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# A NON-LAMINAR DYNAMICAL GREEN CURRENT

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ABSTRACT. A holomorphic endomorphism $f$ of $\mathbb{CP}^2$ admits a Julia set $J_1$, defined as usual to be the locus of non-normality of its iterates $(f^n)_{n\geq 0}$, and a (typically) smaller Julia set $J_2$, which is essentially the closure of the set of repelling periodic orbits. The question has been raised whether $J_1 \setminus J_2$ is filled (possibly in a measure-theoretic sense) with "Fatou subvarieties" along which the dynamics is locally equicontinuous. In this article we construct examples showing that this is not the case in general.

## 1. Introduction

1.1. **Background.** Let $f$ be a holomorphic endomorphism of $\mathbb{CP}^2$, of degree $d \geq 2$. The Julia set $J_1$ (or simply $J$) of $f$ is classically defined as the locus where the iterates $(f^n)_{n\geq 0}$ do not locally form a normal family. Contrary to the one-dimensional case, the closure of the set of repelling periodic orbits is typically smaller than $J_1$. For instance, for the endomorphisms induced by polynomial mappings on $\mathbb{C}^2$, the Julia set is unbounded in $\mathbb{C}^2$ while repelling periodic orbits stay in a compact subset.

Let $J_2$ be the support of the so-called *equilibrium measure* of $f$, which is the unique measure of maximal entropy. Repelling periodic orbits are dense in $J_2$ (see Briend-Duval [4]). It turns out that $J_2$ is better behaved as an analogue of the classical Julia set than the closure of repelling orbits, which may have isolated points (see Hubbard-Papadopol [15, p. 345]).

Let $T$ be the dynamical Green current of $f$, defined by $T = \lim_{n\to\infty} d^{-n}[f^{-n}(L)]$, where $L \subset \mathbb{P}^2$ is a generic line. The Julia set $J_1$ coincides with $\mathrm{Supp}(T)$, and the self-intersection measure $T \wedge T$ is the measure of maximal entropy of $f$ (see Fornæss-Sibony [12]).

It has been a long standing problem in higher dimensional holomorphic dynamics to describe the structure of $J_1 \setminus J_2$. A popular picture is that $J_1 \setminus J_2$ should be "foliated" (in some appropriate sense) by holomorphic disks $D$ along which $(f^n|_D)_{n\geq 0}$ is a normal family. This happens to be the case in all the examples which have been analyzed so far. Such disks will be referred to as *Fatou disks*. For instance, in [12], Fornæss and Sibony define a set $J_2'$ (actually denoted by $J_1$ in their paper) as follows: $x$ belongs to $(J_2')^c$ if there exists a neighborhood $N \ni x$ such that for every $y \in N$ there is a germ of Fatou disk through $y$. They show that $J_2 \subset J_2'$ and ask whether equality holds.

In [10], we gave the following picture of the infinitesimal dynamics on $J_1 \setminus J_2$: for $\sigma_T$-a.e. $x \in J_1 \setminus J_2$ ($\sigma_T$ is the trace measure of $T$), there exists a *Fatou direction* $\mathcal{F}_x$ in $T_x\mathbb{P}^2$ along which $df^n$ is not expanding, while $df^n$ expands exponentially in the remaining directions. The question of the integrability of this field of Fatou directions into a field of Fatou disks was left open in that paper.

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A related question is whether the Green current $T$ is *laminar* in $J_1 \setminus J_2$. Recall that a positive current in $\Omega \subset \mathbb{C}^2$ is laminar if it expresses as an integral of integration currents over a measurable family of compatible holomorphic disks (compatible here means that these disks have no isolated intersections). These disks are then automatically of Fatou type [12]. The reader is referred to [2, 8] for basics on laminar currents. In his thesis De Thélin came quite close to a positive answer to this question (see [5, 6]). First, he showed that $T$ is laminar on $J_1 \setminus J_2$ when $f$ is post-critically finite. In the general case, he proved the following two results:

- If $(C_n)$ is a sequence of complex submanifolds (i.e. curves) in some open set $\Omega \subset \mathbb{C}^2$ such that $\mathrm{Area}(C_n) = d^n$ and $\mathrm{genus}(C_n) = O(d^n)$, then any cluster limit of the sequence of currents $d^{-n}[C_n]$ is laminar in $\Omega$.
- Under a generic condition on $f$ and $L$, if $\Omega \subset \mathbb{P}^2$ is an open set such that $\overline{\Omega}$ is disjoint from $J_2$, then

$$\limsup_{n\to\infty} \frac{1}{n} \log \mathrm{genus}(f^{-n}(L) \cap \Omega) \leq \log d \tag{1}$$

(recall that $f^{-n}(L)$ is smooth for generic $L$ and that $\mathrm{Area}(f^{-n}(L)) = d^n$).

Altogether, this appears as strong evidence in favor of the laminarity of $T$ outside $J_2$.

The main result in this paper is that, somewhat surprisingly, the answer to the laminarity problem is "no" in general. We construct explicit examples of endomorphisms of $\mathbb{P}^2$ (actually polynomial on $\mathbb{C}^2$) such that the Green current is *not* laminar outside $J_2$, even in a very weak sense. In particular this shows that the estimate in (1) cannot be improved to $O(d^n)$. For these examples, we also have that $J_2 \neq J_2'$, thereby solving by the negative the question of Fornæss and Sibony.

1.2. **Setting and main result.** Let $f(z, w) = (p(z), q(z, w))$ be a polynomial skew product in $\mathbb{C}^2$ It is convenient to view the second component as a non-autonomous dynamical system on $\mathbb{C}$, denoted by $q_z(\cdot)$, so that

$$f^n(z, w) = (p^n(z), q_{z_{n-1}} \circ \cdots \circ q_z(w)), \text{ where } z_j = p^j(z).$$

We let $q_z^n = q_{z_{n-1}} \circ \cdots \circ q_z$. The reader is referred to Jonsson [16] for basics on the dynamics of polynomial skew products in $\mathbb{C}^2$ (see also Sester [21]).

We assume that $p(z) = z^d + \text{l.o.t.}$ and $q(z, w) = w^d + \text{l.o.t.}$, so that $f$ extends to a holomorphic self-mapping on $\mathbb{P}^2$. Such polynomial mappings will be referred to as *regular*. We note that in this setting, the Green current was shown to be laminar in the basin of the line at infinity by Bedford and Jonsson [1]. Our methods require that $d \geq 3$, conjecturally one should be able to treat the case $d = 2$ using similar ideas. We assume that $p$ admits a linearizable irrational fixed point at $z = 0$, that is, $p(z) = e^{2\pi i\theta} z + \cdots + z^d$, where the rotation number $\theta$ satisfies the Brjuno condition ($\theta = (\sqrt{5} - 1)/2$ would do). Then $p$ admits a Siegel disk $\Delta$ centered at the origin.

Since $f$ is polynomial on $\mathbb{C}^2$ its dynamical Green function is defined by

$$G(z, w) = \lim_{n\to\infty} d^{-n} \log^+ \|f^n(z, w)\|,$$

and the Green current $T$ satisfies $T = dd^cG$. From now on we simply denote the Julia set by $J$ (recall that $J = \mathrm{Supp}(T)$). In every vertical fiber $\{z\} \times \mathbb{C}$ we can define a vertical filled Julia set $K_z$ as the set of points $(z, w)$ where the sequence $(q_z^n(w))_{n\geq 1}$ is bounded. The vertical Julia set is by definition $J_z = \partial K_z$, and it coincides with the locus of non-normality of $(q_z^n)_{n\geq 1}$. Notice that the sets $K_z$ are locally uniformly bounded. For polynomial skew-products, $J_2$ can be characterized as being the closure of the set of repelling periodic orbits [16, §4], and $J_2 = \overline{\bigcup_{z\in J_p} J_z}$. Therefore in our situation, $J \cap (\Delta \times \mathbb{C})$ is disjoint from $J_2$. In particular $T \wedge T = 0$ in $\Delta \times \mathbb{C}$.

The following set of assumptions on an endomorphism $f$ will be denoted by (A):

(A1) There exists an invariant hyperbolic compact and connected subset $E_0 \subset J(q_0)$, relative to $q_0$ (not reduced to a point).

(A2) There exists a critical point $c$ for $q_0$ and an integer $k$ such that $c \notin E_0$ and $q_0^k(c)$ is a repelling periodic point $m$ belonging to $E_0$.

(A3) There is a local component $V$ of the critical set $\mathrm{Crit}(f)$ at $(0, c)$ such that $f^k(V)$ is smooth at $(0, m)$ and not periodic.

In Proposition 2.1 below we show that such a mapping exists for every $d \geq 3$.

We say that a closed positive current in $\Omega \subset \mathbb{C}^2$ is *quasi-laminar* if for $\sigma_T$-a.e. $x$, there exists a germ of holomorphic disk $D$ containing $x$ such that $u|_D$ is harmonic, where $u$ is any local potential for $T$. If $T$ is the Green current for an endomorphism of $\mathbb{P}^2$, the harmonicity of its potential $G$ on $D$ is equivalent to the equicontinuity of $f^n$ along $D$ [12]. Therefore, the quasi-laminarity of $T$ in some open set means that the field of Fatou directions is integrable (in a measure-theoretic sense) to a field of Fatou disks.

Here is the precise statement of our main theorem.

**Theorem 1.1.** *Let $f$ be a regular polynomial skew product in $\mathbb{C}^2$ of the form $f(z, w) = (p(z), q_z(w))$, where $p(z) = e^{2\pi i\theta}z + \cdots + z^d$ has a linearizable fixed point at the origin, and satisfying the above assumptions (A). Then there exists a neighborhood $N$ of 0 such that the Green current $T$ is not quasi-laminar in any open subset of $N \times \mathbb{C}$.*

Equivalently, if $\Omega \subset N \times \mathbb{C}$ is any open subset, there exists a set $E \subset \Omega$ of positive trace measure such that for every $x \in E$, there us no Fatou disk through $x$.

The following corollary could easily be deduced from the theorem together with the fact that $T \wedge T = 0$ in $\Delta \times \mathbb{C}$. We will actually give a direct proof.

**Corollary 1.2.** *Let $f$ and $N$ be as in Theorem 1.1. The Green current $T$ is not laminar in any open subset of $N \times \mathbb{C}$.*

The first (non-dynamical) examples of non-laminar positive closed currents in $\mathbb{C}^2$ with continuous potential and vanishing self-intersection were constructed in [9]. Since $T \wedge T = 0$ in $\Delta \times \mathbb{C}$, $T$ can thus be considered as a kind of dynamical version of these examples. Notice however that, as opposed to the "Wermer examples" of [9], the Julia sets that we construct contain many holomorphic disks.

Recall that the set $J_2'$ was defined to be the set of points such that locally there does not exists a neighborhood that is filled with germs of Fatou disks. In the particular context of polynomial skew products in $\mathbb{C}^2$, the question whether $J_2$ equals $J_2'$ is raised in [16, §8]. The following corollary is an immediate consequence of Theorem 1.1.

**Corollary 1.3.** *Let $f$ and $N$ be as in Theorem 1.1. Then $J \cap (N \times \mathbb{C})$ is contained in $J_2'$, so in particular $J_2 \subsetneq J_2'$.*

Indeed, if $\Omega \subset N \times \mathbb{C}$ was an open set intersecting $J$ with the property that through every point in $\Omega$ there exists a Fatou disk, then $T$ would be quasi-laminar in $\Omega$, a contradiction.

Theorem 1.1 and Corollary 1.2 will be established in §2. In §3, we discuss the higher dimensional case, and comment about a question raised by Fornæss and Sibony in [13].

## 2. Proof of the main results

2.1. **Examples.** We start by exhibiting examples satisfying the assumptions (A) of the main theorem.

**Proposition 2.1.** *Let $d \geq 3$ and fix $p(z) = e^{2\pi i\theta}z + \cdots + z^d$ as in Theorem 1.1. Then there exists a polynomial $q(z, w)$ such that the skew product $f(z, w) = (p(z), q_z(w))$ satisfies (A).*

*Proof.* To carry out the few computations to come, we need to find $q_0$ in a rather explicit form. For this it is convenient to use the work of P. Roesch [20]. Assume that $g(w)$ is a polynomial of degree $d$ with a super-attracting point of multiplicity at least $d-1$ at the origin, and denote its immediate basin by $B(0)$. We denote by $\mathcal{S}$ the parameter space of such mappings. Since polynomials can be parameterized by specifying their critical points, $\mathcal{S}$ is one-dimensional, and parameterized by the remaining critical point (this differs from the presentation in [20]). In this situation, $\partial B(0)$ is a Jordan curve (we will not use this fact) and it may happen that the free critical point $c$ satisfies $c \notin \overline{B(0)}$ and $g(c) \in \partial B(0)$. Indeed consider the set $\mathcal{H} \subset \mathcal{S}$ of such mappings $g$ such that $g^n(c)$ converges to 0. Then $\mathcal{H} = \bigcup_{i\geq 0} \mathcal{H}_i$, where $g \in \mathcal{H}_i$ if and only if $i$ is the first integer such that $g^i(c) \in B(0)$. For every $i$, $\mathcal{H}_i$ is non-empty. Roesch shows that if $i > 0$, if $\Omega$ is a component of $\mathcal{H}_i$, and if $g \in \partial\Omega$, then $c \notin \overline{B(0)}$ and $g^i(c) \in \partial B(0)$. In particular it is clear that $g$ admits no parabolic periodic points neither any recurrent critical point. Hence from Mañé's Theorem [18], the invariant compact set $\partial B(0)$ is hyperbolic, in particular it is locally persistent and repelling periodic points are dense there. Since $g^i(c)$ does not persistently belong to $\partial B(0)$, by perturbing $g$ within $\mathcal{S}$, we may further assume that $g^i(c)$ is a repelling periodic point. We fix $q_0$ to be such a $g$, with, say, $i = 1$, which thus satisfies (A1) and (A2).

It remains to check the third assumption (A3) that the component of the critical set containing $(0, q_0(c))$ in $\mathbb{C}^2$ is not periodic. In order to use some results from [11], we introduce the parameterization of $\mathcal{S}$ given by

$$\mathcal{S} = \left\{ \frac{w^d}{d} - c\frac{w^{d-1}}{d-1}, c \in \mathbb{C} \right\},$$

where $c$ is both the parameter and the free critical point. We choose $q_z(w)$ of the form

$$q_z(w) = \frac{w^d}{d} - c\frac{w^{d-1}}{d-1} + \beta z,$$

where $c$ is fixed so that $q_0$ is as above, and $\beta \in \mathbb{C}$ is a free parameter (of course then we can go back to the initial normalization $q_z(w) = w^d + \text{l.o.t.}$ by a linear conjugacy). Let us show that (A3) is fulfilled for generic $\beta$.

The critical set of $f$ is

$$\begin{aligned}\operatorname{Crit}(f) &= (\operatorname{Crit}(p) \times \mathbb{C}) \cup \left\{ (z, w),\ \frac{\partial q}{\partial w}(z, w) = 0 \right\} \\ &= (\operatorname{Crit}(p) \times \mathbb{C}) \cup (\mathbb{C} \times \{0\}) \cup (\mathbb{C} \times \{c\}),\end{aligned}$$

and we need to prove that $\mathbb{C} \times \{c\}$ is not preperiodic for generic $\beta$. Notice that $f^i(\Delta \times \mathbb{C})$ is smooth, because the image of a graph over $\Delta$ is a graph (see the proof of Lemma 2.2 for some details on this).

Let $z_1$ be a fixed point of $p$, distinct from 0. We work in the fixed fiber $\{z_1\} \times \mathbb{C}$. It is enough to prove that $c$ is not persistently preperiodic in this fiber. For this, we show that it escapes for large $\beta$. To stick with the notation of [11], put $\beta z_1 = a^d$. Then the restriction of $f$ to $\{z_1\} \times \mathbb{C}$ becomes $P_{c,a} = \frac{w^d}{d} - c\frac{w^{d-1}}{d-1} + a^d$. We have that $P_{c,a}(0) = a^d$ and $P_{c,a}(c) = -\frac{c^d}{d(d-1)} + a^d$. By [11, Prop. 6.3], since $c$ is fixed,

$$G(c,a) = \max\left(g_{P_{c,a}}(0), g_{P_{c,a}}(c)\right) = \log^+ \max(|a|, |c|) + O(1) = \log^+ |a| + O(1)$$

($g_{P_{c,a}}$ is the dynamical Green function of $P_{c,a}$). Now by [11, Lemma 6.5], for every $z \in \mathbb{C}$ we have that

$$\max(g_{P_{c,a}}(z), G(c,a)) \geq \log\left|z - \frac{c}{d-1}\right| - \log 4.$$

Applying this to $z = P_{c,a}(c) = a^d + O(1)$, we see that

$$\max(g_{P_{c,a}}(P_{c,a}(c)), G(c,a)) \geq d \log|a| + O(1) \text{ as } a \to \infty.$$

Thus when $|a|$ is large, $g_{P_{c,a}}(P_{c,a}(c)) > G(c,a)$ and we deduce that $P_{c,a}(c)$, hence $c$, escapes, which was the desired result. The proof is complete. $\square$

2.2. **Proofs.** We now prove Theorem 1.1 and Corollary 1.2. Let $f$ be as in the statement of the theorem. Replacing it by an iterate we may assume that the integer $k$ in (A) equals 1 and that $(0,m)$ is fixed. By convention, we only consider neighborhoods $N \times \mathbb{C}$ of the central fiber such that $N$ is invariant under $p$, that is, $N$ is a disk in the linearizing coordinate. Throughout the proof we freely reduce $N$ without changing its notation. We denote by $\pi_1$ the first coordinate projection in $\mathbb{C}^2$.

*Step 1: geometry of the Julia set.* We first study the persistence of $E_0$ in the fibers close to $\{0\} \times \mathbb{C}$.

**Lemma 2.2.** *There exists a neighborhood $N$ of 0 and a $f$-equivariant holomorphic motion $E$ of $E_0$ over $N$, that is, if $\gamma_w$ denotes the graph of the motion through $(0,w)$, we have $f(\gamma_w) = \gamma_{q_0(w)}$.*

*Furthermore, $f$ is vertically expanding along $E$, that is, there exists $C > 0$ and $\beta > 1$ such that if $(z,w) \in E$, $\left\|df^n_{(z,w)}(0,1)\right\| \geq C\beta^n$.*

Notice that $E_0$ is not a hyperbolic set for $f$ in $\mathbb{C}^2$ so we cannot simply invoke hyperbolicity here. Still, the argument is based on standard ideas.

*Proof.* Since $E_0$ is a hyperbolic set for $q_0$, for every $w \in E$ there exists a disk $D_w$ centered at $w$ such that $q_0$ is invertible on $D_w$ and if $q_{0,w}^{-1}$ denotes the inverse branch of $q_0$ sending $q_0(w)$ to $w$, then $q_{0,w}^{-1}(D_{q_0(w)}) \Subset D_w$. Hence if $N$ is a sufficiently small $p$-invariant neighborhood of 0, for every $(0,w) \in \{0\} \times E_0$, $f$ is invertible in $N \times D_w$ and if as before $f_w$ is the corresponding inverse branch, $f_w^{-1}(N \times D_{q_0(w)})$ is horizontally contained in the "topological bidisk" $N \times D_w$. Recall that this means that if $L$ is any vertical line in $N \times D_w$, $f_w^{-1}(N \times D_{q_0(w)}) \cap L \Subset L$. Thus the situation is similar to that of a crossed mapping (see Hubbard and Oberste-Vorth

[14]) with the difference that since $f$ preserves the vertical fibers, it maps the vertical piece of the boundary of $f_w^{-1}(N \times D_{q_0(w)})$ to the vertical boundary of $N \times D_{q_0(w)}$, that is,

$$f\left(\overline{f_w^{-1}(N \times D_{q_0(w)})} \cap (\partial N \times D_w)\right) \subset \partial N \times D_{q_0(w)}.$$

In addition, $f|_{N\times D_w}$ has degree 1 in the sense that the image of a graph over $N$ is a graph over $N$. Indeed write this graph as $(t, \gamma(t))$, $t \in N$, whose image is $(p(t), q_t(\gamma(t)))$, and observe that $p : N \to N$ is invertible.

Let now $w_0 \in E_0$ and consider its orbit $(w_n)$ under $q_0$. The sequence $f_{w_0}^{-1} \cdots f_{w_{n-1}}^{-1}(N \cap D_{w_n})$ defines a nested sequence of horizontal topological bidisks in $N \times D_{w_0}$. Exactly as in [14], the contraction property of the vertical Poincaré metric shows that it converges to a graph $\gamma_{w_0}$. It is straightforward to check that the family of graphs $\gamma_w$ is the desired holomorphic motion. The vertical expansion statement follows as well. □

Let us now recall a few facts from the work of Jonsson [17]. For every $z$, let $\mu_z = T \wedge [\{z\} \times \mathbb{C}]$, which is a probability measure supported exactly on $J_z$. Every invariant circle $C$ in the Siegel disk admits a unique $p$-invariant probability measure which we denote by $\lambda_C$. Then the measure $\mu_C = \int \mu_z \lambda_C(dz)$ is $f$-invariant and ergodic. Given such a family of circles and a probability measure $\Lambda$ on this family we can integrate with respect to $C$ to get an invariant measure for $f$. Taking $\Lambda$ to be smooth, we see in particular that there exists invariant probability measures absolutely continuous with respect to $\sigma_T$, more precisely, with respect to $T \wedge idz \wedge d\overline{z}$. Observe that by the slicing formula, $T \wedge idz \wedge d\overline{z}$ is an integral of $\mu_z$, so we infer that the support of $T \wedge idz \wedge d\overline{z}$ equals $\overline{\bigcup_{z\in\mathbb{C}} J_z}$, which is smaller than $J$ in general (this happens for instance for $f(z, w) = (z^2, w^2)$). This is not the case in our situation, as the following lemma shows.

**Lemma 2.3.** *In $N \times \mathbb{C}$, the Julia set coincides with $\overline{\bigcup_{z\in N} J_z}$.*

*Proof.* Let $x = (z, w) \in J \cap (N \times \mathbb{C})$. There exists an arbitrary small holomorphic disk $D$ through $x$ such that $f^n|_D$ is not a normal family. We claim that for large $n$, $f^n(D)$ intersects $E$. This implies the lemma because as $f$ is vertically expanding along $E$, $E$ is contained in $\bigcup_{z\in N} J_z$, and since moreover $\bigcup_{z\in\mathbb{C}} J_z$ is totally invariant, we conclude that $D \cap \bigcup_{z\in N} J_z \neq \emptyset$.

To prove the claim, we use a Kobayashi hyperbolicity argument. Since $f^n|_D$ is not a normal family, by the Zalcmann-Brody lemma, some subsequence can be reparameterized to converge to a non constant entire curve $\phi : \mathbb{C} \to N \times \mathbb{C}$, which must then be contained in a vertical line. Now we claim that there is a pluriharmonic function $h$ in $(N \times \mathbb{C}) \setminus E$ that is bounded from below and tends to infinity as $w \to \infty$. So if for all $n \geq 0$, $f^n(D)$ avoids $E$, $h \circ \phi$ is harmonic and bounded from below on $\mathbb{C}$, hence constant, and we get a contradiction.

To construct the function $h$, we do as follows. Since $E_0$ is a non-trivial continuum, it is not polar so it carries a probability measure $m_0$ with continuous potential. Let $m_z$ be the image of $m_0$ under the holomorphic motion at time $z$. In [8, §6] it is shown that the function

$$u : (z, w) \longmapsto \int \log |w - s| \, dm_z(s)$$

is continuous, psh, and $dd^c u = \int [\gamma_w] dm_0(w)$. Therefore it is enough to choose $h = u$ and we are done. □

*Step 2: persistent intersection with the post-critical set.* Recall that by assumption there is a smooth component $W$ of $f(\mathrm{Crit}(f))$ passing through the fixed point $(0, m) \in E$, and that is not fixed. This means that it does not coincide with the continuation $\gamma_m$ of $(0,m)$. Recall also that since $\theta$ is linearizable, there is a foliation of the (punctured) Siegel disk $\Delta$ by invariant circles.

The next lemma is where we use the connectedness of $E_0$.

**Lemma 2.4.** *Reducing $N$ if needed, the following property holds: if $C \subset N$ is any $p$-invariant circle, there is an intersection between $E$ and $W$ over $C$.*

*Proof.* The simplest situation is when $W$ is locally a graph over the $z$ coordinate and it is transverse to $\gamma_m$. In this case, $W \cap E$ is locally homeomorphic to $E_0$ near $(0,m)$. The curves $\pi_1^{-1}(C)$ define a foliation of $W \setminus \{(0,m)\}$ by nested Jordan curves near $(0,m)$. Since $E_0$ is connected, we infer that all curves $\pi_1^{-1}(C)$ sufficiently close to the origin intersect $E$ and we are done.

The argument is slightly more delicate when $W$ is a graph tangent to $\gamma_m$ at $(0,m)$. Let $k$ be the order of contact between these two curves. By [2, Lemma 6.4], for $w \neq m$ close to $m$, $\gamma_w$ intersects $W$ transversely in exactly $k$ points. There exist local coordinates $(x,y)$ near $(0,m)$ in which $\gamma_m = \{y=0\}$ and $W = \left\{y = x^k\right\}$. As before, the point is to prove that $\pi_1(E \cap W)$ is connected (here the projection $\pi_1$ refers to the new coordinates). By Slodkowski's theorem [22], we extend the holomorphic motion to a neighborhood of $E$. Let $\phi$ be the mapping defined near the origin in $\mathbb{C}$ by

$$\phi : x \mapsto (x, x^k) \mapsto \mathrm{hol}(x, x^k) \in \{y = 0\}\,,$$

where hol is the holonomy map sending $(x,y)$ to the central fiber by following the holomorphic motion. We want to show that $\phi^{-1}(E_0)$ is connected near the origin. The idea is that $\phi$ is a topological deformation of $\phi_0 : x \mapsto x^k$. Indeed, by deforming the Beltrami coefficient (ellipse field) of the holomorphic motion $E$ to a trivial one (field of circles), we find a continuous family of laminations $(E^s)_{s\in[0,1]}$ with $E^1 = E$ and $E^0$ is the foliation by horizontal lines. Now, if $y \neq 0$ and $\gamma_y^s$ denotes the leaf of $E^s$ through $(0,y)$, by [2, Lemma 6.4], for $s \in [0,1]$ the intersection points between $W$ and $\gamma_y^s$ move continuously and without collision, thus we can follow them from $s=1$ to $s=0$. It follows that $\phi^{-1}(E_0)$ is homeomorphic to $\phi_0^{-1}(E_0)$ near the origin. Since $E_0$ is connected and contains 0, $\phi_0^{-1}(E_0)$ is connected near the origin, and we are done.

The last case is when $W$ is tangent to the vertical axis at $(0,m)$. Then it is transverse to the lamination, so $W \cap E$ is connected near $(0,m)$ and so does $\pi_1(W \cap E)$. □

*Step 3': propagation of laminarity and conclusion in the laminar case.* To make the argument easier to understand, we start by giving a direct proof of Corollary 1.2.

Assume that there is an open set $\Omega \subset N \times \mathbb{C}$ in which the Green current $T$ is (nonzero and) laminar, that is, it is an integral of compatible holomorphic disks. Let us first observe that these disks cannot all be contained in vertical fibers[1]. Indeed otherwise in $\Omega$ we would have that $T \wedge idz \wedge d\bar{z} = 0$, thereby contradicting Lemma 2.3 (see the comments preceding the lemma). Thus there exists a disk $U$ in $N$ and a non-trivial uniformly laminar current $S \leq T$ made of graphs over $U$. Saturate $U$ under the dynamics of $p$ to obtain $A = \bigcup_{n\geq 0} p^n(U)$ which is typically an annulus (unless $U$ contains the origin). We now show that the existence of such a $S$ forces $J$ to have a uniform laminar structure in $A \times \mathbb{C}$.

1. It is likely that the set of vertical disks subordinate to $T$ is of zero trace measure but we could not prove it.

**Lemma 2.5.** *There exists a lamination in $A \times \mathbb{C}$, whose leaves are locally graphs over the first coordinate, such that $J \cap (A \times \mathbb{C})$ is a union of leaves. Moreover, every holomorphic disk contained in $J$ must be compatible with this lamination, that is, contained in a leaf.*

*Proof.* Let $\mathcal{L}$ be the lamination by graphs over $U$ underlying $S$. By assumption, $\mathcal{L}$ is of positive trace measure. As already explained, the image of a graph $\Gamma$ over $U$ is a graph over $p(U)$ and moreover $f : \Gamma \to f(\Gamma)$ is of multiplicity 1, so if $\Gamma$ is a leaf of $\mathcal{L}$, for every $n \geq 1$, $f^n(\Gamma)$ is a graph over $p^n(U)$, contained in $J$ (more precisely, in $\bigcup_{z \in N} J_z$). We claim that these graphs are compatible (that is, they are disjoint or coincide). Indeed, write $S$ as $S = \int [\gamma_\alpha] m(d\alpha)$ for some measured family of graphs $(\Gamma_\alpha)$, and for notational ease assume that $n = 1$. Then $f_*S$ expresses as $\int [f(\gamma_\alpha)] m(d\alpha)$, that is, it is a uniformly woven current. Now, observe that $f_*S \leq f_*T = dT$, hence $f_*S$ is a positive closed current in $p(U) \times \mathbb{C}$, with continuous potential (see [2, Lemma 8.2]) and such that $(f_*S)^2 \leq d^2T^2 = 0$. The intersection of uniformly woven currents is geometric (see e.g. [7, Prop. 2.6]), so we infer that for a.e. $(\alpha, \beta)$, $f(\Gamma_\alpha)$ and $f(\Gamma_\beta)$ are compatible. To get the result for every $(\alpha, \beta)$, assume that $\alpha_0$ and $\beta_0$ are such that $f(\Gamma_{\alpha_0})$ and $f(\Gamma_{\beta_0})$ have a non-trivial intersection. Then by the persistence of intersections of curves in $\mathbb{C}^2$, for $\alpha$ (resp. $\beta$) close to $\alpha_0$ (resp. $\beta_0$) we get that $[f(\Gamma_\alpha)] \wedge [f(\Gamma_\beta)] > 0$, which contradicts the generic non-intersection. So we conclude that the leaves of $f(\mathcal{L})$ are disjoint, as asserted. In particular, $f(\mathcal{L})$, being a closed family of disjoint graphs over $p(U)$, is a lamination. The same argument shows that for any two integers $n$ and $m$, the leaves of the laminations $f^n(\mathcal{L})$ and $f^m(\mathcal{L})$ are compatible, that is, they admit no proper intersections.

Reduce $U$ a little bit to obtain an open set $U'$, with corresponding annulus $A'$ and lamination $\mathcal{L}'$. Consider the set $\bigcup_{n \geq 0} f^n(\mathcal{L}')$. This is a uniformly bounded family of compatible graphs over sets of the form $p^n(U')$ in $A'$. Notice that $p^n(U')$ is of uniform size and stays at uniform distance from $\partial(p^n(U))$. Recall that over any $p$-invariant circle $C$, there is an $f$-invariant ergodic probability measure $\mu_C$, and that $T \wedge idz \wedge d\overline{z}$ is equivalent to an integral of measures of this form. Since for every $z \in U'$, $0 < S \wedge [\{z\} \times \mathbb{C}] \leq \mu_z$, we get that for every invariant circle in $A'$, $\mathcal{L}'$ is of positive $\mu_C$ measure. By ergodicity we infer that for every such $C$, $\bigcup_{n \geq 0} f^n(\mathcal{L}')$ is of full $\mu_C$ measure, and integrating with respect to $C$ we deduce that it is of full $(T \wedge idz \wedge d\overline{z})$ measure in $A'$. By Lemma 2.3, $\mathrm{Supp}(T \wedge idz \wedge d\overline{z}) = J$ so we conclude that $\bigcup_{n \geq 0} f^n(\mathcal{L}')$ is dense in $J \cap (A' \times \mathbb{C})$. Finally, $\overline{\bigcup_{n \geq 0} f^n(\mathcal{L}')}$ defines a lamination of $J \cap (A' \times \mathbb{C})$. Indeed for every $x \in J \cap (A' \times \mathbb{C})$ belongs to the cluster set of a family of graphs $\Gamma'_j$ over $p^{n_j}(U')$. Since these graphs extend to $p^{n_j}(U)$ and are uniformly bounded, we obtain a limiting graph through $x$. Persistence of isolated intersections implies that any two limiting graphs must be compatible, and we are done.

The second statement of the lemma is obvious: since $J$ is laminated, every disk contained in $J$ and not contained in a leaf would have transverse intersections with nearby leaves, and by using the holonomy we would get that $J$ has non-empty interior, a contradiction. □

We are now ready to conclude the proof of Corollary 1.2. It follows from the previous lemma that the holomorphic motion $E$ is compatible with the laminar structure of $J$. By Lemma 2.4, there is a component $V$ of $\mathrm{Crit}(f)$ in $\Delta \times \mathbb{C}$ such that $f(V)$ admits a non trivial intersection with a leaf of this holomorphic motion over $A$. Let $x \in V \cap (N \times \mathbb{C})$ such that $f(x) = y$ is such an intersection point, and denote by $\Gamma$ the graph of $E$ through $y$. Since such intersections are persistent, shifting $y$ slightly, we may further assume that $\Gamma$ is not contained in (some other component of) $f(\mathrm{Crit}(f))$. Since $J$ is totally invariant, $f^{-1}(\Gamma)$ must be contained in $J$. Let us show that this is contradictory.

Writing in coordinates $\Gamma = \{(t, \gamma(t)), t \in U\}$ we infer that an equation for $f^{-1}(\Gamma)$ is $h(z,w) = q(z,w) - \gamma(p(z)) = 0$. Since $p$ has no critical points in $\Delta$ and $x \in \mathrm{Crit}(f)$, $\frac{\partial h}{\partial w} = \frac{\partial q}{\partial w} = 0$ at $x$. If $\frac{\partial h}{\partial z}(x) \neq 0$, this means that $f^{-1}(\Gamma)$ has a vertical tangency at $x$, which is impossible by Lemma 2.5. Otherwise $\frac{\partial h}{\partial z}(x) = 0$ and there are two possibilities: if the equation $(h = 0)$ is of multiplicity 1, this implies that $f^{-1}(\Gamma)$ is singular at $x$, which again contradicts Lemma 2.5. The other option is that $(h = 0)$ is smooth with non-trivial multiplicity, but then it must be contained in the critical set, which contradicts our assumptions on $\Gamma$. Thus in any case, we arrive at a contradiction, and the proof of Corollary 1.2 is complete □

*Step 3: conclusion in the general case.* We now handle the general case, that is we prove Theorem 1.1. We will design a different argument for the propagation of laminarity, and obtain a weaker version of Lemma 2.5 which will be sufficient for our purposes.

Assume that $T$ is quasi-laminar in some open subset $\Omega \subset N \times \mathbb{C}$. Recall from [10] that for $\sigma_T$-a.e. $x \in N \times \mathbb{C}$, there is a unique Fatou direction at $x$, along which the dynamics is not expanding. This direction coincides with the tangent space to $T$ at $x$, which is well-defined because $T$ is simple a.e. on $J_1 \setminus J_2$ [10, Thm 3.4]. If $D$ is any Fatou disk through $x$, $T_xD$ must then coincide with the tangent space to $T$ at $x$.

By Lemma 2.3, $T \wedge idz \wedge d\overline{z}$ is non-zero in $\Omega$. We claim that for $(T \wedge idz \wedge d\overline{z})$-a.e. $x$, the tangent space of $T$ at $x$ is not vertical[2]. Indeed, write $T$ as a differential form with measure coefficients $T = \sum_{\alpha,\beta\in\{z,w\}} T_{\alpha,\overline{\beta}}\ id\alpha \wedge d\overline{\beta}$. Then $\sigma_T = T_{z,\overline{z}} + T_{w,\overline{w}}$, and by the Radon-Nikodym theorem there exist measurable functions $h_{\alpha,\overline{\beta}}$ such that for $\alpha, \beta \in \{z, w\}$ $T_{\alpha,\overline{\beta}} = h_{\alpha,\overline{\beta}}\sigma_T$. The matrix $(h_{\alpha,\overline{\beta}})$ has rank 1 $\sigma_T$-a.e. on $J_1 \setminus J_2$, and the tangent vector to $T$ is vertical precisely when $h_{w,\overline{w}} = 0$ (which implies $h_{w,\overline{z}} = h_{z,\overline{w}} = 0$). We conclude by observing that since $T \wedge idz \wedge d\overline{z} = T_{w,\overline{w}} = h_{w,\overline{w}}\sigma_T$, the set $\{h_{w,\overline{w}} = 0\}$ has zero $T \wedge idz \wedge d\overline{z}$ measure.

Thus in $\Omega$ there exists a set of positive trace measure of points $x$ such that any Fatou disk at $x$ must have a non-vertical tangent space at $x$. For a small value of $r_0$ to be determined shortly, we let $L$ be the set of points $x \in J \cap (N \times \mathbb{C})$ such that there exists a Fatou disk through $x$, which is a graph over $D(\pi_1(x), 2r_0)$. For $r > 0$, denote by $\mathcal{G}(r)$ the set of graphs over a disk of radius $r$, that are contained in $\bigcup_{\zeta\in\Delta} K_\zeta$. The compactness of $\Gamma(2r_0)$ implies that $L$ is a compact subset of $J$ which is of positive $T \wedge idz \wedge d\overline{z}$-measure for small enough $r_0$. Given such a $r_0$, there exists $0 < r \leq r_0$ so that for every $z \in N$, and every $n \geq 0$, $p^n(D(z, 2r_0))$ contains $D(p^n(z), 2r)$.

Recall from [16] that $z \mapsto J_z$ is lower-semicontinuous. In particular $\delta = \min_{z\in\overline{N}} \mathrm{diam}(J_z)$ is a well defined positive quantity. By compactness of the set of graphs, if $r$ is small enough and $\gamma$ is a graph over some disk $D(z, 2r)$ that is contained in $\bigcup_{\zeta\in\Delta} K_\zeta$, then $\mathrm{diam}(\gamma|_{D(z,r)}) < \frac{\delta}{4}$. We fix $r_0$ such that the associated $r$ has this property.

To prove uniqueness and disjointness properties for the Fatou disks, we will use an argument based on an expansivity property of the fiberwise dynamics, which itself follows from a form of mixing in the fiber direction. The statement we need is the following.

**Lemma 2.6.** *Let as above $\delta = \min_{z\in\overline{N}} \mathrm{diam}(J_z)$. Then for every $z \in \overline{N}$, and every subset $A \subset J_z$ of positive $\mu_z$-measure, there exists $n \geq 0$ such that $\mathrm{diam}(f^n(A)) > \frac{3\delta}{4}$.*

2. Another argument consists in showing that over every invariant circle, the Lyapunov exponent of $\mu_C$ in the vertical direction is at least $\frac{\log d}{2}$

*Proof.* Fix $z \in \overline{N}$ and put $z_n = p^n(z)$. We first claim that if $\varphi \in L^2(\mu_z)$ and $\psi$ is any test function in $\Delta \times \mathbb{C}$, then

$$\left| \int (\psi \circ f^n)\varphi \, \mu_z - \int \psi \, \mu_{z_n} \int \varphi \, \mu_z \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{2}$$

Indeed observe first that by approximating in $L^2(\mu_z)$, it is enough to establish this for a smooth $\varphi$. Using the relation $f_*(h\mu_z) = \frac{1}{d}(f_*h)f_*\mu_z$ for the slice measures, we rewrite

$$\int (\psi \circ f^n)\varphi \, \mu_z = \int \left(\frac{1}{d^n}(f^n)_*\varphi\right) \psi \, \mu_{z_n},$$

so that (2) boils down to

$$\left| \int \left( \frac{1}{d^n}(f^n)_*\varphi - \int \varphi \, \mu_z \right) \psi \, \mu_{z_n} \right| \underset{n\to\infty}{\longrightarrow} 0. \tag{3}$$

Now for smooth $\varphi$, [17, Prop. 2.6] asserts that

$$\mu_{z_n}\left( \left\{ w \in \{z_n\} \times \mathbb{C}, \ \left| \frac{1}{d^n}(f^n)_*\varphi(w) - \int \varphi \, \mu_z \right| > t \right\} \right) \leq \frac{C \left\|\varphi\right\|_{C^2}}{td^n},$$

from which (3), hence (2), readily follows.

Fix $z_0$ such that $\operatorname{diam}(J_{z_0}) > \frac{3\delta}{4}$. Fix $x_0$ and $x_0'$ in $J_{z_0}$ at distance greater than $\frac{3\delta}{4}$ from each other, and $\psi_0$ and $\psi_0'$ be two bump functions (in $\Delta \times \mathbb{C}$) supported in small respective neighborhoods of $x_0$ and $x_0'$. The continuity of $z \mapsto \mu_z$ implies that if $z$ is close to $z_0$, the integrals $\int \psi_0 \, \mu_z$ and $\int \psi_0' \, \mu_z$ are positive.

To conclude the proof of the lemma, we fix a subsequence $n_j$ such that $z_{n_j} \to z_0$, and set $\varphi = \mathbf{1}_A$. For $\psi = \psi_0$ or $\psi = \psi_0'$, from (2) we get that for large $j$, $\int_A \psi \circ f^{n_j} d\mu_z > 0$. This implies that $f^{n_j}(A)$ intersects both $\operatorname{Supp}(\psi_0)$ and $\operatorname{Supp}(\psi_0')$, and the result follows. □

The basic mechanism deriving laminarity from Lemma 2.6 is contained in the following lemma.

**Lemma 2.7.** *Let $z \in \overline{N}$ and assume that there exists a compact set $A \subset J_z$ of positive $\mu_z$-measure, together with a measurable family of Fatou disks $\{\gamma_x, \ x \in A\}$, that are graphs over $D(z, 2r)$.*

*Then the graphs $\gamma_x$ are disjoint over $D(z, r)$.*

*Proof.* Observe first that if $\gamma$ is a Fatou graph over $D(z, 2r)$ such that $\gamma(z) \in J_z$, then by normality of the iterates, $f^n(\gamma$ is uniformly bounded, hence $\gamma \subset \bigcup_{\zeta\in\Delta} K_\zeta$. Introduce the space $\mathcal{G}_z(2r)$ of graphs over $D(z, 2r)$ that are contained in $\bigcup_{\zeta\in\Delta} K_\zeta$, endowed with the compact-open topology, which is a compact metrizable space. In particular we can project the measure $\mu_z$ by $e : x \mapsto \gamma_x$. This mapping is injective since $\gamma_{(z,w)}(z) = w$, so we infer that the image measure $e_*(\mu_z)$ has no atoms.

Now assume that there exists $x \neq y \in J_z$ such that $\gamma_x$ and $\gamma_y$ intersect over $D(z, r)$. Since $e_*(\mu_z)$ is diffuse, by the continuity of isolated intersections, there exists a set $A' \subset A$ of positive $\mu_z$ measure such that if $y_1 \in A'$ then $\gamma_{y_1}$ and $\gamma_x$ intersect over $D(z, r)$. Therefore for $y_1, y_2 \in A'$ there is a chain $(\gamma_{y_1}, \gamma_x, \gamma_{y_2})$ of three intersecting Fatou graphs connecting $y_1$ and $y_2$. Now recall that by definition of $r$, if $\gamma$ is one of these graphs, then for every $n$, $\operatorname{diam}(f^n(\gamma))$ is smaller than $\frac{\delta}{4}$. From this we deduce that for all $n \geq 0$, $\operatorname{dist}(f^n(y_1), f^n(y_2)) < \frac{3\delta}{4}$, which contradicts Lemma 2.6 and thereby completes the proof. □

Recall our assumption that there exists a set $L$ of positive trace measure of points $x = (z, w)$ admitting a Fatou disk through $x$ that is a graph over $D(z, 2r_0)$. By the slicing formula, $T \wedge idz \wedge d\bar{z}$ is an integral of $\mu_z$, hence there exists a set of positive Lebesgue measure of of invariant circles $C$ such that $\mu_C(L) > 0$.

**Lemma 2.8.** *Let $C$ as above be an invariant circle such that $\mu_C(L) > 0$. Let $z \in C$ and $x = (z, w) \in J_z$. Then there exists a unique Fatou disk $\gamma_x$ through $x$ which is a graph over $D(z, 2r)$. In addition, if $x, x' \in J_z$, are distinct points, then $\gamma_x$ and $\gamma_{x'}$ are disjoint over $D(z, r)$.*

*Proof.* For the existence, we argue as in Lemma 2.5. Indeed, by the ergodicity of $\mu_C$ and the definition of $r$, we infer that for $\mu_C$-a.e. $x = (z, w)$, there exists a Fatou disk through $x$ that is a graph over $D(z, 2r)$. By the compactness of the space of graphs, we extend this family to its closure, thus obtaining for every $z \in C$ and every $w \in J_z$ a Fatou graph over $D(z, 2r)$ containing $(z, w)$.

For the uniqueness, fix a Fatou graph $\gamma$ over $D(z, 2r)$ with $\gamma(z) = w$. Recall that the measure $\mu_z|_{L\cap(\{z\}\times\mathbb{C})}$ is diffuse, so its support has no isolated points. For every $x \in J_z$, let $\mathcal{D}_x \subset \mathcal{G}_z(2r)$ be the set of Fatou graphs over $D(z, 2r)$ through $x$, which by assumption is non-empty. Being bounded and closed, $\mathcal{D}_x$ is compact. Put $\mathcal{D} = \bigcup_{x\in J_z} \mathcal{D}_x$, which is also compact.

Consider the natural continuous map $\phi : \mathcal{D} \to J_z$ defined by $\gamma \mapsto \gamma(z)$. The "measurable axiom of choice" (see [3, Thm. 6.9.7] for the version that we use here) implies that this map admits a Borel section, that is, an injective Borel map $e$ such that $\phi \circ e = \mathrm{Id}$. Without changing these properties we may assume that $e(x) = \gamma$. By Lemma 2.7, the graphs $e(x)$, $x \in J_z$ are disjoint. Now let $(x_n)$ be any sequence in $J_z$ converging to $x$. Since for every $n$, $e(x_n) \cap e(x) = \emptyset$, by the Hurwitz theorem any cluster limit of $e(x_n)$ must coincide with $e(x) = \gamma$. If $\gamma'$ is any other Fatou graph at $x$, we can freely modify $e$ so that $e(x) = \gamma'$, so the limit of $e(x_n)$ also equals $\gamma'$ and we conclude that $\gamma$ is unique, as desired.

The disjointness assertion of the lemma now follows directly from Lemma 2.7 □

It follows from the previous lemma and the $\Lambda$-lemma of Mañé, Sad, and Sullivan [19] that $\bigcup_{x\in J_z} \gamma_x$ is a lamination in $D(z, r)\times\mathbb{C}$. Likewise, a similar argument shows that $\bigcup_{z\in C, x\in J_z} \gamma_x$ forms a lamination over some neighborhood of $C$ (we will not need this result), however we cannot characterize its support nor even show that it is contained in $J$.

Let us now conclude the proof of Theorem 1.1 similarly to the laminar case (Step 3'). Indeed by Lemma 2.4, there exists $z \in C$ and a component $V$ of $\mathrm{Crit}(f)$ through $x = (z, w)$ such that $f(V)$ admits a non trivial intersection with a leaf $\Gamma$ of $E$ over $C$ (recall that $E$ is the holomorphic motion issued from the hyperbolic set $E_0$ on the central fiber). Shifting $C$ slightly if needed, we may assume that $\Gamma$ is not contained in $f(\mathrm{Crit}(f))$. Now consider the subvariety $f^{-1}(\Gamma)$ near $x$, which due to this assumption is of multiplicity 1. Looking at the equation of $f^{-1}(\Gamma)$ as in the laminar case, we conclude that there are two possibilities: either $f^{-1}(\Gamma)$ is smooth with a vertical tangency at $x$, or it is singular at $x$. In any case, $f^{-1}(\Gamma)$ admits an isolated intersection with $\gamma_x$ so by the lamination structure of the set of Fatou disks, there is a set of positive $\mu_z$ measure of $x' = (z, w')$ such that $f^{-1}(\Gamma)$ intersects the graph $\gamma_{x'}$. Since $\Gamma$ is a Fatou graph, the iterates $f^n|_{f^{-1}\Gamma}$, $n \geq 0$, form a normal family. Thus, arguing exactly as in Lemma 2.7 we construct a set of positive measure $A \subset J_z$ such that if $y_1$, $y_2$ in $A$, $\mathrm{dist}(f^n(y_1), f^n(y_2)) < \frac{3\delta}{4}$. This contradiction finishes the proof. □

## 3. Further results and comments

3.1. **Higher dimension.** The laminarity problem makes sense in higher dimension as well. Given a holomorphic endomorphism of $\mathbb{P}^k$ of degree $d$ with Green current $T$, we define its $q^{\text{th}}$ Julia set by $J_q = \text{Supp}(T^q)$. Then it may be asked whether for $1 \leq q \leq k-1$, $J_q \setminus J_{q+1}$ can be filled with Fatou disks of codimension $q$. We obtain a negative answer to this question by simply taking the direct product of the example $f$ of Theorem 1.1 with a polynomial map on $\mathbb{C}^{k-2}$, for instance the monomial map

$$g : (z_1, \ldots, z_{k-2}) \longmapsto (z_1^d, \ldots, z_{k-2}^d).$$

The product map $F = (f, g)$ is a polynomial mapping on $\mathbb{C}^k$ which extends to $\mathbb{P}^k$ and its Green current is given by $T_F = \pi_1^* T_f + \pi_2^* T_g$, where

$$\pi_1 : \mathbb{C}^2 \times \mathbb{C}^{k-2} \to \mathbb{C}^2 \text{ and } \pi_2 : \mathbb{C}^2 \times \mathbb{C}^{k-2} \to \mathbb{C}^{k-2}$$

are the natural projections.

Let $\Omega \subset \mathbb{C}^{k-2}$ be an open set such that $\Omega \cap J_{q-1}(g) \neq \emptyset$ and $\Omega \cap J_q(g) = \emptyset$. This happens for instance if in $\Omega$, $k-1-q$ coordinates have modulus smaller than 1 and the $q-1$ remaining ones cross the unit circle. Then in $\Omega$, $J_{q-1}(g)$ is foliated by Fatou disks of dimension $k-1-q$. Now for the product map $F$, in $N \times \mathbb{C} \times \Omega$ we have that $J_q(F) = J(f) \times J_{q-1}(g)$ and $J_{q+1}(F) = \emptyset$, and it is easy to see that $T_F^q = (\pi_1^* T_f) \wedge (\pi_2^* T_g^{q-1})$ is not laminated by Fatou disks of dimension $k-q$ (i.e. of codimension $q$).

3.2. **Another question of Fornæss and Sibony.** In [13], Fornæss and Sibony ask the following question: if $f$ is an endomorphism of $\mathbb{P}^k$, is it true that the non-wandering set the closure of the union of the set of periodic points and of the set *Siegel varieties*? A Siegel variety is a (local) analytic subset $X$ such that there exists a subsequence $n_j$ such that $f^{n_j}|_X$ converges to the identity. Notice that in the setting of our main theorem, $J \cap (N \times \mathbb{C})$ is contained in the non-wandering set. Indeed, since the dynamics is ergodic over every invariant circle, $(T \wedge idz \wedge d\overline{z})$-almost every point is recurrent. Without solving this problem, Theorem 1.1 says at least that in $N \times \mathbb{C}$, Siegel varieties must be very small. For instance, the set of Siegel varieties which are graphs over a given invariant sub-annulus in $N$ is a nowhere dense set in $J$ of trace measure zero. Indeed, otherwise this would give rise to a set of positive trace measure of Fatou graphs, which is impossible, as the proof of Theorem 1.1 shows.

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