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LYAPUNOV EXPONENTS FOR SURFACE GROUPS REPRESENTATIONS.

BERTRAND DEROIN AND ROMAIN DUJARDIN

ABSTRACT. Let $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of a surface group $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{C})$, where S is a topological (possibly punctured) surface with negative Euler characteristic. Given a structure of Riemann surface of finite type on S we construct a bifurcation current on the parameter space Λ , that is a $(1,1)$ positive closed current attached to the bifurcations of the family. It is defined as the dd^c of the Lyapunov exponent of the representation with respect to the Brownian motion on the Riemann surface S , endowed with its Poincaré metric. We show that this bifurcation current describes the asymptotic distribution of various codimension 1 phenomena in Λ . For instance, the random hypersurfaces of Λ defined by the condition that a random closed geodesic on S is mapped under ρ_λ to a parabolic element or the identity are asymptotically equidistributed with respect to the bifurcation current. The proofs are based on our previous work [DD1], and on a careful control of a discretization procedure of the Brownian motion.

INTRODUCTION

Let G be a finitely generated group and $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of G into the Möbius group $\mathrm{PSL}(2, \mathbb{C})$. Viewing these representations as dynamical systems on \mathbb{P}^1 , we can divide the parameter space Λ into a *stability locus*, where the action of G is locally quasi-conformally structurally stable, and its complement, the *bifurcation locus*.

In a previous article [DD1], under natural assumptions on the family $(\rho_\lambda)_{\lambda \in \Lambda}$, we showed how to design *bifurcation currents* on Λ . These are positive closed currents aiming at an understanding of the measurable and complex analytic structure of the bifurcation locus. This construction relies on probabilistic ideas, let us briefly recall some of its key steps. We start by choosing a probability measure μ on G , which gives rise to a random walk on G . An example is the nearest neighbor random walk on the Cayley graph associated to some symmetric set of generators. Fix any matrix norm $\|\cdot\|$ on $\mathrm{PSL}(2, \mathbb{C})$. Then the theory of random products of matrices implies that for any non-elementary representation ρ , for almost every sample path $(g_n \cdots g_1)_{n \geq 1}$ of the random walk, $\frac{1}{n} \log \|\rho(g_n \cdots g_1)\|$ converges to the *Lyapunov exponent* $\chi_\mu(\rho)$. It is easily seen that the Lyapunov exponent function $\lambda \mapsto \chi_\mu(\rho_\lambda)$ is plurisubharmonic (psh) on Λ , and we define the bifurcation current by the formula $T_{\mathrm{bif}, \mu} = dd^c_\lambda \chi_\mu(\rho_\lambda)$.

It turns out that the support of $T_{\mathrm{bif}, \mu}$ is precisely equal to the bifurcation locus. Also, $T_{\mathrm{bif}, \mu}$ describes the asymptotic distribution of certain dynamically defined sequences of subvarieties in Λ . Indeed, for $t \in \mathbb{C}$, and $g \in G$, let $Z(g, t)$ be the hypersurface in Λ defined by

$$Z(g, t) = \{\lambda \in \Lambda, \mathrm{tr}^2(\rho_\lambda(g)) = t\}$$

(tr stands for the trace). The emblematic value is $t = 4$, corresponding to those parameters where $\rho_\lambda(g)$ becomes accidentally parabolic or equal to the identity, which are known to be

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dense in the bifurcation locus. Then the following equidistribution statement holds: for almost every sample path of the random walk, the sequence of divisors $\frac{1}{2n}Z(g_n \cdots g_1, t)$ converges to $T_{\text{bif}, \mu}$ when $n \rightarrow \infty$.

We note that analogous results were previously known in the context of holomorphic families of rational mappings on \mathbb{P}^1 . An essential motivation in [DD1] was actually to fill the corresponding line in the famous Sullivan dictionary between rational iteration and Kleinian group theory. The reader is referred to [Duj] for an overview on these issues.

In this paper we specialize to the case where $G = \pi_1(S)$ is the fundamental group of an orientable surface of genus g with n punctures, such that $\chi(S) = 2 - 2g - n < 0$. As before we consider a holomorphic family $(\rho_\lambda)_{\lambda \in \Lambda}$ of non-elementary representations of G into $\text{PSL}(2, \mathbb{C})$ (we can equally deal with representations modulo conjugacy) sending simple loops about the punctures to parabolic transformations. Such representations will be referred to as *parabolic*. For the sake of this introduction, let us assume without further notice that these representations are faithful at generic parameters and not conjugate to each other in $\text{PSL}(2, \mathbb{C})$. Our main purpose is to construct a bifurcation current on Λ *associated to the choice of a conformal structure on S* . For this, let X be a structure of Riemann surface of finite type on S (that is, X is biholomorphic to a compact Riemann surface minus n points) and endow X with its unique hyperbolic metric of curvature -1 . Recall that it is of finite volume. This data in turn gives birth to a wealth of mathematical objects:

- a dynamical system: the geodesic flow on the unit tangent bundle T^1X ;
- a stochastic process: Brownian motion on X ;
- an isomorphism between G and a lattice Γ in $\text{PSL}(2, \mathbb{R})$ induced by the uniformization of X

We will demonstrate that our bifurcation current fits naturally with all these structures. Let us already point out that as in the previous case, this bifurcation current will be defined by taking the dd^c of a certain Lyapunov exponent function.

A family of representations of G (modulo conjugacy) naturally associated to the Riemann surface structure is obtained by considering the holonomies of holomorphic projective structures on X . The parameter space for this family of representations is biholomorphic to \mathbb{C}^{3g-3+n} and is the ambient space for the so-called Bers embedding of Teichmüller space. As such it plays an important role in Teichmüller theory (see Dumas [Dum] for a nice introduction to this topic). Notice that the much studied cases $(g = 0, n = 4)$ and $(g = 1, n = 1)$, corresponding to one-dimensional Teichmüller spaces, are associated to punctured surface groups. This is a motivation for allowing for non-compact surfaces, which, as we will see, are a source of technical difficulty.

We see that the space of projective structures carries a natural bifurcation current (resp. Lyapunov exponent function). The Sullivan dictionary predicts that the structure of this space should resemble that of the space of polynomials in holomorphic dynamics. In a companion paper [DD2] we introduce an invariant, the *degree* of a projective structure, and show that it is closely related to the Lyapunov exponent of its holonomy. This enables us to unveil a new analogy between spaces of projective structures and spaces of polynomials.

Let us discuss more precisely the contents of the present paper. As before, the definition of the bifurcation current depends on the construction of a Lyapunov exponent function on Λ , that we now describe. Mark a point \star in X , which will serve as a base point for the fundamental group, and equip X with its Poincaré metric. For a Brownian path $\omega : [0, \infty) \rightarrow$

X starting at $\omega(0) = \star$, we do the following construction: for $t > 0$, we append $\omega|_{[0,t]}$ with a shortest path returning to \star , to get a closed loop ω_t , hence an element $[\omega_t] \in \pi_1(X, \star)$ –we disregard any possible ambiguity in the choice of the returning path.

We then have the following result

Proposition A. *Let X be a hyperbolic Riemann surface of finite type and $\rho : G \rightarrow \mathrm{PSL}(2, \mathbb{C})$ be a non-elementary parabolic representation. Then there exists a positive real number $\chi_{\mathrm{Brown}}(\rho)$ such that for almost every Brownian path ω starting at \star , $\frac{1}{t} \log \|\rho([\omega_t])\|$ converges to $\chi_{\mathrm{Brown}}(\rho)$ when $t \rightarrow \infty$.*

In any holomorphic family $(\rho_\lambda)_{\lambda \in \Lambda}$ of non-elementary representations of G , we thus obtain a Lyapunov exponent function $\lambda \mapsto \chi_{\mathrm{Brown}}(\rho_\lambda)$, and we define the bifurcation current associated to the Riemann surface structure X as $T_{\mathrm{bif}} = dd_\lambda^c(\chi_{\mathrm{Brown}}(\rho_\lambda))$. From now on we will refer to it as the *natural bifurcation current*. We observe that this current depends only (up to a multiplicative constant) on the conformal structure on X , and not on the conformal metric on X which was chosen to generate the Brownian motion (in the case where X has punctures some extra care is needed about the behaviour of the metric near the cusps). Indeed, by conformal invariance of the Brownian motion in dimension 2, the Lyapunov exponent function associated to any other choice of conformal metric will be a constant multiple of χ_{Brown} (see Remark 2.8 for details).

A similar procedure yields a Lyapunov exponent $\chi_{\mathrm{geodesic}} = \chi_{\mathrm{Brown}}$ by using a generic geodesic starting at \star instead of a Brownian path (with our normalization, the Brownian motion in the hyperbolic plane escapes at unit speed). In particular $T_{\mathrm{bif}} = dd^c \chi_{\mathrm{geodesic}}$. A statement analogous to Proposition A in this context was already contained in the work of Bonatti, Gomez-Mont and Viana [BGM] (at least when X is compact).

Yet another way to define the Lyapunov exponent is to use the isomorphism between G and a lattice $\Gamma \leq \mathrm{PSL}(2, \mathbb{R})$ provided by uniformization. Let \mathbb{H} be the hyperbolic plane. For a sequence r_n increasing sufficiently fast to infinity (e.g. $r_n = n^\alpha$ will do), we pick at random an element $\gamma_n \in \Gamma \cap B_{\mathbb{H}}(0, r_n)$, relative to the counting measure. Then almost surely, $\frac{1}{2r_n} \log \|\rho(\gamma_n)\|$ converges to $\chi_{\mathrm{Brown}}(\rho)$.

As in [DD1], the first main result asserts that T_{bif} is indeed a bifurcation current.

Theorem B. *Let X be a hyperbolic Riemann surface of finite type and $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of non-elementary parabolic representations of $G = \pi_1(X, \star)$. Then the support of the natural bifurcation current T_{bif} is equal to the bifurcation locus.*

The strategy of the proof is to rely on the corresponding support theorem in [DD1]. For this, we use a discretization procedure for the Brownian motion, which was devised by Furstenberg [F2] and then refined by Lyons and Sullivan [LS], among others. In some sense, it allows to project the Wiener measure (based, say, at \star), to a probability measure μ on G . We then show that the natural bifurcation current is equal to $T_{\mathrm{bif}, \mu}$, up to a multiplicative constant, from which the result follows. The point is that the support theorem in [DD1] requires an exponential moment condition on μ . When X is compact, this fact is apparently part of the folklore. We show that when X is merely of finite volume and ρ is a parabolic representation, μ still satisfies the following slightly modified moment property

$$\int_G \exp(\beta d_{\mathbb{H}}(0, \rho(g)(0))) d\mu(g) < \infty \text{ for some } \beta > 0,$$

that is enough to apply the results of [DD1] (see Proposition 2.16). The proof is rather technical, and occupies most of §2. We believe that it could find applications beyond this work.

The second main theme of the paper is the equidistribution of random hypersurfaces of the form $Z(g_n, t)$ towards the bifurcation current. Given the Riemann surface structure X , there are at least two natural choices for the group elements g_n :

- A first possibility is to pick randomly an element γ_n in $\Gamma \cap B_{\mathbb{H}}(0, r_n)$ (according to the counting measure), and let $r_n \rightarrow \infty$.
- A second choice is to choose at random a closed geodesic γ_n of length at most r_n on X . This geodesic defines a conjugacy class in G , so, abusing notation slightly, we can define $\text{tr}^2(\rho([\gamma]))$ and it makes sense to consider $Z(\gamma_n, t) = \{\lambda \in \Lambda, \text{tr}^2([\gamma_n]) = t\}$.

In both cases, we show that the expected equidistribution holds. For concreteness, let us state precisely the equidistribution theorem for random closed geodesics (see Theorem 3.1 below for the other case). The notion of a random closed geodesic will be made precise in §3.3. Several natural models are possible, the theorem applies to all of them.

Theorem C. *Let X be a hyperbolic Riemann surface of finite type and $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of non-elementary parabolic representations of $G = \pi_1(X, \star)$ into $\text{PSL}(2, \mathbb{C})$. Fix $t \in \mathbb{C}$. Let (r_n) be a sequence such that for every $c > 0$, the series $\sum_{n \geq 0} e^{-cr_n}$ converges. Let (γ_n) be a sequence of independent random closed geodesics on X of length at most r_n . Then almost surely,*

$$\frac{1}{2 \text{length}(\gamma_n)} [Z(\gamma_n, t)] \xrightarrow[n \rightarrow \infty]{} T_{\text{bif}}.$$

Notice that this result does *not* follow from the equidistribution results in [DD1] since the random elements γ_n are not obtained from a random walk on G . To prove the theorem, we consider the sequence of psh potentials

$$u(\gamma_n, \lambda) = \frac{1}{2 \text{length}(\gamma_n)} \log |\text{tr}^2(\rho_\lambda(\gamma_n)) - t|.$$

We need to show that this sequence of psh functions a.s. converges to $\chi_{\text{Brown}}(\rho_\lambda)$ in $L_{\text{loc}}^1(\Lambda)$.

The main step is to study this convergence for a *fixed* representation.

Theorem D. *Let X be a hyperbolic Riemann surface of finite type and $\rho : G \rightarrow \text{PSL}(2, \mathbb{C})$ be a non-elementary parabolic representation. Let (r_n) and (γ_n) be as in Theorem C. Then almost surely*

$$\frac{1}{2 \text{length}(\gamma_n)} \log |\text{tr}^2(\rho(\gamma_n))| \xrightarrow[n \rightarrow \infty]{} \chi_{\text{Brown}}(\rho).$$

Notice that when ρ is the uniformizing representation (i.e. $\rho = \text{id}$ when G is identified to Γ), $\log |\text{tr}^2([\gamma_n])|$ is nothing but the length of γ_n (up to an error of order $O(e^{-\text{length}(\gamma_n)})$). This theorem thus describes how this quantity evolves as ρ is deformed in the space of representations.

By the Borel-Cantelli lemma, to prove Theorem D we need an estimate on the number of geodesics of length at most r such that $\left| \frac{1}{2 \text{length}(\gamma)} \log |\text{tr}^2(\rho(\gamma))| - \chi_{\text{Brown}}(\rho) \right| \geq \varepsilon$. This counting is based on large deviation estimates for the traces of random products of matrices, together with a precise understanding of the Furstenberg-Lyons-Sullivan discretization procedure. Observe that the exponential moment property of the discretization measure is also crucial here.

Finally, Theorem C is a rather straightforward consequence of Theorem D, by using some standard compactness properties of psh functions.

The outline of the paper is as follows. In §1 after a few facts on hyperbolic geometry and $\mathrm{PSL}(2, \mathbb{C})$ representations, we recall some necessary results from [DD1]. We also discuss the basic properties of Brownian motion on hyperbolic surfaces. In §2, we define the natural Lyapunov exponent and prove Theorem B. For further reference as well as for completeness, we include a detailed treatment of the discretization procedure. The equidistribution theorems are established in §3. We conclude the paper with a few open questions.

The results of this paper were announced in a preliminary form in [DD1, §5.1] and [Duj, §4.5].

1. PRELIMINARIES

Throughout the paper we use the following notation: if u and v are two real valued functions, we write $u \asymp v$ (resp $u \lesssim v$) if there exists a constant $C > 0$ such that $\frac{1}{C}u \leq v \leq Cu$ (resp. $u \leq Cv$). We also use the letter C for a generic positive “constant”, which may change from line to line, independently of a varying parameter which should be clear from the context.

1.1. Plane hyperbolic geometry. To fix notation, we summarize a few basic facts from hyperbolic geometry. See Beardon [Bea] for a systematic account. We let \mathbb{H}^2 (often simply \mathbb{H}) be the hyperbolic plane (resp. \mathbb{H}^3 be the hyperbolic 3-space) of curvature -1 . The hyperbolic distance is denoted by $d_{\mathbb{H}}$. The hyperbolic area in \mathbb{H}^2 is simply denoted by $\mathrm{Area}(\cdot)$, and we recall that the area of a ball of radius r is $4\pi \sinh^2 \frac{r}{2}$. As usual, we may switch from the Poincaré disk (resp. ball) to the upper half plane (resp. space) model depending on the particular situation. The group of orientation preserving isometries of \mathbb{H}^2 (resp. \mathbb{H}^3) will be viewed as $\mathrm{PSL}(2, \mathbb{R})$ (resp. $\mathrm{PSL}(2, \mathbb{C})$). The $\mathrm{SO}(2)$ (resp. $\mathrm{SU}(2)$) invariant point will be denoted by 0 . We let $\|\cdot\|$ be the operator norm associated to the standard Hermitian metric on \mathbb{C}^2 ; recall that $\|A\|$ is the spectral radius of A^*A . The norm of an element $\gamma \in \mathrm{PSL}(2, \mathbb{C})$ is by definition $\|A\|$, where A is any matrix representative of γ in $\mathrm{SL}(2, \mathbb{C})$. We also sometimes use $\|\cdot\|_2$ defined by $\|\gamma\|_2 := (|a|^2 + |b|^2 + |c|^2 + |d|^2)^{1/2}$, where $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Likewise, if $\gamma \in \mathrm{PSL}(2, \mathbb{C})$, $\mathrm{tr}^2(\gamma)$ and $|\mathrm{tr}(\gamma)|$ are well-defined quantities. Norm and displacement are related by the identity

$$(1) \quad \|\gamma\|_2^2 = 2 \cosh(d_{\mathbb{H}}(0, \gamma 0)),$$

in particular $2 \log \|\gamma\| \sim d_{\mathbb{H}}(0, \gamma 0)$ as $\|\gamma\| \rightarrow \infty$. If γ is a hyperbolic (resp. loxodromic) element then we denote its translation length by $\ell(\gamma)$, which satisfies

$$(2) \quad \cosh\left(\frac{\ell(\gamma)}{2}\right) = \frac{1}{2} |\mathrm{tr}(\gamma)|.$$

1.2. Spaces of representations of $\pi_1(S)$ into $\mathrm{PSL}(2, \mathbb{C})$. A good source for the material in this section is [Kap]. Let S be a (topological) surface of finite topological type, that is, homeomorphic to a compact surface of genus g with a finite number n of punctures. Throughout the paper we always assume that $\chi(S) = 2 - 2g - n < 0$. If S is endowed with a complex structure, the corresponding Riemann surface will be denoted by X . Fix a basepoint $\star \in S$ and put

$G = \pi_1(S, \star)$. The space of representations of $\text{Hom}(G, \text{PSL}(2, \mathbb{C}))$ is an affine complex algebraic variety (recall that $\text{PSL}(2, \mathbb{C})$ is affine because $\text{PSL}(2, \mathbb{C}) \simeq \text{SO}(3, \mathbb{C})$). A general holomorphic family of representations is simply a holomorphic mapping $\Lambda \rightarrow \text{Hom}(G, \text{PSL}(2, \mathbb{C}))$. The group $\text{PSL}(2, \mathbb{C})$ acts algebraically by conjugation on $\text{Hom}(G, \text{PSL}(2, \mathbb{C}))$, and the algebraic quotient $\text{Hom}(G, \text{PSL}(2, \mathbb{C})) // \text{PSL}(2, \mathbb{C})$ is the *character variety* $\mathcal{X}(G)$. Coordinates on $\mathcal{X}(G)$ are given by functions of the form $\rho \mapsto \text{tr}^2(\rho(\gamma))$. We also let $\mathcal{X}^{NE}(G) \subset \mathcal{X}(G)$ be the subset corresponding to non-elementary representations.

When S is not compact we actually rather work with the *relative character variety* $\mathcal{X}_{\text{par}}(G)$, which is defined as follows. Let $(h_i)_{i=1\dots n}$ in G be classes of closed loops turning once around the punctures. We define the set of *parabolic* representations by

$$\text{Hom}_{\text{par}}(G, \text{PSL}(2, \mathbb{C})) = \{\rho \in \text{Hom}(G, \text{PSL}(2, \mathbb{C})), \forall i, \rho(h_i) \text{ is parabolic}\}.$$

The algebraic quotient $\mathcal{X}_{\text{par}}(G) = \text{Hom}_{\text{par}}(G, \text{PSL}(2, \mathbb{C})) // \text{PSL}(2, \mathbb{C})$ is the relative character variety. It can be shown that $\mathcal{X}_{\text{par}}(G)$ is of complex dimension $6g - 6 + 2n$, and is smooth at non-elementary representations.

1.3. Bifurcation currents. Here we gather some material from [DD1]. See also [Duj] for a general account on bifurcation currents in one-dimensional dynamics, and [Bea, Kap] for basics on Kleinian (and more generally Möbius) groups.

Let G be as above, Λ be a connected complex manifold and $\rho = (\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of G into $\text{PSL}(2, \mathbb{C})$. We usually work under the following assumptions:

- (R1) the family is non-trivial, in the sense that there exists λ_1, λ_2 such that the representations ρ_{λ_i} , $i = 1, 2$ are not conjugate in $\text{PSL}(2, \mathbb{C})$;
- (R2) there exists $\lambda_0 \in \Lambda$ such that ρ_{λ_0} is faithful;
- (R3) for every $\lambda \in \Lambda$, ρ_λ is non-elementary.

It may sometimes be useful to replace (R3) by the following weaker statement

- (R3') there exists $\lambda_0 \in \Lambda$ such that ρ_{λ_0} is non-elementary.

In that case the subset $E \subset \Lambda$ of elementary representations is a proper real-analytic subset. According to Sullivan [Su2] there exists a partition $\Lambda = \text{Bif} \cup \text{Stab}$ of Λ into an open locus Stab of *stable representations* (in the sense of dynamical systems) and a closed *bifurcation locus* Bif . By definition, a representation ρ is stable if nearby representations in Λ are quasi-conformally conjugate to ρ .

Fix a probability measure μ on G , whose support generates G as a semi-group. We will assume the following locally uniform exponential moment condition:

$$(M) \quad \forall \lambda_0 \in \Lambda, \exists N(\lambda_0) \text{ neighbd of } \lambda_0, \text{ and } \sigma > 0 \text{ s.t. } \forall \lambda \in N(\lambda_0), \int_G \|\rho_\lambda(g)\|^\sigma d\mu(g) < \infty,$$

where $\|\cdot\|$ is any matrix norm on $\text{PSL}(2, \mathbb{C})$. Under such a moment condition one can define a Lyapunov exponent function on Λ by the formula

$$\chi_\mu(\lambda) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|\rho_\lambda(g)\| d\mu^n(g),$$

and the associated bifurcation current $T_{\text{bif}, \mu} = dd^c \chi_\mu$ –we deliberately emphasize the dependence on μ here.

The following result was established in [DD1, Thm. 3.13].

Theorem 1.1. *Let (G, ρ, μ) be a holomorphic family of representations of G , satisfying (R1, R2, R3'), endowed with a measure μ , generating G as a semi-group, and satisfying (M). Then $\text{Supp}(T_{\text{bif}, \mu})$ is equal to the bifurcation locus.*

A comment is in order here: the theorem was stated in [DD1] under the more restrictive moment condition

$$(3) \quad \text{there exists } s > 0 \text{ s.t. } \int_G \exp(s \text{ length}(g)) d\mu(g) < \infty,$$

where the length is relative to some finite system of generators of G . But, as explained in the beginning of [DD1, §3.3], it still holds under (M).

It is important to observe that these notions also make sense on (resp. relative) character varieties. Indeed let Λ be a piece of complex submanifold of the set of characters of non elementary representations of G into $\text{PSL}(2, \mathbb{C})$, $\Lambda \subset \mathcal{X}^{NE}(G)$ (resp. $\Lambda \subset \mathcal{X}_{\text{par}}^{NE}(G)$). Then it can be lifted to a holomorphic family of representations, transverse to the foliation by conjugacy classes, so we can define a stability (resp. bifurcation) locus on Λ , which does not depend on the choice of lifting. Likewise, the validity of the moment condition (M), as well as the value of the Lyapunov exponent, are independent of the lift. Hence there is a well-defined bifurcation current on Λ , and Theorem 1.1 holds.

1.4. Brownian motion. More details on the material in this paragraph can be found in [CC, Gri, FL] Let X be a complete hyperbolic Riemann surface, endowed with its hyperbolic metric. Brownian motion on X is defined as the Markov process with transition probability $P_t(x, \cdot) = p_t(x, y) dy$, where the volume element dy is relative to the hyperbolic metric, and the heat kernel p_t satisfies

$$\frac{\partial}{\partial t} p_t(x, \cdot) = \Delta p_t(x, \cdot) \text{ and } \lim_{t \rightarrow 0} p_t(x, \cdot) dx = \delta_x.$$

(It is also common to work with $\frac{1}{2}\Delta$. We choose this normalization to stay in accordance with the usual convention in foliation theory (see [CC])). This process is a diffusion with continuous sample paths so it gives rise for every $x \in X$ to a probability measure W_x (the Wiener measure) on the space Ω_x of continuous paths $\omega : [0, \infty[\rightarrow X$ such that $\omega(0) = x$ (itself endowed with the compact open topology). We also denote by Ω the space of continuous paths $\omega : [0, \infty[\rightarrow X$ and put $W = \int W_x dx$. We often use the classical notation \mathbb{P}_x (resp. \mathbb{E}_x) for the probability (resp. expectation) of some event relative to W_x .

It is useful to think of sample paths of the Brownian motion on $X \simeq \Gamma \backslash \mathbb{H}^2$ as projections of Brownian paths on \mathbb{H}^2 . To say it differently, the Wiener measures associated to a point in X and of one of its lifts to \mathbb{H}^2 are naturally identified. With our normalization, the linear drift of Brownian motion on \mathbb{H}^2 is equal to 1, that is, for W_0 -a.e. Brownian path ω , $d_{\mathbb{H}}(0, \omega(t)) \sim t$ as $t \rightarrow \infty$ (see the appendix in [DDp] for an elementary proof).

We will require the following precise estimate for the radial density $k^0(t, r)$ of the law at time t of Brownian motion issued from 0 (i.e. $P_t(0, \cdot)$) in \mathbb{H}^2

$$(4) \quad k^0(t, r) \asymp (1 + r^{-1})^{-1} t^{-1} (1 + r + t)^{-1} (1 + r) \exp\left(-\frac{(r - t)^2}{4t}\right)$$

(see e.g. [Da, Thm 5.7.2]).

2. THE LYAPUNOV EXPONENT ASSOCIATED TO BROWNIAN MOTION

Let S be a topological surface of genus g with n punctures, with a marked point \star , and such that $\chi(S) < 0$. Recall that $G = \pi_1(S, \star)$. In §2.1 we show that the data of a structure of Riemann surface of finite type on S gives rise to a natural Lyapunov exponent function χ_{Brown} on $\text{Hom}_{\text{par}}(G, \text{PSL}(2, \mathbb{C}))$ (resp. $\mathcal{A}_{\text{par}}(G)$). Proving that $dd^c \chi_{\text{Brown}}$ is a genuine bifurcation current occupies the remainder of this section. Our main tool is a discretization procedure for Brownian motion, which will be discussed thoroughly in §2.2. From this point, the main issue is to obtain estimates for a probabilistic stopping time, associated to the discretization. The case where X is compact is substantially simpler, and will be discussed in §2.3. In §2.4, we deal with the general case. The proof of Theorem 2.7 is given in §2.5. In §2.6, we re-interpret these results from the geodesic point of view, from which Proposition A follows.

2.1. Definition. Let X be a structure of Riemann surface of finite type on S (i.e. a neighborhood of each puncture is biholomorphic to \mathbb{D}^*). The Euler characteristic assumption implies that X is hyperbolic, and we endow X with its hyperbolic metric of curvature -1 . Let (\tilde{X}, \star) be the universal cover of (X, \star) . We fix an isomorphism between \tilde{X} and \mathbb{H}^2 mapping \star to 0. Then the deck transformation group G becomes naturally isomorphic to a lattice $\Gamma \leq \text{PSL}(2, \mathbb{R})$. We let ρ_{can} be the (Fuchsian) uniformizing representation of G such that $\Gamma = \rho_{\text{can}}(G)$, and in the sequel we conveniently identify G and Γ when needed.

Fix a representation $\rho \in \text{Hom}(G, \text{PSL}(2, \mathbb{C}))$. If γ is any loop on X based at \star , and $[\gamma] \in G$ denotes its class (from now on we drop the square brackets), we may consider the quantity $\log \|\rho(\gamma)\|$.

We are now ready to define the natural Lyapunov exponent of a parabolic representation of G , associated to the complex structure on S . Justifying the validity of this definition will occupy the remainder of this subsection.

Definition-Proposition 2.1. *Let X be a hyperbolic Riemann surface of finite type and $\rho : \pi_1(X, \star) \rightarrow \text{PSL}(2, \mathbb{C})$ be a parabolic representation*

Define a family of loops as follows: for $t > 0$, consider a Brownian path ω issued from \star , and concatenate $\omega|_{[0,t]}$ with a shortest geodesic joining $\omega(t)$ and \star , thus obtaining a closed loop $\tilde{\omega}_t$.

Then for W_\star a.e. ω the limit

$$(5) \quad \chi_{\text{Brown}}(\rho) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|\rho(\tilde{\omega}_t)\|$$

exists and does not depend on ω . This number is by definition the Lyapunov exponent of ρ , associated to the complex structure X .

Notice that there is a little ambiguity in the definition for there might be several shortest paths joining $\omega(t)$ and \star . As it will be clear from the proof, this does not affect the existence nor the value of the limit in (5). It will also follow from the proof that χ_{Brown} does not depend on \star . Since with our normalization, the escape rate of the Brownian motion in \mathbb{H}^2 is 1, we deduce from (1) that $\chi_{\text{Brown}}(\rho_{\text{can}}) = \frac{1}{2}$.

To prove the existence of χ_{Brown} and study its properties, we will need to consider the holonomy of non-closed paths on X . For this, we work on the *suspension* of ρ . For any representation $\rho \in \text{Hom}(G, \text{PSL}(2, \mathbb{C}))$, the suspension of ρ is by definition the quotient of $\mathbb{H}^2 \times \mathbb{P}^1$ under the diagonal action of G . To be specific, $M_\rho = \mathbb{H}^2 \times \mathbb{P}^1 / \sim$ where the equivalence relation is defined by $(p, z) \sim (p', z')$ if there exists $g \in G$ such that $(p', z') = (g(p), \rho(g)(z))$.

The natural projection $\pi : M_\rho \rightarrow X$ makes M_ρ a flat \mathbb{P}^1 -bundle over X , and the horizontal foliation on $\mathbb{H}^2 \times \mathbb{P}^1$ descends to M_ρ as a holomorphic foliation with leaves transverse to the \mathbb{P}^1 fibers, whose monodromy representation is ρ .

The point is to study the growth rate of holonomy between vertical fibers in M_ρ along generic Brownian paths in X , measured with respect to some smooth family of spherical metrics on the fibers. When X is compact the choice of the metrics on fibers is harmless. On the other hand in the non-compact case this requires some care, so let us dwell a little bit on this point.

By a spherical metric, we mean a conformal metric on \mathbb{P}^1 of curvature 1. Equivalently, it is the Fubini-Study metric associated to some Hermitian structure on \mathbb{C}^2 . Fix a smooth family of spherical metrics on the fibers of M_ρ . If ω is any continuous path on X , we denote by $h_\rho(\omega)$ its holonomy $h_\rho(\omega) : \pi^{-1}(\gamma(0)) \rightarrow \pi^{-1}(\gamma(1))$. We define $\|h_\rho(\omega)\|$ as $\sup_{x \in \pi^{-1}(\gamma(0))} \|d_x h_\rho(\omega)\|$, where the norm is relative to the choice of conformal metrics on the fibers.

We say that the family of spherical metrics is *normalized* if the induced metric on $\pi^{-1}(\star)$ is the standard Fubini-Study metric. This means that the natural map $\{0\} \times \mathbb{P}^1 \rightarrow \pi^{-1}(\star)$ induced by the definition of M_ρ as a quotient, is an isometry when \mathbb{P}^1 is given the standard Fubini-Study metric. This condition is meant to ensure that for closed loops, the definition of $\|h_\rho(\omega)\|$ coincides with the natural one.

By definition a family of spherical metrics on M_ρ is *Lipschitz* if there exists a constant $C_\rho > 0$ such that for every smooth path ω on X , it holds that

$$(6) \quad \log \|h_\rho(\omega)\| \leq C_\rho \text{length}(\omega),$$

where the length is relative to the hyperbolic metric on X .

When X is compact any smooth family of metrics is of course Lipschitz. In the non compact case we have the following result.

Proposition 2.2. *If ρ is a parabolic representation of G , then there exists a smooth normalized Lipschitz family of spherical metrics on the fibers of M_ρ . Furthermore the constant C_ρ in (6) can be chosen to be locally uniformly bounded on $\text{Hom}_{\text{par}}(G, \text{PSL}(2, \mathbb{C}))$.*

Notice that the result is false for a general representation. Indeed consider a path of length L turning about e^L times around a cusp of X . Let $g \in G$ representing a simple loop around that cusp. Then if $\rho(g)$ is loxodromic, we see that (6) will be violated. On the other hand it can be shown that the proposition (as well as all results in this section) remains true as soon as every such $\rho(g)$ has eigenvalues only on the unit circle.

Proof. The first observation is that the set of spherical metrics on \mathbb{P}^1 is parameterized by \mathbb{H}^3 . This follows from the fact that such a metric is the visual metric $\theta(p)$ from a point $p \in \mathbb{H}^3$ on the sphere at infinity \mathbb{P}^1 . Another way to see it is to observe that a spherical metric is defined by a Möbius map from \mathbb{P}^1 (equipped with the standard Fubini-Study metric) up to precomposition by an isometry, and $\text{PSL}(2, \mathbb{C})/\text{PSU}(2) \simeq \mathbb{H}^3$.

We denote by ds_p the metric associated to $p \in \mathbb{H}^3$. We claim that if $p, q \in \mathbb{H}^3$, we have that

$$(7) \quad \log \left\| \frac{ds_p}{ds_q} \right\|_\infty = d_{\mathbb{H}}(p, q) + O(1).$$

This follows from an elementary computation. Assume first that $q = 0$, so that $ds_q^2 = ds_0^2 = \frac{|dz|^2}{(1+|z|^2)^2}$ is the usual spherical metric. Let $p \in \mathbb{H}^3$, $p = A^{-1}(0)$ for some $A \in \text{PSL}(2, \mathbb{C})$, so

that $ds_p = A^* ds_0$. Putting $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we infer that

$$ds_p = \frac{|dz|}{|az + b|^2 + |cz + d|^2},$$

therefore $\left\| \frac{ds_p}{ds_0} \right\|_\infty = \|A^{-1}\|^2 = \|A\|^2$, and finally

$$\log \left\| \frac{ds_p}{ds_0} \right\|_\infty = 2 \log \|A\| = d_{\mathbb{H}}(0, p) + O(1).$$

To handle the general case, pick an isometry h of \mathbb{H}^3 bringing q back to 0, and observe that since $\frac{ds_p}{ds_q}$ is a function on \mathbb{P}^1 ,

$$\left\| \frac{h^* ds_p}{h^* ds_q} \right\|_\infty = \left\| h^* \frac{ds_p}{ds_q} \right\|_\infty = \left\| \frac{ds_p}{ds_q} \right\|_\infty.$$

A smooth family of spherical metrics on a trivial line bundle $D \times \mathbb{P}^1$ is now simply the data of a smooth map $f : D \rightarrow \mathbb{H}^3$. A smooth family of spherical metrics on the fibers of $\mathbb{H}^2 \times \mathbb{P}^1$ descends to a family of spherical metrics on M_ρ if and only if the associated map f is ρ -equivariant, i.e. for every $g \in G$, $f \circ \rho_{\text{can}}(g) = \rho(g) \circ f$. Furthermore the family is Lipschitz iff f is, and the constant C_ρ is controlled by the Lipschitz constant of f . This follows from (7), since the holonomy of the horizontal foliation on $\mathbb{H}^2 \times \mathbb{P}^1$ is the identity.

To prove the proposition it is thus enough to show that there exists a smooth Lipschitz ρ -equivariant map $f : \mathbb{H}^2 \rightarrow \mathbb{H}^3$. The family of spherical metrics will be normalized provided $f(0) = 0$. To construct such an f , recall that since X is of finite type, there exists a finite number of open disjoint subsets C_1, \dots, C_r of X such that $X \setminus \bigcup_i C_i$ is compact, and each C_i is isometric to the quotient of $\{z \in \mathbb{H}^2 \mid \Im z > \rho_i\}$. by the translation $z \mapsto z + 1$ for some positive ρ_i . For every $i = 1, \dots, r$, we pick some lift $\widetilde{C}_i \subset \mathbb{H}^2$ of C_i , and we first define f on the closure of \widetilde{C}_i . Denote by $g_i \in G$ a generator of the stabilizer of \widetilde{C}_i in G . By assumption, $\rho(g_i)$ is parabolic, hence conjugate to g_i . Thus, there exists an isometric immersion $f : \widetilde{C}_i \rightarrow \mathbb{H}^3$ such that $f \circ g_i = \rho(g_i) \circ f$ (we take the map in $\text{PSL}(2, \mathbb{C})$ which conjugates g_i to $\rho(g_i)$). We extend the map f by ρ -equivariance to a map defined on $\bigcup_i G\widetilde{C}_i$. Then, because \mathbb{H}^3 is contractible, a classical topological argument shows that the map f can be extended as a smooth ρ -equivariant map from \mathbb{H}^2 to \mathbb{H}^3 such that $f(0) = 0$. We claim that this map is Lipschitz. Indeed, $\|Df\|$ is G -invariant (the norm of Df is measured using hyperbolic metrics on source and target), thus descends to a function on X . But since f is an isometric embedding on \widetilde{C}_i , we have $\|Df\| = 1$ on the cusps C_i 's. Hence $\|Df\|$ is bounded, since $X \setminus \bigcup_i C_i$ is compact, and f is indeed our desired smooth Lipschitz ρ -equivariant map. Finally, it is clear that the Lipschitz constant of f can be chosen to be locally uniformly bounded in $\text{Hom}_{\text{par}}(G, \text{PSL}(2, \mathbb{C}))$, so the same is true of the constant C_ρ . \square

The following consequence of the previous proposition will be useful.

Corollary 2.3. *For every parabolic representation ρ there exists a constant β_ρ depending only on the constant C_ρ of (6) such that $\|\rho(g)\| \leq \|\rho_{\text{can}}(g)\|^{\beta_\rho}$.*

Proof. Indeed for $g \in G$, let σ_g be the geodesic segment $[0, \rho_{\text{can}}(g)0]$ in \mathbb{H}^2 . Then in M_ρ , the holonomy of the projection of σ_g in X is $\rho(g)$. By (1), (6) and the normalization of the family of spherical metrics we infer that

$$\log \|h_\rho(\sigma_g)\| = \log \|\rho(g)\| \leq C_\rho \text{length}(\sigma_g) \leq C_\rho \text{argcosh} \frac{\|\rho_{\text{can}}(g)\|_2^2}{2} \leq C_\rho \log \left(2 \|\rho_{\text{can}}(g)\|^2 \right).$$

Here we use the inequalities $\|\gamma\|_2^2 \leq 2 \|\gamma\|^2$ and $\text{argcosh} x \leq \log 2x$. Hence we get that $\|\rho(g)\| \leq (2 \|\rho_{\text{can}}(g)\|)^{2C_\rho}$. To conclude the proof, just note that since $\rho_{\text{can}}(g)$ contains no elliptic transformations, there exists $c > 0$ such that $\|\rho_{\text{can}}(g)\| > 1 + c$, so there exists α such that $2 \leq \|\rho_{\text{can}}(g)\|^\alpha$. The result follows. \square

We have the following analogue of Definition 2.1 in this context.

Proposition 2.4. *Let X be a hyperbolic Riemann surface of finite type, $\rho : \pi_1(X) \rightarrow \text{PSL}(2, \mathbb{C})$ be a parabolic representation, and $\|\cdot\|$ a smooth normalized Lipschitz family of spherical metrics on M_ρ . With notation as above, for W -a.e. path ω , the limit*

$$(8) \quad \chi(\rho; \omega) = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|h_\rho(\omega|_{[0,t]})\|$$

exists and does not depend on ω nor on the choice of $\|\cdot\|$.

Proof. Let σ be the shift semi-group acting on Ω by $\sigma_s : \omega(\cdot) \mapsto \omega(\cdot + s)$. Recall that $W = \int W_x dx$. Since the Riemannian measure is invariant under heat flow, W is invariant under the time shift $\sigma_s : \omega(\cdot) \mapsto \omega(\cdot + s)$. The family of functionals $H_t(\omega) : \omega \mapsto \log \|h_\rho(\omega|_{[0,t]})\|$ defines a sub-additive cocycle. In order to apply Kingman's subadditive ergodic theorem, we need to check that this cocycle is W -integrable, that is, that $\int \log \|dh_\rho(\omega|_{[0,t]})\| dW(\omega) < \infty$ for some (hence every) $t > 0$. For this we use the Lipschitz property of the family of spherical metrics. Let \tilde{X} be a Dirichlet fundamental domain containing 0 for the action of Γ on \mathbb{H}^2 . For a Brownian path ω , let $\tilde{\omega}$ be its lift to \mathbb{H}^2 starting from the lift of $\omega(0)$ lying in \tilde{X} . By (6) we have that

$$\log \|h_\rho(\omega|_{[0,t]})\| \lesssim d_{\mathbb{H}}(\tilde{\omega}(0), \tilde{\omega}(t)) + d_{\mathbb{H}}(0, \tilde{\omega}(0)).$$

Thus

$$\int \log \|dh_\rho(\omega|_{[0,t]})\| dW(\omega) \lesssim \int_{\tilde{X}} \mathbb{E}_x (d_{\mathbb{H}}(\tilde{\omega}(0), \tilde{\omega}(t))) dx$$

which is finite thanks to the (super)exponential decay of the heat kernel on hyperbolic plane (cf. (4)). \square

The following result finally justifies the validity of Definition 2.1.

Proposition 2.5. *Let X , ρ , and $\|\cdot\|$ be as in Proposition 2.4. Then for W_\star a.e. Brownian path ω , we have that*

$$\lim_{t \rightarrow \infty} \frac{1}{t} \log \|\rho(\tilde{\omega}_t)\| = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|h_\rho(\omega|_{[0,t]})\| = \chi_{\text{Brown}}(\rho).$$

Furthermore the limit in (8) is also equal to $\chi_{\text{Brown}}(\rho)$.

Proof. Let us first show that for W_\star a.e. ω , $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|h_\rho(\omega|_{[0,t]})\|$ exists and has the same value as the one in (8). Indeed the diffusion of the Dirac mass δ_\star at time 1 is absolutely continuous with respect to volume measure on X . Now we simply write $h_\rho(\omega|_{[0,t]}) = h_\rho(\omega|_{[1,t]}) \circ h_\rho(\omega|_{[0,1]})$ and by the Markov property of Brownian motion together with Proposition 2.4, we get that for W_\star a.e. ω , $\frac{1}{t} \log \|\omega|_{[1,t]}\|$ converges to the limit in (8).

To finish the proof, we show that for W_\star a.e. ω ,

$$(9) \quad \frac{1}{t} \left| \log \|\rho(\tilde{\omega}_t)\| - \log \|h_\rho(\omega|_{[0,t]})\| \right| \xrightarrow[t \rightarrow \infty]{} 0.$$

If X is compact this is obvious since $d_X(\omega(t), \star)$ is uniformly bounded. In the general case, by the Lipschitz property we have to show that for W_\star a.e. ω , $d_X(\omega(t), \star) = o(t)$. A way to see this is to use the following two facts: first, a theorem of Sullivan [Su1] asserts that for almost every v in the unit tangent space $S_\star X$ to \star , the unit speed geodesic ray $\gamma(t)$ issued from \star pointing in the direction v satisfies $d_X(\star, \gamma(t)) = O(\log t)$. The other result we use is the classical fact that Brownian paths on hyperbolic space are shadowed by geodesic rays (see e.g. [An]). More precisely, if ω is a Brownian path starting from $0 \in \mathbb{H}^2$, then $\lim_{t \rightarrow \infty} \omega(t) = \omega_\infty$ exists in $\partial\mathbb{H}^2$, and if $\gamma_{x, \omega_\infty}$ denotes the unit speed geodesic ray joining x and ω_∞ , then for W_0 a.e. ω , $d_{\mathbb{H}}(\omega(t), \gamma_{x, \omega_\infty}(t)) = o(t)$ (this is actually $O(\log t)$ by a theorem of Ancona [An]). Note that the distribution of ω_∞ is absolutely continuous on the circle at infinity (with density given by the Poisson kernel). Therefore on X , for W_\star a.e. ω , there exists a unit tangent vector such that $d_X(\omega(t), \gamma_{\star, v}(t)) = o(t)$. Since the distribution of the tangent vector v is induced by that of ω_∞ , it is absolutely continuous with respect to Lebesgue measure on $S_\star X$. So we can combine this estimate with Sullivan's, and conclude that $d_X(\omega(t), \star) = o(t)$ a.s., which implies (9). \square

For further reference, we now make the easy observation that the natural Lyapunov exponent is insensitive to finite coverings.

Proposition 2.6. *Let X be a hyperbolic Riemann surface of finite type, and $\pi : (Y, y_0) \rightarrow (X, \star)$ be a finite covering. Any parabolic representation $\rho : \pi_1(X, \star) \rightarrow \mathrm{PSL}(2, \mathbb{C})$ induces by restriction a parabolic representation ρ_Y of $\pi_1(Y, y_0)$. Then, denoting by $\chi_{\mathrm{Brown}}(\rho_Y, Y)$ the Lyapunov exponent of ρ_Y associated to the Riemann surface structure on Y , we have that $\chi_{\mathrm{Brown}}(\rho, X) = \chi_{\mathrm{Brown}}(\rho_Y, Y)$.*

Proof. This is just a matter of following step by step the proof of the existence of $\chi_{\mathrm{Brown}}(\rho, X)$. Indeed, we observe that the suspension Y_{ρ_Y} is naturally a covering of M_ρ , and the lift of a Lipschitz metric $\|\cdot\|$ on the fibers of M_ρ is a Lipschitz metric on the fibers of Y_{ρ_Y} . As already observed, the Wiener spaces W_\star on X and W_{y_0} are naturally identified to W_0 in \mathbb{H}^2 . Now by construction, if ω is a Brownian path issued from y_0 in Y , we have that $\|h_{\rho_Y}(\omega|_{[0,t]})\| = \|h_\rho(\pi(\omega)|_{[0,t]})\|$. The result follows. \square

Let now $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of parabolic representations of G . Thanks to Definition 2.1 we can define a Lyapunov exponent function χ_{Brown} on Λ , and we put $T_{\mathrm{bif}} = dd^c \chi_{\mathrm{Brown}}$. We will refer to it as the *natural bifurcation current* of the family. The following result asserts that T_{bif} is indeed a bifurcation current.

Theorem 2.7. *Let X be a hyperbolic Riemann surface of finite type, and $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of $G = \pi_1(X)$, satisfying (R1, R2, R3'). Then χ_{Brown} is a nonnegative psh function on Λ , and the positive closed current $T_{\mathrm{bif}} = dd^c \chi_{\mathrm{Brown}}$ satisfies $\mathrm{Supp}(T_{\mathrm{bif}}) = \mathrm{Bif}$.*

If furthermore (R3) holds then χ_{Brown} is positive and locally Hölder continuous.

The proof consists in exhibiting a measure μ on G satisfying (M) and such that $T_{\mathrm{bif}, \mu} = T_{\mathrm{bif}}$ (up to some multiplicative constant). It then suffices to apply Theorem 1.1. It will be carried out in the next subsections.

Remark 2.8. We have constructed the natural bifurcation current by using the hyperbolic metric on X . Let us observe that when X is compact any other conformal metric would essentially yield the same object. Indeed, choosing a different conformal metric on X only changes the “clock” of Brownian motion, that is, sample Brownian paths for the new metric are time-reparameterized Brownian paths relative to the original one. It then follows from an easy ergodicity argument that the associated Lyapunov exponent function is a constant multiple of χ_{Brown} . When X has punctures, the same statement holds, provided one restricts to an appropriate class of conformal metrics, for instance the ones that coincide with the Poincaré metric outside some compact set of X .

2.2. Discretization of the Brownian motion. We will rely on a discretization procedure for the Brownian motion which was introduced by Furstenberg [F2], and subsequently studied by several authors, in particular Lyons and Sullivan [LS] (see also Kaimanovich [Kai], Ballman-Ledrappier [BaLe], Ancona [An]). Since we will need precise moment estimates, we include a detailed treatment of the discretization, which is close (but not identical to) that of Lyons and Sullivan.

Let Γ be a finite covolume lattice in $\text{PSL}(2, \mathbb{R})$, and $X = \Gamma \backslash \mathbb{H}$. Let $0 \in \mathbb{H}$ and $R > 0$ such that the balls $B(\gamma 0, R)$ are disjoint, for $\gamma \in \Gamma$. Since the Brownian motion on X is recurrent [Gri], for $0 < r \leq R$ the set $\partial_r = \bigcup_{\gamma} \partial B(\gamma 0, r)$ is recurrent, namely a.e. Brownian path starting at a point x hits ∂_r in finite time. We denote by T_r (resp. T_R) the first hitting time to ∂_r (resp. ∂_R). Since ∂_r (resp. ∂_R) is closed, T_r (resp. T_R) is a stopping time.

2.2.1. Sub-balayage. Recall that the balayage of a finite measure m on a closed recurrent set E is defined to be the image of the measure $\int W_x dm(x)$ under the map $\omega \mapsto \omega(T_E)$ where $T = T_E(\omega)$ is the first moment where ω hits E .

We denote by B_ρ, S_ρ respectively the ball and sphere centered at 0 and of radius ρ . If y is a point interior to B_R , the balayage of the Dirac mass at the point x on S_R is a probability measure absolutely continuous with respect to the spherical measure ds on S_R (conveniently normalized so that ds is a probability measure), the density being given by the Poisson kernel. Hence, there is a constant $0 < p < 1$, such that if x is chosen in the sphere S_r , its balayage on S_R is bounded from below by pds .

The following lemma will allow us to select a “good” set of paths from S_r to S_R . For every $x \in S_r$, a.e. Brownian path issued from x intersects S_R . The hitting time is $T_R(\omega)$ and we denote by $e(\omega) = \omega(T_R(\omega))$ the exit point. We use the standard notation $(\mathcal{F}_t)_{t \geq 0}$ for the Brownian filtration.

Lemma 2.9. *For every $x \in S_r$ there exists a \mathcal{F}_{T_R} -measurable set G_x , with $\mathbb{P}(G_x) = p$ such that $e_*(1_{G_x} W_x) = pds$.*

In addition, reducing p slightly if necessary, G_x can be chosen so that there exists a constant M such that for every $x \in S_r$ and $\omega \in G_x$, $T_R(\omega) \leq M$.

Proof. Let us first establish the lemma without the bound M on the exit time. Denote by $b_x(y)ds(y)$ the balayage measure of δ_x on S_R . Notice that $b_x(y) > p$ for every $y \in S_R$. For $0 \leq t \leq 1$ consider the piece of circle

$$C_t = \left\{ \frac{R+r}{2} e^{i2\pi\theta}, 0 \leq \theta \leq 1-t \right\}.$$

We use C_t as a “continuously opening barrier” for Brownian paths going from x to S_R . For $0 \leq t \leq 1$, let $E_t \subset \Omega_x$ be the set of paths starting at x that avoid C_t before hitting S_R . For

$t = 0$ the barrier is closed so $E_t = \emptyset$ while for $t = 1$ it is open and $W_x(E_t) = 1$; observe also that for $t < t'$, $C_t \supset C_{t'}$ so $E_t \subset E_{t'}$. Therefore, for every $y \in S_R$, $f_y : t \mapsto W_x(E_t | e^{-1}(y))$ is a continuous increasing function with $f_y(0) = 0$ and $f_y(1) = 1$. The continuity of f_y follows from the fact that the set of Brownian paths (starting from x and exiting at y), intersecting $E_{t'} \setminus E_t$ has small measure when $t' - t$ is small. Let $t(y) \in (0, 1)$ be the unique value of t such that $f_y(t(y)) = \frac{p}{b_x(y)}$. We can now define

$$G_x = \{\omega \in \Omega_x, \omega|_{[0, T_R(\omega)]} \cap C_{t(e(\omega))} = \emptyset\}.$$

Clearly, this set is \mathcal{F}_{T_R} measurable (recall that $e(\omega) = \omega(T_R(\omega))$). Since by definition $G_x \cap e^{-1}(y) = E_{t(y)} \cap e^{-1}(y)$ we infer that

$$\begin{aligned} e_*(1_{G_x} W_x) &= e_* \int W_x(G_x | e^{-1}(e(\omega))) dW_x(\omega) = e_* \int W_x(E_{t(e(\omega))} | e^{-1}(e(\omega))) dW_x(\omega) \\ &= W_x(E_{t(y)} | e^{-1}(y)) e_* W_x = W_x(E_{t(y)} | e^{-1}(y)) b_x(y) ds(y) = p ds, \end{aligned}$$

which was the desired property.

To prove the second claim, observe that for every $p_1 < p$, if $A_x(M) \subset W_x$ denotes the set of paths such that $T_R(\omega) \leq M$, then for M sufficiently large, $e_*(1_{A_x(\tau_1)} W_x) \geq p_1 ds$, uniformly in x —this is a consequence of Dini's Theorem. From this point, to get the result it suffices to repeat the above proof with $1_{A_x(\tau_1)} W_x$ instead of W_x . \square

We extend by equivariance the definition of the sets G_x to all of ∂_r by setting $G_{\gamma x} = \gamma G_x$.

2.2.2. A stopping time. For every point $x_0 \in \mathbb{H}$ and every $\omega \in \Omega_{x_0}$, we define a sequence of stopping times

$$U_0 = T_R(\omega) < u_1 = u_1(\omega) < U_1 = U_1(\omega) < \dots < u_n(\omega) < U_n(\omega) < \dots$$

as follows: $U_0 = T_R(\omega)$, and by induction, u_n is the first time after U_{n-1} at which ω hits ∂_r , that is $u_n(\omega) = U_{n-1}(\omega) + T_r(\sigma_{U_{n-1}}(\omega))$, where as before σ_t denotes time shift by t . Likewise, U_n is the first time after u_n at which ω hits ∂_R , i.e. $U_n(\omega) = u_n(\omega) + T_R(\sigma_{u_n}(\omega))$.

By the strong Markov property of Brownian motion, almost surely there exists an integer n such that $\sigma_{u_n} \omega$ belongs to $G_{\omega(u_n)}$. We define $T(\omega) := U_k(\omega)$, where $k = k(\omega)$ is the first such integer. The measurability property of the sets G_x ensures that T is a stopping time (intuitively at time t we know if we already went from ∂_r to ∂_R through some G_x). We let $x_1(\omega) = \omega(T(\omega))$ and $r_1(\omega)$ be the element of Γ such that $x_1 \in \partial B(r_1 \cdot 0, R)$. The law of x_1 defines a transition probability $\Pi(x_0, \cdot)$. By construction it satisfies the equivariance property $\Pi(\gamma x_0, \cdot) = \gamma_* \Pi(x_0, \cdot)$ for $\gamma \in \Gamma$.

Repeating this process independently gives rise to sequences $(x_n)_{n \geq 0} \in (\mathbb{H}^2)^\mathbb{N}$ and $(r_n)_{n \geq 1} \in \Gamma^\mathbb{N}$.

2.2.3. The discretization measure. It follows from the construction that whatever the distribution of the point x_0 is, the distribution of the point x_1 is a convex combination of the spherical measures $ds_\gamma = \gamma_* ds$ on ∂_R . Hence, if x_0 has distribution ds on S_R (observe that $U_0 = 0$ in this case), then x_1 has distribution $\mu * ds$ for a certain measure μ on Γ .

We now define a Markov chain (x_n) with initial distribution given by the normalized spherical measure ds on S_R and transition kernel Π . By the equivariance property of Π , the distribution of x_n is $\mu^n * ds$.

Furthermore, since the transition kernel is defined by stopping the Brownian motion, by the strong Markov property we infer that the map $\omega \mapsto (r_n(\omega))$ sends the measure $\int_{S_R} W_x ds(x)$ to the path measure \mathbb{P} on $\Gamma^{\mathbb{N}}$ of the right random walk on Γ induced by μ .

By definition μ is the discretization measure (notice that it is not unique).

2.3. The cocompact case. The core of the proof of Theorem 2.7 is the fact that the discretization measure satisfies the exponential moment condition (M). When X is compact, this fact is well-known to the specialists (for instance it is announced without proof in [An]), but also not so easy to find in print. Let us explain this case first. Recall that $\Gamma = \rho_{\text{can}}(G)$.

Proposition 2.10. *If the lattice Γ is cocompact, then the discretization measure μ satisfies the exponential moment condition (3) with respect to word length in G .*

Proof. By the stronger assertion of Lemma 2.9, for every $p_1 < p$ there exists a deterministic time τ_1 such that for every $x \in S_r$ there exists a \mathcal{F}_{T_R} measurable set G_x such that $e_*(1_{G_x} W_x) = p_1 ds$ and moreover for every $\omega \in G_x^1$, $T_R(\omega) \leq \tau_1$.

On the other hand, since $X = \Gamma \setminus \mathbb{H}$ is compact, there exists $0 < p_2 < 1$ and $\tau_2 > 0$ such that for every $x \in \mathbb{H}$, the probability that a Brownian path starting at x hits ∂_r in time less than τ_2 is bounded from below by p_2 .

Therefore, by the Markov property we infer that for every $x \in \mathbb{H}$ and every t_0 ,

$$\mathbb{P}_x(T \geq t_0 + \tau_1 + \tau_2) \leq (1 - p_1 p_2) \cdot \mathbb{P}_x(T \geq t),$$

where T is the stopping time defined in §2.2.2. From this we deduce that there exist constants $C, \alpha > 0$ independent of x such that

$$(10) \quad \mathbb{P}_x(T \geq t) \leq C e^{-\alpha t}.$$

Since the discretization measure is the image of $\int_{S_R} W_x ds(x)$ under $\omega \mapsto \omega(T(\omega))$, and the distance between $r_1(\omega) \cdot 0$ and $\omega(T(\omega))$ is R , for every $\delta > 0$, we have that

$$\mu(\gamma \in \Gamma, d(\gamma 0, 0) \geq \delta) \leq \int \mathbb{P}_x(d(\omega(T(\omega)), 0) \geq \delta - R) ds(x),$$

where $d(\cdot, \cdot)$ denotes hyperbolic distance. Fix $K > 1$ as in Lemma 2.11 below. For every $x \in S_R$, we have that

$$\begin{aligned} \mathbb{P}_x(d(\omega(T), 0) \geq \delta - R) &\leq \mathbb{P}_x(d(\omega(T), x) \geq \delta - 2R) \\ &\leq \mathbb{P}_x\left(\sup_{t \leq (\delta - 2R)/K} d(\omega(t), x) \geq (\delta - 2R)\right) + \mathbb{P}_x\left(T \geq \frac{\delta - 2R}{K}\right). \end{aligned}$$

By Lemma 2.11 below the first term decreases exponentially fast when δ tends to infinity, while the second does because of (10). By the homogeneity of hyperbolic plane, these estimates are uniform for $x \in S_R$, and we conclude that $\mu(\gamma \in \Gamma, d(\gamma 0, 0) \geq \delta)$ decreases exponentially with δ . Finally, since $\Gamma = \rho_{\text{can}}(G)$ is cocompact, $(\Gamma 0, d)$ is quasi-isometric to any Cayley graph of G and we are done. \square

The following result is presumably well-known but we could not locate a precise reference.

Lemma 2.11. *There exists a positive constant K such that*

$$\mathbb{P}_0\left(\sup_{0 \leq s \leq t} d(0, \omega(s)) \geq Kt\right)$$

decreases exponentially with t .

Proof. Let us work in the disk model of hyperbolic plane. The expression for the hyperbolic distance is $\exp(d(0, z)) = \frac{1+|z|}{1-|z|}$. Since $x \mapsto \frac{1+x}{1-x}$ is convex and increasing, $z \mapsto \exp(d(0, z))$ is a subharmonic function. From this we infer that if $\omega(t)$ is the Brownian motion starting at 0, the process defined by $(\exp(d(0, \omega(t))))_{t>0}$ is a positive submartingale.

From the asymptotic estimate for the heat kernel given in §1.4, we infer that $\mathbb{E}_0(\exp(d(0, \omega(t))))$ is finite for every t . Now we have that

$$\begin{aligned}
(11) \quad \mathbb{E}_0(\exp(d(0, \omega(t)))) &\leq \mathbb{E}_0(\exp(d(0, \omega(\lceil t \rceil)))) \text{ by the submartingale property} \\
&\leq \mathbb{E}_0 \left(\exp \left(\sum_{i=0}^{\lceil t \rceil - 1} d(\omega(i), \omega(i+1)) \right) \right) \text{ by the triangle inequality} \\
&= \mathbb{E}_0(\exp(d(0, \omega(1)))^{\lceil t \rceil}) \text{ by the Markov property} \\
&\leq \mathbb{E}_0(\exp(d(0, \omega(1)))^{t+1})
\end{aligned}$$

(for the equality on the third line we take successive conditional expectations and use the homogeneity of the hyperbolic plane). The Doob martingale inequality applied to $\exp(d(0, \omega(t)))$ asserts that for every $\alpha > 0$,

$$(12) \quad \mathbb{P}_0 \left(\sup_{0 \leq s \leq t} \exp d(0, \omega(s)) \geq \alpha \right) \leq \frac{1}{\alpha} \mathbb{E}_0(\exp(d(0, \omega(t))))$$

Hence if K is any real number larger than $\mathbb{E}_0(\exp(d(0, \omega(1))))$, applying inequality (12) with $\alpha = \exp(Kt)$ together with (11) finishes the proof. \square

Remark 2.12. It is likely that the result holds for any $K > 1$. A possible way to achieve this would be to refine the subdivision of the interval $[0, t]$ in (11).

2.4. The finite volume case. We will need the following estimate on the hitting time of the Brownian motion on a given subset of X . We thank J. Franchi for explaining it to us.

Proposition 2.13. *Let X be a hyperbolic Riemann surface of finite type, endowed with its Poincaré metric, and Liouville measure dx . Let $K \subset X$ be a compact set of positive area and define a stopping time for the Brownian motion by*

$$T_K(\omega) = \inf \{t > 0, \omega(t) \in K\}.$$

Then there exists a constant $\alpha > 0$ such that for every $x \in X$, $\mathbb{E}_x(e^{\alpha T_K}) < +\infty$. Furthermore, the function $x \mapsto \mathbb{E}_x(e^{\alpha T_K}) < +\infty$ is locally bounded and in $L^2(X, dx)$.

Of course this uniformity is highly non-trivial for some paths go deeply inside a cusp before reaching K .

Proof. Let $(\Pi_t)_{t \geq 0}$ be the Brownian semigroup on X , associated to Δ . The spectral theory of the Laplacian on X implies that Π_t is strongly contracting, specifically there exists a constant $c > 0$ such that for every $f \in L^2(X)$ of zero mean, $\|\Pi_t f\|_{L^2} \leq e^{-ct} \|f\|_{L^2}$ (see e.g. [FL] for an elementary approach to the Poincaré inequality on X , which classically implies a positive lower bound for the first eigenvalue of the Laplacian). Therefore the family of operators defined by $Q_T = \frac{1}{T} \int_0^T \Pi_t$ converges strongly to $f \mapsto \int_X f = \int f(x) dx$ in L^2 . More precisely there exists a function $c(T)$ with $\lim_{+\infty} c(T) = 0$ such that

$$\left\| Q_T f - \int_X f \right\|_{L^2} \leq c(T) \|f\|_{L^2}.$$

Indeed write $f = f_0 + \int f$ with $\int f_0 = 0$. Notice that $\|f_0\|_{L^2} \leq 2\|f\|_{L^2}$. Then we have that

$$\left\| Q_T f - \int f \right\|_{L^2} = \|Q_T f_0\|_{L^2} = \left\| \frac{1}{T} \int_0^{T_0} \Pi_t f_0 + \frac{1}{T} \int_{T_0}^T \Pi_t f_0 \right\| \leq \frac{2}{T} \|f\|_{L^2} + \frac{T - T_0}{T} e^{-cT} \|f\|_{L^2}.$$

In this setting, a theorem by Carmona and Klein [CK, Thm 1] asserts that $x \mapsto \mathbb{E}_x(e^{\alpha T_K})$ belongs to $L^2(X)$, in particular it is finite a.e. Furthermore, elaborating on [CK, Rmk 1], we see that $\mathbb{E}_x(e^{\alpha T_K})$ is locally bounded. Indeed let $t_0 > 0$, and use the Markov property to write

$$\begin{aligned} \mathbb{E}_x(e^{\alpha T_K}) &= \int_{\{T_k \leq t\}} e^{\alpha T_K} dW_x + \int_{\{T_k > t_0\}} e^{\alpha T_K} dW_x \leq e^{\alpha t_0} + e^{\alpha t_0} \int \mathbb{E}_y(e^{\alpha T_K}) p_{t_0}(x, y) dy \\ &\leq e^{\alpha t_0} + e^{\alpha t_0} \|\mathbb{E}_y(e^{\alpha T_K})\|_{L^2} \|p_{t_0}(x, \cdot)\|_{L^2}, \end{aligned}$$

where $\Pi_t(x, y) dy$ is the transition kernel. Now, as it easily follows from the estimates in §1.4, for fixed t_0 the heat kernel $p_{t_0}(x, \cdot)$ is in L^2 , locally uniformly with respect to x . The result follows. \square

From this proposition we get estimates on the stopping times T_r and T_R in \mathbb{H} .

Corollary 2.14. *There exists uniform constants α_0 and M such that for every $x \in \partial_R$, $\mathbb{E}_x(\exp \alpha_0 T_r) \leq M$ (resp. for every $x \in \partial_r$, $\mathbb{E}_x(\exp \alpha_0 T_R) \leq M$).*

Proof. To prove the corollary we work on X . Abusing notation we set $\partial_r = \pi S_r$, and similarly for ∂_R , where $\pi : \mathbb{H}^2 \rightarrow X$ is the natural projection. Recall that by assumption $B(0, R)$ projects to a ball on X . Assume that $x \in \partial_R$, and fatten ∂_r on the side opposite to x to make it a closed annulus $\tilde{\partial}_r$. Then the hitting times of ∂_r and $\tilde{\partial}_r$ are the same, and we conclude by Proposition 2.13. The other case is similar. \square

The next step is to show that the stopping time T of 2.2.2 defining the discretization admits an exponential moment.

Proposition 2.15. *With notation as in §2.2, there exist constants α and M such that for every $x \in S_R$,*

$$\mathbb{E}_x(\exp(\alpha T)) \leq M.$$

Proof. Let $x \in S_R$. Recall that for a.e. $\omega \in W_x$ we defined in §2.2.2 the integer $k(\omega)$ as the first time at which $\omega(u_n(\omega))$ enters the good set $G_{\omega(u_n)}$. For a constant α to be determined later, we decompose the integral $\int \exp(\alpha T) dW_x$ as a sum

$$\sum_K \int_{\Omega_x^K} \exp(\alpha T) dW_x$$

where $\Omega_x^K = \{\omega \mid k(\omega) = K\}$. We define a sequence of shifted paths ω_l , $0 \leq l \leq 2K - 1$ by $\omega_0 = \omega$, $\omega_1 = \sigma_{u_1}(\omega)$, $\omega_2 = \sigma_{U_1}(\omega)$, etc. We also define for $1 \leq l \leq 2K - 1$, $y_l(\omega) = \omega_l(0)$, so that for even l (resp. odd l), $y_l \in S_R$ (resp. $y_l \in S_r$). Notice that by definition of K , for $l = 2j - 1$, $j < K$, $\omega_l \notin G_{y_l}$ while $\omega_{2K-1} \in G_{y_{2K-1}}$.

Writing $T(\omega) = T_r(\omega_0) + T_R(\omega_1) + \cdots + T_R(\omega_{2K-1})$, taking successive conditional expectations with respect to the events $T_\bullet(\omega_l) = y_{l+1}$ (with $\bullet = r$ or R) and applying the strong

Markov property yields

$$(13) \quad \int_{\Omega_x^k} \exp(\alpha T) dW_x = \int_{\Omega_x} \exp(\alpha T_r(\omega_0)) dW_x(\omega_0) \cdots \\ \cdots \int_{G_{y_{2j-1}}^c} \exp(\alpha T_R(\omega_{2j-1})) dW_{y_{2j-1}}(\omega_{2j-1}) \int_{\Omega_{y_{2j}}} \exp(\alpha T_r(\omega_{2j})) dW_{y_{2j}}(\omega_{2j}) \cdots \\ \cdots \int_{G_{y_{2K-1}}} \exp(\alpha T_R(\omega_{2K-1})) dW_{y_{2K-1}}(\omega_{2K-1}).$$

Now we claim that for every $\beta > 0$ there exists $\alpha > 0$ such that putting $q = 1 - p$ we have

- (i) $\forall x \in S_R, \int \exp(\alpha T_r) dW_x \leq e^\beta$
- (ii) $\forall y \in S_r, \int_{G_y^c} \exp(\alpha T_R) dW_y \leq qe^\beta$ and $\int_{G_y} \exp(\alpha T_R) dW_y \leq pe^\beta$.

From this and (13) we immediately get that

$$(14) \quad \int_{\Omega_x^K} \exp(\alpha T) dW_x \leq e^{2K\beta} q^{K-1} p.$$

Thus, choosing β such that $e^{2\beta} q < 1$ and summing over K , we obtain that

$$\int \exp(\alpha T) dW_x \leq \frac{p}{q} \sum_K (e^{2\beta} q)^K < +\infty,$$

and the result follows.

It remains to prove our claim. We only deal with (ii), the other case being analogous. Let $0 < \alpha < \alpha_0/2$, where α_0 is in Corollary 2.14, let $\beta' = \beta/2$ and set $E = G_y$ or $E = G_y^c$. Applying the Chebychev inequality, Corollary 2.14 implies that for every $y \in \partial_r$, we have

$$W_y(T_R \geq t) \leq M \exp(-\alpha_0 t).$$

We partition the set E as

$$E = (E \cap \{\alpha T_R \leq \beta'\}) \cup \bigcup_{n=0}^{\infty} (E \cap \{\beta' + \alpha n < T_R < \beta' + \alpha(n+1)\})$$

and decompose the integral $\int_E \exp(\alpha T_R) dW_y$ accordingly, which yields

$$\int_E \exp(\alpha T_R) dW_y \leq e^\beta W_y(E) + \sum_n \mathbb{P}_x \left(T_R \geq \frac{\beta'}{\alpha} + n \right) e^{\beta' + \alpha(n+1)} \\ \leq e^{\beta'} W_x(E) + C \exp \left(-\frac{\alpha_0 \beta'}{\alpha} \right)$$

with $C = \frac{e^{\alpha_0 + \beta'} M}{1 - e^{-\alpha_0/2}}$. Note that $W_x(E)$ equals p or q , that are both fixed positive numbers, so that if α is sufficiently small we have that

$$\int_E \exp(\alpha T_R) dW_y \leq e^{\beta'} W_x(E) + C \exp \left(-\frac{\alpha_0 \beta'}{\alpha} \right) \leq e^\beta W_x(E).$$

and claim (ii) follows. The proof is complete. \square

We can now show that the discretization measure satisfies the moment condition (M).

Proposition 2.16. *Let X be a hyperbolic Riemann surface of finite type, ρ be a parabolic representation of $G = \pi_1(X)$ and μ be the discretization of the Brownian motion. Then there is a constant $\alpha = \alpha_\rho > 0$ such that*

$$(15) \quad \int \|\rho(g)\|^\alpha d\mu(g) < \infty.$$

Furthermore, the constant α_ρ can be chosen to be locally uniform in $\text{Hom}_{\text{par}}(\Gamma, \text{PSL}(2, \mathbb{C}))$. In this case the value of the integral in (15) is also locally uniform.

Proof. For $\rho = \rho_{\text{can}}$, given Proposition 2.15, the proof is identical to that of Proposition 2.10. However in this case we cannot transfer the moment condition in Γ to a moment condition with respect to word length in G (see Remark 2.17 below). Instead we use the Lipschitz property of the spherical metrics on the fibers of M_ρ to obtain moment estimates for $\rho(G)$.

Let us be more precise. The proof of Proposition 2.10 shows that the exponential decay property (10) implies that

$$\mu(\gamma \in \Gamma, d(\gamma(0), 0) \geq \delta) \leq e^{-c\delta}$$

for some positive constant c . Since $\|\gamma\| \asymp \log d(\gamma(0), 0)$ we infer that there exists $\alpha > 0$ such that

$$\int_G \|\rho_{\text{can}}(g)\|^\alpha d\mu(g) < \infty.$$

Applying Corollary 2.3 thus finishes the proof. \square

Remark 2.17. When X is not compact, the moment condition in (15) cannot be replaced by (3). Actually, μ does not even have a finite moment with respect to word length in G , i.e. $\int_G \text{length}(g) d\mu(g) = \infty$. This was shown (somehow implicitly) by Guivarc'h and Le Jan in [GL]. See also [DKN, Corollary 1.22] for a different approach, as well as Gruet [Gru, Rmk p. 500] for a neat computation in the case of the modular surface.

2.5. Proof of Theorem 2.7. Let μ be the discretization measure and χ_μ be the associated Lyapunov exponent. We showed in Proposition 2.16 that μ satisfies the moment condition (M). Furthermore, it is easy to see that $\text{Supp}(\mu) = G$. Indeed, recall the balayage construction of μ . We start with the circle measure ds on S_R and “sub-balayage” it successively on $\partial_R = \Gamma \cdot S_R$. The first step of the process is to balayage ds onto $\partial_r = \Gamma \cdot S_r$. It is clear that the image measure has positive mass on every circle γS_r , $\gamma \in \Gamma$. Now for each circle γS_r , there is a set of positive measure of good paths, that leaves some measure on γS_R (for the remaining paths, we repeat the process over again). Therefore μ puts some mass on $\{\gamma\}$.

By Theorem 1.1, $\text{Supp}(dd^c \chi_\mu) = \text{Bif}$. To prove the theorem, we show that there exists a positive constant c such that $\chi_{\text{Brown}} = c\chi_\mu$, hence $\text{Supp}(T_{\text{bif}}) = \text{Supp}(T_{\text{bif}, \mu}) = \text{Bif}$. Under (R3), the positivity and Hölder continuity properties of χ_{Brown} then follow from classical results due to Furstenberg [F1] and Le Page [Le].

Let T be the stopping time defining μ , and T_n be the sequence of times obtained by repeating the stopping process. Let W_{S_R} be defined by $W_{S_R} = \int_{S_R} W_x ds(x)$. For W_{S_R} a.e. ω , $\omega(T_n) \in r_n(\omega)S_R$, with $(r_n) \in \Gamma^{\mathbb{N}}$ as in §2.2.2.

Identify Γ and G through ρ_{can} and consider a general parabolic representation ρ . Let as before σ_{r_n} denote the geodesic segment joining 0 and $r_n(0)$. Since $d(r_n(0), \omega(T_n)) = R$ is constant we infer that

$$\frac{1}{n} \left| \log \|h_\rho(\sigma_{r_n(\omega)})\| - \log \|h_\rho(\omega(T_n))\| \right| = \frac{1}{n} \left| \log \|\rho(r_n)\| - \log \|h_\rho(\omega(T_n))\| \right| \longrightarrow 0.$$

Since the law of r_n is μ^n we thus deduce that

$$(16) \quad \chi_\mu(\rho) = \lim_{n \rightarrow \infty} \frac{1}{n} \int_G \log \|\rho(g)\| d\mu^n(g) = \lim_{n \rightarrow \infty} \frac{1}{n} \int \log \|h_\rho(\omega(T_n))\| dW_{S_R}(\omega).$$

To conclude the proof, it remains to show that there exists a constant $\tau > 0$ such that $\frac{T_n}{n} \rightarrow \tau$ for W_{S_R} a.e. ω . This, together with Proposition 2.5, implies that the right hand side of (16) equals $\tau \chi_{\text{Brown}}$

To prove this, we view S_R as a subset of X . Let θ_T be the time shift by $T(\omega)$, acting on the set of Brownian paths ω issued from S_R , equipped with the measure W_{S_R} . By construction the law of $\omega(T)$ is $ds(x)$. From this fact and the strong Markov property we see that $(\theta_T)_* W_{S_R} = W_{S_R}$. Furthermore there cannot be any measurable θ_T -invariant subset, again by the strong Markov property together with the fact that for every $x \in S_R$, the law of $\omega(T)$, conditioned to start at x , is $ds(x)$. Therefore θ_T is W_{S_R} -ergodic

To conclude, observe that $T_n(\omega) = \sum_{k=0}^{n-1} T(\theta_T^k \omega)$ and recall from Proposition 2.15 that T is integrable. Thus, property (ii) follows from Birkhoff's ergodic theorem. The proof is complete. \square

2.6. The Lyapunov exponent associated to the geodesic flow. As before let X be a Riemann surface of finite type with a marked point \star , endowed with its hyperbolic metric. For $v \in T_\star^1 X$ a unit tangent vector, we let γ_v be the associated unit speed geodesic ray. These geodesics form another natural class of paths on X . We also denote by $\tilde{\gamma}_{v,t}$ the loop obtained by appending to $\gamma_v|_{[0,t]}$ a shortest path returning to \star (as before, we disregard the possible ambiguities happening here). This point of view allows to define a Lyapunov exponent function, similarly to Definition 2.1. It turns out that it coincides with χ_{Brown} up to a multiplicative constant. Proposition A immediately follows.

Theorem 2.18. *Let X be a hyperbolic Riemann surface of finite type, endowed with its Poincaré metric. Let $\rho : \pi_1(X, \star) \rightarrow \text{PSL}(2, \mathbb{C})$ be a parabolic representation, and $\|\cdot\|$ a smooth Lipschitz family of spherical metrics on the suspension M_ρ . Then for a.e. v the limit*

$$\chi_{\text{geodesic}}(\rho) := \lim_{t \rightarrow \infty} \frac{1}{t} \log \|h_\rho(\gamma_v|_{[0,t]})\| = \lim_{t \rightarrow \infty} \frac{1}{t} \log \|h_\rho(\tilde{\gamma}_{v,t})\|$$

exists and is equal to $\chi_{\text{Brown}}(\rho)$ (in particular it does not depend on v).

We leave the reader check that the same result holds for geodesics rays issued from generic $(x, v) \in T^1 X$ (relative to the Liouville measure), that is, $\lim_{t \rightarrow \infty} \frac{1}{t} \log \|h_\rho(\gamma_{(x,v)}|_{[0,t]})\| = \chi_{\text{Brown}}(\rho)$.

Proof. The work was already done in Proposition 2.5. It follows from (6) that if ω_1 and ω_2 are two paths on X with $\omega_1(0) = \omega_2(0)$ and such that the respective lifts $\tilde{\omega}_1$ and $\tilde{\omega}_2$ to \mathbb{H}^2 , starting from the same point, satisfy $d_{\mathbb{H}}(\tilde{\omega}_1(t), \tilde{\omega}_2(t)) = o(t)$, then

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\log \|h_\rho(\omega_1(t))\| - \log \|h_\rho(\omega_2(t))\|| = 0.$$

As already said, generic Brownian paths on \mathbb{H}^2 are shadowed by geodesic rays. That is, if ω is a Brownian path starting from $0 \in \mathbb{H}^2$, then $\lim_{t \rightarrow \infty} \omega(t) = \omega_\infty$ exists in $\partial\mathbb{H}^2$, and $d_{\mathbb{H}}(\omega(t), \gamma_{0, \omega_\infty}(t)) = o(t)$, where if $\gamma_{0, \omega_\infty}$ is the unit speed geodesic ray joining 0 and ω_∞ .

To prove the theorem, we use this reasoning in the reverse direction. Taking endpoints of semi-infinite geodesics gives the natural “visibility” identification between $T_0^1 X$ and $\partial\mathbb{H}^2$, sending circle measure on $T_0^1 X$ to the harmonic measure ν_0 associated to 0 on $\partial\mathbb{H}^2$. For

$v \in T_\star^1 X$, let $\omega_\infty(v)$ be the associated endpoint. For a.e. Brownian path ω conditioned to converge to $\omega_\infty(v)$ we have that

$$\lim_{t \rightarrow \infty} \frac{1}{t} |\log \|h_\rho(\gamma_v(t))\| - \log \|h_\rho(\omega(t))\|| = 0.$$

Now by Proposition 2.4, for W_\star a.e. ω we have that $\lim \frac{1}{t} \log \|h_\rho(\omega(t))\| = \chi_{\text{Brown}}$. Hence $\lim \frac{1}{t} \log \|h_\rho(\gamma_v(t))\| = \chi_{\text{Brown}}(\rho)$. To deal with the loops $\tilde{\gamma}_{v,t}$, as in Proposition 2.5 we use Sullivan's result that almost surely, $d_X(\gamma_v(t), \star) = O(\log t)$. The proof is complete. \square

Remark 2.19. In [BGM], Bonatti, Gomez-Mont and Viana study $\text{PSL}(n, \mathbb{C})$ cocycles over hyperbolic and symbolic dynamical systems, with applications to suspensions of fundamental groups of compact surfaces. In particular they show, that if $\rho : \pi_1(S) \rightarrow \text{PSL}(2, \mathbb{C})$ is a non-elementary representation, then the foliated geodesic flow of M_ρ admits non-zero Lyapunov exponents (i.e. $\chi_{\text{geodesic}}(\rho) > 0$). We see that the above methods (discretization of Brownian motion, etc.) give a new approach to this result, as it was recently observed by Alvarez [A] (for compact S). The novelty here is that we are able to deal with parabolic representations on Riemann surfaces of finite type as well.

3. EQUIDISTRIBUTION IN PARAMETER SPACE

Let as before X be a structure of Riemann surface of finite type on S . Identifying (\tilde{X}, \star) with $(\mathbb{H}, 0)$ makes $G = \pi_1(S, \star)$ isomorphic to a lattice $\Gamma \leq \text{PSL}(2, \mathbb{R})$. From now on, we consistently identify G and Γ . We endow Γ with the metric structure induced by \mathbb{H} on $\Gamma \cdot 0$, that is for $\gamma \in \Gamma$ we put $d(\gamma) = d_{\mathbb{H}}(0, \gamma 0)$; observe that $d(\gamma) \sim 2 \log \|\gamma\|$. Notice that when X is not compact, this distance is *not* quasi-isometric to word-length in G . We also put $B_\Gamma(r) = \{\gamma \in \Gamma, d(\gamma) < r\}$.

This metric structure gives rise to a natural notion of a random diverging sequence in Γ : fix a sequence $(r_n)_{n \geq 1}$ increasing to infinity and independently choose $\gamma_n \in B_\Gamma(r_n)$ relative to the counting measure. For technical reasons we will assume that the series $\sum_{n \geq 0} e^{-cr_n}$ converges for every $c > 0$. The resulting random sequences (γ_n) will be qualified as *admissible*.

Let now Λ be a holomorphic family of parabolic representations of G into $\text{PSL}(2, \mathbb{C})$ (resp. a holomorphic family of representations modulo conjugacy) satisfying (R1, R2, R3). For $t \in \mathbb{C}$, we consider the subvariety in Λ defined by

$$Z(\gamma, t) = \{\lambda \in \Lambda, \text{tr}^2(\rho_\lambda(\gamma)) = t\}$$

(which is viewed as a variety, that is, possibly with multiplicity), and the associated integration current $[Z(\gamma, t)]$. By convention if $Z(\gamma, t) = \Lambda$, we put $[Z(\gamma, t)] = 0$.

Our purpose in this section is to establish the following theorem.

Theorem 3.1. *Let Γ be a torsion-free lattice in $\text{PSL}(2, \mathbb{R})$, and $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of parabolic representations of Γ into $\text{PSL}(2, \mathbb{C})$ satisfying (R1, R2, R3). Let $T_{\text{Bif}} = dd^c \chi_{\text{Brown}}$ be the natural bifurcation current, associated to Brownian motion on $X = \Gamma \backslash \mathbb{H}$. If (γ_n) is an admissible random sequence in Γ , then almost surely the following convergence holds:*

$$\frac{1}{2d(\gamma_n)} [Z(\gamma_n, t)] \xrightarrow[n \rightarrow \infty]{} T_{\text{bif}}.$$

As already said in the introduction, this result cannot be deduced from the equidistribution results in [DD1]. The main step of the proof, achieved in §3.1, is to show that if $\rho \in \text{Hom}_{\text{par}}(\Gamma, \text{PSL}(2, \mathbb{C}))$ is a *fixed* representation, then the exponential growth rate of

$\text{tr}^2(\rho(\gamma_n))$ is a.s. given by $\chi_{\text{Brown}}(\rho)$. This is based on the detailed understanding of the asymptotic properties of random product of matrices associated to discretized Brownian motion.

Then in §3.2 we use Fubini's theorem and subharmonicity to get the almost sure L_{loc}^1 convergence of $\lambda \mapsto \log |\text{tr}^2(\rho_\lambda(\gamma_n)) - t|$. From this, Theorem 3.1 immediately follows. This simple argument was suggested to us by D. Chafaï. Notice that the same approach leads to a simplification of the almost sure equidistribution theorems in [DD1] as well (see Theorems B and 4.1 there).

In §3.3 we show that this result can be nicely interpreted in terms of the holonomies of random sequences of closed geodesics on X (Theorems D and C).

3.1. Distribution of traces in $\rho(\Gamma)$. Let Γ be a lattice in $\text{PSL}(2, \mathbb{R})$. If Γ is torsion-free and ρ is a non-elementary type-preserving representation of Γ into $\text{PSL}(2, \mathbb{C})$, the results of §2 enable to define the natural Lyapunov exponent $\chi_{\text{Brown}}(\rho)$. If now Γ has torsion, by the Selberg Lemma, we pick a finite index torsion-free subgroup $\Gamma' \leq \Gamma$, and define $\chi_{\text{Brown}}(\rho)$ to be equal to $\chi_{\text{Brown}}(\rho|_{\Gamma'})$. By Proposition 2.6, the value of the Lyapunov exponent does not depend on the choice of Γ' .

Our main result in this paragraph is the following.

Theorem 3.2. *Let Γ be a lattice in $\text{PSL}(2, \mathbb{R})$, and ρ be a non-elementary parabolic representation of Γ . Then for every $\varepsilon > 0$ there exists $c(\varepsilon) > 0$ such that for large enough $r > 0$,*

$$\# \left\{ \gamma \in B_\Gamma(r), \left| \frac{1}{2r} \log |\text{tr}^2(\rho(\gamma))| - \chi_{\text{Brown}}(\rho) \right| > \varepsilon \right\} \leq e^{(1-c(\varepsilon))r}.$$

The delicate point in this theorem is the following: several times so far we have used the idea that closing a long geodesic or Brownian path by some path of bounded or short length does not affect much the norm of the holonomy. There is no such result for the trace: if γ (resp. a) is a “large” (resp. “small”) element in Γ , $\text{tr}^2(\rho(\gamma))$ may be very different from $\text{tr}^2(\rho(\gamma a))$ (even for the identity representation).

The idea of the proof is to rely on the very precise understanding of generic random products $\gamma_1 \cdots \gamma_n$, where the γ_i are chosen according to the discretization measure μ defined in §2.2. In Proposition 3.3 we show that we can perturb these elements in $\rho(\Gamma)$ by keeping track of the traces. We then use the properties of Brownian motion to compare μ^n and the counting measure, so as to cover most of a large ball in Γ .

Throughout the proof $c(\varepsilon)$ stands for a quantity which may change from line to line, dependent on ε (and ρ) but not on n .

Our first result does not depend on Brownian motion.

Proposition 3.3. *Let Γ be as above and $\rho : \Gamma \rightarrow \text{PSL}(2, \mathbb{C})$ be a non-elementary representation. Fix a probability measure μ on Γ , generating Γ as a semi-group, and satisfying the exponential moment condition $\int_\Gamma \|\rho(\gamma)\|^\alpha d\mu(\gamma) < \infty$ for some $\alpha > 0$.*

Then there exists a constant $c = c(\mu, \rho) > 0$ such that for every $\varepsilon > 0$, when n is large enough

$$(17) \quad \mu^n \left(\left\{ \gamma, \exists a \in \Gamma \text{ s.t. } \|a\| \leq e^{c\varepsilon n} \text{ and } \left| \frac{1}{n} \log |\text{tr} \rho(\gamma a)| - \chi_\mu \right| > \varepsilon \right\} \right) \leq \exp(-c\varepsilon n).$$

We note that in this statement we could replace Γ by an abstract finitely generated group, by replacing the condition “ $\|a\| \leq e^{c\varepsilon n}$ ” in (17) by “ $\text{length}(a) \leq c\varepsilon n$ ”.

Proof. Without the perturbative term a , this was proven in [DD1, Cor. A.2] (see also [Ao]). It turns out that for fixed a , the proof is essentially identical to that of Theorem A.1 in [DD1], which we briefly reproduce for convenience. For a sequence $\gamma \in \Gamma^{\mathbb{N}}$, we put $\ell_n(\gamma) = \gamma_1 \cdots \gamma_n$. We recall that if $g \in \mathrm{PSL}(2, \mathbb{C})$ has large norm, there exists two balls $A(g)$ and $R(g)$ (the attracting and repelling balls respectively) on \mathbb{P}^1 , of diameter about $\frac{1}{\|g\|}$, such that $g(R(g)^c) = A(g)$. We also recall that for $g \in \mathrm{PSL}(2, \mathbb{C})$,

$$\frac{1}{2} \log |\mathrm{tr}^2(g) - 4| = \log \|g\| + \log \delta(g) + O(1),$$

where $\delta(g)$ is the distance between the fixed points of g . Thus to control the trace from the norm it is enough to control the distance between fixed points.

Fix $\varepsilon > 0$, small with respect to χ . Fix $a \in \Gamma$, with $\|a\| \leq \frac{\varepsilon}{2}n$. We have to estimate the probability that for large n $\frac{1}{n} \log \mathrm{tr}^2(\rho(\ell_n(\gamma)a)) \notin [\chi_\mu - \varepsilon, \chi_\mu + \varepsilon]$. By the large deviations theorem for the distribution of the norms [BL, §V.6] we infer that outside a set of exponentially small probability, $|\frac{1}{n} \log \|\rho(\ell_n(\gamma))\| - \chi_\mu| < \frac{\varepsilon}{4}$. Since $\|a\| \leq e^{\frac{\varepsilon}{2}n}$, we get that with high probability,

$$\left| \frac{1}{n} \log \|\rho(\ell_n(\gamma)a)\| - \chi_\mu \right| < \frac{3\varepsilon}{4}.$$

Therefore we need to show that the probability that $\delta(\rho(\ell_n(\gamma)a)) < e^{-\frac{\varepsilon}{4}n}$ decreases like $e^{-c(\varepsilon)n}$.

For this, let $m = \lfloor (1 - \varepsilon)n \rfloor$. We split $\ell_n(\gamma)a$ as $\ell_n(\gamma)a = (\gamma_n \cdots \gamma_{m+1})(\gamma_m \cdots \gamma_1 a)$. By the large deviations theorem for the norms, and the facts that $\|a\| \leq e^{\frac{\varepsilon}{2}n}$ and $m = \lfloor (1 - \varepsilon)n \rfloor$, we get that

$$\mathbb{P}\left(\left|\frac{1}{m} \log \|\rho(\ell_m(\gamma)a)\| - \chi_\mu\right| \geq \varepsilon\right) + \mathbb{P}\left(\log \|\rho(\gamma_n \cdots \gamma_{m+1})\| \geq 2\chi_\mu(n - m)\right) \leq e^{-c(\varepsilon)n}.$$

By independence, we may assume that $\gamma_1, \dots, \gamma_m$ are fixed and satisfy $|\frac{1}{m} \log \|\rho(\ell_m(\gamma)a)\| - \chi_\mu| < \varepsilon$ and estimate the probability that $\delta(\rho(\ell_n(\gamma)a)) < e^{-\frac{\varepsilon}{4}n}$ among the $\gamma_{m+1}, \dots, \gamma_n$ satisfying $\log \|\rho(\gamma_n \cdots \gamma_{m+1})\| < 2\chi_\mu(n - m) \sim 2\chi_n \varepsilon$. Put $h = \rho(\ell_m(\gamma)a)$ and consider the balls $A(h)$ and $R(h)$. If we let

$$\tilde{A}(\rho(\ell_n(\gamma)a)) = \rho(\gamma_n \cdots \gamma_{m+1})A(h) \text{ and } \tilde{R}(\rho(\ell_n(\gamma)a)) = R(h)$$

then by definition we have that

$$\rho(\ell_n(\gamma)a) \left(\tilde{R}(\rho(\ell_n(\gamma)a))^c \right) = \tilde{A}(\rho(\ell_n(\gamma)a))$$

and the diameters of $\tilde{A}(\rho(\ell_n(\gamma)a))$ and $\tilde{R}(\rho(\ell_n(\gamma)a))$ are not greater than $e^{-(\chi_\mu + O(\varepsilon))n}$. So if we can show that these balls are separated by a distance $\geq e^{-\frac{\varepsilon}{4}n}$, we then conclude that $\rho(\ell_n(\gamma)a)$ is loxodromic with a fixed point in each ball, so the distance between its fixed points is $\geq e^{-\frac{\varepsilon}{4}n}$ and we are done. But if this does not happen, this means that $\rho(\gamma_n \cdots \gamma_{m+1})$ maps the ball $A(h)$ very close (i.e. at distance $\leq e^{-\frac{\varepsilon}{4}n}$) to the ball $R(h)$. Since h is fixed here, the probability of this event is governed by the properties of the unique stationary measure on \mathbb{P}^1 , relative to the action of $(\rho(\Gamma), \mu)$. It follows that this probability is of order $e^{-\theta \varepsilon n}$ for some $\theta = \theta(\rho) > 0$ (see [DD1, Lemma A.3] for details) and the desired result follows.

So far we have shown that there exists $c_1 > 0$ (which we may assume is smaller than $\frac{1}{4}$) such that if a is a fixed element in Γ such that $\|a\| \leq e^{\frac{\varepsilon}{4}n}$, then

$$\mathbb{P}\left(\frac{1}{n}\log|\mathrm{tr}\rho(\ell_n(\gamma)a)| \notin [\chi_\mu - \varepsilon, \chi_\mu + \varepsilon]\right) \leq e^{-c_1\varepsilon n}.$$

Since the number of elements of norm not greater than e^{cn} is equivalent to e^{cn} , we get that for $c_2 \leq c_1$,

$$\mathbb{P}\left(\exists a \in \Gamma, \text{ s.t. } \|a\| \leq e^{c_2\varepsilon n}, \text{ and } \left|\frac{1}{n}\log|\mathrm{tr}\rho(\ell_n(\gamma)a)| - \chi_\mu\right| > \varepsilon\right) \leq e^{(c_2 - c_1)n}.$$

By choosing $c_2 = \frac{c_1}{2}$, we conclude that (17) holds with $c = \frac{c_1}{2}$. \square

Proof of Theorem 3.2. Let us first assume that Γ is torsion-free. From now on μ is the discretization measure for the Brownian motion on X , as defined in §2.2. We let $S = S_R$ be the circle centered at 0 and of radius R and work in the probability space of paths issued from S in \mathbb{H} , endowed with the probability measure $W_S = \int_S W_x ds(x)$. For a generic such Brownian path, we let $T = T(\omega)$ be the stopping time defining μ , $T_n = T_n(\omega)$ be the sequence of times obtained by repeating the stopping process, and τ be the average stopping time, which is the almost sure limit $\tau = \lim \frac{T_n}{n}$ (see the proof of Theorem 2.7). Observe that it is enough to prove the theorem for the sequence of radii $r_n = n\tau$.

The first step is the following large deviations estimate, which will be proven afterwards.

Lemma 3.4. *Let $T_n(\omega)$ be as above. Then for large enough n we have that*

$$W_S\left(\left\{\omega, \left|\frac{T_n(\omega)}{n} - \tau\right| > \varepsilon\right\}\right) \leq e^{-c(\varepsilon)n}.$$

For fixed n , we now define a good set G_n of paths starting from S by specifying its complement B_n . Declare that the elements $\gamma \in \Gamma$ appearing in (17) are bad. From the discretization procedure, this corresponds to a set of paths of exponentially small measure, relative to W_S , which we denote by B_n^0 . Now by definition ω belongs to B_n^1 if $|T_n(\omega) - n\tau| > \frac{c\tau}{2K}\varepsilon$ where c is as in Proposition 3.3 and K is as in Lemma 2.11. By Lemma 2.11 and the strong Markov property if $\omega \notin B_n^1$, the probability that $d_{\mathbb{H}}(\omega(T_n), \omega(n\tau))$ is larger than $\frac{c\tau}{2}n\varepsilon$ is exponentially small. We let B_n^2 be the corresponding set of paths. Finally we put $B_n = B_n^0 \cup B_n^1 \cup B_n^2$, and $G_n = B_n^c$. Observe that $W_S(B_n) \leq e^{-c(\varepsilon)n}$.

We claim that if ω is good, that is $\omega \in G_n$, then for every $\gamma \in \Gamma \cap B_{\mathbb{H}}(\omega(n\tau), \frac{c\tau}{2}n\varepsilon)$ we have that

$$\left|\frac{1}{n}\log|\mathrm{tr}(\rho(\gamma))| - \chi_\mu(\rho)\right| < \tau\varepsilon, \text{ i.e. } \left|\frac{1}{2\tau n}\log|\mathrm{tr}^2(\rho(\gamma))| - \chi_{\mathrm{Brown}}(\rho)\right| < \varepsilon$$

(recall from the proof of Theorem 2.7 that $\chi_\mu(\rho) = \tau\chi_{\mathrm{Brown}}(\rho)$). Indeed by Proposition 3.3, if $\omega \notin B_n$, then the element $\alpha \in \Gamma$ corresponding to $\omega(T_n)$ (that is, such that $\omega(T_n) \in \alpha(S)$) has the property that for every $\beta \in B_\Gamma(\alpha, c\tau n\varepsilon)$, $\left|\frac{1}{n}\log|\mathrm{tr}(\rho(\beta))| - \chi_\mu(\rho)\right| < \tau\varepsilon$. Now, if $\omega \in G_n$, $\omega(n\tau)$ lies in $B_\Gamma(\alpha, \frac{c\tau}{2}n\varepsilon)$, whence the result.

Let E be the set of group elements $\gamma \in B_\Gamma\left(\frac{n\tau}{2}\right)$ such that

$$\left|\frac{1}{2\tau n}\log|\mathrm{tr}^2(\rho(\gamma))| - \chi_{\mathrm{Brown}}(\rho)\right| \geq \varepsilon.$$

To complete the proof of the theorem, we have to show that $\#E \leq e^{-c(\varepsilon)n} \#B_\Gamma(n\tau)$. For this, let \tilde{G}_n be the image of G_n under Brownian motion at time $n\tau$, that is $\tilde{G}_n = \{\omega(n\tau), \omega \in G_n\}$. The next lemma follows from classical estimates for the heat kernel on \mathbb{H} (see below for the proof).

Lemma 3.5. *The hyperbolic area of $(\tilde{G}_n)^c \cap B_\mathbb{H}(0, n\tau)$ is exponentially small with respect to the area of $B_\mathbb{H}(0, n\tau)$, that is, there exists $c(\varepsilon) > 0$ such that for large n ,*

$$\text{Area}\left((\tilde{G}_n)^c \cap B_\mathbb{H}(0, n\tau)\right) \leq \exp(n\tau - c(\varepsilon)n).$$

We have shown above that if $\gamma \in E$, then $B_\mathbb{H}(\gamma, \frac{c\tau\varepsilon}{2}n)$ is disjoint from \tilde{G}_n . Let $\delta = \frac{c(\varepsilon)}{2}$, where $c(\varepsilon)$ is as in the previous lemma. By the Vitali covering lemma, there exists a covering of E by a finite collection of balls $\{B_\mathbb{H}(\gamma_i, \delta n), i = 1, \dots, N\}$ such that the balls $B_\mathbb{H}(\gamma_i, \frac{\delta}{5}n)$ are disjoint. Thus we infer that

$$(18) \quad \sum_{i=1}^N \text{Area}\left(B_\mathbb{H}\left(\gamma_i, \frac{\delta}{5}n\right)\right) = 4\pi N \sinh^2\left(\frac{\delta}{10}n\right) \leq \text{Area}\left((\tilde{G}_n)^c\right).$$

Now, a well known theorem due to Margulis [M] (see also [EM]) asserts that when $r \rightarrow \infty$,

$$(19) \quad \#B_\Gamma(r) \sim \frac{\text{Area}(B_\mathbb{H}(0, r))}{\text{Area}(X)} \sim \frac{e^r}{\text{Area}(X)}.$$

It follows that there is a constant C such that for every r , $\#B_\Gamma(r) \leq Ce^r$, hence by (18) we conclude that

$$\#E \lesssim Ne^{\delta n} \lesssim \frac{e^{n\tau - c(\varepsilon)n}}{\sinh^2\left(\frac{\delta}{10}n\right)} e^{\delta n} \lesssim e^{\tau n - (c(\varepsilon) - \frac{4}{5}\delta)n} \leq Ce^{\tau n - \frac{c(\varepsilon)}{2}n},$$

and the proof of the theorem is complete in the torsion free case. \square

Proof of Lemma 3.4. The almost sure convergence $\frac{T_n}{n} \rightarrow \tau$ was established in the proof of Theorem 2.7, the point here is to make this convergence more precise. As we did there we slightly shift our point of view and see our Brownian paths as sitting on X . We want to formalize the idea of a Brownian path issued from S as obtained from the concatenation of path segments starting and ending on S . For this we let Ω_S be the set of continuous paths $[0, \infty) \rightarrow X$ starting on S , endowed as usual with the probability measure W_S . We define a Markov chain with values in Ω_S as follows: the initial distribution is W_S , and for a.e. ω the transition kernel is given by $p(\omega, \cdot) = W_{\omega(T(\omega))}$, where $T(\omega)$ is our stopping time. This Markov chain yields a sequence of random paths (ω_n) in Ω_S . By the strong Markov property of Brownian motion, the law of the random path obtained by concatenating the path segments $\omega_n|_{[0, T(\omega_n)]}$ is W_S . Associated to the Markov chain there is a real random variable $T(\omega_n)$ and we see that the law of $T_n(\omega)$ is the same as that of $\sum_{k=1}^n T(\omega_k)$.

It is easy to see that the random variables (ω_n) have the same law W_S . However they are not independent since ω_{n+1} must start at $\omega_n(T(\omega_n))$. The main observation is that independence holds after two iterations. Indeed for every $x \in S$,

$$p(W_x, \cdot) = \int p(\alpha, \cdot) W_x(d\alpha) = \int W_{\alpha(T(\alpha))} W_x(d\alpha) = \int W_y ds(y) = W_S$$

because by construction the law of $\alpha(T(\alpha))$ is the circle measure ds . Thus we infer that for a.e. $\omega \in S$,

$$p^2(\omega, \cdot) = \int p(\alpha, \cdot) p(\omega, d\alpha) = \int p(\alpha, \cdot) W_{\omega(T(\omega))}(d\alpha) = W_S.$$

It follows that the random variables $\omega_1, \omega_3, \omega_5, \dots$ (resp. $\omega_2, \omega_4, \dots$) are independent. Furthermore, by Proposition 2.15, there exists $s > 0$ such that $\mathbb{E}(e^{sT(\omega)}) < \infty$. Therefore the desired estimate follows from the classical large deviation theorem for partial sums of i.i.d. random variables (applied separately to $\sum_{\text{even } k} T(\omega_k)$ and $\sum_{\text{odd } k} T(\omega_k)$). \square

Proof of Lemma 3.5. Let ν_t (resp. ν_t^0) be the law at time t of Brownian motion with initial distribution ds (resp. δ_0). For notational ease, we put $F = \tilde{G}_n^c \cap B_{\mathbb{H}}(0, n\tau)$. We know that $\nu_{n\tau}(F) \leq W_S(B_n) \leq e^{-c(\varepsilon)n}$, and from this we want to derive an estimate for the hyperbolic area of F . Let $\delta > 0$ to be specified later, and decompose F as

$$F = (F \cap B_{\mathbb{H}}(0, n\tau - n\delta)) \cup (F \cap A_{\mathbb{H}}(n\tau - n\delta, n\tau))$$

where $A_{\mathbb{H}}(r, s)$ denotes the annular region centered at 0 with inner radius r and outer radius s . The first observation is that

$$\text{Area}(F \cap B_{\mathbb{H}}(0, n\tau - n\delta)) \leq \text{Area}(B_{\mathbb{H}}(0, n\tau - n\delta)) \lesssim \exp(n\tau - n\delta).$$

The probability measure ν_t^0 is radial, and from the estimate for its radial density $k^0(t, r)$ given in (4), we deduce that for $t = n\tau$ and $n\tau - n\delta \leq r \leq n\tau$,

$$\frac{1}{\sqrt{n}} e^{-n\frac{\delta^2}{4\tau}} \lesssim k^0(n\tau, r) \lesssim \frac{1}{\sqrt{n}}.$$

Since $\nu_{n\tau}$ is an average of bounded translates of $\nu_{n\tau}^0$, the same estimate holds for the radial density $k(n\tau, r)$ of $\nu_{n\tau}$. On the other hand the radial density of hyperbolic area is $r \sinh r$, so we infer that

$$\begin{aligned} \text{Area}(F \cap A_{\mathbb{H}}(n\tau - n\delta, n\tau)) &\leq \int_{F \cap A_{\mathbb{H}}(n\tau - n\delta, n\tau)} r(\sinh r) dr d\theta \\ &\lesssim \int_{F \cap A_{\mathbb{H}}(n\tau - n\delta, n\tau)} r(\sinh r) k(n\tau, r) \sqrt{n} e^{n\frac{\delta^2}{4\tau}} dr d\theta \\ &\lesssim n^{3/2} \exp\left(n\tau + \frac{\delta^2}{4\tau}n\right) \nu_{n\tau}(F) \leq n^{3/2} \exp\left(n\tau + \frac{\delta^2}{4\tau}n - c(\varepsilon)n\right). \end{aligned}$$

Finally we get that

$$\text{Area}(F) \lesssim \exp(n\tau - n\delta) + n^{3/2} \exp\left(n\tau + \frac{\delta^2}{4\tau}n - c(\varepsilon)n\right),$$

so by choosing $\delta = \sqrt{\tau c(\varepsilon)}$ we are done. \square

Let us now assume that Γ has torsion, and briefly explain how to adapt the proof of the theorem. We pick a torsion-free subgroup $\Gamma' \leq \Gamma$ of finite index, and apply the first part of the proof to Γ' . In this way we get a discretization measure μ on Γ' . We slightly modify Proposition 3.3 to allow for a perturbation a belonging to Γ (this only slightly affects the counting in the last part of the proof), so that its conclusion now reads

$$\mu^n \left(\left\{ \gamma \in \Gamma', \exists a \in \Gamma \text{ s.t. } \|a\| \leq e^{c\varepsilon n} \text{ and } \left| \frac{1}{n} \log |\text{tr } \rho(\gamma a)| - \chi_\mu \right| > \varepsilon \right\} \right) \leq \exp(-c\varepsilon n).$$

Then we argue exactly as in the torsion free case, by introducing the set E of $\gamma \in \Gamma \cap B_{\mathbb{H}}(n\tau)$ such that $\left| \frac{1}{2r_n} \log |\operatorname{tr}^2(\rho(\gamma))| - \chi_{\text{Brown}}(\rho) \right| \geq \varepsilon$. To the discretized Brownian motion on Γ' , we associate the sets G_n and \widetilde{G}_n exactly as before, and we estimate $\#E$ by arguing that, from the modified version of Proposition 3.3, if $\gamma \in E$, then $B_{\mathbb{H}}(\gamma, \frac{c\tau\varepsilon}{2}n)$ is disjoint from \widetilde{G}_n . This completes the proof of the theorem. \square

The following corollary, of independent interest, is the key step in the proof of Theorem 3.1.

Corollary 3.6. *Let Γ be a lattice in $\operatorname{PSL}(2, \mathbb{R})$, and ρ be a non-elementary parabolic representation of Γ . If (γ_n) is an admissible random sequence in Γ , then almost surely,*

$$\frac{1}{2d(\gamma_n)} \log |\operatorname{tr}^2(\rho(\gamma_n))| \xrightarrow[n \rightarrow \infty]{} \chi_{\text{Brown}}(\rho).$$

Proof. Recall that admissible means that the γ_n are chosen randomly in $B_{\Gamma}(r_n)$, relative to the counting measure, where r_n is a sequence such that $\sum e^{-cr_n}$ converges for every $c > 0$.

Fix $\varepsilon > 0$. We will show that

$$(20) \quad \frac{1}{\#B_{\Gamma}(r_n)} \# \left\{ \gamma \in B_{\Gamma}(r_n), \left| \frac{1}{2d(\gamma_n)} \log |\operatorname{tr}^2(\rho(\gamma_n))| - \chi_{\text{Brown}}(\rho) \right| > \varepsilon \right\} \lesssim e^{-c(\varepsilon)r_n}.$$

We then deduce from the Borel-Cantelli lemma that a.s. for large n , $\frac{1}{2d(\gamma_n)} \log |\operatorname{tr}^2(\rho(\gamma_n))|$ is ε -close to $\chi_{\text{Brown}}(\rho) < \varepsilon$, and the result follows.

The first observation is that in $B_{\Gamma}(r_n)$, by the Margulis estimate (19) on $\#B_{\Gamma}(r)$, for every $\delta > 0$, we have that

$$(21) \quad \# \{ \gamma, d(\gamma_n) < (1 - \delta)r_n \} \lesssim e^{(1-\delta)r_n}.$$

We choose $\delta = \min\left(\frac{1}{2}, \frac{\varepsilon}{4\chi}\right)$. Now if

$$d(\gamma_n) \geq (1 - \delta)r_n \text{ and } \left| \frac{1}{2d(\gamma_n)} \log |\operatorname{tr}^2(\rho(\gamma_n))| - \chi_{\text{Brown}}(\rho) \right| > \varepsilon,$$

an elementary computation shows that

$$\left| \frac{1}{2r_n} \log |\operatorname{tr}^2(\rho(\gamma_n))| - \chi_{\text{Brown}}(\rho) \right| > (1 - \delta)\varepsilon - \delta\chi_{\text{Brown}}(\rho) > \frac{\varepsilon}{4}.$$

Therefore the desired estimate (20) follows from (21), Theorem 3.2 and the fact that $\#B_{\Gamma}(r_n)$ grows like e^{r_n} . \square

Remark 3.7. This corollary yields yet another definition of the natural Lyapunov exponent, where the choice of Riemann surface structure is reflected in the fact of counting in the lattice Γ . Similarly, we could replace the trace by the norm, and assert that generically, $\frac{1}{2d(\gamma_n)} \log \|\rho(\gamma_n)\|$ converges to $\chi_{\text{Brown}}(\rho)$. The proof is identical to that of Theorem 3.2 (observe that Proposition 3.3 becomes much simpler in this case).

3.2. Proof of Theorem 3.1. Let m_n be the normalized counting measure on $B_{\Gamma}(r_n)$ and $m = \prod_{n \geq 1} m_n$. By considering the potentials of the currents $\frac{1}{2d(\gamma_n)}[Z(\gamma_n, t)]$, it is enough to show that for m -a.e. sequence $\gamma = (\gamma_n)$ in $\Gamma^{\mathbb{N}}$, the sequence of psh functions $(\lambda \mapsto u(\lambda, \gamma, n))_{n \geq 1}$ on Λ defined by

$$u(\lambda, \gamma, n) = \frac{1}{2d(\gamma_n)} \log |\operatorname{tr}^2(\rho_{\lambda}(\gamma_n)) - t|$$

converges in $L^1_{\text{loc}}(\Lambda)$ to $\lambda \mapsto \chi_{\text{Brown}}(\rho_\lambda)$. Corollary 3.6 above implies that for every *fixed* parameter λ , $u(\lambda, \gamma, n)$ converges m -almost surely to $\chi_{\text{Brown}}(\rho_\lambda)$. Now clearly the set $E \subset \Lambda \times \Gamma^{\mathbb{N}}$ of parameters (λ, γ) such that $u(\lambda, \gamma, n)$ converges to $\chi_{\text{Brown}}(\rho_\lambda)$ is measurable, so by Fubini's theorem it has full $d\lambda \otimes m$ measure. Reverting the order of integration (Fubini's theorem again), we infer that for m a.e. γ , $u(\lambda, \gamma, n)$ converges to $\chi_{\text{Brown}}(\rho_\lambda)$ $d\lambda$ -a.s.

In addition, the sequence $u(\lambda, \gamma, n)$ is locally uniformly bounded from above on Λ . Indeed we have that

$$\begin{aligned} \frac{1}{2d(\gamma_n)} \log |\text{tr}^2(\rho_\lambda(\gamma_n)) - t| &\leq \frac{1}{2d(\gamma_n)} \log \|\rho_\lambda(\gamma_n)\|^2 + O(1) \\ &\leq \frac{\beta_{\rho_\lambda}}{d(\gamma_n)} \log \|\gamma_n\| + O(1) \leq \frac{\beta_{\rho_\lambda}}{2} + O(1) \end{aligned}$$

where the constant β_{ρ_λ} is locally uniform in Λ (see Corollary 2.3). By the standard compactness theorems for sequences of subharmonic functions (see Theorems 3.2.12, 3.2.13 and 3.4.14 in [Hö]), we conclude that $u(\lambda, \gamma, n)$ converges to $\chi_{\text{Brown}}(\rho_\lambda)$ in $L^1_{\text{loc}}(\Lambda)$. Indeed, if γ is such that convergence holds a.e in Λ , we can extract a subsequence converging to some v in L^1_{loc} , and $\limsup_{n \rightarrow \infty} u(\lambda, \gamma, n) = v$ outside a polar set, therefore $v = \chi_{\text{Brown}}$. This finishes the proof. \square

Remark 3.8. As for Theorem 3.2, the statement of Theorem 3.1 may be adapted to the case where Γ admits non-trivial torsion.

3.3. Random closed geodesics. Let $\text{CG}(X)$ be the set of closed geodesics in X , which is a countable set. We allow a geodesic to be travelled several times. If this does not happen, the geodesic is said to be *primitive*.

Let $\rho \in \text{Hom}_{\text{par}}(G, \text{PSL}(2, \mathbb{C}))$. If $\gamma \in \text{CG}(X)$ the holonomy $h_\rho(\gamma)$ is defined up to conjugacy in $\rho(\Gamma)$, so it makes sense to talk about its (squared) trace $\text{tr}^2 h_\rho(\gamma)$. If now (ρ_λ) is family of parabolic representations, for $t \in \mathbb{C}$ we may consider the subvariety in Λ defined by

$$Z(\gamma, t) = \{\lambda \in \Lambda, \text{tr}^2 h_{\rho_\lambda}(\gamma) = t\}.$$

Here we first briefly review several notions of “random closed geodesics” on X , and interpret Theorem 3.1 in these terms.

3.3.1. Thurston model. To start with, assume that X is compact. The following model for random geodesics on X was apparently popularized by Thurston (see e.g. [Bo]): for $(x, v) \in SX$ let as before $\gamma_{(x,v)}$ be the associated unit speed geodesic ray. For $t > 0$, close the path $\gamma_{(x,v)}|_{[0,t]}$ with a shortest path returning to x . Notice that there may be a choice involved here, which we may disregard for it happens only for a set of (x, v) of zero measure. We denote by $\text{cg}(x, v)(t)$ the unique closed geodesic freely homotopic to the obtained loop. Notice that its length may be much smaller than t . Nevertheless it follows from Theorem 3.2 applied to the identity representation that this phenomenon happens only on a set of exponentially small measure.

In this way we have constructed a family of projections $v \mapsto \text{cg}(x, v)(t)$ from SX to $\text{CG}(X)$ indexed by t . Projecting the (normalized) Liouville measure gives rise to a family of probability measures $(m_t^{\text{geodesic}})_{t>0}$ on $\text{CG}(X)$.

Several variants of this construction may be considered:

- Variant 1: a first possibility is to restrict to geodesic paths starting from the marked point \star , and close them so as to obtain loops based at \star .

- Variant 2: allow for paths starting from any (x, v) , but append them at the two endpoints with shortest paths joining them to \star .

Notice that in these two variants, we naturally obtain a family of measures on the fundamental group, which in turn projects to a family of measures on $\text{CG}(X)$. Notice also that in the second variant, the family of measures that we obtain in $\text{CG}(X)$ is the same as in the original Thurston construction (since the corresponding loops are conjugate in Γ). In any case, the measures $(m_t^{\text{geodesic}})_{t>0}$ may be thought of as measures on Γ . A good intuitive picture for m_t^{geodesic} is that of being loosely equidistributed on an annulus of bounded width about the circle of radius t in $\Gamma \subset \mathbb{H}$.

These variants are interesting because they also make sense when X is only of finite volume. Indeed, using these constructions we obtain a family of measures $(m_t^{\text{geodesic}})_{t>0}$ on G . Now $\gamma \in \Gamma$ projects onto a closed geodesic if and only if it is not parabolic. Again by Theorem 3.2 this happens outside a set of exponentially small measure. So what we do is simply to restrict to loxodromic elements, renormalize the mass and project to $\text{CG}(X)$ we also obtain in this way a reasonable model for random closed geodesics on X .

The following lemma asserts that the measures (m_t^{geodesic}) are well spread in Γ .

Lemma 3.9. *Let X be a hyperbolic Riemann surface of finite type endowed with its hyperbolic metric, and let $(m_t^{\text{geodesic}})_{t>0}$ be any of the family of measures on Γ constructed above. Then for every $\delta > 0$ there exist constants K and c such that for all $t > 0$, for every subset $A \subset \Gamma$ with cardinality $\#A \leq e^{(1-\delta)t}$, we have that $m_t^{\text{geodesic}}(A) \leq Ke^{-ct}$.*

Proof. Let us treat the case of paths starting from \star (Variant 1 above). If v is a unit tangent vector at \star , we denote by γ_v the unit speed geodesic ray starting from \star in the direction of v .

The first claim is that there exists a constant K_1 depending only on X such that the probability that $d_X(\gamma_v(t), \star)$ is larger than $\frac{\delta}{2}t$ is not greater than $K_1e^{-\delta t/2}$. A proof of this fact is outlined in [Su1, §9], and discussed with much greater detail and generality in [KM].

For $t > 0$, let E_t be the set of tangent vectors $v \in T_\star^1 X$ such that $d_X(\gamma_v(t), \star) \leq \frac{\delta}{2}t$, which by our previous claim has measure $\geq 1 - K_1e^{-\delta t/2}$. We lift the situation to \mathbb{H} , so that our geodesics are now rays of hyperbolic length t issued from 0. If a lattice point $\gamma \in \Gamma \cdot 0$ is the endpoint of $\tilde{\gamma}_v(t)$ for some $v \in E_t$, then $d_{\mathbb{H}}(\gamma, \gamma_v(t)) \leq \frac{\delta}{2}t$. We apply the first cosine rule [Bea, §7.12] in the triangle $T(0, \gamma, \tilde{\gamma}_v(t))$ to get an estimate of the angle θ at the origin between the rays $\gamma_v(t)$ and $[0, \gamma]$. It follows that $\cos(\theta) = 1 - Ce^{-(2+\alpha)t} + O(e^{-3t})$, with $|\alpha| \leq \delta$, from which we deduce that $\theta \leq K_2e^{-(1-\delta/2)t}$. Hence for any subset A of Γ , the set of $v \in E_t$ such that $\tilde{\gamma}_v(t)$ belongs to A has measure at most $K_2(\#A)e^{-(1-\delta/2)t}$.

We conclude that if A is as in the statement of the lemma, the measure of the set of tangent vectors v such that $\tilde{\gamma}_v(t)$ belongs to A is not greater than $(K_1 + K_2)e^{-\frac{\delta}{2}t}$ and the result follows. \square

It is an entertaining exercise in hyperbolic geometry to show that the cardinality of the set of $\gamma \in B_\Gamma(t)$ which are proper powers of elements of Γ is of order of magnitude $e^{t/2}$. It follows that a generic geodesic with respect to the measure m_t^{geodesic} is primitive.

3.3.2. Brownian Thurston model. We can do exactly the same construction by using Brownian paths instead of geodesic ones. As before, several variants are possible, by choosing the initial point to be either fixed or distributed according to the Poincaré volume element on X , and by choosing the paths to be closed to become a loop based at \star or not. In this way we

obtain families of measures on Γ or on $\text{CG}(X)$, which, abusing slightly, will be given the same notation $(m_t^{\text{Brownian}})_{t>0}$. In the unit disk picture, the corresponding measures on Γ are induced by the heat flow on \mathbb{H}^2 , so they now look like Gaussian measures concentrated around the circle of radius t .

Again, these measures are well spread on Γ .

Lemma 3.10. *Let X be as above and $(m_t^{\text{Brownian}})_{t>0}$ be the family of measures on Γ obtained from the Brownian model for random geodesics. Then for every $\delta > 0$ there exist constants K and c such that for all $t > 0$, for every subset $A \subset \Gamma$ with cardinality $\#A \leq e^{(1-\delta)t}$, it holds that $m_t^{\text{Brownian}}(A) \leq Ke^{-ct}$.*

The proof uses arguments similar to those of Theorem 3.2, and will only be sketched. Details will be left as an exercise to the reader.

We need the following estimate.

Lemma 3.11. *Let (X, \star) be as above. There exists a positive constant c such that*

$$\mathbb{P}_\star(d_X(\star, \omega(t)) \geq ct) \lesssim e^{-ct/2}.$$

Proof (sketch). Let f be the density of the diffusion of δ_\star at time 1, relative to the hyperbolic area element on X . The classical Li-Yau estimates on the heat kernel (see e.g. [CC, §B.7]) imply that $f \in L^\infty(X) \subset L^2(X)$. Let $(\Pi_t)_{t>0}$ be the Brownian semigroup on X , and put $V_t = \{x \in X, d_X(x, \star) \geq ct\}$. We have to show that $\int_{V_t} \Pi_t f$ is exponentially small with t . For this, using the Schwarz inequality we write

$$\left| \int_{V_t} \Pi_t f \right| \leq \|\Pi_t f\|_{L^2(X)} \text{vol}(V_t)^{1/2},$$

and we conclude using the fact that $\|\Pi_t f\|_{L^2(X)}$ converges, while $\text{vol}(V_t)$ decreases exponentially with t . \square

Proof of Lemma 3.10 (sketch). We lift the situation to \mathbb{H} , and look at Brownian paths issued from 0. Fix a constant $c > 0$. By the estimates (4) for the heat kernel on \mathbb{H} and the above lemma, the probability that $\omega(t) \notin A_{\mathbb{H}}(t(1-c), t(1+c))$ or $d_{\mathbb{H}}(\omega(t), \Gamma) \geq ct$ decreases exponentially with t . We discard this set of paths, so that it is enough to work with $A' = A \cap A_{\mathbb{H}}(t(1-3c), t(1+3c))$ instead of A . Now if we denote by $\tilde{\omega}_t$ the concatenation of $\omega|_{[0,t]}$ with a shortest path joining $\omega(t)$ to Γ , we infer that $\mathbb{P}(\tilde{\omega}_t \in A') \leq \nu_t(\bigcup_{a \in A'} B(a, ct))$, where ν_t is the law of Brownian motion at time t . We compare this probability to the hyperbolic area of $\bigcup_{a \in A'} B(a, ct)$ by using (4) again, and get an estimate of the form $\mathbb{P}(\tilde{\omega}_t \in A') \lesssim \exp((-\delta + 5c)t)$. Finally, choosing the constant c small enough with respect to δ finishes the proof. \square

3.3.3. Length-based model. Let $\text{CG}_t(X)$ be the set of primitive geodesics of length at most t on X , and let m_t^{length} be simply the normalized counting measure on $\text{CG}_t(X)$.

To apply Theorem 3.1 we need to lift m_t^{length} to Γ . For this, we choose a shortest path from \star to a given geodesic to make it a loop based at \star . Observe that the length of this path is uniformly bounded. Indeed, recall from Proposition 2.2 that there are neighborhoods C_i of the cusps such that each geodesic must enter the compact region $X \setminus \bigcup C_i$.

We then infer that the lifted measure is supported in an annulus of bounded width about the circle of radius t in Γ . Also, the lifted measure is well spread in Γ in the sense of Lemmas 3.9 and 3.10 because of the well-known asymptotic estimate $\#\text{CG}_t(X) \sim \frac{e^t}{t}$ as $t \rightarrow \infty$. This

estimate originates in the works of Huber and Selberg (references covering the non-compact case include [He, I])

3.3.4. Equidistribution. We now choose any of the above models for random geodesics, and denote by (m_t) the associated family of measures on $\text{CG}(X)$. Let us recall the statement of the equidistribution theorem on Λ associated to random closed geodesics (Theorem C).

Theorem 3.12. *Let X be a hyperbolic Riemann surface of finite type, and $(\rho_\lambda)_{\lambda \in \Lambda}$ be a holomorphic family of representations of $G = \pi_1(X)$, satisfying (R1, R2, R3). Let $T_{\text{bif}} = dd^c \chi_{\text{Brown}}$ be the natural bifurcation current.*

Let (t_n) be a sequence of positive numbers such that for every $c > 0$ the series $\sum e^{-ct_n}$ converges. Let (γ_n) be a random sequence of closed geodesics on X , according to the product measure $\prod m_{t_n}$. Then almost surely we have that

$$\frac{1}{2 \text{length}(\gamma_n)} [Z(\gamma_n, t)] \xrightarrow[n \rightarrow \infty]{} T_{\text{bif}}.$$

The proof of the theorem is identical to that of Theorem 3.1, using the fact that the lifted measures m_{t_n} are well spread on Γ . Indeed for fixed λ , a Borel-Cantelli argument based on Theorem 3.2 together with Lemma 3.9 (Thurston model), Lemma 3.10 (Brownian Thurston model) or the estimate $\#\text{CG}_t(X) \sim \frac{e^t}{t}$ (length-based model) implies that almost surely

$$\frac{1}{2 \text{length}(\gamma_n)} \log |\text{tr}^2(\rho(\gamma_n))| \xrightarrow[n \rightarrow \infty]{} \chi_{\text{Brown}}(\rho),$$

which is the statement of Theorem D. Theorem 3.12 (i.e. Theorem C) then follows exactly as in §3.2.

4. FURTHER REMARKS, OPEN PROBLEMS

4.1. A first natural problem is to study the dependence of the natural Lyapunov exponent χ_{Brown} as a function of the Riemann surface structure X . It is likely that this function is continuous in (ρ, X) . It would be interesting to study the possible subharmonicity properties of this function as X varies in Teichmüller space.

4.2. (cf. [DD1, §5.2.3]). The varieties $Z(\gamma, 4)$ appearing in the equidistribution theorems are made of accidental parabolics and new relations. The question is to determine if one of the two types dominates. At first sight one might guess that the varieties of accidental relations $\{\lambda, \rho_\lambda(\gamma) = \text{id}\}$ are of codimension 3 in the representation variety. Unfortunately this is wrong. Indeed, if $\text{tr}^2(\rho(\gamma)) = 0$ (a codimension 1 condition) then $\rho(\gamma^2) = \text{id}$. More generally for any k we can obtain hypersurfaces of representations such that $\rho(\gamma^k) = \text{id}$. On the other hand, as observed in §3.3, random geodesics are generically primitive, so this phenomenon is negligible (see [Lu] for a wide generalization of this fact).

Is it true that for a generic sequence (γ_n) of closed geodesics, $\{\lambda, \rho_\lambda([\gamma_n]) = \text{id}\}$ is asymptotically of codimension greater than 1 in the representation variety?

4.3. Our results may be recast in the more general framework of holomorphic families of $\text{PSL}(2, \mathbb{C})$ -cocycles over a discrete or continuous time dynamical system, endowed with some invariant measure. A general question in this area of research is how the values of this cocycle over the set of periodic orbits approximates the Lyapunov exponent (see e.g. [Kal]). What Theorem D says is that we can have very precise such results (with exponential deviation estimates) in the particular case of the geodesic flow in constant negative curvature. It

would be interesting to know how this can be generalized to a more general (say, uniformly hyperbolic) base dynamics. One possible way of doing this would be to use the theory of Markov driven random products of matrices, which was developed in particular by Guivarc'h and his collaborators (see [Gu] for a review).

4.4. What can be said about the distribution of $[Z(\gamma_n, 4)]$ when the γ_n are *simple* closed geodesics on X ?

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