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Moment estimators of the extreme value index for randomly censored data in the Weibull domain of attraction

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Abstract This paper addresses the problem of estimating the extreme value index in presence of random censoring for distributions in the Weibull domain of attraction. The methodologies introduced in [Worms 2014], in the heavy-tailed case, are adapted here to the negative extreme value index framework, leading to weighted versions of the popular moments of relative excesses. This leads to the definition of two families of estimators (with an adaptation of the so called Moment estimator as a particular case), for which the consistency is proved under a first order condition. Illustration of their performance, coming from an extensive simulation study, are provided.

Keywords Extreme value index · Tail inference · Random censoring · Kaplan-Meier integration

Mathematics Subject Classification (2000) 62G32 (Extreme value statistics) · 62N02 (Estimation for censored data)

1 Introduction

Extreme value statistics is an active domain of research, with numerous fields of application, and which benefits from an important literature in the context of i.i.d. data, dependent data, and (more recently) multivariate or spatial data. By contrast, methodological articles in the case of randomly censored data are quite recent and few : [Einmahl et al. (2008)] presents a general method for adapting estimators of the extreme value index in a censorship framework (a

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methodology based on a previous work [Beirlant et al. (2007)], [Worms 2014] proposes a more survival analysis-oriented approach restricted to the heavy tail case, and [Diop et al. (2014)] extends the framework to data with covariate information. Other existing works on the topic of extremes for censored data are [Brahimi et al. (2013)] and the review paper [Gomes and Neves (2011)].

In this paper, the topic of extreme value statistics for randomly censored data with negative extreme value index is addressed. Our purpose is to extend the ideas of [Worms 2014] (in which the Hill estimator was adapted to the censoring framework) in order to propose weighted versions of the popular moments of the relative excesses, and therefore define competitive estimators of the extreme value index in this censoring situation, for distributions in the Weibull maximum domain of attraction. Let us first define more precisely the framework, the data, and the notations.

In the classical univariate framework of i.i.d. data, a central task is to estimate the extreme value index γ , which captures the main information about the behavior of the tail distribution of the data. More precisely, a distribution function (d.f.) F is said to be in the maximum domain of attraction of H_γ (noted $F \in D(H_\gamma)$) with

$$H_\gamma(x) := \begin{cases} \exp(-(1+\gamma x)^{-1/\gamma}) & \text{for } \gamma \neq 0 \text{ and } 1+\gamma x > 0 \\ \exp(-\exp(-x)) & \text{for } \gamma = 0 \text{ and } x \in \mathbb{R}, \end{cases}$$

if there exist two normalizing sequences $(a_n) \subset \mathbb{R}^+$ and $(b_n) \subset \mathbb{R}$ such that

$$F^n(a_n x + b_n) \xrightarrow{n \rightarrow \infty} H_\gamma(x) \quad (\forall x \in \mathbb{R}).$$

We consider in this paper two independent i.i.d. non-negative samples $(X_i)_{i \leq n}$ and $(C_i)_{i \leq n}$ with respective continuous distribution functions F and G (with end-points τ_F and τ_G , where $\tau_F := \sup\{x, F(x) < 1\}$). In the context of randomly right-censored observations, one only observes, for $1 \leq i \leq n$,

$$Z_i = X_i \wedge C_i \quad \text{and} \quad \delta_i = \mathbb{I}_{X_i \leq C_i}.$$

We denote by H the distribution function of the Z -sample, satisfying

$$1 - H = (1 - F)(1 - G)$$

and by $Z_{1,n} \leq \dots \leq Z_{n,n}$ the associated order statistics. In the whole paper, $\delta_{1,n}, \dots, \delta_{n,n}$ denote the δ 's corresponding to $Z_{1,n}, \dots, Z_{n,n}$, respectively. F and G are assumed to be in the maximum domains of attraction $D(H_{\gamma_X})$ and $D(H_{\gamma_C})$ respectively, where γ_X and γ_C are real numbers, which implies that $H \in D(H_\gamma)$, for some $\gamma \in \mathbb{R}$.

Our goal is to estimate the extreme value index γ_X in this context of right censoring. The most interesting cases, described in [Einmahl et al. (2008)], are

the following :

$$\begin{aligned}
\text{case 1:} \quad & \gamma_X > 0, \gamma_C > 0 & \text{in this case } \gamma &= \frac{\gamma_X \gamma_C}{\gamma_X + \gamma_C} \\
\text{case 2:} \quad & \gamma_X < 0, \gamma_C < 0, \tau_F = \tau_G & \text{in this case } \gamma &= \frac{\gamma_X \gamma_C}{\gamma_X + \gamma_C} \\
\text{case 3:} \quad & \gamma_X = \gamma_C = 0, \tau_F = \tau_G = +\infty & \text{in this case } \gamma &= 0.
\end{aligned}$$

In [Worms 2014], the heavy-tail case 1 above was considered and an adaptation of the so-called Hill estimator to the right censoring framework was proposed. In this paper, our aim is to consider case 2 above and adapt the approach leading to the so-called Moment Estimator to this censored situation. An adaptation of this estimator was already proposed in [Einmahl et al. (2008)] which consists in dividing the classical Moment Estimator $\hat{\gamma}_n^Z$ of γ (calculated from the Z -sample) by the ultimate proportion

$$\hat{p} := k_n^{-1} \sum_{i=1}^{k_n} \delta_{n-i+1,n}$$

of uncensored data, where k_n is the number of upper order statistics retained. Note that $\hat{\gamma}_n^Z$ is an appropriate combination of the following moments :

$$\mathcal{M}_{n,k_n}^{(\alpha)} := \frac{1}{k_n} \sum_{i=1}^{k_n} \log^\alpha \left(\frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right),$$

for $\alpha = 1$ or 2 (where $\log^\alpha(x)$ stands for $(\log(x))^\alpha$).

Our goal is to show that relying on usual strategies in the survival analysis literature leads to estimators of γ_X which are often sharper than those obtained by simply dividing an estimator of γ by the proportion of uncensored observations. By “usual” strategy we mean using “Kaplan-Meier”-like random weights (we refer to [Worms 2014] for more detailed motivations for the random weights appearing in the formulas below). We define, for any given $\alpha \geq 1$, the following two versions of randomly weighted moments of the log relative excesses :

$$M_{n,k_n}^{(\alpha)} := \frac{1}{n(1 - \hat{F}_n(Z_{n-k_n,n}))} \sum_{i=1}^{k_n} \frac{\delta_{n-i+1,n}}{1 - \hat{G}_n(Z_{n-i+1,n}^-)} \left(\log^\alpha \left(\frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right) \right) \quad (1)$$

and

$$\widetilde{M}_{n,k_n}^{(\alpha)} := \frac{1}{n(1 - \hat{F}_n(Z_{n-k_n,n}))} \sum_{i=1}^{k_n} \frac{1}{1 - \hat{G}_n(Z_{n-i+1,n}^-)} \xi_{i,n} \quad (2)$$

where

$$\xi_{i,n} := i \left(\log^\alpha \left(\frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right) - \log^\alpha \left(\frac{Z_{n-i,n}}{Z_{n-k_n,n}} \right) \right) \quad (3)$$

and (k_n) is a sequence of integers satisfying, as n tends to $+\infty$,

$$k_n \rightarrow +\infty \quad \text{and} \quad k_n = o(n). \quad (4)$$

Above, \hat{F}_n and \hat{G}_n naturally denote the Kaplan-Meier estimators of F and G , respectively, defined as follows : for $t < Z_{n,n}$,

$$1 - \hat{F}_n(t) = \prod_{Z_{i,n} \leq t} \left(\frac{n-i}{n-i+1} \right)^{\delta_{i,n}} \quad \text{and} \quad 1 - \hat{G}_n(t) = \prod_{Z_{i,n} \leq t} \left(\frac{n-i}{n-i+1} \right)^{1-\delta_{i,n}}.$$

In section 2 below, assumptions are presented and discussed, convergence results for the weighted moments $M_{n,k_n}^{(\alpha)}$ and $\widetilde{M}_{n,k_n}^{(\alpha)}$ are stated, and we describe how classes of estimators of γ_X can be deduced by combining these moments for different values of α . In Section 3, performance of these estimators will be presented on the basis of simulations. Section 4 provides some words of conclusion, Section 5 is devoted to the proofs of Theorems 1 and 2 below, and finally the Appendix includes standard (but central to our proofs) results on regularly varying functions, as well as the proofs of the different lemmas which were useful in Section 5.

2 Results

2.1 Assumptions

In addition to (4), our results need the following minimal assumption :

$$(A) \quad F \in D(H_{\gamma_X}), \quad G \in D(H_{\gamma_C}) \quad \text{with} \quad \gamma_X < 0, \gamma_C < 0 \quad \text{and} \quad x^* := \tau_F = \tau_G.$$

As noted earlier, this assumption implies that $H \in D(H_\gamma)$ with $\tau_H = x^*$ and

$$\gamma = \frac{\gamma_X \gamma_C}{\gamma_X + \gamma_C} < 0.$$

If we note $U(t) = H^{\leftarrow}(1 - 1/t)$ the quantile function associated to H , then $x^* = U(\infty)$ and $H \in D(H_\gamma)$ is equivalent to the existence of some positive function a such that

$$\lim_{t \rightarrow +\infty} \frac{\log U(tx) - \log U(t)}{a(t)/U(t)} = \frac{x^\gamma - 1}{\gamma}, \quad \forall x > 0, \quad (5)$$

which, since $\gamma < 0$, is itself equivalent to

$$\lim_{t \rightarrow +\infty} \frac{U(\infty) - U(tx)}{U(\infty) - U(t)} = x^\gamma, \quad \forall x > 0. \quad (6)$$

This means that the function $U(\infty) - U$ is regularly varying (at $+\infty$) with index γ (see the appendix for the definition of regular variation at $+\infty$). A reference for the equivalence of conditions (5) and (6) to (A) is [Haan and Ferreira (2006)] (respectively relation (3.5.4) and Corollary 1.2.10).

In order to obtain the asymptotic behavior of $M_{n,k_n}^{(\alpha)}$, we may need two additional regularity assumptions on the function $p(z) := \mathbb{P}(\delta = 1 | Z = z)$,

already used in [Worms 2014] : denoting by $p := \lim_{z \rightarrow \infty} p(z)$, which equals $\gamma_C/(\gamma_X + \gamma_C)$ in our context, we will suppose that

$$\frac{1}{k_n} \sum_{i=1}^{k_n} \left| p \left(H^{\leftarrow} \left(1 - \frac{i}{n} \right) \right) - p \right| \xrightarrow{\mathbb{P}} c \in \mathbb{R} \quad (7)$$

$$\sup_{(s,t) \in C_n} |p(H^{\leftarrow}(t)) - p(H^{\leftarrow}(s))| \rightarrow 0, \text{ for all } C > 0 \quad (8)$$

where $C_n = \{(s, t) \text{ such that } s < 1, 1 - k_n/n \leq t < 1, |t - s| \leq C\sqrt{k_n}/n\}$. Note that these two assumptions are not needed for the asymptotic behavior of $\widetilde{M}_{n,k_n}^{(\alpha)}$.

Finally, we will need some very mild additional assumption on (k_n)

(K) there exists some $\delta > 0$, or some $\delta \geq \frac{\gamma_X - \gamma_C}{\gamma_X + \gamma_C}$ if $\gamma_C \geq \gamma_X$, such that

$$-\log(k_n/n)/k_n = O(n^{-\delta}). \quad (9)$$

2.2 Asymptotic results

Let us introduce the notation $a_{n,k} := a(n/k_n)/U(n/k_n)$ (see the previous section for the definition of functions U and a), where $a_{n,k} \rightarrow 0$ (cf equation (3.5.5) in [Haan and Ferreira (2006)]). In the paper, $Beta(\cdot, \cdot)$ denotes the usual Beta function, $Beta(a, b) = \int_0^1 t^{a-1}(1-t)^{b-1} dt$ ($a > 0, b > 0$).

Theorem 1 *Under assumption (A) and conditions (4), (7), (8) and (K), for any $\alpha \geq 1$ we have, as n tends to ∞ ,*

$$\frac{M_{n,k_n}^{(\alpha)}}{(a_{n,k})^\alpha} \xrightarrow{\mathbb{P}} |\gamma_X|^{-1} |\gamma|^{-\alpha} Beta(|\gamma_X|^{-1}; \alpha + 1).$$

Theorem 2 *Under assumption (A) and conditions (4) and (K), for any $\alpha \geq 1$ we have, as n tends to ∞ ,*

$$\frac{\widetilde{M}_{n,k_n}^{(\alpha)}}{(a_{n,k})^\alpha} \xrightarrow{\mathbb{P}} |\gamma_X|^{-1} |\gamma|^{-\alpha} Beta(|\gamma_X|^{-1}; \alpha + 1).$$

The following corollary states the consistency of our two different adaptations of the Moment estimator to this censored framework.

Corollary 1 *Under conditions of Theorem 1 or 2 respectively, as $n \rightarrow \infty$,*

$$\hat{\gamma}_{n,Mom} := M_{n,k_n}^{(1)} + 1 - \frac{1}{2} \left(1 - \frac{(M_{n,k_n}^{(1)})^2}{M_{n,k_n}^{(2)}} \right)^{-1} \xrightarrow{\mathbb{P}} \gamma_X$$

and

$$\tilde{\gamma}_{n,Mom} := \tilde{M}_{n,k_n}^{(1)} + 1 - \frac{1}{2} \left(1 - \frac{(\tilde{M}_{n,k_n}^{(1)})^2}{\tilde{M}_{n,k_n}^{(2)}} \right)^{-1} \xrightarrow{\mathbb{P}} \gamma_X.$$

In fact, by using the elementary properties of the Beta function, the weighted moments $M_n^{(\alpha)}$ or $\tilde{M}_n^{(\alpha)}$ can be combined in different ways, leading to the definition of two different classes of consistent estimators of γ_X , parametrized by $\alpha \geq 1$ (proofs of the 3 corollaries are easy and omitted). In the next section, we study their finite sample performance.

Corollary 2 *Under conditions of Theorem 1 or 2 respectively, as $n \rightarrow \infty$,*

$$\hat{\gamma}_{n,1}^{(\alpha)} := (V_{n,\alpha}^{-1} + \alpha + 1)^{-1} \xrightarrow{\mathbb{P}} \gamma_X$$

and

$$\tilde{\gamma}_{n,1}^{(\alpha)} := (\tilde{V}_{n,\alpha}^{-1} + \alpha + 1)^{-1} \xrightarrow{\mathbb{P}} \gamma_X$$

where

$$V_{n,\alpha} := 1 - \frac{\alpha + 2}{\alpha + 1} \frac{(M_n^{(\alpha+1)})^2}{M_n^{(\alpha)} M_n^{(\alpha+2)}} \quad \text{and} \quad \tilde{V}_{n,\alpha} := 1 - \frac{\alpha + 2}{\alpha + 1} \frac{(\tilde{M}_n^{(\alpha+1)})^2}{\tilde{M}_n^{(\alpha)} \tilde{M}_n^{(\alpha+2)}}.$$

Corollary 3 *Under conditions of Theorem 1 or 2 respectively, as $n \rightarrow \infty$,*

$$\hat{\gamma}_{n,2}^{(\alpha)} := \frac{1 - (\alpha + 1)R_{n,\alpha}}{(\alpha + 1)(1 - R_{n,\alpha})} \xrightarrow{\mathbb{P}} \gamma_X$$

and

$$\tilde{\gamma}_{n,2}^{(\alpha)} := \frac{1 - (\alpha + 1)\tilde{R}_{n,\alpha}}{(\alpha + 1)(1 - \tilde{R}_{n,\alpha})} \xrightarrow{\mathbb{P}} \gamma_X,$$

where

$$R_{n,\alpha} := \frac{M_n^{(1)} M_n^{(\alpha)}}{M_n^{(\alpha+1)}} \quad \text{and} \quad \tilde{R}_{n,\alpha} := \frac{\tilde{M}_n^{(1)} \tilde{M}_n^{(\alpha)}}{\tilde{M}_n^{(\alpha+1)}}.$$

Remark 1 *It is straightforward to see that $\hat{\gamma}_{n,2}^{(\alpha)}$ with $\alpha = 1$ equals $1 - \frac{1}{2}(1 - R_{n,1})^{-1}$, which is very close to $\hat{\gamma}_{n,Mom}$, since $M_{n,k_n}^{(1)} \rightarrow 0$ in our finite endpoint framework.*

Remark 2 *If $\mathcal{M}_{n,k_n}^{(\alpha)}$ denotes the unmodified moments defined in the introduction, it can be proved that under (A) and (4), for $\alpha \geq 1$,*

$$\frac{\mathcal{M}_{n,k_n}^{(\alpha)}}{(a_{n,k})^\alpha} \xrightarrow{\mathbb{P}} |\gamma|^{-\alpha-1} \text{Beta}(|\gamma|^{-1}; \alpha + 1).$$

Therefore, it is easy to check that combining those moments as described in Corollaries 2 and 3 leads to consistent estimators of γ , and thus dividing the

latter by \hat{p} (defined in the introduction) leads to 2 classes of consistent estimators $\check{\gamma}_{n,1}^{(\alpha)}$ and $\check{\gamma}_{n,2}^{(\alpha)}$ of γ_X . We also define $\check{\gamma}_{n,Mom}$ as the estimator of γ_X obtained by dividing the classical Moment estimator of γ by the proportion \hat{p} . A finite-sample comparison of those estimators with our new competitors is presented in the following section.

3 Finite sample behavior

The goal of this Section is to present our results concerning the finite sample performances of our new estimators of the extreme value index in presence of random censoring, presented in Corollaries 1, 2 and 3. In each case considered, 2000 random samples of size $n = 500$ were generated, and the median bias and mean squared error (MSE) of the different estimators of γ were plotted against the number k_n of excesses used.

A great variety of situations can be (and has been) considered in our simulation study : various values of γ_X and γ_C (and therefore various censoring rates in the tail), various families of underlying distributions (Reverse Burr, generalized Pareto, Beta), and choice of the value of α . It is impossible to illustrate here the different possible combinations of these features : we will therefore try to draw some general conclusions from the many different situations we have observed, and provide a partial illustration with 3 particular cases.

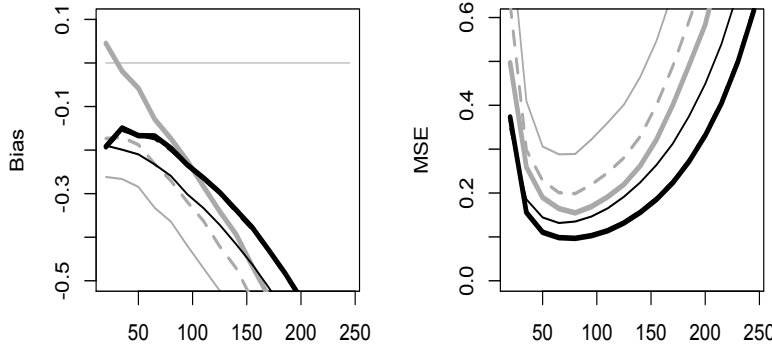


Fig. 1 Comparison between $\hat{\gamma}_{n,1}^{(2)}$ (thick black), $\check{\gamma}_{n,1}^{(2)}$ (dashed black), $\check{\gamma}_{n,1}^{(2)}$ (thin black), $\hat{\gamma}_{n,Mom}$ (thick grey), $\check{\gamma}_{n,Mom}$ (dashed grey) and $\check{\gamma}_{n,Mom}$ (thin grey) for a RevBurr(1, 1, 1, 10) censored by a RevBurr(10, 2/3, 1, 10) ($\gamma_X = -1 < \gamma_C = -3/2$, weak censoring)

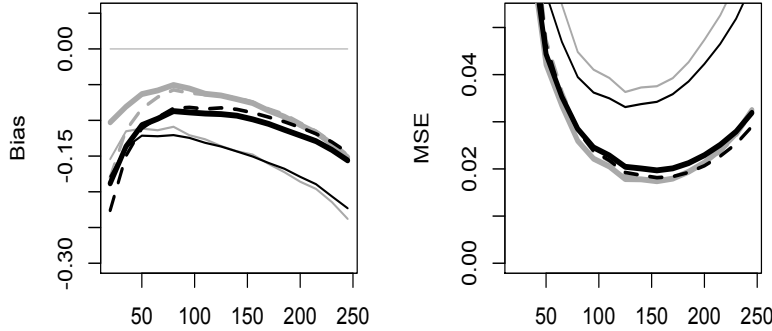


Fig. 2 Comparison between $\hat{\gamma}_{n,2}^{(2)}$ (thick black), $\tilde{\gamma}_{n,2}^{(2)}$ (dashed black), $\check{\gamma}_{n,2}^{(2)}$ (thin black), $\hat{\gamma}_{n,Mom}$ (thick grey), $\tilde{\gamma}_{n,Mom}$ (dashed grey) and $\check{\gamma}_{n,Mom}$ (thin grey) for a $\text{RevBurr}(1, 8, 1/2, 10)$ censored by a $\text{RevBurr}(10, 4, 1/2, 10)$ ($\gamma_X = -1/4 < \gamma_C = -1/2$, weak censoring)

Concerning the choice of the tuning parameter α , we did not find a value which seemed preferable in every situation : nonetheless, in general, for small values of k_n , a value of α around 1 or 2 yields better MSE, whereas for high values of k_n , the MSE is lower for values of α greater than 2. We decided not to include this preliminary study in this article, and chose (almost arbitrarily) the value $\alpha = 2$ in all our subsequent simulations.

Let us now settle the vocabulary used in this section. We will call *Moment estimators* the estimators $\hat{\gamma}_{n,Mom}$ and $\tilde{\gamma}_{n,Mom}$ appearing in Corollary 1, as well as the estimator $\check{\gamma}_{n,Mom}$ introduced in Remark 2 above. We will call *type 1 (resp. type 2) estimators* the estimators $\hat{\gamma}_{n,1}^{(\alpha)}$ and $\tilde{\gamma}_{n,1}^{(\alpha)}$ (resp. $\hat{\gamma}_{n,2}^{(\alpha)}$ and $\tilde{\gamma}_{n,2}^{(\alpha)}$) appearing in Corollary 2 (resp. 3), as well as the estimator $\check{\gamma}_{n,1}^{(\alpha)}$ (resp. $\check{\gamma}_{n,2}^{(\alpha)}$) introduced in Remark 2.

We will also consider names for the different methods : the KM method (for Kaplan-Meier-like weights, appearing in the definition of $M_{n,k_n}^{(\alpha)}$), leading to $\hat{\gamma}$ estimators, the L method (for Leurgans-like weights) leading to $\tilde{\gamma}$ estimators (the name comes from the mathematician Sue Leurgans who inspired the weights, see [Worms 2014] for details and a reference), and the EFG method (for constant weighting by \hat{p}), leading to $\check{\gamma}$ estimators (the names comes from the initials of the authors of [Einmahl et al. (2008)]).

There are two main questions addressed in this empirical study : is one of the 3 methods preferable to the others (and in which conditions) and is there a better choice for the type of estimator (type 1 , type 2, or classical Moment

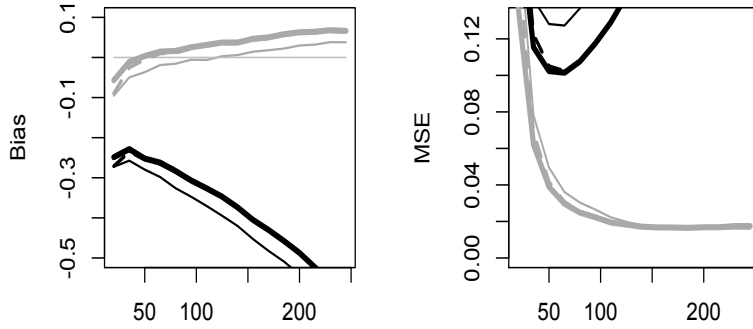


Fig. 3 Comparison between $\hat{\gamma}_{n,2}^{(2)}$ (thick black), $\tilde{\gamma}_{n,2}^{(2)}$ (dashed black), $\check{\gamma}_{n,2}^{(2)}$ (thin black), $\hat{\gamma}_{n,Mom}$ (thick grey), $\tilde{\gamma}_{n,Mom}$ (dashed grey) and $\check{\gamma}_{n,Mom}$ (thin grey) for a Beta(1,4) censored by a RevBurr(10,2,1,1) ($\gamma_X = -1/4 < \gamma_C = -1/2$, weak censoring)

estimator) ? Unsurprisingly, after our intensive simulation study, we may say that the answer is no for the 2 questions, if an overall superiority is looked for. However, we can make some partial comments concerning the choice of the method and of the estimator type, whether the censoring is strong or weak, or the value of $|\gamma|$ is small or not.

Note first that, if the censoring rate in the tail is very low (say $1 - p$ lower than 10%), we observed that there was not much difference between the 3 methods (KM, L, EFG), and that it was just a question of choosing between type 1, type 2, and moment estimator. This is why, in the following, we only consider cases where the censoring rate is larger than $1/4$, and talk about strong censoring in the tail when this rate is greater than $1/2$ (*i.e.* $\gamma_X \leq \gamma_C$), and weak censoring otherwise (when $\gamma_X > \gamma_C$).

For “high” values of γ_X , *i.e.* lower than $-1/2$, we have most of the time observed better performance of the KM and L methods with respect to the EFG method, in strong or weak censoring frameworks. In this context, the type 1 estimators are generally preferable to the type 2 estimators, and comparable or preferable to the moment estimator.

For values of γ_X between $-1/2$ and 0 (sometimes called the “regular” case, and which is the most frequently encountered in practice), there exists a great variety of situations. We observed that the moment estimators were generally better than the type 2 estimators, which were themselves generally better than the type 1 ones. Concerning the choice of the method, for the moment estimator, it seems difficult to suggest a particular one, between the KM, L,

and EFG methods (even though in many cases, at least one among the KM and L methods was better than the EFG method). Concerning the inferiority of types 1 and 2 versus the moment estimator, it should be noted that it is mainly due to the bias, which contributes the most to the MSE (in fact, we clearly noticed that the variances of the types 1 and 2, for $\alpha = 2$, are almost always lower than the variance of the moment estimator).

Let us now present the 3 particular situations we chose as illustrations of the comments above. They involve 2 classes of distributions : the Reverse Burr family $RevBurr(\beta, \tau, \lambda, x^*)$ (with $\beta, \tau, \lambda > 0$) with survival function

$$\mathbb{P}(X > x) = (1 + \beta^{-1}(x^* - x)^{-\tau})^{-\lambda},$$

and which extreme value index is $-1/(\lambda\tau)$, and the standard Beta distribution family $B(a, b)$ ($a, b > 0$) with density function $(Beta(a, b))^{-1}x^{a-1}(1-x)^{b-1}$ on $[0, 1]$, which endpoint is 1 and extreme value index is $-1/b$.

In Figure 1, the value of γ_X is lower than $-1/2$, and therefore, as motivated above, for readability purposes we only kept the type 1 estimators on the graph, whereas for the other two figures, the value of γ_X is between $-1/2$ and 0 and we therefore only kept the type 2 estimator illustrated. Remind here that these 3 examples are only 3 particular cases of the numerous combinations of features we have considered in our simulation study. For instance, the KM and L method yield very similar results in the 3 graphics, but this is not always the case.

4 Conclusion

In this paper, we applied the methodology introduced in [Worms 2014] to construct new estimators of the extreme value index for randomly-censored data with distributions in the Weibull domain of attraction. We proposed, in particular, a new adaptation of the famous Moment estimator. Our intensive simulation study shows that the proposed estimators are competitive even if, in many cases, the bias would need to be reduced. A future possible work would be to exploit our weighting methodology in order to estimate other parameters of the tail (for reducing the bias, for example) as well as extreme quantiles. The asymptotic normality remains a question to be addressed (difficulties come from the control of the Kaplan-Meier estimates in the tail).

5 Proofs

5.1 Proof of Theorem 1

The proof of Theorem 1 has structural similarities with the proof of Theorem 1 in [Worms 2014]. We shall refer to the latter when necessary. We first need to state the following technical Lemmas.

Lemma 1 *Let (A) hold and $a \geq 1$ and $b < 0$ be given. With all the notations previously introduced, for $1 \leq i \leq k_n$, set*

$$Q_{i,n}^{a,b} = \frac{1}{(a_{n,k})^a} \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^a \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^b.$$

If ϵ' and ϵ'' are arbitrary two positive real numbers, then there exists some deterministic sequence (c_n) tending to 1 and some $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, we have for every $1 \leq i \leq k_n$,

$$c_n B_n^- \leq Q_{i,n}^{a,b} \leq c_n B_n^+$$

where B_n^+ and B_n^- are positive random variables satisfying

$$\begin{aligned} B_n^- &\stackrel{d}{=} (1 + \epsilon'')^b \left(\frac{\tilde{Y}_{k_n-i+1,k_n}^\gamma - 1}{\gamma} \right)^a \tilde{Y}_{k_n-i+1,k_n}^{b(\gamma+\epsilon'')} + \epsilon' R_{i,n}^- \\ B_n^+ &\stackrel{d}{=} (1 - \epsilon'')^b \left(\frac{\tilde{Y}_{k_n-i+1,k_n}^\gamma - 1}{\gamma} \right)^a \tilde{Y}_{k_n-i+1,k_n}^{b(\gamma-\epsilon'')} + \epsilon' R_{i,n}^+ \end{aligned}$$

with $(\tilde{Y}_i)_{1 \leq i \leq k_n}$ denoting an i.i.d. sample of standard Pareto random variables, and, for every $1 \leq i \leq k_n$ and some constant $c > 0$,

$$\max\{|R_{i,n}^-|, |R_{i,n}^+|\} \leq c \tilde{Y}_{k_n-i+1,k_n}^{\gamma+\epsilon'+b(\gamma-\epsilon'')}.$$

Lemma 2 *If Y is a standard Pareto random variable, then for every $a > -1$ and $a' < 1$*

$$\mathbb{E} \left(\left(\frac{Y^\gamma - 1}{\gamma} \right)^a Y^{a'} \right) = |\gamma|^{-a-1} \text{Beta} \left(\frac{1-a'}{|\gamma|} ; a+1 \right).$$

We now proceed to the proof of Theorem 1. We have the decomposition

$$M_{n,k_n}^{(\alpha)} = A_n (\overline{W}_n + R_n),$$

where

$$\begin{aligned} A_n &:= \frac{1 - \hat{G}_n(Z_{n-k_n,n})}{1 - G(Z_{n-k_n,n})} & ; & \quad \overline{W}_n := \frac{1}{k_n} \sum_{i=1}^{k_n} W_{in} \\ R_n &:= \frac{1}{k_n} \sum_{i=1}^{k_n} (C_{in} - 1) W_{in} & ; & \quad C_{in} := \frac{1 - G(Z_{n-i+1,n})}{1 - \hat{G}_n(Z_{n-i+1,n})} \end{aligned}$$

and

$$W_{in} := \delta_{n-i+1,n} \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \frac{1 - G(Z_{n-k_n,n})}{1 - G(Z_{n-i+1,n})}.$$

Since $A_n \xrightarrow{\mathbb{P}} 1$ as $n \rightarrow \infty$ (see Theorem 2 in [Csörgő (1996)]), we need to prove that $R_n = o_{\mathbb{P}}(a_{n,k}^\alpha)$ and, using the relation $\gamma/p = \gamma_X$, that

$$\frac{\overline{W}_n}{a_{n,k}^\alpha} \xrightarrow{\mathbb{P}} l_\alpha := p|\gamma|^{-\alpha-1} \text{Beta}(p/|\gamma| ; \alpha+1). \quad (10)$$

5.1.1 Proof of (10)

Since $G \in D(H_{\gamma_C})$, with $\gamma_C < 0$, is equivalent to $t \rightarrow 1 - G(x^* - t)$ being regularly varying at 0 with index $-1/\gamma_C$ (see the appendix for the definition of regular variation at 0), the bounds (15) in Corollary 4 (in the appendix) applied to $f = G$, $x = (x^* - Z_{n-i+1,n})/(x^* - Z_{n-k_n,n})$ and $t = x^* - Z_{n-k_n,n}$, yield, for $\epsilon > 0$, n sufficiently large and every $1 \leq i \leq k_n$,

$$(1 - \epsilon)W_{i,n}^- \leq W_{i,n} \leq (1 + \epsilon)W_{i,n}^+,$$

where

$$W_{i,n}^+ := \delta_{n-i+1,n} \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^{\gamma_C^{-1} - \epsilon},$$

$$W_{i,n}^- := \delta_{n-i+1,n} \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^{\gamma_C^{-1} + \epsilon}.$$

We thus need to prove that, for every given $\epsilon > 0$, the sequences $\frac{1}{k_n} \sum_{i=1}^{k_n} W_{i,n}^+ / a_{n,k}^\alpha$ and $\frac{1}{k_n} \sum_{i=1}^{k_n} W_{i,n}^- / a_{n,k}^\alpha$ converge in probability to some deterministic limit $l_{\alpha,\epsilon}$, which itself tends to l_α as ϵ goes to 0. Since the proof for $W_{i,n}^-$ is similar, we will only treat the case of $W_{i,n}^+$.

Let us introduce, as in the proof of Theorem 1 in [Worms 2014], a sequence $(\tilde{\delta}_i)$ of i.i.d. Bernoulli(p) random variables independent of the sequence (Z_i) and write $\frac{1}{k_n} \sum_{i=1}^{k_n} W_{i,n}^+ = J_n^1 + J_n^2$, where

$$J_n^1 := \frac{1}{k_n} \sum_{i=1}^{k_n} \tilde{\delta}_{n-i+1,n} \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^{\gamma_C^{-1} - \epsilon},$$

$$J_n^2 := \frac{1}{k_n} \sum_{i=1}^{k_n} (\delta_{n-i+1,n} - \tilde{\delta}_{n-i+1,n}) \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^{\gamma_C^{-1} - \epsilon}.$$

We shall treat J_n^1 and J_n^2 separately.

- Lemma 1, applied with $a = \alpha \geq 1$ and $b = \gamma_C^{-1} - \epsilon < 0$, yields, by the independence of the sequences (Z_i) and $(\tilde{\delta}_i)$, for $\epsilon, \epsilon', \epsilon''$ positive real numbers and n sufficiently large,

$$c_n (1 + \epsilon'')^{\gamma_C^{-1} - \epsilon} \xi_n^- \leq \frac{J_n^1}{a_{n,k}^\alpha} \leq c_n (1 - \epsilon'')^{\gamma_C^{-1} - \epsilon} \xi_n^+ \quad (11)$$

where

$$\xi_n^+ := \frac{1}{k_n} \sum_{i=1}^{k_n} \tilde{\delta}_i \left(\frac{\tilde{Y}_i^{\gamma-1}}{\gamma} \right)^\alpha \tilde{Y}_i^{(\gamma_C^{-1} - \epsilon)(\gamma - \epsilon'')} + \epsilon' O_{\mathbb{P}}(1),$$

$$\xi_n^- := \frac{1}{k_n} \sum_{i=1}^{k_n} \tilde{\delta}_i \left(\frac{\tilde{Y}_i^{\gamma-1}}{\gamma} \right)^\alpha \tilde{Y}_i^{(\gamma_C^{-1} - \epsilon)(\gamma + \epsilon'')} + \epsilon' O_{\mathbb{P}}(1),$$

and (\tilde{Y}_i) are standard Pareto random variables independent of $(\tilde{\delta}_i)$. Therefore, applying the weak Law of Large Numbers (using the independence just mentioned) and Lemma 2 with $a = \alpha \geq 1$ and $a' = (\gamma_C^{-1} - \epsilon)(\gamma \pm \epsilon'') < 1$,

and afterwards, making ϵ' and ϵ'' go to 0, proves that, according to (11), $J_n^1/a_{n,k}^\alpha$ converges in probability to

$$p \times |\gamma|^{-\alpha-1} \text{Beta} \left(\frac{1-\gamma(\gamma_C^{-1}-\epsilon)}{|\gamma|} ; \alpha+1 \right).$$

Since $(1-\gamma/\gamma_C)/|\gamma| = |\gamma_X|^{-1} = p/|\gamma|$, this limit tends to l_α as ϵ goes to 0, which is the desired result concerning J_n^1 .

- Let us now prove that $J_n^2 = o_{\mathbb{P}}(a_{n,k}^\alpha)$, which will end the proof of (10). If $p > 1$ and $q > 1$ are such that $\frac{1}{p} + \frac{1}{q} = 1$, then Hölder's inequality yields:

$$|J_n^2| \leq \left(\frac{1}{k_n} \sum_{i=1}^{k_n} |\delta_{n-i+1,n} - \tilde{\delta}_{n-i+1,n}| \right)^{1/p} \left(\frac{1}{k_n} \sum_{i=1}^{k_n} (T_{n-i+1,n})^q \right)^{1/q},$$

$$\text{where } T_{n-i+1,n} := \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^{\gamma_C^{-1} - \epsilon}.$$

According to the proof of Theorem 1 in [Worms 2014] (see page 350 there), we know that (7) and (8) guarantee that $\frac{1}{k_n} \sum_{i=1}^{k_n} |\delta_{n-i+1,n} - \tilde{\delta}_{n-i+1,n}| = o_{\mathbb{P}}(1)$. So it remains to prove that $\frac{1}{k_n} \sum_{i=1}^{k_n} (T_{n-i+1,n}/a_{n,k}^\alpha)^q = O_{\mathbb{P}}(1)$, for an appropriate $q > 1$.

If we choose $q < 1 + \gamma_C/\gamma_X$, and we use once again Lemma 1 (with, this time, $a = \alpha q > 1$ and $b = (\gamma_C^{-1} - \epsilon)q < 0$), the law of Large Numbers combined with Lemma 2 (applied with $a = \alpha q > 1$ and $a' = (\gamma_C^{-1} - \epsilon)(\gamma - \epsilon'')q$ which is < 1 with this choice of q and ϵ small enough) yield the desired result for J_n^2 .

5.1.2 Proof of $R_n = o_{\mathbb{P}}(a_{n,k}^\alpha)$

Let us use the same decomposition as in the proof of the negligibility of the term R_n in [Worms 2014] (see subsection 5.1.2 there). In other words : we define, for some $\delta' > 0$,

$$\tilde{C}(t) := \int_0^t \frac{dG(x)}{(1-G(x))^2(1-F(x))} \quad \text{and} \quad h_{in} := (\tilde{C}(Z_{n-i+1,n}))^{-\frac{1}{2}-\delta'},$$

and we readily have $|R_n| \leq T_n^1 T_n^2$, where

$$T_n^1 := \sup_{1 \leq i \leq k_n} \sqrt{n} |h_{in}(C_{in} - 1)| \quad \text{and} \quad T_n^2 := \frac{1}{k_n} \sum_{i=1}^{k_n} W_{in} h_{in}^{-1} n^{-\frac{1}{2}}.$$

Using sharp results of the survival analysis litterature, we have already proved in [Worms 2014] that $T_n^1 = O_{\mathbb{P}}(1)$. It remains to prove that

$$T_n^2/a_{n,k}^\alpha = o_{\mathbb{P}}(1).$$

First, from the definition of h_{in} and \tilde{C} , since $(1 - H) = (1 - F)(1 - G)$ we clearly have

$$h_{in}^{-1} < \left(\frac{-\log(1 - G(Z_{n-i+1,n}))}{1 - H(Z_{n-i+1,n})} \right)^{\frac{1}{2} + \delta'}.$$

Moreover, under assumption (A), $1 - H(x^* - \cdot)$ is regularly varying at zero with index $-1/\gamma$ and $-\log(1 - G(x^* - \cdot))$ is slowly varying at 0 : therefore, it can be seen that for any given positive ϵ and ϵ' and n sufficiently large, the application of bound (14) to $-\log(1 - G(x^* - \cdot))$, and of bound (15) to $f = G$ and $f = H$, implies that $T_n^2 \leq 4P_n Q_n$ where

$$P_n := n^{-\frac{1}{2}} \left(\frac{-\log(1 - G(Z_{n-k_n,n}))}{1 - H(Z_{n-k_n,n})} \right)^{\frac{1}{2} + \delta'}$$

$$Q_n := \frac{1}{k_n} \sum_{i=1}^{k_n} \left(\log \frac{Z_{n-i+1,n}}{Z_{n-k_n,n}} \right)^\alpha \left(\frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}} \right)^\beta,$$

where $\beta = (2\gamma)^{-1} + \gamma_C^{-1} + \epsilon'''$, for some $\epsilon''' > 0$ (arbitrarily small for appropriate small values of ϵ , ϵ' and δ').

We thus need to prove that $P_n Q_n / a_{n,k}^\alpha = o_{\mathbb{P}}(1)$. Lemma 1, applied with $a = \alpha \geq 1$ and $b = \beta < 0$, yields, for n sufficiently large, and arbitrary small ϵ' and $\epsilon'' > 0$,

$$Q_n / a_{n,k}^\alpha \leq c_n Q'_n \text{ where } Q'_n \stackrel{d}{=} \tilde{Q}_n \quad (12)$$

and

$$\tilde{Q}_n = (1 - \epsilon'')^\beta \frac{1}{k_n} \sum_{i=1}^{k_n} \left(\frac{\tilde{Y}_i^\gamma - 1}{\gamma} \right)^\alpha \tilde{Y}_i^{\beta(\gamma - \epsilon'')} + c\epsilon' \frac{1}{k_n} \sum_{i=1}^{k_n} \tilde{Y}_i^{\beta(\gamma - \epsilon'')} \leq c' \frac{1}{k_n} \sum_{i=1}^{k_n} \tilde{Y}_i^{\beta(\gamma - \epsilon'')}$$

with $c' = |\gamma|^{-\alpha} (1 - \epsilon'')^\beta + c\epsilon'$ and (\tilde{Y}_i) denoting a sample of standard Pareto random variables. We now have to distinguish the case $\gamma_C \leq \gamma_X$ from the case $\gamma_C > \gamma_X$ (respectively weak and strong censoring in the tail).

(i) Case $\gamma_C \leq \gamma_X$

First of all, assumption (K) implies that $P_n = o_{\mathbb{P}}(1)$ (see relation (20) in [Worms 2014]). Since $\beta(\gamma - \epsilon'')$ can be made < 1 in this case (for appropriate values of ϵ'' and ϵ''' , since $1/2 + \gamma/\gamma_C \leq 1$), the upper bound of \tilde{Q}_n above converges in probability : consequently $T_n^2 = o_{\mathbb{P}}(a_{n,k}^\alpha)$, and R_n as well.

(ii) Case $\gamma_C > \gamma_X$

In this case $\beta(\gamma - \epsilon'') > 1$, which means that \tilde{Q}_n is unbounded in probability. However, by relying on the Marcinkiewicz-Zygmund law of large numbers, it can be seen that $k_n^{1-q} \tilde{Q}_n = o_{\mathbb{P}}(1)$ where $q = \beta(\gamma - \epsilon'') + \eta$ for any given $\eta > 0$. Therefore

$$T_n^2 / a_{n,k}^\alpha \leq k_n^{q-1} P_n o_{\mathbb{P}}(1)$$

and using assumption (K) ends the proof (for more details, see relation (20) in [Worms 2014]). \square

5.2 Proof of Theorem 2

Let us first state some preliminary lemmas. The notation $\xi_{i,n}$ was introduced in (3) and we note

$$\tilde{Z}_{i,n} := \frac{x^* - Z_{n-i+1,n}}{x^* - Z_{n-k_n,n}}.$$

Lemma 3 *Let $\alpha \geq 1$ and $u_i = u_{i,n} = \frac{i}{k_n+1}$. Under assumptions (A) and (4) :*

(i) *if $0 < a < 1$ then*

$$\frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a} \frac{\xi_{i,n}}{a_{n,k}^\alpha} \xrightarrow{\mathbb{P}} (1-a)|\gamma|^{-\alpha-1} \text{Beta} \left(\frac{1-a}{|\gamma|} ; \alpha+1 \right),$$

(ii) *if $a > 1$, then for any $\delta' > 0$,*

$$\frac{1}{k_n^{a+\delta'}} \sum_{i=1}^{k_n} u_i^{-a} \frac{\xi_{i,n}}{a_{n,k}^\alpha} \xrightarrow{\mathbb{P}} 0.$$

Lemma 4 *For any given positive exponents θ and $\theta' > 0$, there exist constants $c > 1$, $c' < 1$ both arbitrarily close to 1, and $a_+ > 0$, $a_- > 0$, arbitrarily close to $\gamma\theta$ and to $\gamma\theta'$ respectively, such that*

$$\lim_{n \rightarrow \infty} \mathbb{P} \left(\max_{i \leq k_n} \frac{\tilde{Z}_{i,n}^\theta}{cu_i^{-a_+}} > 1 \right) = \lim_{n \rightarrow \infty} \mathbb{P} \left(\min_{i \leq k_n} \frac{\tilde{Z}_{i,n}^{\theta'}}{c'u_i^{-a_-}} < 1 \right) = 0.$$

We are now ready to proceed to the proof of Theorem 5.2. Similarly to $M_{n,k_n}^{(\alpha)}$, the weighted moment $\tilde{M}_{n,k_n}^{(\alpha)}$ can be written as follows :

$$\tilde{M}_{n,k_n}^{(\alpha)} = A_n(\overline{W}_n + R_n),$$

where A_n and R_n are defined as before (see Subsection 5.1) but here

$$W_{in} := \frac{1 - G(Z_{n-k_n,n})}{1 - G(Z_{n-i+1,n})} \xi_{in}.$$

Recall that $A_n \xrightarrow{\mathbb{P}} 1$ (see beginning of the proof of Theorem 1). We shall deal with \overline{W}_n and R_n separately. Let $\tilde{Z}_{i,n}$ be defined as in the statement of Lemma 4, and l_α denote the limit in the statement of Theorem 2.

5.2.1 Proof of $\overline{W}_n/a_{n,k}^\alpha \xrightarrow{\mathbb{P}} l_\alpha$

Applying bounds (15) to $f = G$ and $x = \tilde{Z}_{i,n}$ and $t = x^* - Z_{n-k_n,n}$ yields, for $\epsilon > 0$ and n sufficiently large,

$$(1 - \epsilon) \xi_{i,n} \tilde{Z}_{i,n}^{\gamma_C^{-1} + \epsilon} \leq W_{i,n} \leq (1 + \epsilon) \xi_{i,n} \tilde{Z}_{i,n}^{\gamma_C^{-1} - \epsilon}.$$

Let $\eta > 0$. We first write, for ϵ sufficiently small,

$$\mathbb{P}(\overline{W}_n/a_{n,k}^\alpha - l_\alpha > \eta) \leq \mathbb{P}(k_n^{-1} \sum_{i=1}^{k_n} \tilde{Z}_{i,n}^{\gamma_C^{-1} - \epsilon} \frac{\xi_{i,n}}{a_{n,k}^\alpha} - l_\alpha > \frac{\eta}{2}),$$

$$\mathbb{P}(l_\alpha - \overline{W}_n/a_{n,k}^\alpha > \eta) \leq \mathbb{P}(l_\alpha - k_n^{-1} \sum_{i=1}^{k_n} \tilde{Z}_{i,n}^{\gamma_C^{-1} + \epsilon} \frac{\xi_{i,n}}{a_{n,k}^\alpha} > \frac{\eta}{2}).$$

Let us now consider constants $c > 1$ and $c' < 1$, both arbitrary close to 1, and $a_+ > 0$ and $a_- > 0$ both arbitrary close to γ/γ_C . These constants come from the application of Lemma 4 above with $\theta = \gamma_C^{-1} - \epsilon$ and $\theta' = \gamma_C^{-1} + \epsilon$. Using positivity of $\xi_{i,n}$, it comes

$$\begin{aligned} \mathbb{P}\left(\frac{\overline{W}_n}{a_{n,k}^\alpha} - l_\alpha > \eta\right) &\leq \mathbb{P}\left(\max_{i \leq k_n} \frac{\tilde{Z}_{i,n}^{\gamma_C^{-1} - \epsilon}}{c u_i^{-a_+}} > 1\right) \\ &\quad + \mathbb{P}\left(c k_n^{-1} \sum_{i=1}^{k_n} u_i^{-a_+} \frac{\xi_{i,n}}{a_{n,k}^\alpha} - l_\alpha > \frac{\eta}{2}\right) \\ \mathbb{P}\left(l_\alpha - \frac{\overline{W}_n}{a_{n,k}^\alpha} > \eta\right) &\leq \mathbb{P}\left(\min_{i \leq k_n} \frac{\tilde{Z}_{i,n}^{\gamma_C^{-1} + \epsilon}}{c' u_i^{-a_-}} < 1\right) \\ &\quad + \mathbb{P}\left(l_\alpha - c' k_n^{-1} \sum_{i=1}^{k_n} u_i^{-a_-} \frac{\xi_{i,n}}{a_{n,k}^\alpha} > \frac{\eta}{2}\right) \end{aligned}$$

where $u_i = u_{i,n} = i/(k_n + 1)$ for $1 \leq i \leq k_n$. If we call $l_{\alpha,a}$ the limit in the statement of Lemma 3 (i), and if we apply Lemma 4 as indicated previously, we have

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbb{P}\left(\frac{\overline{W}_n}{a_{n,k}^\alpha} - l_\alpha > \eta\right) &\leq \limsup_{n \rightarrow \infty} \mathbb{P}\left(k_n^{-1} \sum_{i=1}^{k_n} u_i^{-a_+} \frac{\xi_{i,n}}{a_{n,k}^\alpha} - l_{\alpha,a_+} > \frac{1}{c}(\frac{\eta}{2} + l_\alpha) - l_{\alpha,a_+}\right) \\ \limsup_{n \rightarrow \infty} \mathbb{P}\left(l_\alpha - \frac{\overline{W}_n}{a_{n,k}^\alpha} > \eta\right) &\leq \limsup_{n \rightarrow \infty} \mathbb{P}\left(l_{\alpha,a_-} - k_n^{-1} \sum_{i=1}^{k_n} u_i^{-a_-} \frac{\xi_{i,n}}{a_{n,k}^\alpha} > \frac{1}{c'}(\frac{\eta}{2} - l_\alpha) + l_{\alpha,a_-}\right) \end{aligned}$$

Since $l_\alpha = l_{\alpha,\gamma/\gamma_C}$, and both a_+ and a_- are arbitrary close to $\gamma/\gamma_C < 1$, it is easy to see that convergence in probability of \overline{W}_n to l_α thus comes from the application of Lemma 3 (i) to $a = a_+$ and $a = a_-$.

5.2.2 Proof of $R_n = o_{\mathbb{P}}(a_{n,k}^\alpha)$

Most of the proof is identical to the case of the first theorem. As in Subsection 5.1.2, we have $|R_n| \leq T_n^1 T_n^2$ where T_n^1 is left unchanged and is $O_{\mathbb{P}}(1)$. The factor T_n^2 is bounded by $4P_n Q_n$, where P_n is defined as in Subsection 5.1.2 but, this time,

$$Q_n := \frac{1}{k_n} \sum_{i=1}^{k_n} \xi_{in} \tilde{Z}_{i,n}^\beta,$$

where, as before, $\beta = (2\gamma)^{-1} + \gamma_C^{-1} + \epsilon''$, for some $\epsilon'' > 0$. We now proceed as in Subsection 5.2.1 to control $P_n Q_n / a_{n,k}^\alpha$.

Let $\eta > 0$ and consider constants $c > 1$ arbitrarily close to 1 and $a_+ > 0$ arbitrarily close to $\gamma\beta = \frac{1}{2} + \frac{\gamma}{\gamma_C} + \gamma\epsilon'' < \frac{1}{2} + \frac{\gamma}{\gamma_C}$.

$$\mathbb{P}\left(\frac{P_n Q_n}{a_{n,k}^\alpha} > \eta\right) \leq \mathbb{P}\left(\max_{i \leq k_n} \frac{\tilde{Z}_{i,n}^\beta}{c u_i^{-a_+}} > 1\right) + \mathbb{P}\left(P_n k_n^{-1} \sum_{i=1}^{k_n} u_i^{-a_+} \frac{\xi_{in}}{a_{n,k}^\alpha} > \frac{\eta}{c}\right). \quad (13)$$

First, Lemma 4 (ii) is applied with $\theta = \beta$ and thus the first term of the right-hand side of (13) tends to 0.

Next, Lemma 3 is applied with $a = a_+$. As in subsection 5.1.2, we need here to distinguish the case $\gamma_C \leq \gamma_X$ (for which $\gamma\beta < 1$) from the case $\gamma_C > \gamma_X$ (for which $\gamma\beta > 1$ when $\epsilon'' > 0$ gets small).

(i) Case $\gamma_C \leq \gamma_X$

Recall that under assumption (K), $P_n = o_{\mathbb{P}}(1)$. Since $a_+ < 1$ in this case, Lemma 3 (ii) implies that $\frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a_+} \frac{\xi_{in}}{a_{n,k}^\alpha} = O_{\mathbb{P}}(1)$ and consequently the second term of the right hand-side of (13) tends to 0.

(ii) Case $\gamma_C > \gamma_X$

In this case $a_+ > 1$, therefore Lemma 3 (ii) implies that, for any given $\delta' > 0$, $k_n^{-(a_+ + \delta')} \sum_{i=1}^{k_n} u_i^{-a_+} \frac{\xi_{in}}{a_{n,k}^\alpha} = o_{\mathbb{P}}(1)$. Moreover, assumption (K) implies that, for $\delta' > 0$ small enough and a_+ sufficiently close to $1/2 + \gamma/\gamma_C$, we have $k_n^{a_+ + \delta' - 1} P_n = O_{\mathbb{P}}(1)$. Hence, the second term of the right hand-side of (13) also tends to 0.

□

6 Appendix

6.1 Regular variation and Potter-type bounds

Definition 1 An ultimately positive function $f : \mathbb{R}^+ \rightarrow \mathbb{R}$ is regularly varying (at infinity) with index $\alpha \in \mathbb{R}$, if

$$\lim_{t \rightarrow +\infty} \frac{f(tx)}{f(t)} = x^\alpha \quad (\forall x > 0).$$

This is noted $f \in RV_\alpha$. If $\alpha = 0$, f is said to be slowly varying.

Remark 3 Regular variation (and slow variation) can be defined at zero as well. A function f is said to be regularly varying at zero with index α if the function $x \rightarrow f(1/x)$ is regularly varying at infinity, with index $-\alpha$.

Proposition 1 (See [Haan and Ferreira (2006)] Proposition B.1.9)

Suppose $f \in RV_\alpha$. If $x > 0$ and $\delta_1, \delta_2 > 0$ are given, then there exists $t_0 = t_0(\delta_1, \delta_2)$ such that for any $t \geq t_0$ such that $tx \geq t_0$,

$$(1 - \delta_1)x^\alpha \min(x^{\delta_2}, x^{-\delta_2}) < \frac{f(tx)}{f(t)} < (1 + \delta_1)x^\alpha \max(x^{\delta_2}, x^{-\delta_2}).$$

If $x < 1$ and $\epsilon > 0$, then there exists $t_0 = t_0(\epsilon)$ such that for every $t \geq t_0$,

$$(1 - \epsilon)x^{\alpha+\epsilon} < \frac{f(tx)}{f(t)} < (1 + \epsilon)x^{\alpha-\epsilon}$$

and if $x \geq 1$,

$$(1 - \epsilon)x^{\alpha-\epsilon} < \frac{f(tx)}{f(t)} < (1 + \epsilon)x^{\alpha+\epsilon}.$$

Corollary 4 If f is a positive function with end-point x^* , such that $t \rightarrow 1 - f(x^* - t)$ is regularly varying at 0 with index α , i.e.

$$\frac{1 - f(x^* - tx)}{1 - f(x^* - t)} \rightarrow x^\alpha, \text{ as } t \rightarrow 0,$$

for some $\alpha \in \mathbb{R}$, then for every $\epsilon > 0$, there exists $t_0 > 0$ such that, $\forall 0 < t < t_0$, $\forall 0 < x < 1$,

$$(1 - \epsilon)x^{\alpha+\epsilon} \leq \frac{1 - f(x^* - tx)}{1 - f(x^* - t)} \leq (1 + \epsilon)x^{\alpha-\epsilon} \quad (14)$$

and

$$(1 - \epsilon)x^{-\alpha+\epsilon} \leq \frac{1 - f(x^* - t)}{1 - f(x^* - tx)} \leq (1 + \epsilon)x^{-\alpha-\epsilon} \quad (15)$$

Below, U corresponds to the quantile function associated to H introduced in paragraph 2.1.

Corollary 5 If U satisfies condition (6), then for every $\epsilon > 0$, there exists $t_0 > 0$ such that, $\forall 0 < t < t_0$, $\forall x \geq 1$,

$$(1 - \epsilon)x^{\gamma-\epsilon} \leq \frac{U(\infty) - U(tx)}{U(\infty) - U(t)} \leq (1 + \epsilon)x^{\gamma+\epsilon}. \quad (16)$$

Proposition 2 (see [Haan and Ferreira (2006)] Theorem B.2.18)

If U satisfies condition (5) with the positive function a , then there exists a function q_0 equivalent to a/U at infinity such that $\forall \epsilon > 0$, $\exists t_0 > 0$, $\forall t \geq t_0$, $\forall x \geq 1$,

$$\frac{x^\gamma - 1}{\gamma} - \epsilon x^{\gamma+\epsilon} \leq \frac{\log U(tx) - \log U(t)}{q_0(t)} \leq \frac{x^\gamma - 1}{\gamma} + \epsilon x^{\gamma+\epsilon}. \quad (17)$$

6.2 Proof of Lemma 1

Let $c_n := (q_0(n/k_n)/a_{n,k})^a$, where function q_0 is defined in Proposition 2 of the Appendix.

If $(Y_i)_{1 \leq i \leq n}$ is an i.i.d. sequence of standard Pareto random variables, then $Q_{i,n}^{a,b}$ equals in distribution $c_n(LL_{i,k})^a(UU_{i,k})^b$, where

$$LL_{i,k} := \frac{\log(U(Y_{n-i+1,n})) - \log(U(Y_{n-k_n,n}))}{q_0(Y_{n-k_n,n})} \text{ and } UU_{i,k} := \frac{U(\infty) - U(Y_{n-i+1,n})}{U(\infty) - (Y_{n-k_n,n})}.$$

We can write

$$(LL_{i,k})^a(UU_{i,k})^b = (QQ_{i,k})^a(UU_{i,k})^b + ((LL_{i,k})^a - (QQ_{i,k})^a)(UU_{i,k})^b, \quad (18)$$

where

$$QQ_{i,k} := ((Y_{n-i+1,n}/Y_{n-k_n,n})^\gamma - 1)/\gamma.$$

On one hand, using bounds (17) for some $\epsilon' > 0$, with $t = Y_{n-k_n,n}$ and $x = Y_{n-i+1,n}/Y_{n-k_n,n} > 1$, and relying on the mean value theorem, we easily prove that, since $a \geq 1$ and $\gamma < 0$,

$$|(LL_{i,k})^a - (QQ_{i,k})^a| \leq c \epsilon' \left(\frac{Y_{n-i+1,n}}{Y_{n-k_n,n}} \right)^{\gamma+\epsilon'} \leq c \epsilon' \quad (19)$$

for some constant c (close to $a|\gamma|^{1-a}$). On the other hand, using Potter bounds (16) for some $\epsilon'' > 0$, we have (since $b < 0$)

$$(1 + \epsilon'')^b \left(\frac{Y_{n-i+1,n}}{Y_{n-k_n,n}} \right)^{b(\gamma+\epsilon'')} \leq (UU_{i,k})^b \leq (1 - \epsilon'')^b \left(\frac{Y_{n-i+1,n}}{Y_{n-k_n,n}} \right)^{b(\gamma-\epsilon'')}. \quad (20)$$

Moreover, it is known that

$$(Y_{n-i+1,n}/Y_{n-k_n,n})_{1 \leq i \leq k_n} \stackrel{d}{=} (\tilde{Y}_{k_n-i+1,k_n})_{1 \leq i \leq k_n}, \quad (21)$$

where $\tilde{Y}_{1,k_n}, \dots, \tilde{Y}_{k_n,k_n}$ are the ascending order statistics of k_n i.i.d random variables $\tilde{Y}_1, \dots, \tilde{Y}_{k_n}$ with standard Pareto distribution. Combining (19), (20) and (21) in (18) finally gives the desired result. \square

6.3 Proof of Lemma 2

Let U be a uniform $[0; 1]$ random variable, then

$$\begin{aligned} \mathbb{E} \left(\left(\frac{Y^\gamma - 1}{\gamma} \right)^a Y^{a'} \right) &= |\gamma|^{-a} \mathbb{E}(U^{-a'} (1 - U^{|\gamma|})^a), \\ &= |\gamma|^{-a} \int_0^1 u^{-a'} (1 - u^{|\gamma|})^a du. \end{aligned}$$

The change of variable $v = u^{|\gamma|}$ yields the result. \square

6.4 Proof of Lemma 3

Let $c_n := (q_0(n/k_n)/a_{n,k})^\alpha$ (which tends to 1 as $n \rightarrow \infty$) and (Y_i) be a sequence of i.i.d standard Pareto random variables. Let also $LL_{i,k}$ and $QQ_{i,k}$ be defined as in the proof of Lemma 1. For every $1 \leq i \leq k_n$, we thus have

$$\begin{aligned} c_n^{-1} \frac{\xi_{i,n}}{a_{n,k}^\alpha} &\stackrel{d}{=} i((LL_{i,k})^\alpha - (LL_{i+1,k})^\alpha) \\ &= \alpha_{i,n} + \beta_{i,n} \end{aligned}$$

where

$$\begin{aligned} \alpha_{i,n} &:= i((QQ_{i,k})^\alpha - (QQ_{i+1,k})^\alpha) \\ \beta_{i,n} &:= i(B_{k,n}(i) - B_{k,n}(i+1)) \\ B_{k,n}(i) &:= (LL_{i,k})^\alpha - (QQ_{i,k})^\alpha, \end{aligned}$$

with $B_{k,n}(k_n + 1) = 0$.

Our first step will be to prove that (with $\delta' > 0$)

$$\frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a} \beta_{i,n} \quad \text{or} \quad \frac{1}{k_n^{a+\delta'}} \sum_{i=1}^{k_n} u_i^{-a} \beta_{i,n} \quad \text{is } o_{\mathbb{P}}(1) \text{ whether } 0 < a < 1 \text{ or } a > 1.$$

Since $\alpha \geq 1$, from relation (19) with α replacing a , it comes that $|B_{k,n}(i)| = o_{\mathbb{P}}(1)$, uniformly on $1 \leq i \leq k_n$. Since $|B_{k,n}(i)| = \left| \sum_{j=i}^{k_n} \frac{\beta_{j,n}}{j} \right|$, we thus have, when $0 < a < 1$,

$$\begin{aligned} \left| \frac{1}{k_n+1} \sum_{i=1}^{k_n} u_i^{-a} \beta_{i,n} \right| &= \left| \sum_{i=1}^{k_n} \frac{\beta_{i,n}}{i} u_i^{1-a} ds \right| \\ &= (1-a) \left| \sum_{i=1}^{k_n} \frac{\beta_{i,n}}{i} \int_0^{u_i} s^{-a} ds \right| \\ &\leq (1-a) \sum_{i=1}^{k_n} \left| \sum_{j=i}^{k_n} \frac{\beta_{j,n}}{j} \right| \int_{u_{i-1}}^{u_i} s^{-a} ds \\ &= o_{\mathbb{P}}(1) \sum_{i=1}^{k_n} (u_i^{1-a} - u_{i-1}^{1-a}) \\ &\leq o_{\mathbb{P}}(1) \frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a} \\ &= o_{\mathbb{P}}(1). \end{aligned}$$

The proof for $a > 1$ is similar (see end of subsection 5.2.1 in [Worms 2014] for more details, with the difference that now $B_{k,n}(i) = o_{\mathbb{P}}(1)$ uniformly in i).

Since we have dealt with the $\beta_{i,n}$ part, the lemma will be proved as soon as we obtain that, when $0 < a < 1$,

$$\frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a} \alpha_{i,n} \xrightarrow{\mathbb{P}} (1-a) |\gamma|^{-\alpha-1} \text{Beta} \left(\frac{1-a}{|\gamma|}; \alpha+1 \right) \quad (22)$$

and, when $a > 1$,

$$\frac{1}{k_n^{a+\delta'}} \sum_{i=1}^{k_n} u_i^{-a} \alpha_{i,n} \xrightarrow{\mathbb{P}} 0. \quad (23)$$

From now on we may write k instead of k_n . Let (E_i) be a sequence of i.i.d standard exponential random variables. According to (21), and by applying the mean value theorem, there exist some random variables $E_{i,k}^* \in [E_{k-i,k}, E_{k-i+1,k}]$ such that (remind below that γ is < 0 and $\alpha \geq 1$)

$$\begin{aligned} \alpha_{i,n} &\stackrel{d}{=} i \left(\left(\frac{e^{\gamma E_{k-i+1,k}} - 1}{\gamma} \right)^\alpha - \left(\frac{e^{\gamma E_{k-i,k}} - 1}{\gamma} \right)^\alpha \right) \\ &= \alpha i (E_{k-i+1,k} - E_{k-i,k}) e^{\gamma E_{i,k}^*} \left(\frac{1 - e^{\gamma E_{i,k}^*}}{|\gamma|} \right)^{\alpha-1} \\ &= \alpha |\gamma|^{1-\alpha} \times i (E_{k-i+1,k} - E_{k-i,k}) \left\{ u_i^{|\gamma|} (1 - u_i^{|\gamma|})^{\alpha-1} + \Delta_{i,n} \right\} \quad (24) \end{aligned}$$

where

$$\Delta_{i,n} := e^{\gamma E_{i,k}^*} (1 - e^{\gamma E_{i,k}^*})^{\alpha-1} - u_i^{|\gamma|} (1 - u_i^{|\gamma|})^{\alpha-1}.$$

We will prove later that

$$\max_{i \leq k_n} |\Delta_{i,n}| = o_{\mathbb{P}}(1). \quad (25)$$

For the moment, note that $(i (E_{k-i+1,k} - E_{k-i,k}))_{i \leq k_n} \stackrel{d}{=} (f_i)_{i \leq k_n}$ due to the Renyi representation, where (f_i) denotes a sequence of i.i.d standard exponential random variables. Moreover, application of the law of large numbers for triangular arrays of independent random variables (cf [Chow and Teicher (1997)] ; details are omitted) implies that, when $0 < a < 1$,

$$\frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a} f_i = O_{\mathbb{P}}(1) \quad \text{and} \quad \frac{1}{k_n} \sum_{i=1}^{k_n} u_i^{-a} f_i u_i^{|\gamma|} (1 - u_i^{|\gamma|})^{\alpha-1} \xrightarrow{\mathbb{P}} \int_0^1 x^{|\gamma|-a} (1-x^{|\gamma|})^{\alpha-1} dx \quad (26)$$

and, when $a > 1$ (and $\delta' > 0$ is given),

$$\frac{1}{k_n^{a+\delta'}} \sum_{i=1}^{k_n} u_i^{-a} f_i = o_{\mathbb{P}}(1). \quad (27)$$

Considering first the situation $0 < a < 1$, combining (24), (25) and (26) shows that relation (22) will hold as soon as

$$\int_0^1 x^{|\gamma|-a} (1-x^{|\gamma|})^{\alpha-1} dx = \alpha^{-1} (1-a) |\gamma|^{-2} \text{Beta} \left(\frac{1-a}{|\gamma|}; \alpha+1 \right).$$

Use of the formulas $\text{Beta}(u, v) = \Gamma(u)\Gamma(v)/\Gamma(u+v)$ and $u\Gamma(u) = \Gamma(u+1)$ proves the latter relation. When $a > 1$, relation (23) is a consequence of (24), (25) and (27).

It remains to prove relation (25). For this purpose, we introduce the sequence (V_i) of i.i.d. standard uniform random variables such that $V_{i,k} = e^{-E_{k-i+1,k}}$, and we note $V_{i,k}^* := e^{-E_{i,k}^*}$. If we set $W_i := V_{i,k}^{|\gamma|}$, $W_i^* := (V_{i,k}^*)^{|\gamma|}$ and $v_i := u_i^{|\gamma|}$, then relation (25) is now

$$M_n := \max_{1 \leq i \leq k_n} |W_i^* (1 - W_i^*)^{\alpha-1} - v_i (1 - v_i)^{\alpha-1}| = o_{\mathbb{P}}(1),$$

with $0 < W_i \leq W_i^* \leq W_{i+1} < 1$. Unfortunately, the function $x \mapsto x(1-x)^{\alpha-1}$ is not uniformly continuous on $]0, 1[$ if α is smaller than 2. Until the end of the proof we will note \max_i instead of $\max_{1 \leq i \leq k_n}$. Since $|x - a| \leq \max\{|y - a|, |z - a|\}$ whenever $|x - y| \leq |z - y|$ (this yields the first inequality below), we have

$$\begin{aligned} M_n &\leq \max \left\{ \max_i |W_i(1 - W_{i+1})^{\alpha-1} - v_i(1 - v_i)^{\alpha-1}| ; \right. \\ &\quad \left. \max_i |W_{i+1}(1 - W_i)^{\alpha-1} - v_i(1 - v_i)^{\alpha-1}| \right\} \\ &\leq \max \left\{ \max_i |W_i(1 - W_{i+1})^{\alpha-1} - W_i(1 - W_i)^{\alpha-1}| ; \right. \\ &\quad \left. \max_i |(W_{i+1} - W_i)(1 - W_i)^{\alpha-1}| ; \max_i |W_i(1 - W_i)^{\alpha-1} - v_i(1 - v_i)^{\alpha-1}| \right\} \\ &\leq \max \left\{ \max_i |(1 - W_{i+1})^{\alpha-1} - (1 - W_i)^{\alpha-1}| ; \max_i |W_{i+1} - W_i| ; \right. \\ &\quad \left. \max_i |W_i - v_i| ; \max_i |(1 - W_i)^{\alpha-1} - (1 - v_i)^{\alpha-1}| \right\} \\ &=: \max \{ M_{n,1} ; M_{n,2} ; M_{n,3} ; M_{n,4} \} . \end{aligned}$$

Now, since $|x^a - y^a| \leq (a \wedge 1)|x - y|^{a \vee 1}$ whenever x and y belong to $[0, 1]$ and $a > 0$, we have $M_{n,1} \leq cM_{n,2}^{c'}$ and $M_{n,4} \leq cM_{n,3}^{c'}$ for some positive constants c and c' . On the other hand, $M_{n,2}$ is bounded by $2M_{n,3} + \max_i |v_{i+1} - v_i| = 2M_{n,3} + o(1)$. Therefore, it remains to prove that

$$M_{n,3} = \max_{1 \leq i \leq k_n} |V_i^{|\gamma|} - u_i^{|\gamma|}| = o_{\mathbb{P}}(1).$$

This property is proved in details in [Beirlant et al. (2002)] (page 164, with $-\rho$ instead of $|\gamma|$), so we do not reproduce it here. \square

6.5 Proof of Lemma 4

Once again, if $Y_{1,n}, \dots, Y_{n,n}$ denote the ascending order statistics of n i.i.d standard Pareto random variables, we have

$$\tilde{Z}_{i,n} \stackrel{d}{=} \frac{U(\infty) - U(Y_{n-i+1,n})}{U(\infty) - U(Y_{n-k_n,n})}.$$

Applying bounds (16), it comes, for some given $\epsilon' > 0$, and n sufficiently large,

$$\begin{aligned} \left(\frac{U(\infty) - U(Y_{n-i+1,n})}{U(\infty) - U(Y_{n-k_n,n})} \right)^{\theta} &\leq (1 + \epsilon')^{\theta} \left(\frac{Y_{n-i+1,n}}{Y_{n-k_n,n}} \right)^{a_+} \\ \left(\frac{U(\infty) - U(Y_{n-i+1,n})}{U(\infty) - U(Y_{n-k_n,n})} \right)^{\theta'} &\geq (1 - \epsilon')^{\theta'} \left(\frac{Y_{n-i+1,n}}{Y_{n-k_n,n}} \right)^{a_-} \end{aligned}$$

where

$$a_+ = (\gamma + \epsilon')\theta \quad \text{and} \quad a_- = (\gamma - \epsilon')\theta'.$$

We finish the proof as for Lemma 1 of [Worms 2014]. \square

References

- [Beirlant et al. (2002)] J. Beirlant, G. Dierckx, A. Guillou and C. Stărică . On exponential representations of log spacings of order statistics. In *Extremes* **5**, pages 157-180 (2002)
- [Beirlant et al. (2007)] J. Beirlant, G. Dierckx, A. Guillou and A. Fils-Villetard . Estimation of the extreme value index and extreme quantiles under random censoring. In *Extremes* **10**, pages 151-174 (2007)
- [Beirlant et al. (2010)] J. Beirlant, A. Guillou and G. Toulemonde . Peaks-Over-Threshold modeling under random censoring. In *Comm. Stat. : Theory and Methods* **39**, pages 1158-1179 (2010)
- [Brahimi et al. (2013)] B. Brahimi, D. Meraghni and A. Necir . Approximations to the tail index estimator of a heavy-tailed distribution under random censoring and application. working paper, arXiv:1302.1666 (2013)
- [Chow and Teicher (1997)] Y.S. Chow and H. Teicher . Probability theory. Independence, interchangeability, martingales. *Springer* (1997)
- [Csörgő (1996)] S. Csörgő . Universal Gaussian approximations under random censorship. In *Annals of statistics* **24** (6), pages 2744-2778 (1996)
- [Diop et al. (2014)] A. Diop, J-F. Dupuy and P. Ndao. Nonparametric estimation of the conditional tail index and extreme quantiles under random censoring. In *Computational Statistics & Data Analysis* **79**, pages 63-79 (2014)
- [Einmahl et al. (2008)] J. Einmahl, A. Fils-Villetard and A. Guillou . Statistics of extremes under random censoring. In *Bernoulli* **14**, pages 207-227 (2008)
- [Gomes and Neves (2011)] M.I. Gomes and M.M. Neves . Estimation of the extreme value index for randomly censored data. In *Biometrical Letters* **48** (1), pages 1-22 (2011)
- [Haan and Ferreira (2006)] L. de Haan and A. Ferreira . Extreme Value Theory : an Introduction. *Springer Science + Business Media* (2006)
- [Worms 2014] J. Worms and R. Worms New estimators of the extreme value index under random right censoring, for heavy-tailed distributions. In *Extremes* **17** (2), pages 337-358 (2014)