping each \tilde{E} to E via r_E , and each $\tilde{\tilde{E}}$ to \tilde{E} via $d_{\tilde{E}}$. By Lemmas 2.19 and 3.8 all the constructions used to build $\langle \hat{\tilde{f}}', \hat{\tilde{g}}' \rangle : \tilde{R} \rightarrow \tilde{S} \otimes \tilde{T}$ all commute with these projections. It follows that

$$l_{S \otimes T} \circ \langle \hat{\tilde{f}}', \hat{\tilde{g}}' \rangle : \tilde{R} \rightarrow \tilde{S} \otimes \tilde{T} = \langle \hat{f}', \hat{g}' \rangle \circ l_R$$

$$r_{S \otimes T} \circ \langle \hat{\tilde{f}}', \hat{\tilde{g}}' \rangle : \tilde{R} \rightarrow \tilde{S} \otimes \tilde{T} = \langle \hat{f}', \hat{g}' \rangle \circ r_R$$

So $\langle \hat{f}', \hat{g}' \rangle$ preserves symmetry as needed.

It remains to prove that the mediating arrow is unique up to symmetry. Suppose there are two mediating arrows:



Then we know that for all $z \in \mathcal{C}(R)$, $\theta_z = \{(f(r), \Pi_1 \circ k(r)) \mid r \in z\}$ and $\theta'_z = \{(f(r), \Pi_1 \circ k'(r)) \mid r \in z\}$ are in the isomorphism family of S . It follows that for all $z \in \mathcal{C}(R)$, the isomorphism

$$\psi_z = \{(\Pi_1 \circ k(r), \Pi_1 \circ k'(r)) \mid r \in z\}$$

is in the isomorphism family of S as well, since it is obtained by composition of θ_z and θ'_z . So there is a map:

$$\psi : R \rightarrow \tilde{S}$$

such that $l_S \circ \psi = \Pi_1 \circ k$ and $r_S \circ \psi = \Pi_1 \circ k'$. By the same reasoning there is a map $\psi' : R \rightarrow \tilde{T}$ such that $l_T \circ \psi' = \Pi_2 \circ k$ and $r_T \circ \psi' = \Pi_2 \circ k'$. Moreover, we have $\tilde{\sigma} \circ \psi = \tilde{\tau} \circ \psi'$: this obviously holds once post-composed by l_A and r_A , so this equation follows by joint monicity. By the universal property of $\tilde{S} \otimes \tilde{T}$, there is a unique map $h : S \rightarrow \tilde{S} \otimes \tilde{T}$ such that $\tilde{\Pi}_1 \circ h = \psi$ and $\tilde{\Pi}_2 \circ h = \psi'$. By post-composing with the left and right maps we get:

$$\begin{aligned} \Pi_1 \circ l_{S \otimes T} \circ h &= \Pi_1 \circ k \\ \Pi_1 \circ r_{S \otimes T} \circ h &= \Pi_1 \circ k' \\ \Pi_2 \circ l_{S \otimes T} \circ h &= \Pi_2 \circ k \\ \Pi_2 \circ r_{S \otimes T} \circ h &= \Pi_2 \circ k' \end{aligned}$$

It follows from the universal property of the pullback for $S \otimes T$ that $l_{S \otimes T} \circ h = k$ and $r_{S \otimes T} \circ h = k'$, so $k \sim k'$ as needed. \square

3.5 Composition of \sim -strategies

We now proceed to define the composition of \sim -strategies. Given two \sim -strategies $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$, we compose them as before by first (forgetting polarities) taking the pullback $(S \parallel C) \otimes (A \parallel T)$, which exists by Lemma 3.14 and has symmetry $(\tilde{S} \parallel \tilde{C}) \otimes (\tilde{A} \parallel \tilde{T})$. Then, we define:

$$V = \{p \in (S \parallel C) \otimes (A \parallel T) \mid (\sigma \parallel C) \circ \Pi_1 p \notin B\}$$

This set of events is closed under symmetry (see Definition 2.20), as it is the left projection of \tilde{V} defined similarly on $(\tilde{S} \parallel \tilde{C}) \otimes (\tilde{A} \parallel \tilde{T})$. So by Proposition 2.21 there is an event structure with symmetry:

$$\mathcal{T} \odot \mathcal{S} = (S \parallel C) \otimes (A \parallel T) \downarrow V$$

Note that by definition, this is exactly the same as the composition $T \odot S$ of $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$ regarded as strategies without symmetry, equipped with the symmetry $\tilde{T} \odot \tilde{S}$ obtained as the composition of $\tilde{\sigma} : \tilde{S} \rightarrow \tilde{A}^\perp \parallel \tilde{B}$ and $\tilde{\tau} : \tilde{T} \rightarrow \tilde{B}^\perp \parallel \tilde{C}$ regarded as well as strategies without symmetry. In other words, we have the following equality:

$$\widetilde{T \odot S} = \tilde{T} \odot \tilde{S}$$

between esps, which reflects the earlier $\widetilde{\mathbb{C}}_A = \mathbb{C}_{\widetilde{A}}$. Interestingly, it can be checked that the equality above also holds when $\widetilde{T}, \widetilde{S}$ and $\widetilde{T \odot S}$ are regarded as event structures with symmetry equipped with the free higher symmetry (see Subsection 2.3.2).

Since in particular composition following this definition is compatible with usual composition without symmetry, it follows that the composition of \sim -strategies is a strong-receptive courteous pre- \sim -strategy; we check below that it is thin as well and hence is a \sim -strategy.

Lemma 3.17. *Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be \sim -strategies, then $\tau \odot \sigma : \mathcal{T} \odot \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{C}$ is a \sim -strategy.*

Proof. We only have to check that $\tau \odot \sigma$ is thin, i.e. that $l_{T \odot S}$ reflects positive compatibility. Take a configuration of $\widetilde{T} \odot \widetilde{S}$ (so also a configuration of $(\widetilde{S} \parallel \widetilde{C}) \otimes (\widetilde{A} \parallel \widetilde{T})$) represented as a commuting diagram:

$$\begin{array}{ccc} x_1 & \xrightarrow{\underline{g}} & \sigma x_1 & = & \tau y_1 & \xrightarrow{\underline{\tau}} & y_1 \\ \theta_1 \downarrow_{\mathcal{S} \parallel \mathcal{C}} & & & & & & \theta'_1 \downarrow_{\mathcal{A} \parallel \mathcal{T}} \\ x'_1 & \xrightarrow{\underline{g}} & \sigma x'_1 & = & \tau y'_1 & \xrightarrow{\underline{\tau}} & y'_1 \end{array}$$

Suppose that, as a configuration of $\widetilde{T} \odot \widetilde{S}$, it extends positively to two other versions of this diagram, with components labeled 2 and 3 respectively, such that the left projections $(x_2, y_2$ and $x_3, y_3)$ are compatible. Regarded as configurations of $(\widetilde{S} \parallel \widetilde{C}) \otimes (\widetilde{A} \parallel \widetilde{A})$, these extensions are not necessarily positive; however the new visible events have positive polarity in $\mathcal{A}^\perp \parallel \mathcal{C}$. For simplicity suppose the extensions to x_2, y_2 and x_3, y_3 to be atomic, the general case will follow by immediate induction. If the extensions of x_1 to x_2, x_3 have different polarities, then the extensions of θ_1 to θ_2, θ_3 and of θ'_1 to θ'_2, θ'_3 are compatible as well by race-preservation of $\mathcal{S} \parallel \mathcal{C}^\perp$ and $\mathcal{A} \parallel \mathcal{T}$ (which follows from Lemma 3.13). If the extensions of x_1 to x_2, x_3 are both positive, then they cannot both be in \mathcal{C}^\perp as that would contradict the fact that new visible events have positive polarity. If they are both in \mathcal{S} , then the extensions of θ_1 to θ_2, θ_3 are compatible since \mathcal{S} is thin, and the extensions of θ'_1 to θ'_2, θ'_3 are compatible by Lemma 3.12 since $\mathcal{A} \parallel \tau$ is courteous and strong-receptive. If one (say x_2) is in \mathcal{S} and the other (say x_3) is in \mathcal{C} , then θ_2 and θ_3 are compatible by definition of \parallel , and θ'_2 and θ'_3 are compatible by Lemma 3.12 since $\mathcal{A} \parallel \tau$ is strong-receptive and courteous. \square

So, from the results of [RW11] we know that tcgs and \sim -strategies up to isomorphism form a bicategory. However, as argued before, for this setting of strategies up to symmetry to make sense, we need to check that weak equivalence is preserved by composition. In proving so we will use crucially Proposition 3.16. Note that Proposition 3.16 was proved for pullbacks of the form $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}^\perp$ where both σ and τ admit receptive thin sub-symmetries – this is the case for the composition pullback, since concurrent games with symmetry were assumed to have receptive thin sub-symmetries.

Lemma 3.18. *Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$, $\sigma' : \mathcal{S}' \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ with $\sigma \simeq \sigma'$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$, $\tau' : \mathcal{T}' \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ with $\tau \simeq \tau'$. Then:*

$$\tau \odot \sigma \simeq \tau' \odot \sigma'$$

Proof. We show that if $\sigma \simeq \sigma'$ then $\tau \odot \sigma \simeq \tau \odot \sigma'$, the general case easily follows from that. Let us write $P = (\mathcal{S} \parallel \mathcal{C}) \otimes (\mathcal{A} \parallel \mathcal{T})$ and $P' = (\mathcal{S}' \parallel \mathcal{C}) \otimes (\mathcal{A} \parallel \mathcal{T})$ the two composition pullbacks. By definition of weak equivalence, there are maps $f : \mathcal{S} \rightarrow \mathcal{S}'$ and $g : \mathcal{S}' \rightarrow \mathcal{S}$ such that $\sigma' \circ f \sim \sigma$, $\sigma \circ g \sim \sigma'$, $f \circ g \sim \text{id}_{\mathcal{S}'}$ and $g \circ f \sim \text{id}_{\mathcal{S}}$. Therefore, forgetting polarities, the following square commutes up to symmetry:

$$\begin{array}{ccc}
 & \mathcal{P}' & \\
 (f \parallel \mathcal{C}) \circ \Pi'_1 \swarrow & & \searrow \Pi'_2 \\
 \mathcal{S}' \parallel \mathcal{C} & \sim & \mathcal{A} \parallel \mathcal{T} \\
 \sigma' \parallel \mathcal{C} \searrow & & \swarrow \mathcal{A} \parallel \tau \\
 & \mathcal{A} \parallel \mathcal{B} \parallel \mathcal{C} &
 \end{array}$$

But besides being a pullback, \mathcal{P} is a bipullback by Proposition 3.16. Indeed, although the courteous strong-receptive pre- \sim -strategy $\sigma \parallel \mathcal{C}^\perp : \mathcal{S} \parallel \mathcal{C}^\perp \rightarrow \mathcal{A}^\perp \parallel \mathcal{B} \parallel \mathcal{C}^\perp$ is not thin (because $\text{id}_{\mathcal{A}} : \mathcal{A} \rightarrow \mathcal{A}$ is not thin in general), its symmetry has a receptive thin sub-symmetry $\tilde{S} \parallel \tilde{C}_-$, and likewise $\tilde{A} \parallel \tilde{T}$ has a receptive thin sub-symmetry $\tilde{A}_+ \parallel \tilde{T}$.

From there it is immediate to apply the universal property of bipullbacks to obtain $f' : \mathcal{P} \rightarrow \mathcal{P}'$ and $g' : \mathcal{P}' \rightarrow \mathcal{P}$ satisfying the required commutations up to symmetry, and the weak equivalence between the two compositions follow by restriction. \square

We have defined a structure TCG, with:

- Objects: concurrent games with symmetry,
- Morphisms from \mathcal{A} to \mathcal{B} : \sim -strategies $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$, with an identity \sim -strategy $\alpha_{\mathcal{A}} : \mathbb{C}_{\mathcal{A}} \rightarrow \mathcal{A}^\perp \parallel \mathcal{A}$,
- 2-cells: weak equivalences.

For $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ a morphism in TCG, we will sometimes write $\sigma : \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{B}$.

Since our construction extends [RW11] conservatively, we get that copycat is neutral with respect to composition up to isomorphism, hence up to weak equivalence. In fact the compositional structure is completely similar to [RW11]. except that our composition is performed by an universal property up to \sim rather than strict. For the same reasons as in [RW11], it follows that the laws of a bicategory are satisfied up to symmetry; we call that a \sim -bicategory. In fact as in [RW11, CCW14] we also have:

3.6 Compact closed structure

We show that TCG satisfies the laws of a compact closed category up to \simeq .

Bifunctor. We first remark that parallel composition \parallel on cgs extends to a functor:

$$\parallel_s : \text{TCG}/\simeq \times \text{TCG}/\simeq \rightarrow \text{TCG}/\simeq$$

For \sim -strategies $\sigma_1 : \mathcal{S}_1 \rightarrow \mathcal{A}_1^\perp \parallel \mathcal{B}_1$ and $\sigma_2 : \mathcal{S}_2 \rightarrow \mathcal{A}_2^\perp \parallel \mathcal{B}_2$, we set:

$$\sigma_1 \parallel_s \sigma_2 : \mathcal{S}_1 \parallel \mathcal{S}_2 \rightarrow (\mathcal{A}_1 \parallel \mathcal{A}_2)^\perp \parallel (\mathcal{B}_1 \parallel \mathcal{B}_2)$$

to be the obvious map of essps. It is direct to show the bifunctor laws, which are in fact satisfied up to isomorphisms.

Lifting. In order to give the structural maps of the compact closed structure, it will be useful to be able to automatically lift maps of essps to \sim -strategies. For that, we start by noticing that given a strong-receptive, courteous map of essps $f^\perp : \mathcal{B}^\perp \rightarrow \mathcal{A}^\perp$, one can always lift it to a \sim -strategy $\bar{f} : \mathcal{A} \dashrightarrow \mathcal{B}$, defined by:

$$\begin{aligned} \bar{f} : \mathbb{C}_B &\rightarrow \mathcal{A}^\perp \parallel \mathcal{B} \\ c &\mapsto (f^\perp \parallel \mathcal{B}) \circ \alpha_B(c) \end{aligned}$$

The \sim -pre-strategy \bar{f} is courteous and strong-receptive because these properties are stable under composition in \mathcal{ESSP} . It is also thin, which is automatically inherited from \mathbb{C}_B . Moreover, we note that if $f, g : \mathcal{B} \rightarrow \mathcal{A}$ are symmetric then \bar{f} and \bar{g} are weakly equivalent; indeed the identity $\text{id}_{\mathbb{C}_B}$ in both directions provides a weak equivalence.

We now characterise the configurations and symmetries corresponding to the composition of \sim -strategies with lifted maps.

Lemma 3.19. *Let $f^\perp : \mathcal{B}^\perp \rightarrow \mathcal{A}^\perp$ be a strong-receptive, courteous map of essps with \mathcal{A}, \mathcal{B} tcgs, and $\sigma : \mathcal{S} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be a \sim -strategy. Then $\sigma \circ \bar{f} : \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{C}$ is isomorphic (in $\mathcal{ESSP}/\mathcal{A}^\perp \parallel \mathcal{C}$) to $(f^\perp \parallel \mathcal{C}) \circ \sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{C}$.*

Proof. Consider the pullback $(\mathbb{C}_B \parallel \mathcal{C}) \otimes (\mathcal{A} \parallel \mathcal{S})$ (along the maps $(f \parallel \mathcal{B} \parallel \mathcal{C}) \circ (\alpha_B \parallel \mathcal{C})$ and $\mathcal{A} \parallel \sigma$) involved in the definition of $\sigma \circ \bar{f}$.

For each $x \in \mathcal{C}(S)$ we have $\sigma(x) = x_B + x_C$, we write $\sigma_B = x_B$ and $\sigma_C = x_C$. By Proposition 2.13, configurations of the pullback correspond to composite secured bijections:

$$x_B \parallel \sigma_B y \parallel \sigma_C y \cong f(x_B) \parallel \sigma_B y \parallel \sigma_C y \cong f(x_B) \parallel y$$

with $x_B \sqsubseteq \sigma_B y \in \mathcal{C}(B)$ (where \sqsubseteq , defined as $\supseteq^+ \sqsubseteq^-$ is the Scott order – see e.g. [Win11]) and $y \in \mathcal{C}(S)$. Clearly, this space of configurations is isomorphic to the space of configurations of the pullback $(\mathbb{C}_B \parallel \mathcal{C}) \otimes (\mathcal{B} \parallel \mathcal{S})$ (along the maps $(\alpha_B \parallel \mathcal{C}) \otimes (\mathcal{B} \parallel \sigma)$), described by the composite secured bijections:

$$x_B \parallel \sigma_B y \parallel \sigma_C y \cong x_B \parallel \sigma_B y \parallel \sigma_C y \cong x_B \parallel y$$

with $x_B \sqsubseteq \sigma_B y$. This isomorphism of configurations preserves $-C$ and compatible unions, so it is an isomorphism of event structures between the corresponding pullbacks. Additionally it sends visible events to visible events, so it yields an isomorphism (of \mathcal{ESP}) making the following diagram commute:

$$\begin{array}{ccc}
S \odot \bar{f} & \xrightarrow{f'} & S \odot \mathbb{C}_B \\
\downarrow \sigma \odot \bar{f} & & \downarrow \sigma \odot \alpha_B \\
A^\perp \parallel C & \xleftarrow{f^\perp \parallel C} & B^\perp \parallel C
\end{array}$$

By composition with the usual isomorphism $\sigma \odot \alpha_B \cong \sigma$, this yields the required isomorphism between $\sigma \odot \bar{f}$ and $(f^\perp \parallel C) \circ \sigma$. This isomorphism preserves symmetry, since the very same construction can be performed the symmetries in a way preserving projections. \square

Note that from the lemma above already follows a useful property of lifting: it is functorial.

Lemma 3.20. *Let $f^\perp : C^\perp \rightarrow B^\perp$ and $g^\perp : B^\perp \rightarrow A^\perp$ be strong-receptive, courteous maps of essps. Then, we have an isomorphism in $\mathcal{ESSP} / A^\perp \parallel C$:*

$$\bar{f} \odot \bar{g} = \overline{g \circ f}$$

Proof. Direct application of Lemma 3.19. \square

Compact closed structure. In \mathcal{ESSP} , parallel composition has a unit up to isomorphism, namely the empty game 1. In fact equipped with \parallel , \mathcal{ESSP} has a symmetric monoidal structure given by the following natural isomorphisms:

$$\begin{array}{lll}
\rho_{\mathcal{A}} & : & \mathcal{A} \parallel 1 \rightarrow \mathcal{A} \\
\lambda_{\mathcal{A}} & : & 1 \parallel \mathcal{A} \rightarrow \mathcal{A} \\
s_{\mathcal{A}, \mathcal{B}} & : & \mathcal{A} \parallel \mathcal{B} \rightarrow \mathcal{B} \parallel \mathcal{A} \\
\alpha_{\mathcal{A}, \mathcal{B}, \mathcal{C}} & : & (\mathcal{A} \parallel \mathcal{B}) \parallel \mathcal{C} \rightarrow \mathcal{A} \parallel (\mathcal{B} \parallel \mathcal{C})
\end{array}$$

Then, these lift to:

$$\begin{array}{lll}
\overline{\rho_{\mathcal{A}^\perp}^{-1}} & : & \mathcal{A} \parallel 1 \xrightarrow{\text{TCG}} \mathcal{A} \\
\overline{\lambda_{\mathcal{A}^\perp}^{-1}} & : & 1 \parallel \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{A} \\
\overline{s_{\mathcal{A}^\perp, \mathcal{B}^\perp}^{-1}} & : & \mathcal{A} \parallel \mathcal{B} \xrightarrow{\text{TCG}} \mathcal{B} \parallel \mathcal{A} \\
\overline{\alpha_{\mathcal{A}^\perp, \mathcal{B}^\perp, \mathcal{C}^\perp}^{-1}} & : & (\mathcal{A} \parallel \mathcal{B}) \parallel \mathcal{C} \xrightarrow{\text{TCG}} \mathcal{A} \parallel (\mathcal{B} \parallel \mathcal{C})
\end{array}$$

which satisfy the required equations by Lemma 3.20.

Note also the duality: for $\sigma : \mathcal{A} \xrightarrow{\text{TCG}} \mathcal{B}$ we also have $\sigma^\perp : \mathcal{B}^\perp \xrightarrow{\text{TCG}} \mathcal{A}^\perp$, such that $(\tau \odot \sigma)^\perp = \sigma^\perp \odot \tau^\perp$. Using that, the fact that the essp maps above are isomorphisms and Lemma 3.19, it is straightforward to show that the liftings above are natural.

Finally, we have the obvious maps:

$$\begin{aligned} \eta_{\mathcal{A}} &: \mathbb{C}_{\mathcal{A}} \rightarrow 1^\perp \parallel (\mathcal{A}^\perp \parallel \mathcal{A}) \\ \epsilon_{\mathcal{A}} &: \mathbb{C}_{\mathcal{A}} \rightarrow (\mathcal{A} \parallel \mathcal{A}^\perp)^\perp \parallel 1 \end{aligned}$$

and it is a direct variant of the neutrality of copycat under composition to check that these obey the required unit and co-unit laws, so we have:

Proposition 3.21. *The category TCG/\simeq is compact closed.*

4 Concurrent Hyland-Ong games

We have built a \sim -bicategory TCG , which has the structure of a compact closed category and satisfies its laws up to weak equivalence. Using it as underlying linear category, we will construct a sub- \sim -bicategory of strategies playing on (tcgs generated from) standard Hyland-Ong arenas, and show how the usual ccc of innocent strategies arises as a subcategory.

4.1 Arenas and expanded games

First, we give (an alternative presentation of) the usual notion of arenas [HO00]. For A an event structure, write $\min(A)$ for its set of minimal events.

Definition 4.1. *An arena is a countable esp A which is:*

- *A forest:* for all $a, a' \leq a''$ we have either $a \leq a'$ or $a' \leq a$,
- *conflict-free:* $\mathcal{P}_f(A) \subseteq \text{Con}_A$,
- *alternating:* for all $a \rightarrow a'$, $\text{pol}(a) \neq \text{pol}(a')$.

An arena A is *negative* iff $\text{pol}(\min(A)) = \{-\}$. We can also add a Questions/Answers labeling, but this is outside the scope of this paper.

We will now be interested in \sim -strategies playing on a tcg $!A$ thought as “ A with replications”, derived from an arena A . Its underlying set of events will be the set of *index functions* on A : functions $\alpha : [a] \rightarrow \omega$. For such an index function, we write $\text{lbl } \alpha = a$ for the maximal element of its domain, and $\text{ind } \alpha = \alpha(\text{lbl } \alpha)$ for the copy index of its domain.

Lemma 4.2. *Let A be an arena. There is an tcg $!A$ having:*

- *Events:* indexing functions $\alpha : [a] \rightarrow \omega$.

- *Causality*: $\alpha \leq \alpha'$ defined as $\text{lbl } \alpha \leq \text{lbl } \alpha'$ and α, α' agree on their common domain,
- *Consistency*: trivial – there is no conflict.
- *Symmetry*: comprises order-isomorphisms $\theta : x \cong y$ with $x, y \in \mathcal{C}(!A)$ such that θ preserves labels.

Events $\alpha \in !A$ also inherit polarities from A .

Proof. Clearly this set contains the identity and is stable under composition, inverse and restriction. If we have $x \cong_{\theta}^y$ and $x \xrightarrow{\alpha_1: [a] \rightarrow \omega} C$, then since A is a forest there is a unique $a' \rightarrow a$, and necessarily $\alpha'_1 = \alpha_1 \upharpoonright [a']$ satisfies $\alpha'_1 \rightarrow \alpha_2$. Take $\alpha'_2 = \theta(\alpha'_1)$. Necessarily, since y is finite, there is $n \in \omega$ such that $\alpha_2 = \alpha'_2 \cup \{a \mapsto n\} \notin y$, it is direct to check that extending θ with (α_1, α_2) gives an order-isomorphism preserving labels as required.

The $\text{essp } !A$ is race-preserving, since it is conflict-free. For receptive thin sub-symmetries, pick the isomorphism families:

$$\begin{aligned} \widetilde{!A}_+ &= \{x \cong_{!A}^{\theta} y \mid \forall \alpha^+ \in x, \text{ind } \alpha = \text{ind } (\theta(\alpha))\} \\ \widetilde{!A}_- &= \{x \cong_{!A}^{\theta} y \mid \forall \alpha^- \in x, \text{ind } \alpha = \text{ind } (\theta(\alpha))\} \end{aligned}$$

which obviously satisfy the requirements. \square

Arenas are closed under parallel composition $A \parallel B$ – sometimes written $A \times B$ for compatibility with [HO00] and since it will be the product of our cartesian closed category. They also support the dual operation $(-)^{\perp}$, and include the empty arena 1.

A \sim -strategy playing on an arena A is a \sim -strategy $\sigma : \mathcal{S} \rightarrow !A$, and a \sim -strategy from $!A$ to $!B$ is a \sim -strategy $\sigma : \mathcal{S} \rightarrow !A^{\perp} \parallel !B$. Restricting TCG to objects of the form $!A$ gives a full sub- \sim -category of TCG. We will now restrict this sub- \sim -bicategory further to obtain one satisfying the laws of a cartesian closed category up to weak equivalence.

4.2 A sub- \sim -bicategory of TCG

To get a cartesian closed structure, we will restrict to *negative* arenas, and only consider as morphisms \sim -strategies that are *negative*, and *single-threaded* in the following sense.

Definition 4.3. An $\text{esp } A$ is *single-threaded* iff it satisfies:

- (1) For any $a \in A$, $[a]$ has exactly one minimal event.
- (2) For any $x \in \mathcal{C}(A)$ such that $x \xrightarrow{a_1} C, x \xrightarrow{a_2} C$ and $x \cup \{a_1, a_2\} \notin \mathcal{C}(A)$, then $[a_1] \cap [a_2] \neq \emptyset$. (alternatively, they share the same minimal event).

An esp \mathcal{A} is single-threaded iff A is. For a \sim -strategy $\sigma : \mathcal{S} \rightarrow \mathcal{A}$, we say that it is single-threaded iff \mathcal{S} is.

The \sim -bicategory Cho has:

- *Objects*: Negative arenas,
- *Morphisms from A to B* : \sim -strategies

$$\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$$

that are **negative** (i.e. \mathcal{S} is negative), and **single-threaded**. We write $\sigma : A \xrightarrow{\text{Cho}} B$. Negative single-threaded strategies are called Cho-strategies.

- *2-cells*: weak equivalences.

For that to typecheck, we need to check that copycat satisfies these conditions and that they are stable under composition.

Lemma 4.4. *For \mathcal{A}, \mathcal{B} conflict-free single-threaded tcgs, the strategies obtained by lifting maps $f : \mathcal{B}^\perp \rightarrow \mathcal{A}^\perp$ are negative and single-threaded.*

Proof. We just have to check that $\mathbb{C}_\mathcal{B}$ for any conflict-free tcg \mathcal{B} is negative and single-threaded. Clearly minimal events of $\mathbb{C}_\mathcal{B}$ are negative. Let us first check condition (1). Take a prime configuration $x \in \mathcal{C}(\mathbb{C}_\mathcal{B})$, we need to check that x has a unique minimal event. We know that x must correspond to:

$$x_1 \supseteq^+ x_2 \subseteq^- x_3$$

Suppose first that the maximal event of x corresponds to $b \in x_1$. Then we make two observations: firstly, $x_1 = [b]$ – this follows from a straightforward analysis of causal dependencies in $\mathbb{C}_\mathcal{B}$. Secondly, necessarily $x_2 = x_3$ since by definition of $\mathbb{C}_\mathcal{B}$ no event in $x_3 \setminus x_2$ could be below b . Any minimal event of x must correspond to a minimal (negative) event in x_3 by negativity of $\mathbb{C}_\mathcal{B}$. Therefore if there were two such $a_1, a_2 \in x_2 = x_3$ they would appear (and be minimal) as well in x_1 , which is absurd since $x_1 = [b]$ and \mathcal{B} is single-threaded.

Suppose now that the maximal event of x corresponds to $b \in x_3$. Then for the same reason as above $x_3 = [b]$. Since minimal events of x must be in x_3 , there must be only one by single-threadedness of \mathcal{B} .

For condition (2), note that since \mathcal{B} has no conflict, $\mathbb{C}_\mathcal{B}$ has none either. \square

In particular, from Lemma 4.4 it follows that $\alpha_{!A} : A \xrightarrow{\text{Cho}} A$ is negative and single-threaded. We now show that Cho-strategies are stable under composition.

Lemma 4.5. *Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{B}$ and $\tau : \mathcal{T} \rightarrow \mathcal{B}^\perp \parallel \mathcal{C}$ be negative single-threaded \sim -strategies with $\mathcal{A}, \mathcal{B}, \mathcal{C}$ negative tcgs. Then, $\tau \circ \sigma : \mathcal{T} \circ \mathcal{S} \rightarrow \mathcal{A}^\perp \parallel \mathcal{C}$ is negative and single-threaded.*

Proof. Note first that this is independent from symmetry, so we shall just reason on strategies $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$.

Let $P = (S \parallel C) \circledast (A \parallel T)$ be the pullback (in \mathcal{ES}) of $\sigma \parallel C$ and $A \parallel \tau$, with projections Π_1 and Π_2 . By construction we have $P = \text{Pr}(\mathcal{F})$ with \mathcal{F} the stable family:

$$\mathcal{F} = \mathcal{C}(S \parallel C) \times \mathcal{C}(A \parallel T) \upharpoonright R$$

where $R = \{(e_1, e_2) \mid (\sigma \parallel C)(e_1) = (A \parallel \tau)(e_2)\}$.

In order to prove condition (1) of single-threadedness, we are going to prove by induction on $<_P$ that for all $p \in P$, there exists a unique $p' \in \min(P)$, necessarily of the shape $[\!(\!(2, c), (2, t))\!]_z$, such that $p' \leq p$. If $p \in P$ is minimal it has the form $[\!(\!(2, c), (2, t))\!]_z$ for some $z \in \mathcal{F}$, indeed any other case would contradict negativity of either S or T . Otherwise p is not minimal, *i.e.* it has an immediate cause $q_1 \rightarrow p$. By induction hypothesis, there is a unique q'_1 minimal such that $q'_1 \leq q_1 \leq p$. So if there is another $q'_2 \leq p$ minimal, there must be another $q_2 \rightarrow p$ such that $q'_2 \leq q_2$. We now distinguish cases, depending on the form of p .

- If p has the form $[\!(\!(1, s), (1, a))\!]_z$. Then, by Lemma 2.11 it is straightforward to show that q_i must have the form $[\!(\!(1, s_i), -)\!]_z$. Necessarily, this implies that $s_i \rightarrow s$ as well, and in particular $s_i \leq s$. By single-threadedness of S it follows that there is a unique $s' \in \min(S)$ such that $s' \leq s_1$ and $s' \leq s_2$. By definition of the pullback it follows that there is $t \in T$ such that $(\!(1, s'), (2, t)) \in z$, and we have:

$$[\!(\!(1, s'), (2, t))\!]_z \leq q_i$$

By induction hypothesis, the unique minimal cause of q_1 and q_2 must be the same as for $[\!(\!(1, s'), (2, t))\!]_z$, so we must have $q'_1 = q'_2$.

- If p has the form $[\!(\!(1, s), (2, t))\!]_z$ where $\text{pol}(s) = +$, then by Lemma 2.11 it is straightforward to show that q_i must have the form $[\!(\!(1, s_i), e)\!]_z$. Then we argue that, as above, we must have $s_i \leq s$. Indeed from Lemma 2.11 we also know that we have either $s_i \rightarrow s$, or e has the shape $(2, t_i)$ with $t_i \rightarrow t$. But by courtesy the latter would imply that $\tau t_i \rightarrow \tau t$, so $s_i \leq s$ as well since σ is a map of event structures. Then the same reasoning as in the previous case applies.
- If p has the form $[\!(\!(1, s), (2, t))\!]_z$ where $\text{pol}(s) = -$, then for the same reason as above, q_i must have the form $[\!(\!(-, (2, t_i))\!]_z$. The reasoning is dual to the previous case.
- If p has the form $[\!(\!(2, c), (2, t))\!]_z$, then q_i must have the form $[\!(\!(-, (2, t_i))\!]_z$ with $t_i \leq t$ and the same reasoning applies.

Note in passing that from the characterisation of minimal events of the pullback above it follows that $T \odot S$ is negative.

Let us now show condition (2). Take an immediate conflict in $T \odot S$. Necessarily it must originate in a conflict in P :

$$w_1 \supset \text{---} z \text{---} \subset w_2 \quad \begin{array}{c} [(s,t)]_{\cup w_1} \\ [(s',t')]_{\cup w_2} \end{array}$$

Necessarily by Lemma 2.12 this conflict must originate from one of the projections. Let us suppose w.l.o.g. that the following is a conflict in $S \parallel !C$.

$$\Pi_1 w_1 \supset \text{---}^s \Pi_1 z \text{---}^s \subset \Pi_1 w_2$$

Since σ is single-threaded, there must be $s'' \in S \parallel C$ such that $s'' \leq s$ and $s'' \leq s'$. Necessarily, there is $t'' \in A \parallel T$ such that $(s'', t'') \in z$. Take a minimal predecessor $p \leq [(s'', t'')]_z$; it is necessarily visible (and in C) since S and T are negative. It is also below $[(s, t)]_{\cup w_1}$ and $[(s', t')]_{\cup w_2}$ by construction, so $\tau \odot \sigma$ is single-threaded. \square

So, we have constructed our \sim -bicategory. We now prove that it is cartesian.

4.3 A cartesian category

First, let us note that the empty arena 1 is such that $!1 = 1$ (with the right hand side 1 denoting the empty cgs). Since the objects of Cho are negative arenas and the strategies are negative as well, any morphism in Cho :

$$\sigma : S \rightarrow !A^\perp \parallel !B$$

is such that the minimal events of S must map to $!B$. But in the particular case where $B = 1$, this means that S cannot have any minimal events, so it is necessarily the unique trivial map from the empty set to $!A^\perp \parallel !B$. It follows that 1 is a terminal object.

Projections. We start by noting the following isomorphism.

Lemma 4.6. *Let A and B be arenas. Then there is an isomorphism in \mathcal{ESSP} :*

$$!(A \parallel B) \stackrel{\gamma_{A,B}}{\cong} !A \parallel !B$$

Proof. We define $\gamma_{A,B}$ as the map:

$$\gamma_{A,B} : !(A \parallel B) \rightarrow !A \parallel !B \quad \begin{array}{l} \alpha \mapsto (1, \alpha') \\ \beta \mapsto (2, \beta') \end{array} \quad \left\{ \begin{array}{l} \alpha' : [a] \rightarrow \omega \quad (\text{lbl } \alpha = (1, a)) \\ \mapsto \alpha(1, a') \\ \beta' : [b] \rightarrow \omega \quad (\text{lbl } \beta = (2, b)) \\ \mapsto \alpha(1, b') \end{array} \right.$$

This is clearly a map of \mathcal{ESP} s, and it has an obvious inverse $\gamma_{A,B}^{-1}$. Both preserve labels and as isomorphism of \mathcal{ESP} s they are rigid, so they transport isomorphisms on $!(A \parallel B)$ (order-isomorphisms preserving labels) to isomorphisms on $!A \parallel !B$. Therefore they are actually isomorphisms in \mathcal{ESSP} . \square

From that it follows that we have maps:

$$\begin{aligned} i_A & : !A \rightarrow !(A \times B) \\ & \quad \alpha \mapsto \gamma_{A,B}^{-1}(1, \alpha) \\ i_B & : !B \rightarrow !(A \times B) \\ & \quad \beta \mapsto \gamma_{A,B}^{-1}(2, \beta) \end{aligned}$$

which are such that i_{A^\perp} and i_{B^\perp} are strong-receptive and courteous whenever A and B are negative. Therefore we define by lifting:

$$\begin{aligned} \omega_A & = \overline{i_{A^\perp}} : A \times B \xrightarrow{\text{Cho}} A \\ \omega_B & = \overline{i_{B^\perp}} : A \times B \xrightarrow{\text{Cho}} B \end{aligned}$$

Pairing. For A, B, C negative we need to define the pairing of:

$$\begin{aligned} \sigma & : \mathcal{S} \rightarrow !C^\perp \parallel !A \\ \tau & : \mathcal{T} \rightarrow !C^\perp \parallel !B \end{aligned}$$

We would like to set:

$$\begin{aligned} \langle \sigma, \tau \rangle & : \mathcal{S} \parallel \mathcal{T} \rightarrow !C^\perp \parallel !(A \times B) \\ (1, s) & \mapsto (!C^\perp \parallel i_A) \circ \sigma \\ (2, t) & \mapsto (!C^\perp \parallel i_B) \circ \tau \end{aligned}$$

However this might fail local injectivity, since σ and τ might have a common codomain on $!C^\perp$. So we need to reindex their moves to make sure that no such collision can occur. In order to do that we introduce two maps:

$$\begin{aligned} \iota_o & : !A \rightarrow !A \\ \alpha & \mapsto \alpha' \quad \text{where } \begin{cases} \alpha'(a) = 2\alpha(a) + 1 & (\text{if } a \text{ minimal}) \\ \alpha'(a) = \alpha(a) & (\text{otherwise}) \end{cases} \\ \iota_e & : !A \rightarrow !A \\ \alpha & \mapsto \alpha' \quad \text{where } \begin{cases} \alpha'(a) = 2\alpha(a) & (\text{if } a \text{ minimal}) \\ \alpha'(a) = \alpha(a) & (\text{otherwise}) \end{cases} \end{aligned}$$

It is immediate to check that ι_e and ι_o are maps of essp. Moreover, they have disjoint codomains, ι_e^\perp and ι_o^\perp are strong-receptive (since we only change the copy index of minimal events) and courteous, and they are both symmetric with the identity: $\iota_e \sim \text{id}_{!A}$ and $\iota_o \sim \text{id}_{!A}$.

It follows that we have weak equivalences:

$$\begin{aligned} (\iota_o^\perp \parallel !A) \circ \sigma & \simeq \sigma \\ (\iota_e^\perp \parallel !B) \circ \tau & \simeq \tau \end{aligned}$$

And we can now form the pairing:

$$\begin{aligned} \langle \sigma, \tau \rangle : \mathcal{S} \parallel \mathcal{T} &\rightarrow !C^\perp \parallel !(A \times B) \\ (1, s) &\mapsto (t_o^\perp \parallel i_A) \circ \sigma \\ (2, s) &\mapsto (t_e^\perp \parallel i_B) \circ \tau \end{aligned}$$

It is then direct – a straightforward variant of the neutrality of copycat under composition – to check:

$$\begin{aligned} \omega_A \odot \langle \sigma, \tau \rangle &\simeq \sigma \\ \omega_B \odot \langle \sigma, \tau \rangle &\simeq \tau \end{aligned}$$

In order to have a cartesian category, we still have to prove:

Lemma 4.7 (Surjective pairing). *For any negative arenas A, B and negative single-threaded \sim -strategy $\sigma : C \xrightarrow{\text{Cho}} A \times B$ we have the weak equivalence:*

$$\sigma \simeq \langle \omega_A \odot \sigma, \omega_B \odot \tau \rangle$$

Proof. Let us write $\sigma : \mathcal{S} \rightarrow !C^\perp \parallel !(A \times B)$. By condition (1) of single-threadedness, each event $s \in S$ has a unique minimal dependency $s' \in S$. By negativity of arenas and strategies s' must map either to A (more formally to an event of the form $\{(1, a) \mapsto n\}$ for $a \in \min(A)$) or to B (more formally to an event of the form $\{(2, b) \mapsto n\}$ for $b \in \min(B)$). Write S_A for the subset of S comprising events whose minimal dependency map to A , and similarly write S_B . Then S_A and S_B are down-closed, otherwise that would contradict condition (1) of single-threadedness. Therefore, they are event structures. The same reasoning holds for symmetries, so we have the decomposition $\mathcal{S} = S_A \cup S_B$.

Take $x \in \mathcal{C}(S)$. For the reason above, x decomposes uniquely in $x_A \cup x_B$, with $x_A \in \mathcal{C}(S_A)$ and $x_B \in \mathcal{C}(S_B)$. From x_A and x_B we can (by an immediate variation on the neutrality of copycat) build $x'_A \in \mathbb{C}_{!A} \odot S$ (composing the \sim -strategies ω_A – which has “internal” event structure $\mathbb{C}_{!A}$ – and σ) and $x'_B \in \mathbb{C}_{!B} \odot S$, so

$$x'_A \parallel x'_B \in \mathcal{C}(\mathbb{C}_{!A} \odot S \parallel \mathbb{C}_{!B} \odot T)$$

which is the event structure for the \sim -strategy $\langle \omega_A \odot \sigma, \omega_B \odot \tau \rangle$. This operation satisfies by construction the conditions of Lemma 2.7, so induce a map. The operation lifts smoothly on symmetries, so the map preserves symmetry as well. This construction preserves the projection to the game on the nose.

Reciprocally take $x'_A \in \mathbb{C}_{!A} \odot S$ and $x'_B \in \mathbb{C}_{!B} \odot S$. Again by an immediate variation on the neutrality of copycat we can get corresponding $x_A \in \mathcal{C}(S)$ and $x_B \in \mathcal{C}(S)$ in a way preserving the projection to the game, the covering relation and compatible union of configurations. Now, it actually follows that $x_A \cup x_B \in \mathcal{C}(S)$ as well: indeed events in x_A and x_B cannot share their minimal dependencies, since for x_A they map to A and to B for x_B . Therefore by condition (2) of single-threadedness x_A and x_B are compatible. The correspondence from x'_A, x'_B to $x_A \cup x_B$ preserves covering and compatible unions so by Lemma

2.7 induces a map, and one can show that it preserves symmetry by performing the exact same construction at the level of symmetries. The projection to the game is again preserved on the nose.

Finally the correspondence generates an isomorphism, since we have only transformed the events via the correspondence between $\mathbb{C}_{!A} \odot S_A$ and S_A , which is an isomorphism. \square

So putting all of this together, we have proved

Proposition 4.8. *The \sim -bicategory Cho satisfies the laws of a cartesian category up to weak equivalence.*

4.4 A cartesian closed category

Cho is a sub- \sim -bicategory of TCG which is compact closed, so we could expect the closed structure of Cho to correspond to the compact closed structure of TCG; however we run into the issue that the arena $A^\perp \times B$ is not an object of Cho, since it is not negative. So we proceed as in [HO00], and replace it with a negative variant where minimal events of A^\perp are set to depend on minimal events of B .

So, we start by recalling the arrow arena construction of [HO00]. For two negative arenas A and B , we define the arrow arena $A \Rightarrow B$ as having:

- Events, $\{(1, (b, a)) \mid a \in A \ \& \ b \in \min(B)\} \cup \{(2, b) \mid b \in B\}$.
- Causality,

$$\begin{aligned} & \{((1, (b, a_1)), (1, (b, a_2))) \mid a_1 \leq a_2 \ \& \ b \in \min(B)\} \cup \\ & \{((2, b_1), (2, b_2)) \mid b_1 \leq b_2\} \cup \\ & \{((2, b), (1, (b, a))) \mid a \in A \ \& \ b \in \min(B)\} \end{aligned}$$

- Polarity, $pol((1, (b, a))) = -pol(a)$ and $pol((2, b)) = pol(b)$.

To show that this gives a closed structure, we need to relate it to the compact closed structure of TCG. To that effect, we will now define a map of essps:

$$\zeta_{A,B} : !(A \Rightarrow B) \rightarrow !A^\perp \parallel !B$$

For events $b \in B$ we use $\#b$ for the natural number associated to b by the countability of B . We also use $\langle -, - \rangle : \omega^2 \rightarrow \omega$ for any injective function; the collision with the pairing operation should not generate any confusion.

We set:

$$\begin{aligned} \chi_{A,B} : \quad & !(A \Rightarrow B) && \rightarrow && !A^\perp \parallel !B \\ & (\alpha : [(1, (b, a))] \rightarrow \omega) && \mapsto && \alpha' \\ & (\beta : [(2, b)] \rightarrow \omega) && \mapsto && \beta' \end{aligned}$$

where:

$$\begin{array}{lcl} \alpha' : [a] & \rightarrow & \omega \\ a' & \mapsto & \langle \#b, \alpha((2, b)), \alpha((1, (b, a'))) \rangle \quad (\text{if } a' \in \min(A)) \\ a' & \mapsto & \alpha((1, (b, a'))) \quad (\text{otherwise}) \end{array}$$

and:

$$\begin{array}{lcl} \beta' : [b] & \rightarrow & \omega \\ b' & \mapsto & \beta((2, b')) \end{array}$$

With this definition $\chi_{A,B}$ preserves symmetry, is strong-receptive (it does not change the copy indices of negative events, since minimal events of A^\perp are positive) and courteous (it only breaks immediate causal links from minimal events of B to minimal events of A^\perp , so from negative to positive). This allows us, from $\sigma : \mathcal{S} \rightarrow !C^\perp \parallel !(A \Rightarrow B)$, to define:

$$\begin{aligned} \Phi(\sigma) & : \mathcal{S} \rightarrow !C^\perp \parallel !(A^\perp \parallel !B) \\ & = (!C^\perp \parallel \chi_{A,B}) \circ \sigma \end{aligned}$$

Reciprocally, we can go in the other direction using the following lemma.

Lemma 4.9. *Any single-threaded map of ess $\sigma : \mathcal{S} \rightarrow !C^\perp \parallel !(A^\perp \parallel !B)$ such that minimal events of \mathcal{S} map to $!B$ (as in a negative strategy in the presence of polarities) factors uniquely through $!C^\perp \parallel \chi_{A,B}$ up to weak equivalence.*

Proof. We define $\sigma' : \mathcal{S} \rightarrow !C^\perp \parallel !(A \Rightarrow B)$. For $s \in \mathcal{S}$, then if $\sigma(s) = (1, \gamma)$ we set $\sigma'(s) = (1, \gamma)$ still.

If $\sigma(s) = (2, (2, \beta))$ with $\beta : [b] \rightarrow \omega$, then we set $\sigma'(s) = (2, \beta')$ with

$$\begin{array}{lcl} \beta' : [(2, b)] & \rightarrow & \omega \\ (2, b') & \mapsto & \beta(b') \end{array}$$

If $\sigma(s) = (2, (1, \alpha))$ with $\alpha : [a] \rightarrow \omega$, then by condition (1) of single-threadedness there is a unique minimal $s' \leq s$. Since S is negative, s' must be negative as well, but the only negative minimal events of $!C^\perp \parallel !(A^\perp \parallel !B)$ have the form $(2, (2, \beta))$ with $\beta = \{b \mapsto n\}$. Therefore we set:

$$\begin{array}{lcl} \alpha' : [(1, (b, a))] & \rightarrow & \omega \\ (1, (b, a')) & \mapsto & \alpha(a') \\ (2, b) & \mapsto & n \end{array}$$

and we define $\sigma'(s) = (2, \alpha')$.

It is routine to check that this preserves symmetry, is strong-receptive and courteous, and that its composition with $\chi_{A,B}$ is weakly equivalent to σ – copy indices have been heavily modified, but labels and causal dependency remains the same. Uniqueness up to symmetry follows from the fact that arena labels have to be preserved, and the dependency annotations in $!(A \Rightarrow B)$ are forced by the causality in S . Only copy indices can be modified, and they do not matter up to symmetry. \square

From that, we deduce the following.

Proposition 4.10. *There is a bijection Φ up to weak equivalence, preserving and reflecting weak equivalence, between:*

- Negative, single-threaded \sim -strategies $\sigma : \mathcal{S} \rightarrow \mathcal{C}^\perp \parallel !(A \Rightarrow B)$,
- Negative, single-threaded \sim -strategies $\sigma' : \mathcal{S} \rightarrow \mathcal{C}^\perp \parallel (!A^\perp \parallel !B)$.

Moreover this bijection is compatible with pre-composition: for all $\tau : \mathcal{T} \rightarrow !D^\perp \parallel !C$, we have:

$$\Phi(\sigma) \circ \tau \simeq \Phi(\sigma \circ \tau)$$

Proof. On the one hand $\Phi(\sigma)$ is obtained as $(!C^\perp \parallel \chi_{A,B}) \circ \sigma$, while $\Phi^{-1}(\sigma')$ is obtained by the unique factorisation of Lemma 4.9. The bijection up to weak equivalence follows from Lemma 4.9 as well.

To check that Φ commutes with composition, we consider the two following pullbacks:

$$\begin{array}{ccc} \mathcal{P} & \longrightarrow & \mathcal{T} \parallel !(A \Rightarrow B) \\ \downarrow \lrcorner & & \downarrow \tau \parallel !(A \Rightarrow B) \\ !D \parallel \mathcal{S} & \xrightarrow{!D \parallel \Phi(\sigma)} & !D \parallel !C \parallel !(A \Rightarrow B) \end{array} \quad \begin{array}{ccc} \mathcal{P}' & \longrightarrow & \mathcal{T} \parallel (!A^\perp \parallel !B) \\ \downarrow \lrcorner & & \downarrow \tau \parallel (!A^\perp \parallel !B) \\ !D \parallel \mathcal{S} & \xrightarrow{!D \parallel \sigma'} & !D \parallel !C \parallel (!A^\perp \parallel !B) \end{array}$$

The maps:

$$\begin{aligned} \mathcal{T} \parallel \chi_{A,B} & : \mathcal{T} \parallel !(A \Rightarrow B) \rightarrow \mathcal{T} \parallel (!A^\perp \parallel !B) \\ !D \parallel \mathcal{S} & : !D \parallel \mathcal{S} \rightarrow !D \parallel \mathcal{S} \\ !D \parallel !C \parallel \chi_{A,B} & \rightarrow !D \parallel !C \parallel !A^\perp \parallel !B \end{aligned}$$

map the base of the left hand side pullback to the base of the right hand side pullback, which generate a map $h : \mathcal{P} \rightarrow \mathcal{P}'$. Reciprocally, we have seen in the proof of Lemma 4.5 that the composition pullback \mathcal{P}' is single-threaded. Using that observation, we can define maps from \mathcal{P}' to $\mathcal{T} \parallel !(A \Rightarrow B)$ as in Lemma 4.9. This map along with the identity on $!D \parallel \mathcal{S}$ induces an inverse $h^{-1} : \mathcal{P}' \rightarrow \mathcal{P}$ for h . It follows from uniqueness in Lemma 4.9 that h and h^{-1} are inverse of each other up to symmetry, yielding a weak equivalence. After projection to visible events, this gives a weak equivalence between $\tau \circ \Phi(\sigma)$ and $\Phi(\tau \circ \sigma)$. \square

From that, it is now straightforward to deduce the cartesian closed structure.

Theorem 4.11. *The \sim -bicategory Cho satisfies the laws of a cartesian closed category up to weak equivalence.*

Proof. We already know that it is cartesian.

Firstly, evaluation is defined as:

$$\begin{aligned} ev_{A,B} & : A \times (A \Rightarrow B) \xrightarrow{\text{Cho}} B \\ & = \overline{\lambda_{!A^\perp}^{-1}} \circ \epsilon_{!A} \circ \overline{\alpha_{!A^\perp, !A, !B^\perp}} \circ (!A \parallel_s \Phi(\alpha_{!(A \Rightarrow B)})) \circ \overline{\gamma_{A^\perp, (A \Rightarrow B)}^{-1}} \end{aligned}$$

Likewise curryfication of $\sigma : \mathcal{S} \rightarrow !(A \times C)^\perp \parallel !B$ is obtained by:

$$\begin{aligned} \Lambda(\sigma) &: !C \xrightarrow{\text{Cho}} !(A \Rightarrow B) \\ &= \Phi((!A^\perp \parallel_s (\sigma \odot \overline{\gamma_{A^\perp, C^\perp}})) \odot \overline{\alpha_{!A^\perp \parallel !A^\perp \parallel !C}^{-1}} \odot (\eta_{!A} \parallel_s !C)) \end{aligned}$$

The required verifications are routine calculations, using the fact that Φ is a bijection and the compact closed structure of TCG – all the structural isomorphisms eliminate each other. \square

4.5 Recursion

4.5.1 A dcpo of \sim -strategies

We compute least upper bounds of \sim -strategies using this concrete partial ordering:

Definition 4.12 (Inclusion). *Let \mathcal{A}, \mathcal{B} be event structures with symmetry and polarities. We say that $\mathcal{A} \leq \mathcal{B}$ whenever $A \subseteq B$, all data on \mathcal{A} (causality, consistency, polarities, symmetry) is the restriction of that of \mathcal{B} , and the inclusion induces a map of essps $\mathcal{A} \hookrightarrow \mathcal{B}$ (so A is down-closed in B).*

The relation \leq is clearly a partial order on event structures with symmetry and polarities. We set:

Definition 4.13 (Inclusion of \sim -strategies). *Let $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ and $\tau : \mathcal{T} \rightarrow \mathcal{A}$ be strategies. We say that $\sigma \leq \tau$ when $\mathcal{S} \leq \mathcal{T}$ and for all $s \in \mathcal{S}$, $\sigma(s) = \tau(s)$.*

This partial order on \sim -strategies is directed complete. Indeed if \mathbb{D} is a directed set of \sim -strategies on a cgs \mathcal{A} , then we define:

$$\bigsqcup_{(\sigma: \mathcal{S} \rightarrow \mathcal{A}) \in \mathbb{D}} \sigma : (\bigcup_{(\sigma: \mathcal{S} \rightarrow \mathcal{A}) \in \mathbb{D}} \mathcal{S}) \rightarrow \mathcal{A}$$

It is direct to check that this defines a \sim -strategy, which is a least upper bound for \mathbb{D} . Additionally if \sim -strategies in \mathbb{D} are negative or single-threaded, this is still the case of $\bigsqcup \mathbb{D}$.

So, \leq equips the set of \sim -strategies with a structure of dcpo. It does not have a least element though: a minimal strategy $\sigma : \mathcal{S} \rightarrow \mathcal{A}$ has as events in \mathcal{S} an initial negative segment corresponding to that of \mathcal{A} , so there are as many minimal strategies as possible renamings of this segment.

We pick one distinguished minimal \sim -strategy on a cgs \mathcal{A} . Let us write \mathcal{A}^- for the initial negative segment of \mathcal{A} : the maximal purely negative cgs such that $\mathcal{A}^- \leq \mathcal{A}$ (not to be confused with the negative thin sub-symmetry \mathcal{A}_-). We set:

$$\perp_{\mathcal{A}} : \mathcal{A}^- \rightarrow \mathcal{A}$$

acting as the identity map. Clearly $\perp_{\mathcal{A}}$ is a \sim -strategy, negative and single-threaded, and is minimal for \leq because anything strictly smaller would fail receptivity.

Every strategy is isomorphic to a strategy lying over $\perp_{\mathcal{A}}$:

Lemma 4.14. *Let $\sigma : S \rightarrow A$ be a strategy. There exists a strategy $\sigma^\dagger : S^\dagger \rightarrow A$ isomorphic to σ such that $\perp_A \trianglelefteq \sigma^\dagger$.*

Proof. By receptivity S contains an initial segment isomorphic to the minimal negative events of A . S^\dagger is obtained by replacing those events by the corresponding events of A , and renaming the other events of S to avoid any potential collision. It is routine to check that we get an isomorphic strategy on A and we clearly have $\sigma \cong \sigma^\dagger$. \square

The main operations we have for constructing strategies are continuous:

Lemma 4.15. *Composition, parallel composition, pairing, curryfication and the $(-)^{\dagger}$ operation introduced just above are continuous for \trianglelefteq .*

Proof. Direct. \square

4.5.2 Fixpoint combinator in Cho

We are now ready to prove that there is a fixpoint combinator in Cho. For A a cgs, we write \mathcal{D}_A for the pointed dcpo of negative single-threaded \sim -strategies above \perp_A . We consider the following operation on $\mathcal{D}_{!(A \Rightarrow A) \perp \parallel !A}$:

$$F(\sigma) = \left((A \Rightarrow A) \xrightarrow{\langle \sigma, \text{id}_{A \Rightarrow A} \rangle} A \times (A \Rightarrow A) \xrightarrow{ev_{A,A}} A \right)^{\dagger}$$

By Lemma 4.15, F is continuous and because of the outermost dagger, it has indeed value in $\mathcal{D}_{!(A \Rightarrow A) \perp \parallel !A}$. Therefore, we can take its fixpoint:

$$Y_A \in \mathcal{D}_{!(A \Rightarrow A) \perp \parallel !A}$$

By construction, it satisfies

$$Y_A \simeq ev_{A,A} \odot \langle Y, \text{id}_{A \Rightarrow A} \rangle : (A \Rightarrow A) \xrightarrow{\text{Cho}} A$$

As a consequence, we have:

Corollary 4.16. *Let $\sigma : \Gamma \xrightarrow{\text{Cho}} A \Rightarrow A$. Then,*

$$Y_A \odot \sigma \simeq ev_{A,A} \odot \langle Y_A \odot \sigma, \sigma \rangle : \Gamma \xrightarrow{\text{Cho}} A$$

Proof. Direct using the equation above and the cartesian closed laws. \square

Using this structure, one can follow the lines of [HO00] and define the interpretation of PCF in Cho. Of course, \sim -strategies in Cho have many more possible behaviours than those coming from terms of PCF, including complex concurrent and non-deterministic behaviours, but also behaviours typically associated with stateful computation.

In the final subsection we show however how to define in Cho a subcategory corresponding exactly to the cartesian closed category from [HO00] of deterministic innocent strategies (modulo the Questions/Answers distinction).

4.6 Embedding of standard Hyland-Ong innocent strategies

In this final subsection, we construct a sub- \sim -bicategory ChoInn of Cho whose quotient under weak equivalence is isomorphic to the standard category of arenas and innocent strategies. We will first recall the basic definitions of the usual category Inn of arenas and innocent strategies, and then relate Inn and ChoInn .

4.6.1 The category Inn

This section is mainly there for reference and to fix the notions and notations. The construction is standard, so we only give the definitions and the properties, but skip all proofs. From now on, all arenas are considered negative.

Plays, strategies. First, we recall the notions of plays and strategies on an arena.

Definition 4.17. Let Σ be an alphabet. A **pointing string** on Σ is a sequence $m_0 m_1 \dots m_n$ of elements of Σ when each m_i may be equipped with a pointer to an earlier m_j (i.e. $j < i$). In that case we write $m_i \rightarrow m_j$.

Definition 4.18. Let m be a pointing string on Σ , and $\Sigma' \subseteq \Sigma$. The restriction $m' = m \upharpoonright \Sigma'$ of m to Σ' is obtained by removing in m all moves from $\Sigma \setminus \Sigma'$. The move m'_i points to m'_j if there is a pointer chain between the corresponding moves in m , passing only through $\Sigma \setminus \Sigma'$.

The definition extends trivially if $\Sigma' \hookrightarrow \Sigma$ is an injection instead of an inclusion.

Notations. We write $|m|$ for the length of m , $m' \sqsubseteq m$ for the prefix ordering, $m' \sqsubseteq^P m$ if additionally m' is P -ending.

Definition 4.19. If A is an arena, then a **legal play** on A is a pointing string m on A such that:

- If m_i has no pointer, then $m_i \in \min(A)$,
- If $m_i \rightarrow m_j$, then $m_j \vdash_A m_i$,
- For all $0 \leq i \leq |m| - 2$, $\text{pol}(m_i) \neq \text{pol}(m_{i+1})$.

Let \mathcal{L}_A be the set of legal plays on A .

Definition 4.20. A (**deterministic**) **strategy** $s : A$ on A is a non-empty, Opponent-branching, even-prefix closed set of even-length legal plays on A .

Copycat, composition.

Definition 4.21. *The copycat strategy is defined by:*

$$\text{id}_A = \{m \in \mathcal{L}_{A_1 \Rightarrow A_2} \mid \forall m' \sqsubseteq^P m, m' \upharpoonright A_1 = m' \upharpoonright A_2\}$$

Definition 4.22. *Let $\mathfrak{s} : A \Rightarrow B$ and $\mathfrak{t} : B \Rightarrow C$ be two strategies. Their **interaction** is:*

$$\mathfrak{s} \parallel \mathfrak{t} = \{u \in (A \Rightarrow B) \Rightarrow C \mid u \upharpoonright A, B \in \mathfrak{s} \& u \upharpoonright B, C \in \mathfrak{t} \& u \upharpoonright A, C \in \mathcal{L}_{A \Rightarrow C}\}$$

Their **composition** is then defined by:

$$\mathfrak{t} \odot \mathfrak{s} = \{u \upharpoonright A, C \mid u \in \mathfrak{s} \parallel \mathfrak{t}\}$$

It is well-known that composition is associative and forms a category.

Visibility, innocence.

Definition 4.23. *Let $m \in \mathcal{L}_A$. Its **P-view** is defined inductively as follows:*

$$\begin{aligned} \lceil m \bar{i} \rceil &= i \\ \lceil m p m' o \rceil &= \lceil m p \rceil o \\ \lceil m p \rceil &= \lceil m \rceil p \end{aligned}$$

where in the first line $i \in \min(A)$, in the second $o \rightarrow p$ and we have $\text{pol}(o) = -$ and $\text{pol}(p) = +$. Pointers are preserved if the target of the pointer is in the P-view, lost otherwise.

For $m \in \mathcal{L}_A$, the P-view $\lceil m \rceil$ is in general just a pointing string on A ; not a legal play since some pointers might be lost. For a play $m \in \mathcal{L}_A$, we say that m is a **P-view** if $\lceil m \rceil = m$; it is easy to see that this is equivalent to the fact that Opponent always points to the previous move.

Definition 4.24. *Let $m \in \mathcal{L}_A$. We say that m is **P-visible** if for all $m' \sqsubseteq m$, $\lceil m' \rceil \in \mathcal{L}_A$. In other words pointers are not lost when computing the P-view of prefixes of m , so Player always points within its P-view.*

A strategy $\mathfrak{s} : A$ is **visible** when all its plays are P-visible.

Visible strategies are stable under composition and form a sub-category of the category of arenas and strategies. We can now define:

Definition 4.25. *Let $\mathfrak{s} : A$ be a visible strategy. It is **innocent** if for all $m, mop, no \in \mathfrak{s}$ with $\lceil mo \rceil = \lceil no \rceil$, then $nop \in \sigma$ as well (with the same pointer).*

Innocent strategies are stable under composition, and form a CCC.

From the definition it follows that innocent strategies are entirely characterized by their set of P-views, so to define an innocent strategy it is sufficient to define a set of P-views, and to compose innocent strategies it is sufficient to compute only interactions that give rise to P-views.

4.6.2 The \sim -bicategory ChoInn

Let us now define the sub- \sim -bicategory of Cho corresponding to it. In this section, we consider negative single-threaded \sim -strategies – however, the conditions introduced here and the corresponding reasonings only concern the underlying strategy $\sigma : S \rightarrow A$ of \sim -strategies $\sigma : S \rightarrow \mathcal{A}$, so we will mostly ignore symmetry in this subsection.

We introduce two further conditions: sequential innocence and determinism.

Definition 4.26. *An esp S is sequential innocent iff it is:*

- (1) Backward sequential: *for all $s \in S$, $[s]$ is a total order,*
- (2) Forward sequential: *for all $s \in S$ with $s \rightarrow s_1^+$, $s \rightarrow s_2^+$, then $[s_1] \cup [s_2] \notin \mathcal{C}(S)$.*

In particular, a strategy $\sigma : S \rightarrow A$ is sequential innocent iff S is.

It is obvious that copycat is sequential innocent, as are maps obtained through lifting. Note that we call that *sequential innocence* and not just innocence, as we have (in a forthcoming companion paper) a notion of innocence that accepts strategies with concurrent behaviour such as that for the parallel or.

We now show that backward sequentiality is stable under composition.

Lemma 4.27. *Let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$ be two backward sequential strategies. Then, $S \otimes T$ is backward sequential as well.*

Proof. It suffices to show that in $S \otimes T$, each event has at most one immediate causal dependency; but that follows immediately from Lemma 2.11, backward sequentiality of S and T and the fact that A is a forest. \square

Proposition 4.28. *If $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$ are backward sequential, so is $\tau \odot \sigma$.*

Proof. By Lemma 4.27, the composition pullback $S \otimes T$ is backward sequential as well. From that it follows from a straightforward induction that events in $T \odot S$ have at most one predecessor as well, so $\tau \odot \sigma$ is backward sequential. \square

We now show that forward sequentiality is stable under composition.

Lemma 4.29. *Let $\sigma : S \rightarrow A^\perp \parallel B$ and $\tau : T \rightarrow B^\perp \parallel C$ be forward sequential. Then, $\tau \odot \sigma$ is forward sequential as well.*

Proof. Let $c \rightarrow_{T \odot S} c_1^+, c_2^+$. By definition of projection, we must have:

$$\begin{array}{c}
 \begin{array}{c}
 c \succ [(s_1, t_1)]_x \succ \dots \succ [(s_i, t_i)]_x \\
 \nearrow \\
 [(s_{i+1}, t_{i+1})]_{x_1} \succ \dots \succ [(s_n, t_n)]_{x_1} \succ c_1 \\
 \searrow \\
 [(s'_{i+1}, t'_{i+1})]_{x_2} \succ \dots \succ [(s'_p, t'_p)]_{x_2} \succ c_2
 \end{array}
 \end{array}$$

in the interaction pullback $P = (S \parallel C) \otimes (A \parallel T)$. Suppose wlog that $\text{pol}(s_i) = -$. By Lemma 2.11, we either have $s_i \rightarrow s_{i+1}$ or $t_i \rightarrow t_{i+1}$. But if $t_i \rightarrow t_{i+1}$, then $\tau t_i \rightarrow \tau t_{i+1}$ as well by courtesy. It follows that $s_i \leq s_{i+1}$, but necessarily $s_i \rightarrow s_{i+1}$ otherwise that would contradict $[(s_i, t_i)]_{x_1} \rightarrow [(s_{i+1}, t_{i+1})]_{x_1}$. For the same reason, $s_i \rightarrow s'_{i+1}$ as well. Therefore by forward sequentiality of σ , $[s_{i+1}] \cup [s'_{i+1}] \notin \mathcal{C}(S)$. But $c_1 \cup c_2$ must contain $[s_{i+1}] \cup [s'_{i+1}]$, so cannot be a valid configuration of the pullback – so $[c_1]_{T \odot S} \cup [c_2]_{T \odot S} \notin \mathcal{C}(T \odot S)$. \square

So, we have a bicategory of sequential innocent strategies. All these constructions being orthogonal with symmetry, we also have a \sim -bicategory of thin concurrent games and sequential innocent strategies – and a \sim -bicategory of negative arenas and sequential innocent strategies. Strategies obtained by lifting are sequential innocent and it is preserved by ccc operations, hence there is a \sim -bicategory satisfying ccc laws up to weak equivalence of negative arenas and single-threaded sequential innocent strategies.

Definition 4.30. For A a conflict-free esp, a strategy $\sigma : S \rightarrow A$ is **deterministic** iff any finite subset of S is consistent.

This is not the standard definition of deterministic concurrent strategies (for that, see [Win12]), but it is a specialization of it in a case – such as here – when the games are conflict-free. Clearly copycat on a conflict-free game is deterministic in the sense above. For self-completeness, we prove that deterministic strategies are stable under composition.

Lemma 4.31. Let A be a conflict-free esp, and let $\sigma : S \rightarrow A$ and $\tau : T \rightarrow A^\perp$ be deterministic. Then, every finite subset of $S \otimes T$ is consistent. Therefore, deterministic strategies are stable under composition.

Proof. First, note that since all finite subsets of events of S and T are consistent it follows that σ and τ are injective. Therefore for any $s \in S$, there is at most one $t \in T$ such that $\sigma s = \tau t$. Take a finite set of configurations of $S \otimes T$, represented as secured bijections:

$$x_i \cong \overset{\sigma}{\sigma} x_i = \tau y_i \cong \overset{\tau}{\tau} y_i$$

for $1 \leq i \leq n$. From the observation above, we have that

$$\bigcup_{1 \leq i \leq n} x_i \cong \overset{\sigma}{\sigma} (\bigcup_{1 \leq i \leq n} x_i) = \tau (\bigcup_{1 \leq i \leq n} y_i) \cong \overset{\tau}{\tau} \bigcup_{1 \leq i \leq n} y_i$$

is still a bijection. Since x_i and y_i , as configurations of respectively S and T , are down-closed, it immediately follows that this bijection is secured, making it a configuration of $S \otimes T$ corresponding to the union of the configurations of $S \otimes T$ that we started with. It also follows that all finite sets of events of the projection are consistent, hence deterministic strategies are stable under composition. \square

Note that although non-deterministic sequential innocent \sim -strategies are not automatically single-threaded – they might fail condition (2) of single-threadedness, deterministic sequential innocent \sim -strategies automatically are and therefore form a sub- \sim -bicategory of Cho satisfying the equations of a ccc up to weak equivalence. We call this \sim -bicategory ChoInn.

4.6.3 Relating ChoInn and Inn

From ChoInn to Inn. First, let us show how to associate an innocent strategy in the usual Hyland-Ong sense to any negative deterministic sequential innocent \sim -strategy $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$. For the rest of this paragraph, we fix σ .

Note first that for any $s \in S$, by sequential innocence $[s]$ is a total order:

$$s_0 \rightarrow s_1 \rightarrow \dots \rightarrow s_n = s$$

to which we associate the sequence:

$$(\text{lbl}(\sigma s_0), \text{lbl}(\sigma s_1), \dots, \text{lbl}(\sigma s_n))$$

Additionally, this sequence is equipped with pointers by setting $\text{lbl}(\sigma s_i)$ to point to $\text{lbl}(\sigma s_j)$ if, either $\sigma s_j \rightarrow \sigma s_i$, or $j = 0$ and σs_i is minimal in $!A$ – the second case is here because we started with $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ rather than $\sigma : \mathcal{S} \rightarrow !(A \Rightarrow B)$. We denote this pointing sequence by $\mathcal{P}(s)$.

Let us now make a few observations on this pointing sequence. First of all it is alternating: indeed by courtesy S can only enrich the causality of $!A^\perp \parallel !B$ with causal links from negative to positive events, and $!A^\perp \parallel !B$ is alternating. Moreover, all Opponent moves in $\mathcal{P}(s)$ must point to the previous move; indeed otherwise there would be s_j^+, s_k^+, s_i^- with $\sigma s_j \rightarrow \sigma s_i$, but $s_j < s_k \rightarrow s_i$. But by courtesy that would mean that $\sigma s_k \rightarrow \sigma s_i$ as well, so $\sigma s_k = \sigma s_j$ since $!A^\perp \parallel !B$ is a forest – contradiction. Moreover all Player moves are equipped with a pointer (since $[s] \in \mathcal{C}(S)$, and for any $s'^+ \leq s$, either $\sigma s'$ is not minimal in $!A^\perp \parallel !B$ and it has an immediate dependency, or it is minimal in $!A^\perp$, in which case it is by definition set to point to $\text{lbl}(\sigma s_0)$). So we have proved that $\mathcal{P}(s)$ is a P-view.

Now, we are in a position to form:

$$\mathcal{P}(\sigma) = \{\epsilon\} \cup \{\mathcal{P}(s) \mid s^+ \in S\}$$

obtaining a set of P-ending P-views. We now wish to show that this set of P-views defines an innocent strategy. For that, it has to be O-branching. For that, we will need the following key lemma.

Lemma 4.32. *Let $s, s' \in S$, such that $||[s]|| = ||[s']||$. Then, the following three propositions are equivalent.*

- (1) $\mathcal{P}(s) = \mathcal{P}(s')$,
- (2) *The unique order-isomorphism $\theta : [s] \cong [s']$ is in the isomorphism family of \mathcal{S} ,*
- (3) *The unique order-isomorphism $\theta : [s] \cong [s']$ is such that $\sigma \theta$ is in the isomorphism family of $!A^\perp \parallel !B$.*

Proof. (1) \Rightarrow (2). By induction on \leq_S . If s, s' are minimal then since $\text{lbl}(\sigma s) = \text{lbl}(\sigma s')$ they can only differ on their copy indices. By definition of the isomorphism family of $!A^\perp \parallel !B$, the unique bijection between the singletons $\{\sigma s\}$ and $\{\sigma s'\}$ is a valid isomorphism. Since σ is negative, it follows that the unique bijection between $\{s\}$ and $\{s'\}$ is a valid isomorphism by strong-receptivity.

Now suppose that $[s_1] \stackrel{\theta}{\cong}_S [s_2]$, and suppose $s_1 \rightarrow s'_1$ and $s_2 \rightarrow s'_2$, with $\mathcal{P}(s'_1) = \mathcal{P}(s'_2)$. If $\text{pol}(s'_1) = \text{pol}(s'_2) = -$, then the same reasoning as above applies. Suppose now that $\text{pol}(s'_1) = \text{pol}(s'_2) = +$. By induction hypothesis we have $[s_1] \stackrel{\theta}{\cong}_S [s_2]$, and $[s_1] \dashv\vdash s'_1$. By the extension property of isomorphism families, there is s''_2 such that $\theta \dashv\vdash (s'_1, s''_2)$, writing $\theta' = \theta \cup \{(s'_1, s''_2)\}$. By property of isomorphism families θ' is an order-isomorphism, therefore $s_2 \rightarrow s''_2$. By forward sequentiality and determinism of σ , we must then have $s'_2 = s''_2$, so the unique order-isomorphism between $[s'_1]$ and $[s'_2]$ is indeed in the isomorphism family of \mathcal{S} .

(2) \Rightarrow (3). Obvious.

(3) \Rightarrow (1). Immediate by definition of \mathcal{P} . □

From this lemma, it follows that $\mathcal{P}(\sigma)$ is O-branching. Indeed, suppose we have $s_1 \rightarrow s'^+_1, s_2 \rightarrow s'^+_2$ with $\mathcal{P}(s_1) = \mathcal{P}(s_2)$. From the lemma above, the unique order-isomorphism θ satisfies

$$[s_1] \stackrel{\theta}{\cong}_S [s_2]$$

by the extension property of isomorphism families, we must have $\theta \dashv\vdash (s'_1, s''_2)$, so $[s_2] \dashv\vdash s''_2$. By determinism, forward sequentiality and the fact that θ is an order-iso, it follows that $s''_2 = s'_2$. But from that we get by σ the iso:

$$\sigma[s_1] \cup \{\sigma s'_1\} \stackrel{\sigma(\theta_{cup}\{(s'_1, s'_2)\})}{\cong}_{!A^\perp \parallel !B} \sigma[s_2] \cup \{\sigma s'_2\}$$

so by definition of the isomorphism family of $!A^\perp \parallel !B$, we get that $\sigma s'_1$ and $\sigma s'_2$ have the same label and dependency in $!A^\perp \parallel !B$, so $\mathcal{P}(s'_1) = \mathcal{P}(s'_2)$.

So, we have established that $\mathcal{P}(\sigma)$ is a non-empty (by construction) set of even-length P-views, obviously even-prefix-closed. So, it induces an innocent strategy in the usual Hyland-Ong sense, still written $\mathcal{P}(\sigma)$.

Let us note in passing that \mathcal{P} is well-defined on weak equivalence classes of strategies. If $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\sigma' : \mathcal{S}' \rightarrow !B^\perp \parallel !C$ and $\sigma \stackrel{\phi}{\simeq} \sigma'$ is a weak equivalence, then $\mathcal{P}(\sigma) = \mathcal{P}(\sigma')$. Indeed for any $s \in \mathcal{S}$, it is immediate to check that $\mathcal{P}(s) = \mathcal{P}(\phi(s))$. We will show later on that the converse also holds: if $\mathcal{P}(\sigma) = \mathcal{P}(\sigma')$, then $\sigma \simeq \sigma'$.

Functoriality of \mathcal{P} . Now we prove that this gives a functor.

Lemma 4.33. *For any arena A , we have:*

$$\mathcal{P}(\alpha_{!A}) = \text{id}_A$$

Proof. \subseteq . By immediate induction on \leq_A , for any $s \in \mathbb{C}_{!A}$, for any $m' \sqsubseteq^P \mathcal{P}(s)$ we have $m' \upharpoonright A_1 = m' \upharpoonright A_2$ (where $\text{id}_A : A_1 \Rightarrow A_2$ is labeled for convenience).

\supseteq . From m a P-ending P-view in id_A , we have $m \upharpoonright A_1 = m \upharpoonright A_2$. Suppose wlog that the last move of m is in A_1 . In fact (by property of P-views of copycat) $m \upharpoonright A_1$ must be a pointer chain, hence we obtain $\alpha^+ \in !A^\perp$ by adding an arbitrary choice of copy indices. By construction we have $\mathcal{P}((1, \alpha)) = m$. \square

Lemma 4.34. *For all $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\tau : \mathcal{T} \rightarrow !B^\perp \parallel !C$ negative deterministic sequential innocent \sim -strategies, we have:*

$$\mathcal{P}(\tau \circ \sigma) = \mathcal{P}(\tau) \circ \mathcal{P}(\sigma)$$

Proof. Let $P = (S \parallel !C) \otimes (!A \parallel T)$ be the pullback involved in the composition of σ and τ . For $p \in P$, by Lemma 4.27 $[p]$ is a total order:

$$p_0 \rightarrow p_1 \rightarrow \dots \rightarrow p_n = p$$

We define $\mathcal{P}(p)$ similarly to the corresponding definition on strategies, as the sequence

$$(\text{lbl}((\sigma \otimes \tau)p_0), \dots, \text{lbl}((\sigma \otimes \tau)p_n))$$

with the pointers defined by causality in $!A \parallel !B \parallel !C$ for events that are non-minimal in the game, $\text{lbl}((\sigma \otimes \tau)p_0)$ for moves minimal in $!B$. For $\text{lbl}((\sigma \otimes \tau)p_i)$ minimal in $!A$, necessarily $\Pi_1 p_i \in S$ is positive, and hence $[\Pi_1 \pi_i]_S$ has a unique minimal event corresponding to some $\Pi_1 p_j$; then $\text{lbl}((\sigma \otimes \tau)p_i)$ is set to point to $\text{lbl}((\sigma \otimes \tau)p_j)$.

Let P_n be its restriction to events whose causal history has exactly n events. Likewise for $\mathfrak{s} : A \Rightarrow B$ and $\mathfrak{t} : B \Rightarrow C$ two innocent strategies in the Hyland-Ong sense, let $\sigma \parallel_n \tau$ be the similarly restricted interaction, comprising pointing strings u on $(A \Rightarrow B) \Rightarrow C$ of length n , such that $u \upharpoonright A, B$ is either in \mathfrak{s} or an immediate Opponent extension of a play in \mathfrak{s} , $u \upharpoonright B, C$ is either in \mathfrak{t} or an immediate extension of a play in \mathfrak{t} . Additionally, we require that $u \upharpoonright A, C \in \mathcal{L}_{A \Rightarrow C}$ is a P-view, *i.e.* that external Opponent moves always point to the previous move.

We show by induction on n that

$$\mathcal{P}(P_n) = \mathcal{P}(\sigma) \parallel_n \mathcal{P}(\tau)$$

\sqsubseteq . Take $p \in P_{n+1}$. If p is an external Opponent move, suppose wlog that it is in A . Take $p' \rightarrow p$. Necessarily $\mathcal{P}(p')$ terminates with a move in A with Player polarity in $A \Rightarrow B$. Then $\mathcal{P}(p) \upharpoonright A, B$ is an Opponent extension of a play in σ by induction hypothesis and $\mathcal{P}(p) \upharpoonright B, C = \mathcal{P}(p') \upharpoonright B, C \in \tau$. Finally $\mathcal{P}(p) \upharpoonright A, C \in \mathcal{L}_{A \Rightarrow C}$, since it consists in an Opponent extension of P-ending $\mathcal{P}(p') \upharpoonright A, C$ which is in $\mathcal{L}_{A \Rightarrow C}$ by induction hypothesis.

Otherwise $p' \rightarrow p$ is positive, either for σ or τ , suppose wlog that it is for σ . Then either $\mathcal{P}(p) \upharpoonright B, C = \mathcal{P}(p') \upharpoonright B, C \in \mathcal{L}_{B \Rightarrow C}$, or it is an immediate Opponent extension of $\mathcal{P}(p') \upharpoonright B, C \in \mathcal{L}_{B, C}$ as required. If p corresponds to an event in $!B$, then obviously $\mathcal{P}(p) \upharpoonright A, C \in \mathcal{L}_{A \Rightarrow C}$. If p corresponds to an event in $!A$, then if p' corresponds to an event in $!A$, $\mathcal{P}(p) \upharpoonright A, C$ is an extension of $\mathcal{P}(p') \upharpoonright A \Rightarrow C$ preserving alternation, so in $\mathcal{L}_{A \Rightarrow C}$. Finally if p' corresponds to an event in $!B$, we still have $\mathcal{P}(p') \in \mathcal{P}(\sigma) \parallel_n \mathcal{P}(\tau)$ terminating by a move with Opponent polarity in B . By standard reasoning on polarities in interactions, it follows that the last move of $\mathcal{P}(p')$ appearing in A (if any) must have a Player polarity in A . Therefore the extension of $\mathcal{P}(p')$ with $\text{lbl}((\sigma \otimes \tau)p)$ preserves alternation in $A \Rightarrow C$ and $\mathcal{P}(p) \upharpoonright A, C \in \mathcal{L}_{A \Rightarrow C}$. So it remains to show that $\mathcal{P}(p) \upharpoonright A, B \in \mathcal{P}(\sigma)$. For that, we need to show that for all $m \sqsubseteq^P \mathcal{P}(p) \upharpoonright A, B$, $\ulcorner m \urcorner \in \mathcal{P}(\sigma)$. For any strict P-ending prefix, it is true by induction hypothesis, so we must just show that it holds for $m = \mathcal{P}(p) \upharpoonright A, B$. We calculate:

$$\ulcorner \mathcal{P}(p) \upharpoonright A, B \urcorner = \ulcorner \mathcal{P}(p') \upharpoonright A, B \urcorner (\text{lbl}((\sigma \otimes \tau)p))$$

where by induction hypothesis, $\ulcorner \mathcal{P}(p') \upharpoonright A, B \urcorner$ is an Opponent extension of a P-view in $\mathcal{P}(\sigma)$, therefore there is $s'^- \in S$ such that $\ulcorner \mathcal{P}(p') \upharpoonright A, B \urcorner = \mathcal{P}(s')$. From the usual arguments about immediate causality in a composition pullback, we have necessarily $s' \rightarrow \Pi_1 p$. From there it is direct to observe that $\mathcal{P}(\Pi_1 p) = \ulcorner \mathcal{P}(p) \upharpoonright A, B \urcorner$ as required, so $\mathcal{P}(p) \in \mathcal{P}(\sigma) \parallel_{n+1} \mathcal{P}(\tau)$.

\supseteq . Take $um \in \mathcal{P}(\sigma) \parallel_{n+1} \mathcal{P}(\tau)$, by induction hypothesis there if $p \in P_n$ such that $\mathcal{P}(p) = u$. If m is an external O-move, then it is straightforward to extend p accordingly, with the unique corresponding move with copy index 0, as ensured by receptivity. Otherwise m is either a σ -move or a τ -move. Let us assume wlog that it is a σ -move. By hypothesis we have

$$\ulcorner um \upharpoonright A, B \urcorner = \ulcorner u \upharpoonright A, B \urcorner m \in \mathcal{P}(\sigma)$$

therefore there is $s' \in S$ such that $\mathcal{P}(s') = \ulcorner u \upharpoonright A, B \urcorner m$. By backward sequentiality, s' has a unique immediate dependency $s \rightarrow s'$, necessarily with $\mathcal{P}(s) = \ulcorner u \upharpoonright A, B \urcorner$. But then, we observe that $\mathcal{P}(\pi_1 p) = \ulcorner \mathcal{P}(p) \upharpoonright A, B \urcorner$, which follows from an immediate induction on \leq_p . Therefore, $\mathcal{P}(s) = \mathcal{P}(\pi_1 p)$. So by Lemma 4.32, the unique order-isomorphism satisfies:

$$[s] \cong_S^\emptyset [\pi_1 p]$$

So using backward sequentiality and the extension property of isomorphism families, $\pi_1 p \rightarrow s''$. By receptivity, there is $c \in !A \parallel S$ such that $p \rightarrow [((1, s''), c)]_x =$

p' , for some x (that is here irrelevant, since all finite sets of events in the pull-back are consistent, σ and τ being deterministic). By construction, we have as required $\mathcal{P}(p') = um$. \square

So, we have constructed a functor:

$$\mathcal{P} : \text{ChoInn} \rightarrow \text{Inn}$$

Note that \mathcal{P} satisfies the laws for a functor, even though its source is a \sim -bicategory rather than a category. We will now show that this functor is full and faithful.

\mathcal{P} is faithful. First, we prove that it is faithful. Take $\sigma : \mathcal{S} \rightarrow !A^\perp \parallel !B$ and $\sigma' : \mathcal{S}' \rightarrow !A^\perp \parallel !B$ negative, deterministic sequential innocent such that $\mathcal{P}(\sigma) = \mathcal{P}(\sigma')$, we want to establish that $\sigma \simeq \sigma'$.

Lemma 4.35. *There is a isomorphism $\phi : \mathcal{S} \cong \mathcal{S}'$, such that $\sigma' \circ \phi \sim \sigma$. In particular, ϕ is a weak equivalence.*

Proof. In a backward innocent event structure S , we say that $s \in S$ have *depth* n if $[s]$ contains exactly n events. We write \mathcal{S}_n for the projection of \mathcal{S} to events of depth less or equal than n . By induction on n , we define an isomorphism of essps $\phi_n : \mathcal{S}_n \cong \mathcal{S}'_n$ such that $\sigma' \circ \phi_n \sim \sigma$, such that ϕ_n preserves depth and preserves the copy index of (the projection to the game of) events of negative polarity. Moreover, we ask that the family $(\phi_n)_n$ is compatible, meaning that ϕ_{n+1} agrees with ϕ_n on events of depth less than n .

For $n + 1$ odd, the extension of ϕ_n is forced by receptivity and the requirement that ϕ_{n+1} preserves the copy index of events of negative polarity. For $n + 1$ even, pick $s \in S$ of depth $n + 1$. Since $n + 1$ even, by alternation of the game and courtesy of σ , $\text{pol}(s) = +$. Pick its immediate dependency $s_0 \rightarrow s$, and $s'_0 = \phi_n(s_0) \in \mathcal{S}'$. Since $\mathcal{P}(\sigma) = \mathcal{P}(\sigma')$, there is $s'' \in \mathcal{S}'$ such that $\mathcal{P}(s) = \mathcal{P}(s'')$. Take $s''_0 \rightarrow s''$. Necessarily, $\mathcal{P}(s_0) = \mathcal{P}(s''_0)$. Now, note that since $\phi_n(s_0) = s'_0$ we have $\mathcal{P}(s_0) = \mathcal{P}(s'_0)$ as well – it follows from $\sigma' \circ \phi_n \sim \sigma$ and Lemma 4.32. Therefore, $\mathcal{P}(s'_0) = \mathcal{P}(s''_0)$. By Lemma 4.32, the unique order-isomorphism satisfies:

$$[s'_0] \cong_{\mathcal{S}'}^{\theta} [s''_0]$$

But from $s''_0 \rightarrow s''$, we know by the extension property that there is $s'_0 \rightarrow s'$ as well such that $\theta \dashv \text{C}(s', s'')$. We set $\phi_{n+1}(s) = s'$. Note that s' is uniquely determined by forward sequentiality and determinism, from that it is easy to deduce that ϕ_{n+1} so defined yields an order-isomorphism, and by construction it satisfies the requirements.

Forming $\phi = \bigsqcup_{n \in \omega} \phi_n$ we get the required weak equivalence. The fact that ϕ preserves symmetry is an obvious consequence of Lemma 4.32. \square

Note that one consequence of that is that for ChoInn-strategies, if σ and σ' are weakly equivalent then the weak equivalence can actually be assumed to be an isomorphism between the corresponding essps \mathcal{S} and \mathcal{S}' .

\mathcal{P} is full. Finally, it remains to show that \mathcal{P} is full. Take $\mathfrak{s} : A \Rightarrow B$ a Hyland-Ong innocent strategy. Write $PV(\mathfrak{s})$ for the set of P-views of \mathfrak{s} , possibly extended by a ending Opponent move pointing to the latest move. For $m \in PV(\mathfrak{s})$, write $O(m)$ for the set of O-moves in m . Define an essp \mathcal{S} having:

- Events, pairs (m, α) where $\alpha : O(m) \rightarrow \omega$,
- Order, $(m, \alpha) \leq (m', \alpha')$ iff $m \sqsubseteq m'$ and α, α' compatible on their common domain.
- Consistency, every finite set.
- Symmetry, the unique order-isomorphism between $[(m, \alpha)]$ and $[(m', \alpha')]$ when $m = m'$.

By induction on n , we now define:

$$\sigma_n : \mathcal{S}_n \rightarrow !A^\perp \parallel !B$$

Take $(m, \alpha) \rightarrow (m', \alpha')$, and assume that $m' = mb$ with b in B , the other case is similar. If m' is O-ending, then we set, with $\sigma_n((m, \alpha)) = (2, \beta)$:

$$\sigma_{n+1}((m', \alpha')) = (2, \beta \cup \{b \mapsto \alpha'(b)\})$$

so the copy index of Opponent is set in the game as it is in the strategy. On the other hand if m' is P-ending (and so $\alpha = \alpha'$), we set:

$$\sigma_{n+1}((m', \alpha)) = (2, \beta \cup \{b \mapsto \langle \sharp m, \sharp \alpha \rangle\})$$

where $\sharp m, \sharp \alpha$ denote natural numbers representing uniquely m and α , and $\langle -, - \rangle : \omega^2 \rightarrow \omega$ is an arbitrary injection.

It is direct to check that $\sigma = \bigsqcup_{n \in \omega} \sigma_n$ defines a \sim -strategy, and by construction we have $\mathcal{P}(\sigma) = \mathfrak{s}$ as needed – so \mathcal{P} is full.

References

- [CCW14] Simon Castellan, Pierre Clairambault, and Glynn Winskel. Symmetry in concurrent games. In *CSL-LICS*, 2014.
- [HO00] J. M. E. Hyland and C.-H. Luke Ong. On full abstraction for PCF: I, II, and III. *Inf. Comput.*, 163(2):285–408, 2000.
- [MM07] Paul-André Melliès and Samuel Mimram. Asynchronous games: Innocence without alternation. In Luís Caires and Vasco Thudichum Vasconcelos, editors, *CONCUR*, volume 4703 of *LNCS*, pages 395–411. Springer, 2007.
- [RW11] Silvain Rideau and Glynn Winskel. Concurrent strategies. In *LICS*, pages 409–418. IEEE Computer Society, 2011.

- [Win07] Glynn Winskel. Event structures with symmetry. *Electr. Notes Theor. Comput. Sci.*, 172:611–652, 2007.
- [Win11] Glynn Winskel. Event structures, stable families and games. *Lecture notes, Aarhus University*, At <http://daimi.au.dk/~gwinskel>, 2011.
- [Win12] Glynn Winskel. Deterministic concurrent strategies. *Formal Asp. Comput.*, 24(4-6):647–660, 2012.