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# THE DYNAMICAL MANIN-MUMFORD PROBLEM FOR PLANE POLYNOMIAL AUTOMORPHISMS

R. DUJARDIN AND C. FAVRE

ABSTRACT. Let  $f$  be a polynomial automorphism of the affine plane defined over a number field. In this paper we consider the possibility for it to possess infinitely many periodic points on an algebraic curve  $C$ . We conjecture that this happens if and only if  $f$  admits a time-reversal symmetry; in particular the Jacobian  $\text{Jac}(f)$  must be a root of unity.

As a step towards this conjecture, we prove that if  $f$  is defined over a number field, then its Jacobian, together with all its Galois conjugates lie on the unit circle in the complex plane. Under mild additional assumptions we are able to conclude that indeed  $\text{Jac}(f)$  is a root of unity.

We use these results to show in various cases that any two automorphisms sharing an infinite set of periodic points must have a common iterate, in the spirit of recent results by Baker-DeMarco and Yuan-Zhang.

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## INTRODUCTION

In this paper we discuss the following problem in the case of polynomial automorphisms of the affine plane.

**Dynamical Manin-Mumford Problem.** *Let  $X$  be a quasi-projective variety and  $f : X \rightarrow X$  be a dominant endomorphism.*

*Describe all positive-dimensional irreducible subvarieties  $C \subset X$  such that the Zariski closure of the set of preperiodic<sup>1</sup> points of  $f$  contained in  $C$  is Zariski-dense in  $C$ .*

In case  $(X, f)$  is the dynamical system induced on an abelian variety (defined over a number field) by the multiplication by an integer  $\geq 2$ , it is a deep theorem originally due to M. Raynaud (and formerly known as the Manin-Mumford conjecture) that any such  $C$  is a translate of an abelian subvariety by a torsion point. Several generalizations of this theorem have appeared since then, concerning abelian and semi-abelian varieties over fields of arbitrary characteristic. We refer to [PR, Roe] for an account on the different approaches to these results.

S.-W. Zhang [Zh95] conjectured that a similar result should hold in the more general setting of polarized<sup>2</sup> endomorphisms. More precisely, he asked whether any subvariety containing a Zariski dense set of periodic points is itself preperiodic. This conjecture was recently disproved by D. Ghioca, T. Tucker and S.-W. Zhang [GTZ], who proposed a modified statement (see also [Pa]). Some positive results on Zhang's conjecture are also available, see for instance [MS].

Our goal is to explore this problem when  $f$  is a polynomial automorphism of the affine plane  $\mathbb{A}^2$ , defined over a field of characteristic zero.

Let us first collect a few facts on the dynamics of these maps. A dynamical classification of polynomial automorphisms was given by S. Friedland and J. Milnor [FM], based on a famous theorem of H. W. E. Jung. They proved that any polynomial automorphism is conjugate to one of the following forms:

- an affine map,
- an elementary automorphism, that is a map of the form  $(x, y) \mapsto (ax + b, y + P(x))$  with  $a \neq 0$ ,  $b$  is a constant and  $P$  is a polynomial,
- a polynomial automorphism  $f$  satisfying  $\deg(f^n) = \deg(f)^n \geq 2$  for every integer  $n \geq 1$ .

In the last case the integer  $\deg(f) \geq 1$  denotes the maximum of the degrees of the components of  $f$  in any set of affine coordinates. An automorphism falling into this category will be referred to as of *Hénon type*. Since the dynamical Manin-Mumford problem is uninteresting for affine and elementary mappings, we restrict our attention to Hénon-type automorphisms.

<sup>1</sup>that is, satisfies  $f^n(p) = f^m(p)$  for some  $n > m \geq 0$ .

<sup>2</sup>This means that  $X$  is projective, and  $f^*L \simeq L^{\otimes q}$  for some ample line bundle  $L \rightarrow X$  and an integer  $q \geq 2$ .

Suppose that  $f$  is an automorphism of Hénon type that is conjugate to its inverse by an involution  $\sigma$  possessing a curve  $C$  of fixed points. Such a map is usually called reversible, see [GM03a, GM03b]. Then any point  $p \in C \cap f^{-n}(C)$  is periodic of period  $2n$ , and we verify in §6 that  $\#(C \cap f^{-n}(C))$  indeed grows to infinity. Thus the pair  $(f, C)$  falls into the framework of the Manin-Mumford problem. On the other hand it is a theorem by E. Bedford and J. Smillie [BS91] that there exists no  $f$ -invariant algebraic curve. These examples motivate the following conjecture.

**Conjecture 1** (Dynamical Manin-Mumford conjecture for complex polynomial automorphisms of the affine plane). *Let  $f$  be a complex polynomial automorphism of Hénon type of the affine plane. Assume that there exists an irreducible algebraic curve  $C$  containing infinitely many periodic points of  $f$ .*

*Then there exists an involution  $\sigma$  of the affine plane whose set of fixed points is  $C$  and an integer  $n \geq 1$  such that  $\sigma f^n \sigma = f^{-n}$ .*

Recall that the Jacobian  $\text{Jac}(f)$  of a polynomial automorphism is a non-zero constant. If  $f$  is reversible then  $\text{Jac}(f) = \pm 1$ . In particular, if  $f^n$  is reversible for some  $n$ ,  $\text{Jac}(f)$  must be a root of unity.

Our first main result can thus be seen as a step towards Conjecture 1 (see Remark 4.4 below for comments about the asserted symmetry in the conjecture).

**Theorem A.** *Let  $f$  be a complex polynomial automorphism of Hénon type of the affine plane. Assume that there exists an algebraic curve containing infinitely many periodic points of  $f$ .*

*Then the Jacobian  $\text{Jac}(f)$  is algebraic over  $\mathbb{Q}$  and all its Galois conjugates have complex modulus 1.*

In particular if  $\text{Jac}(f)$  is an algebraic integer then it is a root of unity.

Using a specialization argument, one can reduce the proof to the case  $f$  and  $C$  are both defined over a number field  $\mathbb{L}$ . Fix an algebraic closure  $\mathbb{L}^{\text{alg}}$  of  $\mathbb{L}$ . Modifying a construction of S. Kawaguchi [Ka06], C.-G. Lee [Le] built a dynamical height function  $h_f : \mathbb{A}^2(\mathbb{L}^{\text{alg}}) \rightarrow \mathbb{R}_+$ . This height is associated to a continuous semi-positive adelic metrization of the ample line bundle  $\mathcal{O}_{\mathbb{P}^2}(1)$  (in the sense of Zhang [Zh95]) and  $h_f(p) = 0$  if and only if  $p$  is periodic. We refer the reader to the survey [CL11] for a detailed account on these concepts.

Theorem A is now a consequence of the following effective statement.

**Theorem A'.** *Let  $f$  be a polynomial automorphism of Hénon type of the affine plane, defined over a number field  $\mathbb{L}$ . Assume that there exists an archimedean place  $v$  such that  $|\text{Jac}(f)|_v \neq 1$ .*

*Then for any algebraic curve  $C$  defined over  $\mathbb{L}$  there exists a positive constant  $\varepsilon = \varepsilon(C) > 0$  such that the set  $\{p \in C(\mathbb{L}^{\text{alg}}), h_f(p) \leq \varepsilon\}$  is finite.*

Let us briefly explain the strategy of the proof. We follow the approach of L. Szpiro, E. Ullmo and S.-W. Zhang [SUZ, Ul, Zh98] to the Bogomolov

conjecture whose statement is the analog of Theorem A' in case  $f$  is the doubling map on an abelian variety.

The first step is to describe the asymptotic distribution of the periodic points lying on  $C$ . Pick any place  $v$  on  $\mathbb{L}$  and denote by  $\mathbb{L}_v$  the completion of the algebraic closure of the completion of  $\mathbb{L}$  relative to the norm  $v$ . Write  $\|(x, y)\|_v = \max\{|x|, |y|\}$  and  $\log^+ = \max\{\log, 0\}$ . Then it can be shown that the sequence of functions  $\frac{1}{d^n} \log^+ \|f^n(x, y)\|_v$  converges uniformly on bounded sets in  $\mathbb{L}_v^2$  to a continuous "Green" function  $G_v^+ : \mathbb{L}_v^2 \rightarrow \mathbb{R}_+$  satisfying the invariance property  $G^+ \circ f = dG^+$ , where  $d = \deg(f)$ . Its zero locus  $\{G^+ = 0\}$  coincides with the set of points in  $\mathbb{L}_v^2$  with bounded forward orbit.

Replacing  $f$  by its inverse, one defines a function  $G^-$  in a similar way and we set  $G = \max(G^+, G^-)$ . These Green functions were first introduced and studied in the context of complex polynomial automorphisms by J. Hubbard [Hu], E. Bedford and J. Smillie [BS91] and J.E. Fornæss and N. Sibony [FS].

The key observation is that the asymptotic distribution of periodic points on  $C$  can be understood by applying suitable equidistribution result for points of small heights on curves. These results were developed by various authors in greater generality and we here use a version due to P. Autissier [Au] and A. Thuillier [Th]. More precisely, we prove that the collection of functions  $\{G_v^+\}_v$  (resp.  $\{G_v^-\}_v$ ) induces continuous semi-positive metrizations on  $\mathcal{O}_C(1)$  at all places. Then the Autissier-Thuillier theorem implies that the probability measures equidistributed over Galois conjugates of periodic points in  $C$  converge to a multiple of  $\Delta G_v^+|_C$  (resp.  $\Delta G_v^-|_C$ ) at any place<sup>3</sup> when the period tends to infinity.

From this one deduces that for each  $v$  the functions  $G_v^+$  and  $G_v^-$  are proportional on  $C$  (up to a harmonic function).

The second step is to use this information on the Green functions to infer that  $f$  is conservative at archimedean places. The argument relies on Pesin's theory and is quite technical so that let us first explain how the mechanism works under a more restrictive assumption.

Suppose indeed that there exists a hyperbolic periodic point  $p$  in the regular locus of  $C$ , with multipliers  $|u| > 1 > |s|$ , and assume moreover that the local unstable manifold  $W_{\text{loc}}^u(p)$ , the local stable manifold  $W_{\text{loc}}^s(p)$ , and the curve  $C$  are pairwise transverse. Using the invariance property of  $G^+$ , we can compute the local Hölder exponent  $\vartheta_+$  of  $G^+$  at  $p$  along  $W_{\text{loc}}^u(p)$  which satisfies the equality  $|u|^{\vartheta_+} = d$ . Using a rescaling argument reminiscent of that used by X. Buff and A. Epstein in [BE] one then shows that this Hölder exponent is actually equal to that of  $G^+|_C$ . Applying the same argument to  $f^{-1}$ , we get that the local Hölder exponent  $\vartheta_-$  of  $G^-$  along the stable

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<sup>3</sup>At a non-archimedean place,  $\Delta$  stands for the Laplacian operator as defined by Thuillier.

manifold satisfies  $|s|^{-\vartheta_-} = d$ . But since  $G^+|_C$  and  $G^-|_C$  are proportional,  $\vartheta_-$  and  $\vartheta_+$  must be equal. This proves that  $|\text{Jac}(f)| = |us| = 1$ .

Unfortunately we cannot ensure the existence of such a saddle point at an archimedean place. It turns out that working at all places (archimedean or not) resolves this difficulty. Using the above argument then leads to our next main result.

**Theorem B.** *Let  $f$  be a polynomial automorphism of Hénon type that is defined over a number field  $\mathbb{L}$ . Assume that there exists an irreducible curve  $C$  containing infinitely many periodic points of  $f$ . We suppose in addition that the following transversality statement is true:*

- (T) *There exists a periodic point  $p \in \text{Reg}(C)$  such that  $T_p C$  is not periodic under the induced action of  $f$ .*

*Then  $\text{Jac}(f)$  is a root of unity.*

Observe that this result lies very much in the spirit of [GTZ, Conjecture 2.4]. Let us also note that if  $C$  contains a saddle point at an archimedean place, then the transversality assumption (T) is superfluous (see Theorem 4.3).

Returning to the proof of Theorem A we get around the issue of the existence of a hyperbolic periodic point on  $C$  and that of the transversality of its invariant manifolds with  $C$  by applying Pesin's theory of non-uniform hyperbolicity, in combination with the theory of laminar currents, in the spirit of the work of E. Bedford, M. Lyubich and J. Smillie [BLS93a]. This allows to estimate the Hölder exponent of  $G^+$  at generic points and relate it to the positive Lyapunov exponent of the so-called *equilibrium measure*  $\mu_f := (dd^c)^2 \max(G^+, G^-)$ . This is an ergodic invariant measure which has remarkable properties; in particular it describes the asymptotic distribution of periodic orbits see [BLS93b].

The proportionality of  $G^+$  and  $G^-$  on  $C$  finally implies that the positive and negative exponents are opposite, thereby showing that  $|\text{Jac}(f)| = 1$ .

The key input of Pesin's theory in our argument is to guarantee the *transversality* of stable and unstable manifolds at a  $\mu_f$ -generic point with the curve  $C$ .

A dual way to state Theorem A is to say that the intersection of the set of periodic points with any curve is finite when  $|\text{Jac}(f)| \neq 1$ . We expect that the following stronger uniform statement holds.

**Conjecture 2.** *Let  $f$  be a complex polynomial automorphism of Hénon type such that  $|\text{Jac}(f)| \neq 1$ . Then for any algebraic curve  $C$ , the cardinality of the set of periodic points of  $f$  lying on  $C$  is bounded from above by a constant depending only on the degree of  $C$  and on the degree of  $f$ .*

We indicate in §3.4 how to adapt the arguments of Theorem A' to confirm a weaker form of this conjecture.

The automatic uniformity statement obtained by T. Scanlon [Sc], based on ideas of E. Hrushovski, implies that such a bound would follow from (a restricted version of) the dynamical Manin-Mumford problem for product maps of the form  $(f, f, \dots, f)$  acting on  $(\mathbb{A}^2)^n$ . Even though this problem seems very delicate, we are able to address some cases of the dynamical Manin-Mumford problem for special product maps of Hénon type, as the following theorem shows.

**Theorem C.** *Let  $f$  and  $g$  be two polynomial automorphisms of Hénon type of the affine plane, defined over a number field.*

*If  $f$  and  $g$  share a set of periodic points that is Zariski dense, then there exists two non-zero integers  $n, m \in \mathbb{Z}$  such that  $f^n = g^m$ .*

Notice that this is indeed a statement about product maps  $(f, g)$  such that the diagonal in  $\mathbb{A}^2 \times \mathbb{A}^2$  admits a Zariski dense subset of periodic points. Theorem A implies that if  $|\text{Jac}(f)| \neq 1$  then it is enough to assume that  $f$  and  $g$  share an infinite set of periodic points to conclude that  $f^n = g^m$ . We also show in §5.3 that Theorem C can be applied to automorphisms sharing infinitely many periodic *cycles*.

Observe that Theorem C is a generalization to our setting of recent results due to M. Baker and L. DeMarco [BdM] and X. Yuan and S.-W. Zhang [YZ13a, YZ13b].

We believe that Theorem C holds under the following weaker assumption.

**Conjecture 3.** *Suppose  $f$  and  $g$  are two complex polynomial automorphisms of Hénon type sharing infinitely many periodic points. Then  $f^n = g^m$  for some non-zero integers  $n$  and  $m$ .*

The proof of Theorem C goes as follows. The hypothesis implies that the equidistribution theorem for points of small height (Yuan [Yu], Lee [Le]) can be applied. Therefore  $f$  and  $g$  have the same equilibrium measure. If it happens that  $f$  and  $g$  are simultaneously conjugate to automorphisms with a nice compactification at infinity, then it is not difficult to see that the Green functions of  $f$  and  $g$  coincide. We can then invoke a theorem of S. Lamy [La] to conclude that  $f$  and  $g$  have a common iterate.

Otherwise we use the equality of equilibrium measures at all places to infer that  $f$  and  $g$ , as well as any automorphism belonging to the group generated by  $f$  and  $g$ , have the same sets of periodic points. Then we use Lamy's geometric group theoretic description of  $\text{Aut}[\mathbb{A}^2]$  to reduce the situation to the previous one.

◇

The plan of the paper is as follows. In §1 we gather a number of facts on the dynamics of polynomial automorphisms over arbitrary metrized fields, including equidistribution theorems for points of small height. Then in §2 we show how these equidistribution results apply in our situation. Theorems

A, B, and C are respectively established in §3, 4 and 5. Finally, §6 is devoted to a discussion on reversible mappings.

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## 1. POLYNOMIAL AUTOMORPHISMS OVER A METRIZED FIELD

We classically write  $\log^+$  for  $\max\{\log, 0\}$ . In the first four sections, we fix an arbitrary complete (non trivially) metrized field  $(L, |\cdot|)$  of characteristic zero that is algebraically closed. In §1.6 and 1.7, we work over a number field  $\mathbb{L}$ .

**1.1. Potential theory over a non-archimedean curve.** In this section we suppose that the norm on  $L$  is non-archimedean and give a brief account on Thuillier's potential theory on curves [Th].

Pick any smooth algebraic curve  $C$  defined over  $L$ . We shall work with the analytification  $C^{\text{an}}$  of  $C$  in the sense of Berkovich [Be, §3.4]. If  $U \subset C$  is a Zariski affine open subset of  $C$ , then its analytification  $U^{\text{an}}$  is defined as the set of multiplicative seminorms on the ring of regular functions  $L[U]$  whose restriction to  $L$  equals  $|\cdot|$ , endowed with the topology of pointwise convergence. Any closed point  $p \in C$  defines a point in  $U^{\text{an}}$  given by  $L[U] \ni \phi \mapsto |\phi(p)| \in \mathbb{R}_+$ .

The space  $C^{\text{an}}$  is then constructed by patching together the sets  $U^{\text{an}}$  where  $U$  ranges over any affine cover of  $C$ . In this way, one obtains a locally compact and connected space. There is a distinguished set of compact subsets of  $C^{\text{an}}$  that forms a basis for its topology and are referred to as strictly  $L$ -affinoid subdomains. We refer to [Be, §3] for a formal definition. For us it will be sufficient to say that each affinoid subdomain  $A$  has a finite boundary, and admits a canonical retraction to its skeleton  $\text{Sk}(A) \subset A$  which is the geometric realization of a finite graph. We write  $r_A : A \rightarrow \text{Sk}(A)$  for this retraction.

Any of these skeletons comes equipped with a canonical integral affine structure, hence with a metric. One can thus make sense of the notion of harmonic function on  $\text{Sk}(A)$ . By definition this is a continuous function that it is piecewise affine, and such that the sum of the directional derivatives at any point (including the endpoints) is zero.

A *harmonic* function  $h : U \rightarrow \mathbb{R}$  defined on a (Berkovich) open subset  $U \subset C^{\text{an}}$  is a continuous function such that for all subdomain  $A$  the map  $h|_{\text{Sk}(A)}$  is harmonic.

For any invertible function  $\phi \in L[U]$  defined on some affine open subset  $U \subset C$ , the function  $\log |\phi|$  is harmonic on  $U^{\text{an}}$ , see [Th, Proposition 2.3.20]. However it is not true that any harmonic function can be locally expressed



as the logarithm of an invertible function, see [Th, Lemme 2.3.22]. This discrepancy with the complex case will however not affect our arguments.

**Proposition 1.1.** *Pick any open subset  $U$  of  $C^{\text{an}}$ , and suppose that  $h_n$  is a sequence of harmonic functions defined on  $U$  that converges uniformly. Then its limit  $\lim_n h_n$  is harmonic.*

*Proof.* The result is a consequence of the following fact: suppose we are given a sequence of convex functions on a real segment that converges locally uniformly. Then the limit is convex and the directional derivatives also converge at each point.  $\square$

**Proposition 1.2.** [Th, Proposition 2.3.13] *Suppose  $u$  is a non-negative harmonic function, such that  $h(p) = 0$  for some point  $p \in U$ .*

*Then  $h$  is constant in a neighborhood of  $p$ .*

Pick any connected open subset  $U \subset C^{\text{an}}$ . An upper-semicontinuous function  $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$  is said to be *subharmonic* if it is not identically  $-\infty$  and satisfies the condition that for any strictly  $L$ -affinoid subdomain  $A$  and any harmonic function  $h$  on  $A$  then  $u|_{\partial A} \leq h|_{\partial A}$  implies  $u \leq h$  on  $A$ .

One can check that the set of subharmonic functions is a positive convex cone that is stable by taking maxima, contains all functions of the form  $\log |\phi|$  for any regular function  $\phi$ , and is stable under decreasing sequences<sup>4</sup>, see [Th, Proposition 3.1.9].

To any subharmonic function  $u$  defined on an open set  $U \subset C^{\text{an}}$  is associated a unique positive Radon measure  $\Delta u$  supported on  $U$  that satisfies the following properties:

- $\Delta(au + v) = a\Delta u + \Delta v$  for any two subharmonic functions  $u, v$  and any positive constant  $a > 0$ ;
- for any regular function  $\phi$ , the Poincaré-Lelong formula holds:

$$\Delta \log |\phi| = \sum_{\phi(p)=0} \text{ord}_p(\phi) \delta_p ;$$

- for any decreasing sequence  $u_n \rightarrow u$ ,  $\Delta u_n$  converges to  $\Delta u$  in the weak sense of measures.

We shall use the following properties of this Laplacian operator.

**Proposition 1.3.** *Let  $u : U \rightarrow \mathbb{R} \cup \{-\infty\}$  be any subharmonic function.*

*Then  $u$  is harmonic iff  $\Delta u = 0$ .*

*Proof.* If  $u$  is harmonic then  $\pm u$  are subharmonic as in [Th, Definition 3.1.5], hence  $\pm \Delta u$  is a positive measure by [Th, Théorème 3.4.8], and  $\Delta u = 0$ .

Conversely, if  $\Delta u = 0$  then  $u$  is harmonic by [Th, Corollaire 3.4.9].  $\square$

---

<sup>4</sup>We shall be concerned *only* with subharmonic functions that are uniform limits of positive linear combinations of maxima of functions of the form  $\log |\phi|$ .

**Proposition 1.4.** *Suppose  $U, V$  are two open subsets of the Berkovich analytification of smooth algebraic curves, and  $f : U \rightarrow V$  is an isomorphism. Let  $u$  be any subharmonic function on  $V$ .*

*Then  $u \circ f$  is subharmonic on  $U$ , and  $\Delta(u \circ f) = f^* \Delta u$ .*

*Proof.* The first statement follows from [Th, Proposition 3.1.13], and the second from [Th, Proposition 3.2.13].  $\square$

**1.2. Dynamics of regular automorphisms.** Following Sibony [Si] we say that a polynomial automorphism of the affine plane  $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$  is *regular* if its extension as a rational map to the projective plane  $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$  contracts the line at infinity  $H_\infty$  to a point  $p_+$  that is *not* indeterminate. It follows that  $p_+$  is a super-attracting fixed point, and that its inverse map contracts  $H_\infty$  to the (single) point of indeterminacy  $p_-$  of  $F$ .

The degree of a polynomial map is the maximum of the degrees of its (two) components. By [FM], up to a linear change of coordinates, any regular polynomial automorphism of degree  $\geq 2$  is the composition of finitely many maps of the form

$$(x, y) \mapsto (ay, x + P(y))$$

where  $a \in L^*$ , and  $P$  is a polynomial of degree  $\geq 2$ .

A polynomial automorphism of  $\mathbb{A}^2$  will be said to be of Hénon type if it is conjugated (in the group of automorphisms) to a regular automorphism of degree  $\geq 2$ . A complex polynomial automorphism has positive topological entropy if and only if it is of Hénon-type.

Let  $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$  be any regular polynomial automorphism of degree  $d \geq 2$  and such that  $p_+ = [0 : 1 : 0]$  and  $p_- = [1 : 0 : 0]$  in homogeneous coordinates in  $\mathbb{P}^2$ .

For any constant  $C > 0$ , define

$$\begin{aligned} V &= \{p = (x, y) \in L^2, \|p\| = \max\{|x|, |y|\} \leq C\}, \\ V^+ &= \{(x, y) \in L^2, |y| \geq \max\{|x|, C\}\}, \\ V^- &= \{(x, y) \in L^2, |x| \geq \max\{|y|, C\}\}. \end{aligned}$$

One can show that for any sufficiently large constant  $C$ , then

$$(1.1) \quad \frac{1}{d} \log |y \circ f| \geq \log |y| - \text{Cst} \geq C$$

for any point in  $V_+$  so that  $f(V^+) \subset V^+$ . The same kind of inequalities hold when  $f$  is replaced by its inverse, and one obtains  $f^{-1}(V^-) \subset V^-$ . This implies  $f(V) \subset V \cup V^+$ .

Let us set

$$\begin{aligned} K &= \{p \in L^2, \sup_{n \in \mathbb{Z}} \|f^n(p)\| < +\infty\}, \\ K^+ &= \{p \in L^2, \sup_{n \geq 0} \|f^n(p)\| < +\infty\}, \\ K^- &= \{p \in L^2, \sup_{n \leq 0} \|f^n(p)\| < +\infty\}. \end{aligned}$$

The next result follows easily from the above properties.

**Lemma 1.5.** *We have*

- $K = K^+ \cap K^- \subset V$ ;
- $L^2 \setminus K^+ = \bigcup_{n \geq 0} f^{-n}(V_+)$ ; and  $K^+ \subset U \cup V_-$ ;
- $L^2 \setminus K^- = \bigcup_{n \geq 0} f^{-n}(V_-)$ ; and  $K^- \subset V \cup V_+$ .

It follows from (1.1) that the sequence  $\frac{1}{d^n} \log^+ \|f^n(p)\|$  converges uniformly on  $L^2$  to a non-negative continuous function that we denote by  $G^+$ . Similarly one can define  $G^-(p) = \lim_{n \rightarrow \infty} \frac{1}{d^n} \log^+ \|f^{-n}(p)\|$ . The next result collects some properties satisfied by these functions, see [BS91] and [Ka13, Theorem A].

**Proposition 1.6.** *The functions  $G^+, G^-$  are continuous non-negative functions on  $L^2$  that satisfy*

- (1)  $G^\pm \circ f^{\pm 1} = dG^\pm$ ;
- (2)  $G^\pm(p) - \log^+ \|p\|$  extend to continuous functions to  $\mathbb{P}^2(L) \setminus \{p_\mp\}$  that are bounded from above;
- (3)  $\{G^\pm = 0\} = K^\pm$ .

The following fact will be crucial in our work. It follows from [BS91, Proposition 4.2] whose proof works over any field.

**Proposition 1.7.** *A polynomial automorphism of Hénon type admits no invariant algebraic curve.*

**1.3. Invariant measures.** We keep notation as in the previous subsection. Our purpose is to construct an invariant measure from the functions  $G^+$  and  $G^-$ .

The next result follows directly from Proposition 1.6 above.

**Proposition 1.8.** *The function  $G = \max\{G^+, G^-\}$  is a continuous non-negative function on  $L^2$  such that:*

- (1)  $G(p) - \log^+ \|p\|$  extends to a continuous function to  $\mathbb{P}^2(L)$ ;
- (2)  $\{G = 0\} = K$ .

Assume first that  $(L, |\cdot|)$  is archimedean, i.e.  $L = \mathbb{C}$  endowed with its standard hermitian norm. In this case,  $G^+$  and  $G^-$  are continuous plurisubharmonic functions on  $\mathbb{C}^2$  and so is  $G$ . Using Bedford-Taylor's theory it is possible to make sense of the Monge-Ampère of  $G$  and define the positive measure  $\mu_f := (dd^c)^2 G$ . It is a  $f$ -invariant probability measure whose support is included in  $K$ . We refer to [BLS93a] for more details on its ergodic properties.

Pick any irreducible algebraic curve  $C$  and denote by  $\text{reg}(C)$  its set of regular points. Then  $\mu_{f,C}$  is the Laplacian of the function  $G$  restricted to  $\text{reg}(C)$ . Since  $G$  is continuous and  $G(p) - \log^+ \|p\|$  is bounded, this measure carries no mass on points and its mass equals  $\text{deg}(C)$ .

When  $(L, |\cdot|)$  is non-archimedean, the analogues of the measures  $\mu_f$  and  $\mu_{f,C}$  have been constructed by A. Chambert-Loir [CL06, CL11].

Indeed the function  $G$  induces a metrization  $|\cdot|_G$  on the line bundle  $\mathcal{O}(1)_{\mathbb{P}^2}$  by setting  $|\sigma|_G := \exp(-G)$  where  $\sigma$  is the section corresponding to the constant function 1 on  $\mathbb{A}^2$ .

Proposition 1.8 together with the fact that  $G^+$  and  $G^-$  are uniform limits of multiples of functions of the form  $\log \max\{|P_1|, |P_2|\}$  with  $P_i \in L[x, y]$  imply that the metrization  $|\cdot|_G$  is a continuous semi-positive metric in the sense of [CL11, §3.1].

The measure  $\mu_f$  is defined as a probability measure on the Berkovich analytic space  $\mathbb{A}_L^{2, \text{an}}$ . This measure is  $f$ -invariant, but the study of its ergodic properties remains to be done.

When the affine plane is replaced by an irreducible curve  $C$ , the measure  $\mu_{f,C}$  is then a positive measure on the analytification  $C^{\text{an}}$  of  $C$  in the sense of Berkovich. It is defined using Thuillier's theory recalled in §1.1 as  $\mu_{f,C} := \Delta G|_C$ . Its mass is again equal to  $\deg(C)$ .

**1.4. Saddle fixed points.** In this subsection we let  $f$  be any analytic germ fixing the point  $0 \in \mathbb{A}_L^2$ . The fixed point  $0$  is said to be a *saddle* when the eigenvalues  $u, s$  of  $Df(0)$  satisfy  $|u| > 1 > |s|$ .

Given a small bidisk  $B$  around  $0$ , we let  $W_{\text{loc}}^s(0)$  (resp.  $W_{\text{loc}}^u(0)$ ) to be the set of points  $p \in B$  such that for every  $n \geq 0$ ,  $f^n(p) \in B$  (resp.  $f^{-n}(p) \in B$ ). It follows that if  $p \in W_{\text{loc}}^s(0)$  (resp.  $p \in W_{\text{loc}}^u(0)$ )  $\lim_{n \rightarrow \infty} f^n(p) \rightarrow 0$  ( $\lim_{n \rightarrow \infty} f^{-n}(p) \rightarrow 0$ ). It is a theorem that  $W_{\text{loc}}^s$  and  $W_{\text{loc}}^u$  are graphs of analytic functions in a neighborhood of  $0$  that intersect transversally. We refer to [HY, Theorem A.1] for a proof that works over any metrized field. We refer to  $W_{\text{loc}}^s(0)$  (resp.  $W_{\text{loc}}^u(0)$ ) as the local stable (resp. unstable) manifold (or curve) of  $0$ .

It is easy that one may always make a change of coordinates such that  $W_{\text{loc}}^u(0) = \{y = 0\}$  and  $W_{\text{loc}}^s(0) = \{x = 0\}$ . We will need the following more precise normal form.

**Lemma 1.9.** *There exist coordinates  $(x, y)$  near  $0$  in which  $f$  assumes the form*

$$f(x, y) = (ux(1 + xyg_1(x, y)), sy(1 + xyg_2(x, y))) ,$$

where  $g_1, g_2$  are analytic functions.

Note that in this set of coordinates  $f$  is linear along the stable and unstable manifolds.

*Proof.* By straightening the local stable and unstable manifold,  $f$  can be put under the form

$$f(x, y) = (ux(1 + \text{h.o.t.}), sy(1 + \text{h.o.t.})), \text{ with } |u| > 1 \text{ and } |s| < 1.$$

and we want to make this expression more precise. First, by the Schröder linearization theorem (which holds for arbitrary  $L$ , see [HY]) we can make a

local change of coordinates depending only on  $x$  in which  $f|_{W_{\text{loc}}^u(0)}$  becomes linear. Doing the same for  $y$ , we reach the form

$$(1.2) \quad f(x, y) = (ux(1 + yg^{(0)}(x, y)), sy(1 + xh^{(0)}(x, y))).$$

Let us now focus on the first coordinate in (1.2). We want to get rid of monomials of the form  $xy^j$  for  $j > 0$ . Re-order the expression of  $f$  so that it writes as

$$f(x, y) = (uxa_1(y) + x^2a_2(y) + \dots, syb_1(x) + y^2b_2(x) + \dots),$$

where the  $a_j$  and  $b_j$  are analytic and  $a_1(0) = b_1(0) = 1$ . We want to find local coordinates in which  $a_1(y) \equiv 1$ . For this, put  $(x', y') = (\varphi(y)x, y)$ , with  $\varphi(0) = 1$ . Notice that in the coordinates  $(x', y')$ ,  $f$  preserves the coordinate axes and remains linear there, so  $f$  is still of the form (1.2). In the new coordinates,  $f$  expresses as

$$(x', y') \mapsto \left( ua_1(y') \frac{\varphi(sy')}{\varphi(y')} x' + O((x')^2), sy' + O(x') \right),$$

thus to achieve  $a_1(y') \frac{\varphi(sy')}{\varphi(y')} = 1$  it is enough to choose  $\varphi(y') = \prod_{n=0}^{\infty} a_1(s^n y')$ , which is well-defined for sufficiently small  $y'$ , since  $|s| < 1$  and  $a_1(0) = 1$ . Doing the same in the second variable, and renaming the coordinates as  $(x, y)$ , we obtain

$$f(x, y) = (ux + x^2a_2(y) + \dots, sy + y^2b_2(x) + \dots).$$

Going back to the form (1.2), we get the desired result.  $\square$

**1.5. Stable manifolds of polynomial automorphisms.** In the case of polynomial automorphisms local stable manifolds can be globalized and have the following structure.

**Proposition 1.10.** *Let  $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$  be any polynomial automorphism of  $\mathbb{A}^2$  and assume that 0 is a saddle fixed point for  $f$ . Denote by  $s$  the eigenvalue of  $Df(0)$  lying in the unit disk.*

*Then the global stable manifold  $W^s(0) := \{p \in L^2, f^n(p) \rightarrow 0\}$  is an immersed affine line. More precisely, there exists an analytic injective immersion  $\phi_s : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  whose image is  $W^s(0)$ , and such that  $f \circ \phi_s(\zeta) = \phi_s(s\zeta)$  for every  $\zeta \in \mathbb{A}^1$ .*

*Proof.* We saw in §1.4 that for a sufficiently small neighborhood  $N$  of 0 the intersection  $W_{\text{loc}}^s(0) = W^s(0) \cap N$  is parameterized by an analytic immersion  $\phi_s : B(0, 1) \rightarrow N$  which satisfies

$$(1.3) \quad f \circ \phi_s(\zeta) = \phi_s(s\zeta).$$

Since  $f$  is an automorphism, it follows that  $W^s(0) = \bigcup_{n \geq 0} W_{\text{loc}}^s(0)$ , and using the functional equation we may extend the analytic immersion  $\phi_s$  by putting

$$\phi_s(\zeta) = f^{-n} \phi_s(s^n \zeta)$$

for every  $\zeta \in L$  and sufficiently large  $n$ .  $\square$

**Proposition 1.11.** *Let  $f : \mathbb{A}^2 \rightarrow \mathbb{A}^2$  be a polynomial automorphism of Hénon type of  $\mathbb{A}^2$  and assume that 0 is a saddle fixed point for  $f$ .*

*Then the restriction of  $G^-$  to  $W^s(0)$  cannot be identically 0. In particular  $G^-|_{W^s(0)}$  cannot vanish identically in a neighborhood of the origin. The same results hold for  $G^+|_{W^u(0)}$ .*

*Proof.* Suppose by contradiction that  $G^- \equiv 0$  on  $W^s(0)$ . By Proposition 1.6 (3), we have  $W^s(0) \subset K^-$ . Since the successive images by  $f$  of any point in  $W^s(0)$  converges to 0 it follows that  $W^s(0) \subset K^+$  hence  $W^s(0) \subset K$ .

We conclude that the image of the analytic map  $\phi_s : \mathbb{A}^1 \rightarrow \mathbb{A}^2$  lies in a bounded domain hence it is a constant by Liouville's theorem, see [Rob] in the non-archimedean case. This contradicts the fact that  $\phi_s$  is an immersion.

The second statement follows from the invariance relation (1.3).  $\square$

**1.6. Heights associated to adelic metrics on the affine space.** We now assume that  $\mathbb{L}$  is a number field. Let  $\mathcal{M}_{\mathbb{L}}$  be the set of places of  $\mathbb{L}$ , that is, of its multiplicative norms modulo equivalence. For each place  $v \in \mathcal{M}_{\mathbb{L}}$  we denote by  $|\cdot|_v$  the unique representative that is normalized in such a way that its restriction to  $\mathbb{Q}$  is either the standard archimedean norm or the  $p$ -adic norm satisfying  $|p| = p^{-1}$  for some prime  $p > 1$ .

Then for any  $x \in \mathbb{L}$ , the *product formula*  $\prod_{v \in \mathcal{M}_{\mathbb{L}}} |x|_v^{n_v} = 1$  holds. Here the integer  $n_v$  is the degree of the field extension of the completion of  $\mathbb{L}$  over the completion of  $\mathbb{Q}$  relative to  $|\cdot|_v$ .

Fix an integer  $d \geq 1$ . We let  $\mathbb{L}_v$  be the completion of the algebraic closure of the completion of  $\mathbb{L}$  relative to the absolute value  $|\cdot|_v$ . For any  $p = (x_1, \dots, x_d) \in \mathbb{L}_v^d$  we shall write  $\|p\| = \max\{|x_1|, \dots, |x_d|\}$ .

Recall that one the standard height of a point  $p \in \mathbb{A}^d(\mathbb{L}^{\text{alg}})$  is defined by the formula

$$h(p) = \frac{1}{\deg(p)} \sum_{v \in \mathcal{M}_{\mathbb{L}}} \sum_{q \in O(p)} n_v \log^+ \|q\|_v$$

where  $O(p)$  denotes the orbit of  $p$  under the absolute Galois group of  $\mathbb{L}$  and  $\deg(p)$  is the cardinality of  $O(p)$ .

As in [CL11] we use heights that are associated to semi-positive adelic metrics on ample line bundles. We present here this notion in a form that is tailored to our needs.

Suppose  $X \subset \mathbb{A}_{\mathbb{L}}^d$  is an irreducible affine variety. For us,  $X$  will always be either a curve or  $\mathbb{A}_{\mathbb{L}}^2$ .

A semi-positive adelic metric on the (ample) line bundle  $\mathcal{O}(1)_X$  is a collection of functions  $\{G_v\}_{v \in \mathcal{M}_{\mathbb{L}}}$ ,  $G_v : X(\mathbb{L}^{\text{alg}}) \rightarrow \mathbb{R}$  such that

- (M1) the function  $G_v(p) - \log^+ \|p\|$  extends continuously to the closure of  $X$  in  $\mathbb{P}^d(\mathbb{L})$  for each place  $v$ ;
- (M2) for all but finitely many places  $G_v(p) = \log^+ \|p\|$ ;
- (M3) for each archimedean place the function  $G_v$  is plurisubharmonic;

(M4) for each non-archimedean place, the function  $G_v$  is a uniform limit of positive multiples of functions of the form  $\log \max\{|P_1|, \dots, |P_r|\}$  with  $P_i \in \mathbb{L}[x_1, \dots, x_d]$ .

To any such semi-positive adelic metric  $\{G_v\}_{v \in \mathcal{M}_{\mathbb{L}}}$  is associated a height defined on  $X(\mathbb{L}^{\text{alg}})$  by

$$h_G(p) = \frac{1}{\deg(p)} \sum_{v \in \mathcal{M}_{\mathbb{L}}} \sum_{q \in \mathcal{O}(p)} n_v G_v(q),$$

and such that  $\sup_{X(\mathbb{L}^{\text{alg}})} |h_G - h| < \infty$ .

For any place  $v$ , one can also associate to the metrization  $G_v$  a positive measure  $\text{MA}(G_v)$  which is defined in the same way as in §1.3.

When  $v$  is archimedean then  $\text{MA}(G_v)$  is the Monge-Ampère measure defined using Bedford-Taylor's theory whose mass is  $\deg(X)$ .

When  $v$  is non-archimedean, then the measure  $\text{MA}(G_v)$  is defined by Chambert-Loir [CL06] as a positive measure of mass  $\deg(X)$  on the analytification of  $X$  over  $\mathbb{L}_v$  in the sense of Berkovich. Its definition relies in an essential way on the condition (M4) above.

When  $X$  is a curve then  $\text{MA}(G_v)$  is alternatively defined as the Laplacian of  $G_v|_C$  (in the sense of Thuillier when  $v$  is a non-archimedean place).

**1.7. Metrizations associated to polynomial automorphisms of Hénon type and equidistribution.** Assume that  $f$  is a regular polynomial automorphism of degree  $\geq 2$  defined over a number field  $\mathbb{L}$ .

The following two results are direct consequences of the definitions and Propositions 1.6 and 1.8.

**Proposition 1.12.** *For any regular polynomial automorphism  $f$  of degree  $\geq 2$  the collection  $\{G_{v,f}\}$  defines a semi-positive adelic metric on  $\mathcal{O}(1)_{\mathbb{P}^2}$ .*

In the sequel we denote by  $h_f$  the height associated to this collection of metrics.

**Proposition 1.13.** *Let  $f$  be a regular polynomial automorphism of degree  $d \geq 2$ . As above, denote by  $p_+$  the fixed point at infinity of its rational extension to  $\mathbb{P}^2$ .*

*Then for any irreducible algebraic curve  $C$  whose Zariski closure  $\bar{C}$  in  $\mathbb{P}^2$  intersects the line at infinity at the single point  $p_+$ , the collection  $\{G_{v,f}^+|_C\}$  defines a semi-positive adelic metric on  $\mathcal{O}(1)_{\bar{C}}$ .*

We will need two versions of the equidistribution theorem for points of small height: one is in restriction to a curve, and the other is in  $\mathbb{A}^2$ .

The first statement follows from a statement due to Autissier [Au, Prop. 4.7.1] at the archimedean place and to Thuillier [Th, Théorème 4.3.6].

**Theorem 1.14** (Equidistribution for points of small height on a curve). *Let  $f$  be a polynomial automorphism of Hénon type defined over  $\mathbb{L}$ . Suppose  $C$*

is an irreducible curve of the affine space  $\mathbb{A}^2$  that is defined over  $\mathbb{L}$  whose Zariski closure in  $\mathbb{P}^2$  intersects the line at infinity at the single point  $p_+$ .

Suppose that we are given a infinite sequence of distinct points  $p_m \in C(\mathbb{L}^{\text{alg}})$  such that  $h_{G^+}(p_m) \rightarrow 0$ .

Then, for any place  $v \in \mathcal{M}_{\mathbb{L}}$ , the convergence

$$(1.4) \quad \frac{1}{\deg(p_m)} \sum_{q \in O(p_m)} \delta_q \longrightarrow \frac{1}{\deg(C)} \Delta(G_v^+|_{C_v})$$

holds in the weak topology of measures, where  $O(x_m)$  is the orbit of  $x_m$  under the action of the absolute Galois group of  $\mathbb{L}$ .

For simplicity, we write  $C_v$  for the analytification of  $C$  over the field  $\mathbb{L}_v$ .

*Proof.* To keep the argument as short as possible we directly apply Yuan's result [Yu, Theorem 3.1]. To do so one needs to check that the height of  $C$  induced by the metrization given by  $\{G_v^+\}_v$  on  $\mathcal{O}(1)_C$  is equal to zero. We refer to [CL11] for a definition of this quantity. Now for a curve it follows from e.g. [Zh95, Theorem 1.10] that this height is equal to

$$e = \text{ess. inf}_C h_{G^+} := \sup_{\#F < \infty} \inf_{p \in C \setminus F} h_{G^+}(p) .$$

Our assumption implies that  $\inf_{p \in C \setminus F} h_{G^+}(p) \leq \liminf_n h_{G^+}(p_n) = 0$ . On the other hand we have  $G_v^+ \geq 0$  at all places hence  $h_{G^+}(p) \geq 0$  for every  $p \in C$ . Therefore  $e = 0$  as required.  $\square$

The next result is based on a theorem of Yuan [Yu] and is due to C.-G. Lee [Le, Theorem A].

**Theorem 1.15** (Equidistribution theorem for periodic points of Hénon maps). *Let  $f$  be an automorphism of Hénon type defined over  $\mathbb{L}$ .*

*Let  $(p_m)_{m \geq 0}$  be any sequence of distinct periodic points such that the set  $\{p_m\} \cap C$  is finite for any irreducible curve  $C \subset \mathbb{A}_{\mathbb{L}}^2$ . Then, for any place  $v \in \mathcal{M}_{\mathbb{L}}$ , the convergence*

$$(1.5) \quad \frac{1}{\deg(p_m)} \sum_{q \in O(p_m)} \delta_q \longrightarrow \text{MA}(G_v)$$

holds in the weak topology of measures, where  $O(x_m)$  is the orbit of  $x_m$  under the action of the absolute Galois group of  $\mathbb{L}$ .

## 2. APPLYING THE EQUIDISTRIBUTION THEOREM

A key step of the proofs of Theorems A and B is the use of equidistribution theorems for points of small height. Doing so we follow the approach to the Manin-Mumford conjecture initiated by Szpiro-Ullmo-Zhang [SUZ]. In our setting this results in the following proposition.



**Proposition 2.1.** *Let  $f$  be a regular polynomial automorphism, and  $C$  be any irreducible algebraic curve in the affine plane that are both defined over a number field  $\mathbb{L}$ . Suppose there exists a sequence of distinct points  $p_n \in C(\mathbb{L}^{\text{alg}})$  such that  $h_f(p_n) \rightarrow 0$ .*

*Then for any place  $v \in \mathcal{M}_{\mathbb{L}}$  there exists a positive rational number  $\alpha := \alpha(C, f)$  and a harmonic function  $H_v : C_v \rightarrow \mathbb{R}$  such that*

$$G_{v,f}^+ = \alpha G_{v,f}^- + H_v \text{ on } C.$$

Let us stress that the constant  $\alpha$  does not depend on the chosen place  $v$ , but only on the curve  $C$  and the automorphism  $f$ . When  $v$  is non-archimedean, the harmonicity of  $H_v$  is understood in the sense of Thuillier, see §1.1.

*Remark 2.2.* When  $C$  has a single place at infinity<sup>5</sup>, then it is easily shown that  $H_v$  must be a constant, hence necessarily zero since it vanishes at any periodic point of  $f$ . Taking  $H_v \equiv 0$  somewhat simplifies the proof of Theorem A. However it seems delicate to prove the vanishing of  $H_v$  in the general case of a curve with several places at infinity.

*Proof.* Let  $p_n$  be a sequence of distinct points in  $C(\mathbb{L}^{\text{alg}})$  with  $h_f(p_n) \rightarrow 0$ , and fix any place  $v \in \mathcal{M}_{\mathbb{L}}$ . To simplify notation we denote by  $[F_n]$  the normalized equidistributed integration measure on the Galois orbit of  $p_n$ . It is a probability measure supported on the analytification  $C_v$  of  $C$  over  $\mathbb{L}_v$  in the sense of Berkovich.

By [BS91, Proposition 4.2] together with Proposition 1.7, there exists an integer  $k \geq 0$  such that  $f^k(C)$  intersects the line at infinity in  $\mathbb{P}_{\mathbb{L}}^2$  only at the super-attracting point  $p_+$ .

By Proposition 1.13, the metrization given by  $\{G_{v,f}^+\}$  is semi-positive adelic. Let  $h_{G^+}$  be the associated height. Since  $0 \leq G_{f,v}^+ \leq G_{f,v} \leq d^k G_{f,v} \circ f^{-k}$  at all places, it follows that

$$h_{G^+}(f^k(p_n)) \leq h_f(f^k(p_n)) \leq d^k h_f(p_n) \rightarrow 0$$

whence  $h_{G^+}(p_n) \rightarrow 0$ . Theorem 1.14 thus applies and we get that the sequence of probability measures  $f_*^k [F_n]$  converges to the unique probability measure  $\mu_k$  that is proportional to  $\Delta(G_v^+|_{f^k(C_v)})$ , that is

$$\mu_k = \frac{1}{\deg(f^k(C))} \Delta(G_{f,v}^+|_{f^k(C_v)}) .$$

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<sup>5</sup>that is,  $C$  intersects the line at infinity in  $\mathbb{P}^2$  at a single point and is analytically irreducible there.

Pulling-back the convergence  $f_*^k[F_n] \rightarrow \mu_k$  by the automorphism  $f^k$ , we get that

$$\begin{aligned} \lim_n [F_n] &= \frac{1}{\deg(f^k(C))} (f^k)^* \Delta \left( G_{f,v}^+|_{f^k(C_v)} \right) \\ &= \frac{1}{\deg(f^k(C))} \Delta \left( G_{f,v}^+ \circ f^k|_{C_v} \right) = \frac{d^k}{\deg(f^k(C))} \Delta \left( G_{f,v}^+|_{C_v} \right). \end{aligned}$$

Proceeding in the same way for  $f^{-1}$  deduce that there exist two non-negative integers  $k, k'$  such that

$$\frac{d^k}{\deg(f^k(C))} \Delta \left( G_{f,v}^+|_{C_v} \right) = \frac{d^{k'}}{\deg(f^{-k'}(C))} \Delta \left( G_{f,v}^-|_{C_v} \right).$$

Therefore, there exists a positive rational number  $\alpha_C$  depending only on  $C$  and  $f$  such that the restriction of the function  $G_{f,v}^+ - \alpha_C G_{f,v}^-$  to  $C_v$  is harmonic. The proof is complete.  $\square$

For completeness, we mention the following partial converse to Proposition 2.1.

**Proposition 2.3.** *Suppose that there exists a positive constant  $\alpha > 0$ , such that  $G_{v,f}^+ = \alpha G_{v,f}^-$  on  $C$  for each place  $v$ .*

*Then there exists a sequence of points  $p_n \in C(\mathbb{L}^{\text{alg}})$  such that  $h_f(p_n) \rightarrow 0$ .*

*Remark 2.4.* If for every place  $v$  there exists a constant  $\alpha_v$  such that  $G_{v,f}^+|_C = \alpha_v G_{v,f}^-|_C$  then  $\alpha_v$  does not depend on  $v$ . This follows from the fact that the mass of  $\Delta G_v^\pm|_{C_v}$  only depends on the geometry of the branches of  $C$  at infinity.

*Proof.* Replacing  $C$  by  $f^n(C)$  for  $n$  large enough, one may suppose that  $\alpha > 1$  and that the completion of  $C$  intersects the line at infinity only at the point  $p_+$ . We claim that the height of  $C$  is zero (see [CL11] for a definition of the height of a curve). Indeed,

$$h_f(C) = \sum_v \int_{C_v} G_v \Delta G_v|_{C_v} + h_f(p_+) = \sum_v \int_{C_v} G_v^+ \Delta G_v^+|_{C_v} = 0$$

The fact that  $h_f(p_+) = 0$  can be obtained from [Le, Theorem 6.5] which asserts that

$$h_f(p) = \lim_n \frac{1}{\deg(f)^n} h_{\text{naive}}(\phi_n(p))$$

where  $\phi_n : \mathbb{P}^2 \rightarrow \mathbb{P}^4$  is the regular map whose restriction to  $\mathbb{A}^2$  is defined by  $\phi_n(p) = (f^n(p), f^{-n}(p))$ . An easy computation shows that  $\phi_n(p_+)$  is independent of  $n$  so the result follows.

We then conclude by the arithmetic Hilbert-Samuel theorem, see [Zh95, Theorem 1.10].  $\square$

### 3. THE DMM STATEMENT IN THE COMPLEX DISSIPATIVE CASE

This section is devoted to the proof of Theorems A' and A. Throughout this section we assume that  $f$  is a regular polynomial automorphism of  $\mathbb{A}^2$  defined over a number field. We use the notation and results of Section 1.

The proof will be based on Pesin's theory of non-uniformly hyperbolic dynamical systems. We refer to [BLS93a] for a presentation adapted to our situation.

**3.1. Proof of Theorem A'.** Recall that  $h_f$  denotes the height associated to the semi-positive adelic metric  $\{G_{f,v}\}$ . We suppose that there exists an irreducible curve  $C$  defined over  $\mathbb{L}$  and a sequence of points  $p_n \in C(\mathbb{L}^{\text{alg}})$  with  $h_f(p_n) \rightarrow 0$ .

We want to prove that  $|\text{Jac}(f)|_v = 1$ . To simplify notation we assume that  $\mathbb{L} \subset \mathbb{C}$ , drop the reference to  $v$  and work directly over  $\mathbb{C}$  endowed with its standard absolute value.

Under our assumptions, we know from Proposition 2.1 that there exists a positive constant  $\alpha$  such that the restrictions of  $G^+$  and  $\alpha G^-$  to  $C$  differ from a harmonic function, which implies that the positive measures  $\mu_C^\pm := dd^c(G^\pm|_C)$  are proportional.

Denote by  $\chi^u$  and  $\chi^s$  the Lyapunov exponents of  $f$  relative to the measure  $\mu = (dd^c)^2 G_f$ . Recall that they are defined by

$$\chi^u = \lim_{n \rightarrow +\infty} \int \log \|Df_p^n\| d\mu(p) \quad \text{and} \quad \chi^s = \lim_{n \rightarrow +\infty} \int \log \|Df_p^{-n}\| d\mu(p).$$

We also put  $\lambda^u = \exp(\chi^u)$  and  $\lambda^s = \exp(-\chi^s)$ . It is known that  $\lambda^{u/s} \geq d > 1$ , and  $\lambda^u \lambda^s = |\text{Jac}(f)|$ , see [BS92]. The main step of the proof of Theorem A' is the following proposition which computes the lower Hölder exponent of continuity of  $G^+$  at a  $\mu_C^+$ -generic point.

**Proposition 3.1.** *For  $\mu_C^+$ -almost every point  $p$  in  $C$ , one has*

$$\liminf_{r \rightarrow 0} \frac{1}{\log r} \log \left[ \sup_{d(p,q) \leq r, q \in C} G^+(q) \right] = \vartheta_+$$

where  $\vartheta_+$  is the unique positive real number satisfying  $(\lambda^u)^{\vartheta_+} = d$ .

Replacing  $f$  by  $f^{-1}$ , we see that a similar result holds for  $G^-$  at  $\mu_C^-$ -a.e. point, with  $\lambda^u$  replaced by  $(\lambda^s)^{-1}$  and  $\vartheta_+$  by  $\vartheta_-$  such that  $(\lambda^s)^{-\vartheta_-} = d$ . Observe that  $\lambda^{u/s} \geq d$  implies that  $\vartheta_\pm \in (0, 1]$ .

If  $\vartheta_+ = \vartheta_- = 1$  then  $|\text{Jac}(f)| = \lambda^u \lambda^s = d \cdot d^{-1} = 1$  and we are done. We may thus assume  $\vartheta_- < 1$ . Recall that  $G^+ = \alpha G^- + H$  with  $\alpha > 0$  and  $H$  a harmonic function on  $C$ . For a  $\mu_C^+$ -generic point  $p$ , the preceding

proposition yields

$$\begin{aligned}\vartheta_+ &= \liminf_{r \rightarrow 0} \frac{1}{\log r} \log \left[ \sup_{d(p,q) \leq r, q \in C} G^+(q) \right] \\ &= \liminf_{r \rightarrow 0} \frac{1}{\log r} \log \left[ \sup_{d(p,q) \leq r, q \in C} (\alpha G^- + H)(q) \right].\end{aligned}$$

Now observe that for any sequence  $r_n \rightarrow 0$  and any sequence of points  $p_n$  at distance  $r_n$  from  $p$  we have

$$\liminf_n \frac{\log G^+(p_n)}{\log r_n} \geq \liminf_n \frac{1}{\log r_n} \log \left[ \sup_{d(p,q) \leq r_n, q \in C} G^+(q) \right] \geq \vartheta_+.$$

Now pick  $p_n$  with  $d(p_n, p) = r_n \rightarrow 0$  such that  $\frac{\log G^-(p_n)}{\log r_n} \rightarrow \vartheta_- < 1$ . This is possible because since  $\mu_C^+$  is proportional to  $\mu_C^-$  the corresponding notions of generic points coincide. Since  $H$  is smooth, if  $\varepsilon$  is so small that  $(1+\varepsilon)\vartheta_- < 1$ , we infer that

$$\alpha G^-(p_n) + H(p_n) \geq \alpha r_n^{(1+\varepsilon)\vartheta_-} + O(r_n) \geq C^{\text{st}} r_n^{(1+\varepsilon)\vartheta_-}$$

for large  $n$ , so that  $1 > (1+\varepsilon)\vartheta_- \geq \vartheta_+$ .

Thus we may apply the same argument by reversing the roles of  $\vartheta_+$  and  $\vartheta_-$  and we conclude that  $\vartheta_+ = \vartheta_-$ , which implies that  $\lambda^u = (\lambda^s)^{-1}$  and  $|\text{Jac}(f)| = 1$ .  $\square$

*Remark 3.2.* It would perhaps be dynamically more significant to prove that the Hausdorff dimension of the measure  $\Delta(G^+|_C)$  at a generic point is equal to the Hausdorff dimension of the measures induced by  $T^+$  along the unstable lamination. The value of this dimension is precisely equal to the above constant  $\vartheta_+$ , in virtue of Lai-Sang Young's formula [Yo].

**3.2. Proof of Proposition 3.1.** The proposition relies on the interplay between Pesin's theory and the laminarity properties of the currents  $T^\pm$ . This is very close in spirit to the main results of [BLS93a].

Since its Lyapunov exponents are both non-zero, the measure  $\mu_f$  is hyperbolic. Pesin's theory then asserts the existence of a family of Lyapunov charts, in which  $f$  expands (resp. contracts) in the horizontal (resp. vertical) direction. The precise statement is as follows [BP, Theorem 8.14]. Let  $\mathbb{B}(r) = \{(x, y), \max\{|x|, |y|\} < r\}$  be the polydisk of radius  $r$  in  $\mathbb{C}^2$ . Then for any given  $\varepsilon > 0$ , there exists an  $f$ -invariant set  $E$  (the set of *regular points*) of full  $\mu_f$ -measure, a measurable function  $\rho : E \rightarrow (0, 1)$  and a family of charts  $\varphi_p : \mathbb{B}(\rho(p)) \rightarrow \mathbb{C}^2$  defined for  $p \in E$  and satisfying

- (i)  $\varphi_p(0) = p; e^{-\varepsilon} < \rho(f(p))/\rho(p) < e^\varepsilon;$
- (ii) if  $f_p := \varphi_{f(p)}^{-1} \circ f \circ \varphi_p$ , then

$$(3.1) \quad f_p(x, y) = (a^u(p)x + xh_1(x, y), a^s(p)y + yh_2(x, y))$$

where  $\lambda^u - \varepsilon \leq |a^u(p)| \leq \lambda^u + \varepsilon$ ,  $\lambda^s - \varepsilon \leq |a^s(p)| \leq \lambda^s + \varepsilon$ , and

$$\sup\{\|h_1\|, \|h_2\|\} < \varepsilon ;$$

(iii) there exists a constant  $B > 0$  and a measurable function  $A : E \rightarrow (0, \infty)$  such that

$$(3.2) \quad B^{-1} \|\varphi_p(q) - \varphi_p(q')\| \leq \|q - q'\| \leq A(p) \|\varphi_p(q) - \varphi_p(q')\|$$

with  $e^{-\varepsilon} < A(f(p))/A(p) < e^\varepsilon$ .

We denote by  $W_{\text{loc}}^u(p)$  (resp.  $W_{\text{loc}}^s(p)$ ) the image by  $\varphi_p$  of  $\{x = 0\}$  (resp.  $\{y = 0\}$ ). These will be referred to as local unstable (resp. stable) manifold at  $p$ . Notice that (3.1) is slightly different from the corresponding statement in [BP] as we have straightened the local stable and unstable manifolds. Observe also that removing a set of measure 0 we may assume that  $f^{-1}(W_{\text{loc}}^u(p)) \subset W_{\text{loc}}^u(f^{-1}(p))$ , and  $f(W_{\text{loc}}^s(p)) \subset W_{\text{loc}}^s(f(p))$ .

For every point  $p \in E$ , we define the global stable and unstable manifolds by  $W^s(p) = \bigcup_{n \geq 0} f^{-n} W_{\text{loc}}^s(f^n(p))$  and  $W^u(p) = \bigcup_{n \geq 0} f^n W_{\text{loc}}^u(f^{-n}(p))$ . These are embedded images of  $\mathbb{C}$  respectively lying in  $K^+$  and  $K^-$  like the stable and unstable manifolds of saddle points as we saw in § 1.4.

The next result follows from [BLS93a, Lemma 8.6], and will be proved afterwards. Recall that  $\mu_C^+ := T^+ \wedge [C]$ .

**Lemma 3.3.** *Let  $E$  denote as above the set of Pesin regular points for  $\mu_f$ . Then for every subset  $A \subset E$  of full  $\mu_f$ -measure there exists  $\bar{A} \subset C$  of full  $\mu_C^+$ -measure such that if  $\bar{p} \in \bar{A}$ , there exists  $p \in A$  such that*

- $\bar{p} \in W^s(p)$ ,
- $W^s(p)$  intersects  $C$  transversally at  $\bar{p}$ .

With notation as in the lemma, pick any  $\bar{p} \in \bar{E}$  and introduce the function

$$\theta_{\bar{p}}(r) = \sup\{G^+(q), q \in C, d(\bar{p}, q) \leq r\} .$$

To prove the proposition we need to show that a.s.

$$\liminf_{r \rightarrow 0} \frac{\log(\theta_{\bar{p}}(r))}{\log r} = \vartheta_+ = \frac{\log d}{\log \lambda^u} .$$

Using Lemma 3.3, let  $p \in E$  such that  $W^s(p)$  intersects  $C$  transversely at  $\bar{p}$ . Then there exists an integer  $N$  such that  $f^N(\bar{p})$  lies in  $W_{\text{loc}}^s(f^N(p))$ . By the invariance relation for  $G^+$  and the differentiability of  $f$ , replacing  $C$  by  $f^N(C)$  if needed, it is no loss of generality to assume that  $N = 0$ .

Choose an integer  $n$ , and pick a point  $q \in B(\rho(p))$  such that  $f^k(q) \in \varphi_{f^k(p)} B(\rho(f^k(p)))$  for all  $0 \leq k \leq n$ . Write  $\varphi_p(q) = (x, y)$  so that  $|x|, |y| \leq \rho(p)$ , and let  $(x_k, y_k) := \varphi_{f^k(p)}^{-1}(f^k(q))$ . It then follows from (3.1) that  $|y_n| \leq (\lambda^s + 2\varepsilon)^n |y_0|$  and

$$(3.3) \quad (\lambda^u - 2\varepsilon)^n |x_0| \leq |x_n| \leq (\lambda^u + 2\varepsilon)^n |x_0| .$$

Conversely, it follows a posteriori from these estimates that any point  $q = \varphi_p(x, y)$  such that

$$|x| \leq \rho(p)(\lambda^u + 2\varepsilon)^{-n} e^{-\varepsilon n}$$

satisfies  $|x_k| \leq |x|(\lambda^u + 2\varepsilon)^k \leq \rho(f^k(p))$ , hence  $f^k(x) \in \varphi_{f^k(p)} B(\rho(f^k(p)))$  for every  $0 \leq k \leq n$ .

We now estimate  $\theta_{\bar{p}}(r)$  for a given small enough  $r$ . To estimate it from below, choose  $q \in C$  such that  $d(p, q) \leq r$  and  $\theta_{\bar{p}}(r) = G^+(q)$ . By (3.2) writing  $\varphi_p(q) = (x, y)$ , we infer that  $|x| \leq Br$ . To ease notation let us assume without loss of generality that  $B = 1$ . Choose  $n$  such that

$$(3.4) \quad \rho(p)(e^\varepsilon(\lambda^u + 2\varepsilon))^{-n} \leq r < \rho(p)(e^\varepsilon(\lambda^u + 2\varepsilon))^{-(n-1)}.$$

From (3.4) and the invariance relation for  $G^+$  we get the upper bound:

$$\begin{aligned} \log \theta_{\bar{p}}(r) &= -\log(d^n) + \log G^+(f^n(q)) \leq -\log(d^n) + \max_{\varphi_p(B(\rho(p)))} G^+ \\ &\leq -n \log d + C^{\text{st}} \leq \frac{\log r - \log \rho(p)}{(\varepsilon + \log(\lambda^u + 2\varepsilon))} \log d + C^{\text{st}}, \end{aligned}$$

where the first inequality on the second line follows from the fact that due to (3.2),  $\bigcup_E \varphi_p(B(\rho(p)))$  is bounded in  $\mathbb{C}^2$ . Letting  $r \rightarrow 0$ , we infer that

$$\liminf_{r \rightarrow 0} \frac{\log \theta_{\bar{p}}(r)}{\log r} \geq \frac{\log d}{\varepsilon + \log(\lambda^u + 2\varepsilon)}.$$

Since this holds for every  $\varepsilon$ , we conclude that

$$(3.5) \quad \liminf_{r \rightarrow 0} \frac{\log \theta_{\bar{p}}(r)}{\log r} \geq \frac{\log d}{\log \lambda^u}.$$

To prove the opposite inequality we proceed as follows. Let us introduce the auxiliary function  $\psi : p \mapsto \sup_{W_{\text{loc}}^u(p)} G^+$ . This is a measurable function that is uniformly bounded from above since  $\bigcup_E \varphi_p(B(\rho(p)))$  is bounded in  $\mathbb{C}^2$ . Likewise  $\psi(p) > 0$  for every  $p \in E$  since  $W_{\text{loc}}^u(p)$  cannot be contained in  $K^+$ , see [BLS93a, Lemma 2.8] (the argument is identical to that of Proposition 1.11).

Now fix a constant  $g_0 > 0$  such that the set  $\{\psi > g_0\}$  has positive  $\mu_f$ -measure. By the Poincaré recurrence theorem, there exists a measurable set  $A \subset E$  of full  $\mu_f$ -measure such that for every  $p \in A$ ,  $\psi(f^{n_j}(p)) > g_0$  for infinitely many  $n_j$ 's.

Let  $\bar{A}$  be as in Lemma 3.3, let  $\bar{p} \in \bar{A}$  and  $p$  be as above. Stable manifold theory shows that there exists  $n_0$  such that for  $n \geq n_0$  the connected component of  $\varphi_{f^n(p)}^{-1}(f^n(C))$  in  $\mathbb{B}(\rho(f^n(p)))$  containing  $\varphi_{f^n(p)}^{-1}(\bar{p})$  is a graph over the first coordinate which converges exponentially fast in the  $C^0$  (hence  $C^1$ ) topology to  $\{y = 0\}$  [BP, §8.2 and Thm 8.13]. Let us denote it by  $C_{n, f^n(\bar{p})}$ .

Since  $p$  belongs to  $A$  we have  $\psi(f^{n_j}(p)) > g_0$  for infinitely many  $n_j$ 's. To ease notation we simply write  $n$  for  $n_j$ . Since  $G^+$  is Hölder continuous, for such an iterate  $n$  we infer that

$$\sup\{G^+(w), w \in C_{n, f^n(\bar{p})}\} \geq g_0 - \delta_n,$$

where  $\delta_n$  is exponentially small. Let  $w_n \in C_{n, f^n(\bar{p})}$  be a point at which  $G^+(w_n) \geq \frac{g_0}{2}$ . Consider now  $f^{-n}(w_n)$  and denote  $\varphi_p^{-1}(f^{-n}(w_n)) = (x_n, y_n)$ . By (3.1), we have that

$$|x_n| \leq \frac{\rho(f^n(p))}{(\lambda^u - 2\varepsilon)^n} \leq \frac{C^{\text{st}}}{(\lambda^u - 2\varepsilon)^n},$$

hence, since  $C$  is transverse to  $W_{\text{loc}}^s(p)$ , from (3.2) we get that

$$d(f^{-n}(w_n), \bar{p}) \leq \frac{C_0}{(\lambda^u - 2\varepsilon)^n},$$

where  $C_0$  does not depend on  $n$ . Therefore, putting  $r_n = \frac{C_0}{(\lambda^u - 2\varepsilon)^n}$  and using the invariance relation for  $G^+$  and the definition of  $w_n$ , we infer that

$$\theta(\bar{p}, r_n) \geq \frac{g_0}{2d^n} = \frac{g_0}{2} \left( \frac{r_n}{C_0} \right)^{\frac{\log d}{\log(\lambda^u - 2\varepsilon)}}.$$

Finally

$$\limsup_{n \rightarrow \infty} \frac{\log \theta(\bar{p}, r_n)}{\log r_n} \leq \frac{\log d}{\log(\lambda^u - 2\varepsilon)},$$

thus

$$\liminf_{r \rightarrow 0} \frac{\log \theta(\bar{p}, r)}{\log r} \leq \frac{\log d}{\log \lambda^u}$$

which, along with (3.5), finishes the proof.

*Proof of Lemma 3.3.* The proof relies on the theory of laminar currents [BLS93a]. It is shown in [BLS93a, Thm 7.4] that the positive closed  $(1, 1)$ -current  $T^+ := dd^c G^+$  is laminar.

Recall that this means the following. First, we say that a current  $S$  in  $\Omega \subset \mathbb{C}^2$  is *locally uniformly laminar* if every point in  $\text{supp}(S)$  admits a neighborhood  $B$  biholomorphic to a bidisk, such that in adapted coordinates,  $S$  locally writes as  $\int [\Delta_a] d\alpha(a)$ , where the  $\Delta_a$  are disjoint graphs over the first coordinate in  $B$  and  $\alpha$  is a positive measure on the space of such graphs. These disks will be said to be *subordinate* to  $S$ . Notice that a locally uniformly laminar current is always closed.

A current is *laminar* if for any  $\varepsilon > 0$  there exists a finite family of disjoint open sets  $\Omega^i$ , and for each  $i$  a locally uniformly laminar current  $T^i \leq T$  such that the mass of  $T - \sum_i T^i$  is smaller than  $\varepsilon$ . If  $R$  is any positive closed current in  $\mathbb{C}^2$  such that the wedge product  $T \wedge R$  is admissible, then slightly abusing notation we define the wedge product  $(\sum_i T^i) \wedge R$  by  $\sum_i (T^i \wedge R)|_{\Omega^i}$ .

The geometric intersection product of a current of integration over a curve  $[M]$  with a uniformly laminar current  $T = \int [\Delta_a] d\alpha(a)$  is defined by

$$T \wedge [M] = \int [\Delta_a \cap M] d\alpha(a)$$

where  $[\Delta_a \cap M]$  is the atomic positive measure putting mass 1 at any intersection of  $\Delta_a$  and  $M$ . If  $T$  has continuous potential then  $T \wedge [M] = T \wedge [M]$ , see [BLS93a, Lemma 6.4].

Pick any open subset  $\Omega \subset \mathbb{C}^2$ , and let  $0 < S^+ \leq T^+$  be any locally uniformly laminar current in  $\Omega$ . Denote by  $\mathbf{M}(\mu)$  the total mass of a given positive measure  $\mu$ . We claim that

$$(3.6) \quad \lim_{n \rightarrow \infty} \mathbf{M} \left( \frac{1}{d^n} (f^n)^* S^+ \wedge [C] \right) = \deg(C) \cdot \mathbf{M}(S^+ \wedge T^-) .$$

To see this, we observe that the current  $S^+$  has continuous potential by [BLS93a, Lemma 8.2] so that

$$\frac{1}{d^n} (f^n)^* S^+ \wedge [C] = \frac{1}{d^n} (f^n)^* S^+ \wedge [C]$$

as positive measures in  $f^{-n}(\Omega)$ . Now  $f^n : f^{-n}(\Omega) \rightarrow \Omega$  is an automorphism so that

$$\mathbf{M} \left( \frac{1}{d^n} (f^n)^* S^+ \wedge [C] \right) = \mathbf{M} (S^+ \wedge d^{-n} (f^n)_* [C]) .$$

Replacing  $C$  by some iterate we may assume that it intersects the line at infinity at  $p_+$  only, hence  $d^{-n} (f^n)_* [C] \rightarrow (\deg(C)) T^-$  by [BS91, FS]. Again since  $S^+$  has continuous potential in  $\Omega$ , the measures  $S^+ \wedge d^{-n} (f^n)_* [C]$  converge to  $\deg(C) S^+ \wedge T^-$ . We conclude that  $\mathbf{M} (d^{-n} (f^n)^* S^+ \wedge [C])$  converges to  $\deg(C) \mathbf{M}(S^+ \wedge T^-)$  as  $n \rightarrow \infty$  as required.

Another result in [BLS93a] is that  $T^+$  and  $T^-$  intersect geometrically (see also [Duj]). This implies that for every  $\varepsilon > 0$  there exists a current  $T_\varepsilon^+ \leq T^+$ , which is a finite sum of uniformly laminar currents in disjoint open sets  $\Omega^i$  as above, and such that  $\mathbf{M}(T_\varepsilon^+ \wedge T^-) \geq 1 - \varepsilon$ . Then from (3.6) we deduce that for large  $n$ , the positive measures

$$\frac{1}{d^n} (f^n)^* T_\varepsilon^+ \wedge [C] := \sum_i \frac{1}{d^n} (f^n)^* T_\varepsilon^+ \wedge [C]|_{f^{-n}\Omega_i}$$

are dominated by  $\mu_C^+$ , and

$$\begin{aligned} \mathbf{M} \left( \frac{1}{d^n} (f^n)^* T_\varepsilon^+ \wedge [C] \right) &\geq (1 - \varepsilon) \deg(C) \sum_i \mathbf{M}(T_\varepsilon^+|_{\Omega_i} \wedge T^-) \geq \\ &\geq (1 - \varepsilon)^2 \deg(C) \mathbf{M}(T^+ \wedge T^-) \geq (1 - 2\varepsilon) \deg(C) . \end{aligned}$$

Now recall that  $T_\varepsilon^+|_{\Omega_i}$  has continuous potential for each  $i$  hence does not charge any curve. By [BLS93a, Lemma 6.4] it follows that only transverse intersections between disks subordinate to  $\frac{1}{d^n} (f^n)^* (T_\varepsilon^+|_{\Omega_i})$  and  $[C]$  need to be taken into account in the computation of the geometric intersection  $\frac{1}{d^n} (f^n)^* (T_\varepsilon^+|_{\Omega_i}) \wedge [C]$ . Further by [BLS93a, Corollary 8.8], almost every disk subordinate to  $T_\varepsilon^+|_{\Omega_i}$  is an open set of some stable curve  $W^s(p)$  for some  $p \in A$ .



In particular, there exists a set  $B_n$  of total mass for the positive measure  $d^{-n}(f^n)^*T_\varepsilon^+ \wedge [C]$  such that for all points  $q \in B_n \subset C$  there exists a point  $p \in A$  such that  $W^s(p)$  intersects  $C$  transversally at  $q$ . Since  $\mu_C^+(B_n) \geq (1 - 2\varepsilon) \deg(C)$  the proof is complete.  $\square$

**3.3. Proof of Theorem A.** Here we assume that  $f$  is a regular polynomial automorphism defined over a number field  $\mathbb{L} \subset \mathbb{C}$ , and  $C$  is an irreducible curve containing infinitely many periodic points. Here the curve  $C$  is only supposed to be defined over  $\mathbb{C}$ . To conclude it is therefore sufficient to prove that  $C$  is actually defined over a number field, and to apply Theorem A'.

The first observation is that since  $f$  do not admit any periodic curve, any of its periodic points is defined over a number field. It follows that  $C$  contains infinitely many  $\mathbb{L}^{\text{alg}}$ -points. Pick any defining equation  $C = \{P = 0\}$  with  $P = \sum a_{ij}x^i y^j \in \mathbb{C}[x, y]$ . Let  $b_1, \dots, b_k$  be a basis of the  $\mathbb{L}^{\text{alg}}$ -vector space  $\sum a_{ij}\mathbb{L}^{\text{alg}} \subset \mathbb{C}$ . We may then write  $P$  in a unique way under the form  $P = \sum_1^k b_l P_l$  with  $P_1, \dots, P_k \in \mathbb{L}^{\text{alg}}[x, y]$ . Since  $(b_l)$  is a basis, evaluating this expression at the points of  $C(\mathbb{L}^{\text{alg}})$  implies that for every  $l$ , the restriction of  $P_l$  to  $C(\mathbb{L}^{\text{alg}})$  is identically zero. It follows that  $C$  is contained in the locus  $\cap_1^k \{P_l = 0\}$  which is a curve defined over  $\mathbb{L}^{\text{alg}}$ .

**3.4. A uniform Theorem A'.** In this section we indicate how our arguments can be modified so as to get the following statement.

**Theorem A''.** *Let  $f$  be a polynomial automorphism of Hénon type of the affine plane, defined over a number field  $\mathbb{L}$ . Assume that there exists an archimedean place  $v$  such that  $|\text{Jac}(f)|_v \neq 1$ .*

*For any integer  $d$ , there exists a positive constant  $\epsilon(d) > 0$  and an integer  $N(d) \geq 1$  such that for any algebraic curve  $C$  defined over  $\mathbb{L}$  of degree at most  $d$ , the set  $\{p \in C(\mathbb{L}^{\text{alg}}), h_f(p) \leq \epsilon(d)\}$  contains at most  $N(d)$  points.*

*Proof.* Pick an automorphism  $f$  of Hénon type defined over  $\mathbb{L}$  and fix any archimedean place  $v$ . We suppose that there exists a sequence of curves  $C_m$  defined over  $\mathbb{L}$  of degree  $d$  and finite sets  $F_m \subset C_m$  invariant under the absolute Galois group of  $\mathbb{L}$  such that  $\#F_m \geq m$  and  $h_f(F_m) \leq \frac{1}{m}$ . Let us show that  $|\text{Jac}(f)|_v = 1$ .

Let  $\mu_m$  be the probability measure equidistributed over  $F_m$ .

**Lemma 3.4.** *Any weak limit of the sequence  $(\mu_m)$  is supported on a curve of degree  $d$ .*

Suppose first that any curve defined over  $\mathbb{L}$  intersect only finitely many  $F_m$ 's. Then Yuan's result [Yu, Theorem 3.1] implies the equidistribution  $\mu_n \rightarrow \mu_{f,v}$ . However  $\mu_{f,v}$  gives no mass to curves so the previous lemma gives a contradiction.

We may thus suppose that there exists a curve  $D$  defined over  $\mathbb{L}$  that contains infinitely many  $F_m$ 's. Theorem A then applies and shows that  $|\text{Jac}(f)|_v = 1$  as required.  $\square$

*Proof of Lemma 3.4.* For each  $m$  pick an equation  $P_m = \sum_{i+j \leq d} a_{ij}^m x^i y^j$  of  $C_m$  such that  $\max\{|a_{ij}^m|\} = 1$ . Replacing  $F_m$  by a suitable subsequence we may assume that each coefficient  $a_{ij}^m$  converges to some  $a_{ij} \in \mathbb{L}_v$  and we set  $P = \sum_{i+j \leq d} a_{ij} x^i y^j$ . Since the height of  $F_m$  is bounded from above,  $\bigcup_m F_m$  is included in a fixed bounded set  $K$  in  $\mathbb{L}_v^2$ . We have  $\sup_K |P_m - P| \rightarrow 0$  and this implies  $\int |P| d\mu = \lim_m \int |P_m| d\mu_m = 0$ . Therefore  $\mu$  is supported on the curve  $\{P = 0\}$ .  $\square$

#### 4. THE DMM STATEMENT UNDER A TRANSVERSALITY ASSUMPTION

This section is devoted to the proof of Theorem B. Let  $f$  be a regular polynomial automorphism of  $\mathbb{A}^2$ , and  $C$  be an irreducible algebraic curve containing infinitely many periodic points, both defined over a number field  $\mathbb{L}$ . By the transversality assumption (T), replacing  $f$  by  $f^N$  if needed we assume that one of these periodic points  $p \in \text{Reg}(C)$  is fixed and satisfies  $Df_p(T_p C) \neq T_p C$ .

We want to show that  $\text{Jac}(f)$  is a root of unity. To do so it will be enough to prove that  $|\text{Jac}(f)|_v = 1$  for each place  $v$ .

If the place  $v$  is archimedean, the equality  $|\text{Jac}(f)|_v = 1$  follows from Theorem A, so we will work at non-archimedean places only.

**Lemma 4.1.** *Let  $f$  and  $C$  be as in Theorem B. Let  $p$  be any fixed point lying on  $C$  and denote by  $\lambda_1, \lambda_2$  the two (possibly equal) eigenvalues of  $Df(p)$ .*

*At any non-archimedean place  $v$ , then either  $|\lambda_1|_v = |\lambda_2|_v = 1$  or  $p$  is a saddle.*

It is also not difficult to see that for all but finitely many places  $v$ , all periodic points are indifferent in the sense that their multipliers have norm 1.

*Proof of Lemma 4.1.* Assume by contradiction that  $|\lambda_1|_v < 1$  and  $|\lambda_2|_v \leq 1$ .

Since  $|\lambda_1|_v \leq |\lambda_2|_v \leq 1$  it is classical that there exists a (bounded) neighborhood  $U$  of  $p$  that is forward invariant, in particular  $U \subset K^+$ . Indeed, performing a linear change of coordinates we can write  $f(x, y) = (\lambda_1 x + \text{h.o.t.}, \lambda_2 y + \text{h.o.t.})$ , and since  $v$  is non-archimedean if  $x$  and  $y$  are small enough, then  $|\lambda_1 x + \text{h.o.t.}| = |\lambda_1| |x|$  and  $|\lambda_2 y + \text{h.o.t.}| = |\lambda_2| |y|$ .

It follows that  $G^+|_U \equiv 0$ , hence from Proposition 2.1 we deduce that  $G^-|_{U \cap C}$  is harmonic, and since  $G^-(p) = 0$  and  $G^- \geq 0$ , by Proposition 1.2 we conclude that  $G^-|_{U \cap C} \equiv 0$  as well. This implies that  $f^{-n}(U \cap C) \subset K$  for all  $n$ , and since  $K$  is bounded the Cauchy inequality implies that the norms  $\|Df^{-n}(p)\|$  stay uniformly bounded in  $n$  along  $U \cap C$ .

If  $p$  is a sink, that is  $|\lambda_1|_v \leq |\lambda_2|_v < 1$ , then  $\|Df^{-n}(p)\|$  must grow exponentially and we readily get a contradiction. The semi-attracting case  $|\lambda_1|_v < |\lambda_2|_v = 1$  requires a few more arguments.

Assume first that  $\lambda_2$  is not a root of unity. Then a theorem by Herman and Yoccoz [HY] asserts that in this case  $\lambda_2$  satisfies a Diophantine

condition, hence so does the pair  $(\lambda_1, \lambda_2)$ , and the fixed point  $p$  is analytically linearizable. Therefore there exist adapted coordinates  $(x, y)$  near  $p$ , in which  $p$  is sent to the origin, and  $f$  takes the form  $f(x, y) = (\lambda_1 x, \lambda_2 y)$ . Since  $\|Df^{-n}(p)\|$  is uniformly bounded in  $n$  along  $U \cap C$ , in these coordinates  $C$  must be tangent to the  $y$  axis, so it locally expresses as a graph  $t \mapsto (\psi(t), t)$ . Now  $Df^{-n}(\psi'(t), 1) = (\lambda_1^{-n}\psi'(t), \lambda_2^{-n})$  is unbounded as  $n \rightarrow \infty$  unless  $\psi \equiv 0$ . From this we conclude that  $C$  locally coincides with the  $y$  axis, so in particular it is invariant. But  $f$  does not admit any invariant algebraic curve so we reach a contradiction.

If  $\lambda_2$  is a root of unity, the argument is similar. Replace  $f$  by some iterate so that  $\lambda_2 = 1$ . To ease notation we work with  $f^{-1}$  instead of  $f$ . A theorem by Jenkins and Spallone [JS, §4] asserts that there are coordinates  $(x, y)$  as above in which  $f^{-1}$  expresses as

$$f^{-1}(x, y) = (\lambda_1^{-1}x(1 + g(y)), h(y)), \text{ with } g(0) = 0 \text{ and } h(y) = y + \text{h.o.t.}$$

(recall that  $|\lambda_1^{-1}| > 1$ ). From this we deduce that

$$f^{-n}(x, y) = (\lambda_1^{-n}x(1 + g_n(y)), h^n(y)) = \left( \lambda_1^{-n}x \prod_{j=0}^{n-1} (1 + g(h^j(y))), h^n(y) \right).$$

For  $\varepsilon$  small enough, if  $|y|_v < \varepsilon$ , using the ultrametric inequality we get that for every  $0 \leq j \leq n$ ,  $|h^j(y)|_v < \varepsilon$ . It follows that  $|1 + g(h^j(y))|_v = 1$ .

Now as in the previous case, in the new coordinates the curve  $C$  must be tangent to the  $y$ -axis. Parameterize it as  $t \mapsto (\psi(t), t)$ , so that the curve  $f^{-n}C$  is parameterized by

$$t \mapsto \left( \lambda_1^{-n}\psi(t) \prod_{j=0}^{n-1} (1 + g(h^j(t))), h^n(t) \right).$$

We see that the only possibility for it to be locally bounded as  $n \rightarrow \infty$  is that  $\psi \equiv 0$ . As before we deduce that  $C$  is equal to the  $y$ -axis, which is invariant, and again we get a contradiction.  $\square$

Let us resume the proof of Theorem B. Pick any non-archimedean place  $v$ , and recall that we wish to prove that  $|\text{Jac}(f)|_v = 1$ . To simplify notation we drop all indices referring to this place  $v$ . Let  $p$  be the fixed point satisfying (T), and  $\lambda_i$  be its eigenvalues. Then  $\text{Jac}(f) = \lambda_1 \lambda_2$ . By the previous proposition either  $|\lambda_1| = |\lambda_2| = 1$  and we are done, or  $p$  is a saddle. For notational consistency we denote by  $u$  (resp.  $s$ ) the unstable (resp. stable) eigenvalue (which can alternatively be  $\lambda_1$  or  $\lambda_2$  depending on the place). By the transversality assumption (T),  $W_{\text{loc}}^u(p)$  and  $W_{\text{loc}}^s(p)$  are not tangent to  $C$  at  $p$ . Indeed since the tangent directions to  $W_{\text{loc}}^u(p)$  and  $W_{\text{loc}}^s(p)$  are given by the eigenvectors of  $Df(p)$ , this transversality does not depend on the place.

By Lemma 1.9 there are adapted coordinates  $(x, y)$  near  $p$  in which  $f$  takes the form

$$(4.1) \quad f(x, y) = (ux(1 + xyg_1(x, y)), sy(1 + xyg_2(x, y))).$$

By scaling the coordinates if necessary we may assume that we work in the unit bidisk  $\mathbb{B}$ .

The following key renormalization lemma will be proven afterwards.

**Lemma 4.2.** *If  $|x|, |y|$  are small enough, then for every  $1 \leq j \leq n$ ,  $f^j\left(\frac{x}{u^n}, y\right) \in \mathbb{B}$  and*

$$f^n\left(\frac{x}{u^n}, y\right) = (x, 0) + O(n\rho^n),$$

uniformly in  $(x, y)$ , with  $\rho := \max\{|u|^{-1}, |s|\} < 1$ .

In the coordinates  $(x, y)$ , we write  $C$  as a graph  $y = \psi(x) = bx + \text{h.o.t.}$  over the first coordinate. Replacing  $y$  by  $by$  we can assume  $b = 1$ . Using Proposition 2.1 we set  $\tilde{G}(x) = G^+(x, \psi(x)) = \alpha G^-(x, \psi(x)) + H(x, \psi(x))$ . By Lemma 4.2, the continuity of  $G^+$  and the invariance relation for  $G^+$  we get that for small enough  $x$ ,

$$(4.2) \quad d^n \tilde{G}\left(\frac{x}{u^n}\right) = G^+ \circ f^n\left(\frac{x}{u^n}, \psi\left(\frac{x}{u^n}\right)\right) \rightarrow G^+(x, 0).$$

Applying Lemma 4.2 to  $f^{-1}$ , for small  $y$  we have the following uniform convergence in a small disk:

$$(4.3) \quad f^{-n}(\psi^{-1}(s^n y), s^n y) \xrightarrow{n \rightarrow \infty} (0, y).$$

Now we claim that  $|us| = 1$ . Indeed assume by contradiction that  $|us| > 1$ . Then putting  $y_n = s^{-n}\psi(x/u^n)$ , we get that  $y_n \rightarrow 0$  when  $n \rightarrow \infty$ . We write

$$(4.4) \quad d^n \tilde{G}\left(\frac{x}{u^n}\right) = \alpha G^- \circ f^{-n}\left(\frac{x}{u^n}, \psi\left(\frac{x}{u^n}\right)\right) + d^n H\left(\frac{x}{u^n}, \psi\left(\frac{x}{u^n}\right)\right)$$

and applying (4.3), we see that

$$G^- \circ f^{-n}\left(\frac{x}{u^n}, \psi\left(\frac{x}{u^n}\right)\right) = G^- \circ f^{-n}(\psi^{-1}(s^n y_n), s^n y_n) \xrightarrow{n \rightarrow \infty} G^-(0, 0) = 0.$$

Thus from (4.2) and (4.4) we infer that

$$d^n H\left(\frac{x}{u^n}, \psi\left(\frac{x}{u^n}\right)\right) \xrightarrow{n \rightarrow \infty} G^+(x, 0),$$

locally uniformly in the neighborhood of the origin. Since a limit of harmonic functions is harmonic (see Proposition 1.1), we conclude that  $G^+$  is harmonic, hence identically zero on  $W_{\text{loc}}^u(p)$ , thereby contradicting Proposition 1.11. This contradiction shows that at the place  $v$  we have  $|us|_v = |\text{Jac}(f)|_v = 1$ , and completes the proof of Theorem B.  $\square$

*Proof of Lemma 4.2.* Recall that we work in the unit bidisk  $\mathbb{B}$ . Scaling the coordinates further, we may assume that the functions  $g_1, g_2$  appearing in (4.1) are as small as we wish, say  $\sup_{\mathbb{B}}\{|g_1|, |g_2|\} \leq \varepsilon$ , where  $\varepsilon$  is a small positive constant whose value will be determined shortly.

Assume  $(x_0, y_0) \in \mathbb{B}$  and denote by  $(x_1, y_1) = f(x_0, y_0), \dots, (x_k, y_k) = f^k(x_0, y_0)$  its successive iterates (whenever defined). Using (4.1) recursively, we obtain

$$x_k = u^k x_0 \prod_{j=0}^{k-1} (1 + x_j y_j g_1(x_j, y_j)) \text{ and } y_k = s^k y_0 \prod_{j=0}^{k-1} (1 + x_j y_j g_2(x_j, y_j)).$$

We claim that if  $|x_0| \leq B|u|^{-n}$  for a suitable constant  $B$  and  $|y_0| \leq 1$  then the  $n$  first iterates of  $(x_0, y_0)$  are well-defined.

Indeed assume by induction that the  $k-1$  first iterates of  $(x_0, y_0)$  stay in  $\mathbb{B}$  for some  $k \leq n-1$ . Then we get that

$$|y_k| \leq |s|^k \prod_{j=0}^{k-1} (1 + |y_j| \varepsilon).$$

This will in turn be bounded by  $A|s|^k$  if  $A$  is any constant satisfying  $A \geq \prod_{j \geq 0} (1 + A|s|^j \varepsilon)$ . We leave the reader check that if  $\varepsilon < (1 - |s|)/10$ , then  $A = 3$  will do. In what follows we work under this assumption.

Now assume that  $|x_0| \leq \frac{1}{4}|u|^{-n}$  and let us show by induction that for small enough  $\varepsilon$ ,  $|x_j| \leq |u|^{j-n}$  for  $0 \leq j \leq n$  (so that in particular  $(x_j, y_j) \in \mathbb{B}$ ). Indeed, if this estimate holds for  $0 \leq j \leq k-1$ , then using the formula for  $x_k$ , we get that

$$\begin{aligned} |x_k| &\leq |u|^k |x_0| \prod_{j=0}^{k-1} (1 + |x_j| 3|s|^j \varepsilon) \leq \frac{1}{4} |u|^{k-n} \prod_{j=0}^{k-1} \left( 1 + 3\varepsilon \frac{|us|^j}{|u|^n} \right) \\ &\leq \frac{1}{4} |u|^{k-n} \left( 1 + \exp \left( \frac{3\varepsilon}{|u|^n} \sum_{j=0}^n |us|^j \right) \right) \\ &\leq \frac{1}{4} |u|^{k-n} \left( 1 + \exp \left( 3\varepsilon \sup_{n \geq 0} \max \left( \frac{|s|^n}{|us| - 1}, \frac{n}{|u|^n} \right) \right) \right). \end{aligned}$$

Hence, choosing  $\varepsilon$  sufficiently small (depending only on  $u$  and  $s$ ), the exponential term is smaller than 2, and we are done.

To get the conclusion of the lemma, we simply reconsider the previous computation for  $k = n$ , and use the inequality

$$\left| \prod (1 + z_j) - 1 \right| \leq \exp \left( \sum |z_j| \right) - 1$$

to obtain that

$$\left| \frac{x_n}{u^n x_0} - 1 \right| = \left| \prod_{j=0}^{n-1} (1 + x_j y_j g_1(x_j, y_j)) - 1 \right| = O \left( \max \left( |s|^n, \frac{n}{|u|^n} \right) \right)$$

and we are done.  $\square$

In the next theorem, we give a direct argument for Theorem B under a more restrictive assumption which is reasonable from the dynamical point of view. We feel interesting to include it as it gives in this case a purely archimedean proof of our main result. Observe that no transversality assumption is required.

**Theorem 4.3.** *Let  $f$  be a polynomial automorphism of the affine plane of Hénon type that is defined over a number field  $\mathbb{L}$ . Assume that there exists an algebraic curve defined over  $\mathbb{L}$  and containing infinitely many periodic points.*

*Suppose that at some archimedean place there exists a saddle point  $p \in C$ . Then  $\text{Jac}(f)$  is a root of unity.*

It follows from [BLS93b] that at the archimedean place most periodic orbits of  $f$  are saddles, which make the assumption of the proposition natural. Still, there exists examples polynomial automorphisms of  $\mathbb{C}^2$  with infinitely many non-saddle periodic orbits, even in a conservative setting, see [Dua].

*Proof.* We do an analysis similar to that of the proof of Theorem B, starting from equation (4.2), and keeping the same notation. For simplicity we write  $\mathbb{L}_v = \mathbb{C}$ , and drop the reference to  $v$ . By assumption there is a saddle periodic point  $p \in C$ , with multipliers  $u$  and  $s$ . From Theorem A we know that  $|us| = 1$ . We work in the local adapted coordinates  $(x, y)$  given by Lemma 1.9. Since we make no smoothness or transversality assumption here, we pick any local irreducible component of  $C$  at  $p$ , and parameterize it by  $\Psi : t \mapsto (t^k, \psi(t))$  with  $\psi(t) = t^l + \text{h.o.t.}$ . By Proposition 2.1 for small  $t \in \mathbb{C}$  we have that

$$\tilde{G}(t) := G^+ \circ \Psi(t) = G^+(t^k, \psi(t)) = \alpha G^-(t^k, \psi(t)) + H(t^k, \psi(t)).$$

Swapping the stable and unstable directions if needed we may assume that  $k \leq l$ . Pick a  $k^{\text{th}}$  root of  $u$ , denoted by  $u^{1/k}$ .

Applying the same reasoning as in (4.2) we get that

$$(4.5) \quad d^n \tilde{G} \left( \frac{t}{u^{n/k}} \right) = G^+ \circ f^n \left( \frac{t^k}{u^n}, \psi \left( \frac{t}{u^{n/k}} \right) \right) \rightarrow G^+(t^k, 0).$$

Since  $k \leq l$  and  $|us| = 1$ , we see that  $|s^k u^l| \geq 1$ , from which we infer that  $\psi\left(\frac{t}{u^{n/k}}\right) = O(s^n)$ . Therefore we can do the same with  $f^{-n}$  to deduce that

$$\begin{aligned}
(4.6) \quad d^n \tilde{G}\left(\frac{t}{u^{n/k}}\right) &= \alpha G^- \circ f^{-n}\left(\frac{t^k}{u^n}, \psi\left(\frac{t}{u^{n/k}}\right)\right) + d^n H\left(\frac{t^k}{u^n}, \psi\left(\frac{t}{u^{n/k}}\right)\right) \\
&= \alpha G^-\left(0, s^{-n} \psi\left(\frac{t}{u^{n/k}}\right)\right) + o(1) + d^n H\left(\frac{t^k}{u^n}, \psi\left(\frac{t}{u^{n/k}}\right)\right) \\
&= \alpha G^-\left(0, \frac{t^l}{(s^k u^l)^{n/k}}\right) + o(1) + d^n H\left(\frac{t^k}{u^n}, \psi\left(\frac{t}{u^{n/k}}\right)\right).
\end{aligned}$$

Arguing exactly as in the proof of Theorem B, we see that this is contradictory unless  $|s^k u^l| = 1$ , that is,  $k = l$  (in particular if  $C$  is smooth at  $p$  it must be transverse to  $W_{\text{loc}}^u(p)$  and  $W_{\text{loc}}^s(p)$ ).

Now since we work in the archimedean setting, we can push the analysis further and proceed to prove that  $us = \text{Jac}(f)$  is a root of unity. Assume by contradiction that this is not the case. Choose any  $\theta$  in the unit circle, and pick a subsequence  $(n_j)$  such that  $(u^k s^k)^{n_j} \rightarrow \theta$ . Observe that  $s^{-n} \psi(x/u^{n/k}) \rightarrow x^k/\theta$ . Then by (4.2) and (4.4) in the smooth case ( $k = 1$ ) and (4.5) and (4.6) in the singular case we get that for small  $t$

$$\begin{aligned}
G^+(t^k, 0) &= \lim_{n_j \rightarrow \infty} G^+ \circ f^{n_j}\left(\Psi\left(\frac{t}{u^{n_j/k}}\right)\right) \\
&= \lim_{n_j \rightarrow \infty} \left[ \alpha G^- \circ f^{-n_j}\left(\Psi\left(\frac{t}{u^{n_j/k}}\right)\right) + d^{n_j} H \circ \Psi\left(\frac{t}{u^{n_j/k}}\right) \right] \\
&= \alpha G^-\left(0, \frac{t^k}{\theta}\right) + \lim_{n_j \rightarrow \infty} d^{n_j} H \circ \Psi\left(\frac{t}{u^{n_j/k}}\right)
\end{aligned}$$

Since  $\theta$  was arbitrary and since a uniform limit of harmonic functions is harmonic, we see that the Laplacian of the function  $t \mapsto G^-(0, t^k)$  is rotation-invariant in a neighborhood of the origin. Observe that this Laplacian expresses as  $\kappa^* \Delta(G^-(0, t))$ , where  $\kappa : t \rightarrow t^k$ . Recall also that the support of  $\Delta(G^-(0, t))$  equals  $\partial(K^- \cap W_{\text{loc}}^s(p))$ , where the boundary is relative to the intrinsic topology on  $W_{\text{loc}}^s(p)$ . Thus we conclude that relative the linearizing coordinate on  $W^s(p)$ ,  $\kappa^{-1}(\partial(K^- \cap W_{\text{loc}}^s(p)))$  is rotation invariant, so  $\partial(K^- \cap W_{\text{loc}}^s(p))$  is rotation invariant as well. But since  $dd^c(G^-|_{W_{\text{loc}}^s(p)})$  gives no mass to points,  $p$  must be an accumulation point of  $\partial(K^- \cap W_{\text{loc}}^s(p))$ . By rotation invariance,  $K^- \cap W_{\text{loc}}^s(p)$  will then contain small circles around the origin. By the Maximum principle this implies that  $G^-|_{W_{\text{loc}}^s(p)}$  vanishes in a neighborhood of  $p$ , which contradicts Proposition 1.11. The proof is complete.  $\square$

*Remark 4.4.* It is a common thread in the dynamical study of plane polynomial automorphisms that the slices of  $T^\pm$  by stable and unstable manifolds (or more generally by any curve) contain a great deal of information about

$f$ . As we saw in §3, the Lyapunov exponents of the maximal entropy measure can be read off this data. The same holds for multipliers of all saddle periodic orbits. See also [BS98b] for a striking application of this circle of ideas.

The proof of Theorem 4.3 (with  $k = 1$ , say) implies that in the adapted coordinates a relation of the form  $G^+(x, 0) = G^-(0, x) + \tilde{H}$  holds, where  $\tilde{H}$  is a harmonic function. So we get that an unstable slice of  $K^+$  is *holomorphically equivalent* to a stable slice of  $K^-$ .

This rigidity suggests a strong form of symmetry between  $f$  and  $f^{-1}$  which gives additional credit to Conjecture 1.

## 5. AUTOMORPHISMS SHARING PERIODIC POINTS

The main goal of this section is to prove Theorem C.

**5.1. The Bass-Serre tree of  $\text{Aut}[\mathbb{A}^2]$ .** Let us recall briefly how the group of polynomial automorphisms of the affine plane naturally acts on a tree. We refer to [La] for details and to [Se] for basics on trees.

Denote by  $A$  (resp.  $E$ ) the subgroup of affine (resp. elementary) automorphisms. The intersection  $A \cap E$  consist of those automorphisms of the form  $(x, y) \mapsto (ax + b, cy + dx + e)$  with  $ac \neq 0$ .

Jung's theorem states that  $\text{Aut}[\mathbb{A}^2]$  is the free amalgamated product of  $A$  and  $E$  over their intersection. This means that any automorphism  $f \in \text{Aut}[\mathbb{A}^2]$  can be written as a product

$$f = e_1 \circ a_1 \dots \circ a_s \circ e_s$$

with  $e_i \in E$  and  $a_i \in A$ , and such a decomposition is unique up to replacing a product  $e \circ a$  by  $(e \circ h^{-1}) \circ (h \circ a)$  with  $h \in A \cap E$ .

The Bass-Serre tree  $\mathcal{T}$  of  $\text{Aut}[\mathbb{A}^2]$  is the simplicial tree whose vertices are left cosets modulo  $A$  or  $E$ . In other words we choose a set of representative  $S_A$  (resp.  $S_E$ ) of the quotient of  $\text{Aut}[\mathbb{A}^2]$  under the right action of  $A$  (resp. of  $E$ ). Then vertices of  $\mathcal{T}$  are in bijection with  $\{hA\}_{h \in S_A} \cup \{hE\}_{h \in S_E}$ . An edge is joining  $hA$  to  $h'E$  if  $h' = h \circ a$  for some  $a \in A$  or  $h = h' \circ e$  for some  $e \in E$ .

We endow  $\mathcal{T}$  with the unique tree metric putting length 1 to all edges. The left action of an automorphism  $h$  on cosets induces an action on  $\mathcal{T}$  by isometries. It sends any vertex of the form  $hA$  (resp.  $hE$ ) to  $fhA$  (resp. to  $fhE$ ).

An automorphism is conjugate to an affine map (resp. to an elementary map) if and only if it fixes a vertex of the form  $hA$  (resp.  $hE$ ) in  $\mathcal{T}$ . On the other hand, a polynomial automorphism  $f$  of Hénon type acts as a hyperbolic element of  $\text{Isom}(\mathcal{T})$ , that is, there exists a unique geodesic<sup>6</sup>  $\text{Geo}(f)$  invariant under the action of  $f$ . Furthermore, both ends of the geodesic are fixed, and  $f$  acts as a non-trivial translation on  $\text{Geo}(f)$ .

<sup>6</sup>that is, a bi-infinite path in  $\mathcal{T}$ .



**5.2. Proof of Theorem C.** We decompose the proof in three steps.

**Step 1:**  $f$  and  $g$  have the same equilibrium measure at any place.

Let us assume that  $f$  and  $g$  share a set  $\{p_m\}$  of periodic points which is Zariski dense in  $\mathbb{A}^2$ . We use a diagonal argument to extract a subset  $\{p'_k\}$  satisfying the requirements of Theorem 1.15. For this, enumerate all irreducible curves  $C_q \subset \mathbb{A}_{\mathbb{L}}^2$ ,  $q = 1, 2, \dots$ . We construct an auxiliary subsequence of  $(p_m)$  as follows. Let  $m_1$  be the minimal integer such that  $p_{m_1} \notin C_1$ , and set  $p'_1 = p_{m_1}$ . Then define  $m_2 > m_1$  to be the minimal integer such that  $p_{m_2} \notin C_1 \cup C_2$ , and set  $p'_2 = p_{m_2}$ . These integers exist since the set  $\{p_m\}$  is Zariski dense. Continuing in this way one defines recursively a sequence of periodic points  $(p'_k)$  with the desired properties. In particular we conclude from Theorem 1.15 that  $\mu_{f,v} = \mu_{g,v}$  for every place  $v$ .

**Step 2:**  $f$  and  $g$  have the same set of periodic points.

The difficulty is that we do not assume  $f$  and  $g$  to be conjugate by the *same* automorphism to a regular map. If this happens, the conclusion follows rather directly from the work of Lamy [La], as we will see in Step 3.

To overcome this problem, we proceed as follows. Fix a place  $v$ , and define  $K_v(f) = \{p \in \mathbb{A}_{\mathbb{L}_v}^{2,\text{an}}, \sup_{n \in \mathbb{Z}} |H^n(p)| < +\infty\}$ .

**Lemma 5.1.** *For any place  $v$ , the set  $K_v(f)$  is the largest compact set in  $\mathbb{A}_{\mathbb{L}_v}^{2,\text{an}}$  such that*

$$\sup_{K_v(f)} |P| = \sup_{\text{supp}(\mu_{f,v})} |P|$$

for all  $P \in \mathbb{L}_v[\mathbb{A}^2]$ .

*Proof.* Since  $\text{supp}(\mu_{f,v}) \subset K_v(f)$ , it is sufficient to prove that the supremum of  $|P|$  over  $K_v(f)$  is attained at a point lying in  $\text{supp}(\mu_{f,v})$ .

Suppose that  $P$  is a polynomial function, and pick any constant  $C_0 > 0$  such that  $\log(|P|/C_0) \leq 0$  on  $\text{supp}(\mu_{f,v})$ . Then  $\tilde{G} := \max\{G, \log(|P|/(C_0 + \varepsilon))\}$  is a continuous non-negative function on  $\mathbb{A}_{\mathbb{L}_v}^2$  that induces a continuous semi-positive metric on  $\mathcal{O}(1)$ . Since  $\tilde{G} = G$  near  $\text{supp}(\mu_{f,v})$ , from Corollary A.2 we deduce the equality of measures  $\text{MA}(\tilde{G}) = \text{MA}(G)$ , and it follows from Yuan-Zhang's theorem, see [YZ13a] that  $\tilde{G} - G$  is a constant, hence  $\tilde{G} = G$ . It follows that  $\log(|P|/(C_0 + \varepsilon)) \leq 0$  on  $K_v(f)$ . By letting  $\varepsilon \rightarrow 0$  we conclude that  $\log(|P|/C_0) \leq 0$  on  $K_v(f)$ .  $\square$

In plain words, the polynomially convex envelope of  $\text{supp}(\mu_{f,v})$  is the set  $K_{f,v}$ . Since  $\mu_{f,v} = \mu_{g,v}$  for all  $v$  we conclude that  $K_{f,v} = K_{g,v}$ .

Now pick any periodic point  $p$  of  $f$ . At the place  $v$ , it belongs to  $K_{f,v}$  hence to  $K_{g,v}$ . Since Lee's height can be computed by summing the local quantities  $G_{g,v} := \max\{G_{g,v}^+, G_{g,v}^-\}$  and since  $\{G_{g,v} = 0\} = K_{g,v}$  we get that the canonical  $g$ -height of  $p$  is zero, hence  $p$  is  $g$ -periodic.

In the sequel we actually need a stronger information.

**Lemma 5.2.** *Suppose  $f$  and  $g$  are two hyperbolic polynomial automorphisms of the affine plane defined over a number field  $\mathbb{L}$ , satisfying the assumptions of Theorem B.*

*Then for all places  $v$  over  $\mathbb{L}$ , and for any Hénon-type automorphism  $h$  belonging to the subgroup generated by  $f$  and  $g$ , one has  $K_{h,v} = K_{g,v} = K_{f,v}$ .*

*Proof.* We already know that  $K_v := K_{g,v} = K_{f,v}$ . Since this compact set is invariant by both  $f$  and  $g$ , it follows that  $h$  also preserves  $K_v$ , and this implies  $K_v \subset K_{h,v}$  for all  $v$ . Now let  $F_n$  denote the set of points of period  $n$  for  $f$ . For all  $v$ , we have  $F_n \subset K_{h,v}$  hence the canonical  $h$ -height of  $F_n$  is equal to 0. Extracting a subsequence if necessary, we may always assume that  $F_n$  is generic since the set of periodic points of a hyperbolic automorphism is Zariski-dense. By Yuan's theorem  $F_n$  is equidistributed with respect to the equilibrium measure of both  $K_v$  and  $K_{h,v}$  and we conclude that  $K_{h,v} = K_v$ .  $\square$

**Step 3:**  $f$  and  $g$  admit a common iterate.

We use the structure of subgroups of the group of polynomial automorphisms of the plane as explained in [La]. Let us assume by contradiction that  $f$  and  $g$  admit no iterate in common.

**Lemma 5.3.** *There exists two Hénon-type elements  $h_1, h_2$  in the subgroup generated by  $f$  and  $g$  and a polynomial automorphism  $\varphi$  such that  $\varphi^{-1} \circ h_1 \circ \varphi$ ,  $\varphi^{-1} \circ h_2 \circ \varphi$  and its commutator are regular automorphisms of  $\mathbb{A}_{\mathbb{L}}^2$  and the subgroup they generate is a free non-abelian group.*

The rest of the argument is now contained in [La, Théorème 5.4]. Let us explain his arguments for the convenience of the reader. We may assume that  $h_1, h_2$  are two regular polynomial automorphisms of  $\mathbb{A}_{\mathbb{L}}^2$ . By Lemma 5.2 we have  $K_v := K_{h_1,v} = K_{h_2,v}$  at all places. Pick any archimedean place  $v$ . Then  $\mu := \mu_{h_1,v} = \mu_{h_2,v}$ . Now since both  $h_1$  and  $h_2$  are regular, it follows that  $G_1 := \max\{G_{h_1}^+, G_{h_1}^-\}$  and  $G_2$  are both equal to the Siciak-Green function of  $K_v$ , and are hence equal.

Replacing  $h_1$  by its inverse if necessary, it follows that  $G_{h_1}^+ = G_{h_2}^+$  on a non-empty open set where the two functions are positive. Since these functions are pluriharmonic where they are non zero, and since for a Hénon-type automorphism  $h$ ,  $\mathbb{C}^2 \setminus K^+$  is connected (indeed  $\mathbb{C}^2 \setminus K^+ = \bigcup_{n \geq 0} h^{-n}(V^+)$ ) we deduce that they coincide everywhere. We conclude that the positive closed (1,1)-currents  $T := dd^c G_{h_1}^+ = dd^c G_{h_2}^+$  are equal. Now consider the commutator  $h_3 = h_1 h_2 h_1^{-1} h_2^{-1}$ . Observe that  $h_3^* T = T$ , and  $h_3$  is regular by the previous lemma. Since the support of  $T$  has a unique point on the line at infinity, replacing  $h_3$  by  $h_3^{-1}$  if needed we may suppose that this point is not an indeterminacy point of  $h_3$ . It then follows that the mass of  $h_3^* T$  equals the degree of  $h_3$  times the mass of  $T$  which is contradictory. This completes the proof of Theorem C.

*Proof of Lemma 5.3.* We need the following two facts, see [La]. Let  $f$  and  $g$  be two polynomial automorphisms of Hénon type.

- (F1) The geodesics  $\text{Geo}(f)$  and  $\text{Geo}(g)$  intersect if and only if there exists  $\varphi \in \text{Aut}[\mathbb{A}_L^2]$  such that both automorphisms  $\varphi^{-1} \circ f \circ \varphi$  and  $\varphi^{-1} \circ g \circ \varphi$  are regular.
- (F2) If  $f$  and  $g$  do not share a non-trivial iterate, then the two geodesics  $\text{Geo}(f)$  and  $\text{Geo}(g)$  have a compact intersection (possibly empty).

Let us first treat the case when  $\text{Geo}(f) \cap \text{Geo}(g)$  is non-empty. Pick a vertex  $\gamma$  in this intersection. By (F1) we may assume  $f$  and  $g$  are regular automorphisms.

Set then  $h_1 = f^N$  and  $h_2 = g^N$  where  $N$  is greater than the diameter of  $\text{Geo}(f) \cap \text{Geo}(g)$ . The invariant geodesics of these two automorphisms are equal to  $\text{Geo}(f)$  and  $\text{Geo}(g)$  respectively. Now pick any non-trivial word

$$h = h_1^{n_p} \circ h_2^{m_p} \circ \dots \circ h_1^{n_1} \circ h_2^{m_1}$$

with  $p \geq 1$  and all  $n_i, m_i \in \mathbb{Z}^*$ . Then a classical ping-pong argument (see [La, Proposition 4.3]) shows that  $\gamma$  lies in the interior of the segment  $[h(\gamma), h^{-1}(\gamma)]$  which implies  $h$  to be of Hénon type and  $\text{Geo}(h) \ni \gamma$ . By (F1) we conclude that  $h$  is also regular. This concludes the proof in this case.

Assume now that  $\text{Geo}(f)$  and  $\text{Geo}(g)$  are disjoint. Then there exists a unique segment  $I = [\gamma_1, \gamma_2]$  in the tree with  $I \cap \text{Geo}(f) = \{\gamma_1\}$  and  $I \cap \text{Geo}(g) = \{\gamma_2\}$ . Pick any element  $\gamma$  lying in the interior of  $I$ , and  $N$  large enough such that the translation length of both  $f^N$  and  $g^N$  are larger than twice the diameter of  $I$ . Then both automorphisms  $h_+ = f^N g^N$  and  $h_- = f^{2N} g^{2N}$  satisfy  $\gamma \in [h_i(\gamma), h_i^{-1}(\gamma)]$  so that we can apply the same argument as in the previous case. The proof is complete.  $\square$

**5.3. Sharing cycles.** Let us start with the following observation.

**Proposition 5.4.** *Let  $f$  be an automorphism of Hénon type defined over a number field  $\mathbb{L}$ . Let  $(F_m)$  be any sequence of disjoint periodic cycles. Then the sequence of probability measures  $(\mu_m)$  equidistributed over the Galois conjugates of  $F_m$  converges weakly to  $\mu_{f,v}$  for all places  $v$ .*

As a consequence we have the following variation on Theorem C.

**Corollary 5.5.** *Let  $f$  and  $g$  be two complex polynomial automorphisms of Hénon type of the affine plane.*

*Then if  $f$  and  $g$  share an infinite set of periodic cycles, there exists two non-zero integers  $n, m \in \mathbb{Z}$  such that  $f^n = g^m$ .*

Indeed to prove the corollary it suffices to repeat the proof of Theorem C starting from Step 2.

*Proof of Proposition 5.4.* We may assume that all  $F_m$  are Galois invariant. The result does not quite follow from Lee's argument of [Le, Theorem A]

since we do not assume the set  $\bigcup_m (F_m \cap C)$  to be finite for every curve  $C$ . We claim however that any algebraic curve  $C$  it holds that

$$(5.1) \quad \#(C \cap F_m) = o(\#F_m) \text{ as } n \rightarrow \infty.$$

One then argues exactly as in [FG, Proof of Theorem 1] to conclude that  $\mu_m$  converges to  $\mu_f$ .

Let us justify (5.1). Suppose by contradiction that there exists  $\varepsilon > 0$  such that

$$\#(C \cap F_m) \geq \varepsilon \#F_m .$$

First observe that the minimal period of all points in  $F_m$  tends to infinity since  $f^n$  admits only finitely many fixed points for any  $n > 0$ . Pick any integer  $N > 1/\varepsilon$  and  $n$  large enough such that the periods of all points in  $F_m$  are larger than  $N$ . We claim that

$$\#\{p \in F_m \cap C, \text{ such that } f^k(p) \in C \text{ for some } 0 < k \leq N\}$$

tends to infinity. But this implies that  $C \cap f^{-k}(C)$  is infinite for some  $0 < k \leq N$ , whence  $f^k(C) = C$ , a contradiction.

To prove the claim, let  $B$  denote the set of points in  $F_m \cap C$  such that  $f^k(p) \notin C$  for all  $1 \leq k \leq N$ , and let  $G$  be its complement in  $F_m \cap C$ . We want to estimate  $\#G$ . For this, we see that  $\#B \times N \leq \#F_m$ , hence

$$\#G \geq \#(F_m \cap C) - \#B \geq \varepsilon \#F_m - \frac{1}{N} \#F_m \rightarrow \infty$$

as required.  $\square$

## 6. REVERSIBLE POLYNOMIAL AUTOMORPHISMS

A polynomial automorphism of  $\mathbb{A}^2$  is said to be *reversible* if there exists a polynomial automorphism  $\sigma$ , which may or may not be an involution, such that  $\sigma^{-1}f\sigma = f^{-1}$ . Any such  $\sigma$  is then called a *reversor*.

Since invariance under time-reversal appears frequently in physical models, such mappings have attracted a lot of attention in the mathematical physics literature. In the context of plane polynomial automorphisms, reversible mappings were classified by Gomez and Meiss in [GM03a, GM03b]. In particular they prove that the reversor  $\sigma$  is either affine or elementary and of finite (even) order. Moreover they show that when  $\sigma$  admits a curve of fixed points, then  $\sigma$  must be an involution conjugate to the affine involution  $t : (x, y) \mapsto (y, x)$ .

Our aim is to prove the following:

**Proposition 6.1.** *Suppose that  $f$  is a reversible polynomial automorphism of Hénon type and that  $\sigma$  is an involution conjugating  $f$  to  $f^{-1}$ .*

*Then any curve of fixed points of  $\sigma$  contains infinitely many periodic points of  $f$ .*

Specific examples include all Hénon transformations of Jacobian 1, that are of the form  $(x, y) \mapsto (p(x) - y, x)$ , for which the reversor is the affine involution  $t$ . So is the Hénon mapping  $(x, y) \mapsto (-y, p(y^2) - x)$ , of Jacobian  $-1$ . More generally, a mapping of the form  $tH^{-1}tH$  is reversible with reversor  $t$ , where  $H$  denotes any polynomial automorphism.

Let us also observe that taking iterates is really necessary in Conjecture 1. Indeed pick any primitive  $n$ -th root of unity  $\zeta$ , and let  $p$  be any polynomial such that  $p(\zeta x) = \zeta p(x)$ . Then the three automorphisms  $(p(x) - y, x)$ ,  $(\zeta x, \zeta y)$  and  $t$  commute and the automorphism defined by  $H := (p(x) - y, x) \circ (\zeta x, \zeta y)$  is not reversible but its  $n$ -th iterate is. Observe also that the jacobian of  $H$  equals  $\zeta^2$ .

*Algebraic proof of Proposition 6.1.* As observed above, we may assume that  $\sigma = t$  so that the curve of fixed points is actually the diagonal  $\Delta = \{x = y\}$ .

Observe now that any point  $p \in \Delta \cap f^n(\Delta)$  satisfies  $f^{-n}(p) = \sigma f^n \sigma(p) = \sigma f^n(p) = f^n(p)$ , and is thus periodic of period  $2n$ .

To conclude, it remains to prove that  $\#\Delta \cap f^n(\Delta) \rightarrow \infty$ . For any  $p \in \Delta \cap f^n(\Delta)$ , we denote by  $\mu_n(p)$  the multiplicity of intersection of  $\Delta$  and  $f^n(\Delta)$  at  $p$ . We rely on the following result of Arnold [Ar], see also [SY].

**Lemma 6.2.** *For any  $p \in \Delta$ , the sequence  $\mu_n(p)$  is bounded.*

Now since  $f$  is a polynomial automorphism of Hénon type we may choose affine coordinates such that  $f$  extends to  $\mathbb{P}^2$  as a regular map. Recall that  $f$  admits a super-attracting point  $p_+$  and a point of indeterminacy  $p_-$  on the line at infinity and that  $p_+ \neq p_-$ . Write  $\overline{\Delta}$  for the closure of the diagonal in  $\mathbb{P}^2$ .

By [BS91], we know that there exists an integer  $n_0 \geq 1$  such that  $f^n(\overline{\Delta}) \ni p_+$  for all  $n \geq n_0$  and  $f^n(\overline{\Delta}) \ni p_-$  for all  $n \leq -n_0$ . It follows that for all  $n \geq n_0$ , the intersection  $f^{-n_0}(\overline{\Delta}) \cap f^n(\overline{\Delta})$  is included in  $\mathbb{A}^2$ . By Bézout's Theorem we infer that

$$\sum_{p \in \Delta} \mu_n(p) = f^{-n_0}(\overline{\Delta}) \cdot f^n(\overline{\Delta}) = \deg(f^{-n_0}(\overline{\Delta})) \times \deg(f^n(\overline{\Delta})) \rightarrow \infty.$$

We conclude that there are infinitely many fixed points of  $f$  on  $\Delta$  for otherwise their multiplicities would have to grow to infinity, contradicting Lemma 6.2.  $\square$

*Analytic proof of Proposition 6.1.* Let us sketch an alternate argument for the fact that  $\#\Delta \cap f^n(\Delta) \rightarrow \infty$ , which is based on intersection theory of laminar currents.

For notational ease, let us assume that  $n = 2k$  is even. Then  $\#\Delta \cap f^n(\Delta) = \#f^{-k}(\Delta) \cap f^k(\Delta)$ . We know from [BS91] that the sequence of positive closed  $(1, 1)$ -currents  $d^{-k}[f^k(\Delta)]$  converges to  $T^-$ , and likewise for  $T^+$ . It follows from [BS98a] that this convergence holds in a geometric sense. Informally this means that we can discard a part of  $d^{-k}[f^k(\Delta)]$  of arbitrary small mass, uniformly in  $k$ , such that the remaining part is made of disks

of uniformly bounded geometry, which geometrically converge to the disks making up the laminar structure of  $T^-$ .

To state things more precisely, we follow the presentation of [Duj]. Fix  $\varepsilon > 0$ . Given a generic subdivision  $\mathcal{Q}$  of  $\mathbb{C}^2$  by affine cubes of size  $r > 0$ , there exists uniformly laminar currents  $T_{\mathcal{Q},k}^- \leq d^{-k}[f^k(\Delta)]$  and  $T_{\mathcal{Q},k}^+ \leq d^{-k}[f^{-k}(\Delta)]$  made of graphs in these cubes and such that the mass of  $d^{-k}[f^{\pm k}(\Delta)] - T_{\mathcal{Q},k}^\pm$  is bounded by  $Cr^2$  for some constant  $C$  [Duj, Prop. 4.4]. Therefore, up to extracting a subsequence, the currents  $T_{\mathcal{Q},k}^\pm$  converge to currents  $T_{\mathcal{Q}}^\pm \leq T^\pm$  such that  $\mathbf{M}(T^\pm - T_{\mathcal{Q}}^\pm) \leq Cr^2$ . Then we infer from [Duj, Thm. 4.2] that if  $r$  is smaller than some  $r(\varepsilon)$ ,  $\mathbf{M}(T^+ \wedge T^- - T_{\mathcal{Q}}^+ \wedge T_{\mathcal{Q}}^-) \leq \varepsilon/2$ . Furthermore, only transverse intersections account for the wedge product  $T_{\mathcal{Q}}^+ \wedge T_{\mathcal{Q}}^-$ . If we denote by  $\hat{\wedge}$  the geometric intersection product for curves, which consists in putting a Dirac mass at any proper intersection, without counting multiplicities, we have the weak convergence

$$T_{\mathcal{Q},k}^+ \hat{\wedge} T_{\mathcal{Q},k}^- \rightarrow T_{\mathcal{Q}}^+ \hat{\wedge} T_{\mathcal{Q}}^- = T_{\mathcal{Q}}^+ \wedge T_{\mathcal{Q}}^- ,$$

where the last equality follows from [Duj, Thm 3.1]. Therefore the mass of  $T_{\mathcal{Q},k}^+ \hat{\wedge} T_{\mathcal{Q},k}^-$  is larger than  $1 - \varepsilon$  for large enough  $k$ , which was the result to be proved.  $\square$

*Remark 6.3.* Observe that the analytic argument implies that  $\Delta$  intersects  $f^{2k}(\Delta)$  transversally at  $\sim d^k$  points. We claim that this implies

$$\#\text{Per}(f^{2k}) \cap \Delta = d^k(1 + o(1)) ,$$

that is, most of  $\Delta \cap f^{2k}(\Delta)$  is made of points of exact period  $2k$ . Indeed, assume that this is not the case. Then there exists  $\varepsilon > 0$  and a sequence  $k_j \rightarrow \infty$  such that  $\varepsilon d^{k_j}$  of these points have a period which is a proper divisor  $N$  of  $d^{2k_j}$ , in particular  $N \leq d^{k_j}$ . Let  $F_j$  be this set of points and  $\nu_j = d^{-k_j} \sum_{p \in F_j} \delta_p$ . By [BLS93b], the measure equidistributed on  $\text{Fix}(f^{k_j})$  (which has cardinality  $d^{k_j}$ ) converges to  $\mu_f$ . Thus any cluster limit  $\nu$  of  $(\nu_j)$  has mass at least  $\varepsilon$  and satisfies  $\nu \leq \mu_f$ . It follows that  $\mu_f$  gives a mass at least  $\varepsilon$  to  $\Delta$ , which is contradictory.

APPENDIX A. A COMPLEMENT ON THE NON-ARCHIMEDEAN  
MONGE-AMPÈRE OPERATOR

Let  $K$  be any complete non-archimedean field. We prove

**Theorem A.1.** *Suppose  $L \rightarrow X$  is an ample line bundle over a smooth  $K$ -variety of dimension  $d$ . Pick any two semi-positive continuous metrics  $|\cdot|_1, |\cdot|_2$  in the sense of Zhang. Then*

$$(A.1) \quad \mathbf{1}_{\{|\cdot|_1 < |\cdot|_2\}} c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d = \mathbf{1}_{\{|\cdot|_1 < |\cdot|_2\}} c_1(L, |\cdot|_1)^d.$$

As in [BFJ, §5] this result implies

**Corollary A.2.** *Suppose  $L \rightarrow X$  is an ample line bundle over a smooth  $K$ -variety of dimension  $d$ . Pick any two semi-positive continuous metrics  $|\cdot|_1, |\cdot|_2$  in the sense of Zhang and suppose that they coincide on an open set  $\Omega$  in the analytification of  $X$  in the sense of Berkovich.*

*Then the positive measures  $c_1(L, |\cdot|_1)^d$  and  $c_1(L, |\cdot|_2)^d$  coincide in  $\Omega$ .*

Recall that in the main body of the text, we were dealing with metrics  $|\cdot|$  on  $\mathcal{O}(1) \rightarrow \mathbb{P}^d$  and the evaluation of the section 1 on the analytification of the affine space  $\mathbb{A}^d \subset \mathbb{P}^d$  defines a function  $G := \log |1|$ . With this identification, one has  $\text{MA}(G) = c_1(\mathcal{O}(1), |\cdot|)^d$ .

*Proof.* Assume first that the two metrics  $|\cdot|_i$  are model metrics. This means that we can find a model  $\mathfrak{X}$  of  $X$  over  $\text{Spec } \mathcal{O}_K$ , and nef line bundles  $\mathfrak{L}_i \rightarrow \mathfrak{X}$  whose restriction to the generic fiber of  $\mathfrak{X}$  is  $L$ .

In that case  $c_1(L, \max\{|\cdot|_1, |\cdot|_2\})^d$  and  $c_1(L, |\cdot|_1)^d$  are both atomic measures, supported on divisorial points corresponding to irreducible components of the special fiber.

If  $E$  is such a component for which  $|\cdot|_1/|\cdot|_2(x_E) < 1$  then  $|\cdot|_1/|\cdot|_2(x_F) \leq 1$  for all irreducible components  $F$  of the special fiber intersecting  $E$ . It follows that  $\mathfrak{L}_1|_E = \mathfrak{L}_2|_E$  as numerical classes on  $E$ , and hence  $c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d\{x_E\}$  and  $c_1(L, |\cdot|_1)^d\{x_E\}$  by the definition of Monge-Ampère measures of model functions.

In the general case, we may assume that we have sequences of model metrics  $|\cdot|_{i,n}$  on  $L$  such that  $|\cdot|_{i,n} \rightarrow |\cdot|_i$  uniformly on  $X^{\text{an}}$ . Observe that  $\Omega := \{|\cdot|_1 < |\cdot|_2\}$  is open since both metrics are continuous. It suffices to prove that

$$\int h c_1(L, \min\{|\cdot|_1, |\cdot|_2\})^d = \int h c_1(L, |\cdot|_1)^d$$

for all continuous functions  $h$  whose support is contained in  $\Omega$  and such that  $0 \leq h \leq 1$ .

Pick  $\varepsilon > 0$  small and rational and write  $\Omega_n := \{|\cdot|_{1,n}e^{-\varepsilon} < |\cdot|_{2,n}\}$ . For  $n \gg 0$ , we have  $\Omega \subseteq \Omega_n$ . Since  $|\cdot|_{1,n}e^{-\varepsilon}$  and  $|\cdot|_{2,n}$  are both model metrics, we have  $c_1(L, \min\{|\cdot|_{1,n}e^{-\varepsilon}, |\cdot|_{2,n}\})^d = c_1(L, |\cdot|_{1,n})^d$  on  $\Omega_n$  by the previous

step. Since  $h$  is supported in  $\Omega \subset \Omega_n$  we get

$$\int h c_1(L, \min\{|\cdot|_{1,n} e^{-\varepsilon}, |\cdot|_{2,n}\})^d = \int h c_1(L, |\cdot|_{1,n} e^{-\varepsilon})^d = \int h c_1(L, |\cdot|_{1,n})^d$$

for all  $n$ , and we conclude letting  $n \rightarrow \infty$  and  $\varepsilon \rightarrow 0$  and using [CL06, Proposition 2.7].  $\square$

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