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# Quasi-Hamiltonian bookkeeping of WZNW defects

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## Abstract

We interpret the chiral WZNW model with general monodromy as an infinite dimensional quasi-Hamiltonian dynamical system. This interpretation permits to explain the totality of complicated cross-terms in the symplectic structures of various WZNW defects solely in terms of the single concept of the quasi-Hamiltonian fusion. Translated from the WZNW language into that of the moduli space of flat connections on Riemann surfaces, our result gives a compact and transparent characterisation of the symplectic structure of the moduli space of flat connections on a surface with  $k$  handles,  $n$  boundaries and  $m$  Wilson lines.

# 1 Introduction

The study of WZNW defects has been quite a hot topic since last ten years [3, 4, 8, 10, 11, 12, 13, 14, 18, 19, 20, 22, 25, 26, 28]. The idea was to modify the standard WZNW dynamics by consistent boundary conditions on the world-sheet or by defect lines in the bulk where the group valued WZNW field is allowed to jump in a particular way. In the presence of such defects the WZNW classical field equations can still be explicitly solved and the corresponding symplectic structure on the classical space of solutions can be derived starting from the classical WZNW action in [16, 27]. The resulting explicit expressions for the symplectic forms turn out to be quite complicated, however.

More conceptual understanding of the WZNW symplectic structures in the presence of defects was proposed in [17], where the language of flat connections on Riemann surfaces was used. This insight was motivated by an older result [7] where the symplectic structure of the bulk WZNW model without defects was identified with that of the moduli space of flat connections on the annulus. In the paper [17], the phase space of the boundary WZNW model was then shown to be symplectomorphic to the moduli space of flat connections on the disc with two Wilson lines inserted. The holonomies of the flat connections around the insertion points lie in some conjugacy classes in the group manifold  $G$  which are interpreted as "D-branes", i.e. as submanifolds of the target space  $G$  on which the open strings end.

Following the same philosophy, symplectic structures of several other defects were identified with those of appropriate moduli spaces of flat connections [27]. Thus the jump of the group valued WZNW field through a defect line on the world-sheet [13, 18] was shown to lead to the moduli space of flat connections on the annulus with one Wilson line insertion [27]. In this case, the holonomy around the insertion point lies in the same conjugacy class as the jump. The dictionary between the WZNW defects and the moduli spaces of flat connections was then enlarged to yet other types of defects still in [27]. For example, the phase space of the boundary WZNW model with one bulk defect line turns out to be the moduli space of flat connections on the disc with three Wilson line insertions. Finally, the last example treated in [27] is that of permutation branes [10, 12, 14, 26, 28] which are the boundary conditions for the  $n$ -fold direct product  $G \times G \times \dots \times G$  WZNW model on a strip world-sheet. It was conjectured in [27] that the relevant moduli space for this situation corresponds to the Riemann surface with  $n$  boundaries and two Wilson line insertions.

Although the book-keeping of the WZNW defects via the moduli spaces of flat connections is very elegant, it is more of conceptual importance than of concrete technical utility. In practice, one rather needs to have a description of the relevant symplectic structures on the moduli spaces in terms of group-valued holonomies of the flat connections since they correspond to the physically interpretable WZNW observables. Such description is, however, quite cumbersome already in the presence of small number of defects, since there arise many cross terms in the symplectic forms which correspond to "interactions" of the defects.

The goal of the present paper is to propose an alternative conceptual bookkeeping of the WZNW defects which would be technically more friendly and would use quantities with direct physical interpretation. Our main inspiration comes from the approach of Ref. [2], where the symplectic

structures of the moduli spaces of flat connections on closed surfaces (i.e. without boundaries) were described in terms of the so called quasi-Hamiltonian fusion. Speaking more precisely, the moduli space of flat connections on the compact closed surface with  $m$  Wilson line insertions and  $k$  handles was identified in [2] as the following symplectic manifold

$$M_{mk} \equiv (C_1^- \otimes C_2^- \otimes \dots \otimes C_m^- \otimes \underbrace{\mathbf{D}(G) \otimes \dots \otimes \mathbf{D}(G)}_{k \text{ times}})_e. \quad (1.1)$$

Here  $C_i^-$  is the conjugacy class to which belongs the holonomy of the connection around the  $i^{\text{th}}$  insertion point (the superscript  $-$  means the inverse of the standard quasi-Hamiltonian structure on the conjugacy class), the symbols  $\mathbf{D}(G)$  stand for the so called internally fused quasi-Hamiltonian double of the structure Lie group  $G$ , the operation  $\otimes$  is the fusion of two quasi-Hamiltonian manifolds and the notation  $(M)_e$  means the symplectic manifold obtained by the quasi-Hamiltonian reduction of the quasi-Hamiltonian manifold  $M$  at the unit level of the moment map.

The big advantage of the expression (1.1) consists in the fact that not only it gives the explicit characterization of the symplectic structures of the moduli spaces in terms of the convenient group-like variables but, at the same time, it remains conceptually neat. Indeed, each handle or defect brings its building block into the expression and all ingredients are glued together using the single concept of the quasi-Hamiltonian fusion.

In what follows, we shall generalize the formula (1.1), by allowing the presence of the boundaries on the Riemann surface. This change involves the transition from the finite dimensional context to an infinite-dimensional one, since the moduli spaces of flat connections in the presence of boundaries are smooth infinite-dimensional symplectic manifolds [6]. Indeed, the flat connections on the closed surfaces correspond roughly to the topological  $G/G$  WZNW model and the surfaces with boundaries take into account the full field theoretical WZNW dynamics. In spite of the infinite-dimensional setting, the result of our generalisation is conceptually as simple as the expression (1.1). Indeed, we shall argue that the moduli space of flat connections on the surface with  $n$  boundaries,  $m$  Wilson lines insertions and  $k$  handles reads:

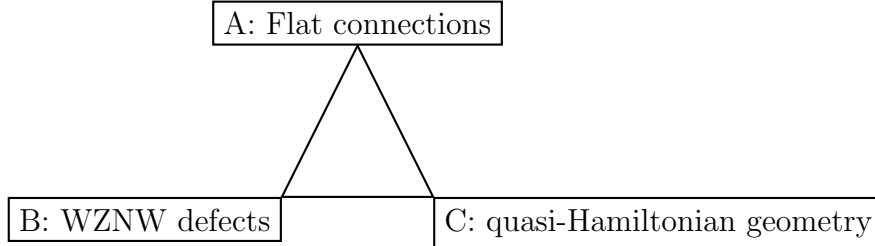
$$M_{nmk} \equiv \underbrace{(W^- \otimes \dots \otimes W^-)}_{n \text{ times}} \otimes C_1^- \otimes C_2^- \otimes \dots \otimes C_m^- \otimes \underbrace{\mathbf{D}(G) \otimes \dots \otimes \mathbf{D}(G)}_{k \text{ times}})_e, \quad (1.2)$$

where  $W^-$  is the particular infinite-dimensional quasi-Hamiltonian manifold the points of which are quasi-periodic maps with values in  $G$ . We shall refer to  $W^-$  as to quasi-Hamiltonian chiral WZNW model. We shall see, in particular, that the quasi-Hamiltonian language of formula (1.2) is very well suited for bookkeeping of multitude of terms appearing in the explicit description of symplectic forms associated to various WZNW defects.

The plan of the paper is as follows: In Section 2, we expose some basic facts about the quasi-Hamiltonian geometry; in particular, we define the quasi-Hamiltonian fusion, quasi-Hamiltonian reduction and explain the contents of the so called equivalence theorem of [2] relating Hamiltonian loop group  $LG$ -spaces to the quasi-Hamiltonian  $G$ -spaces. In Section 3, we define the chiral WZNW model as the quasi-Hamiltonian system and explain how it can be obtained from the full WZNW model via that equivalence theorem just mentioned. Section 4 prepares ingredients

for proving the formula (1.2), namely, it gives an elegant description of the Hamiltonian loop group space associated by the equivalence theorem to *any* quasi-Hamiltonian space. The section 5 and 6 are respectively devoted to the sides  $AC$  and  $BC$  of the following triangle diagram (the side  $AB$  was largely discussed in [27]):

Figure 1:



In particular, in Section 5 we review the definition of the symplectic structures on the moduli space of flat connections and then we prove that those structures are indeed described by the formula (1.2). Finally, in Section 6, we work out the symplectic structures of the bulk, boundary and defect WZNW models starting from the formula (1.2) and find agreement with the WZNW defect symplectic structures obtained in [15, 16, 17, 27] by the detailed analysis of the WZNW dynamics.

## 2 Quasi-Hamiltonian geometry

Quasi-Hamiltonian manifold  $M$  is acted upon by a simple compact connected Lie group  $G$ , it is equipped with an invariant two-form  $\Omega$  and with a moment map  $\mu : M \rightarrow G$  in such a way that four axioms must hold:

1.  $\mu$  intertwines the  $G$  action  $\triangleright$  on  $M$  with the conjugacy action on  $G$ :

$$\mu(g \triangleright x) = g\mu(x)g^{-1}, \quad g \in G, x \in M. \quad (2.3)$$

2. The exterior derivative of  $\Omega$  is given by

$$\delta\Omega = -\frac{1}{12}\mu^*(\theta, [\theta, \theta]). \quad (2.4)$$

3. The infinitesimal action of  $\mathcal{G} \equiv \text{Lie}(G)$  on  $M$  is related to  $\mu$  and  $\Omega$  by

$$\iota(\zeta_M)\Omega = \frac{1}{2}\mu^*(\theta + \bar{\theta}, \zeta), \quad \forall \zeta \in \mathcal{G}. \quad (2.5)$$

4. At each  $x \in M$ , the kernel of  $\Omega_x$  is given by

$$\text{Ker}(\Omega_x) = \{\zeta_M(x) \mid \zeta \in \text{Ker}(\text{Ad}_{\mu(x)} + \text{Id})\}. \quad (2.6)$$

Here  $(\cdot, \cdot)$  is the Killing-Cartan form on  $\mathcal{G}$ ,  $\theta$  and  $\bar{\theta}$  denote, respectively, the left- and right-invariant Maurer-Cartan forms on  $G$  and  $\zeta_M$  stands for the vector field on  $M$  that corresponds to  $\zeta \in \mathcal{G}$ .

Three examples of quasi-Hamiltonian manifolds will be important for us: the conjugacy class in  $G$ , the so called quasi-Hamiltonian double  $D(G)$  of the group  $G$  and the internally fused double  $\mathbf{D}(G)$ . The quasi-Hamiltonian moment map  $\mu$  for a conjugacy class  $\mathcal{C} \subset G$  is just the embedding  $\mathcal{C} \hookrightarrow G$  and the quasi-Hamiltonian form  $\alpha$  evaluated at  $f \in \mathcal{C}$  is defined by the formula [2]

$$\alpha_f^{\mathcal{C}}(v_\xi, v_\eta) = \frac{1}{2} \left( (\eta, \text{Ad}_f \xi) - (\xi, \text{Ad}_f \eta) \right). \quad (2.7)$$

Here  $v_\xi, v_\eta$  are the vector fields corresponding to the infinitesimal actions of  $\xi, \eta \in \mathcal{G}$ . There is another useful way of representing the quasi-Hamiltonian form  $\alpha$  in terms of the following parametrization of the points on the conjugacy class  $\mathcal{C}$ :

$$f = ke^{2\pi i \tau} k^{-1}, \quad (2.8)$$

where  $\tau$  is in the Weyl alcove and  $k \in G$ . We have then

$$\alpha_f^{\mathcal{C}} = \frac{1}{2} (k^{-1} \delta k, e^{-2\pi i \tau} k^{-1} \delta k e^{2\pi i \tau}). \quad (2.9)$$

As a manifold, the double  $D(G)$  is just the direct product  $G \times G$ . It is the quasi-Hamiltonian  $G \times G$  manifold with respect to the  $G \times G$  action

$$(g_1, g_2) \triangleright (a, b) \equiv (g_1 a g_2^{-1}, g_2 b g_1^{-1}), \quad (2.10)$$

moment map  $\mu_D = (\mu_1, \mu_2) : D(G) \rightarrow G \times G$

$$\mu_1(a, b) = ab, \quad \mu_2(a, b) = a^{-1} b^{-1} \quad (2.11)$$

and the quasi-Hamiltonian form  $\Omega_D$  defined by

$$\Omega_D = \frac{1}{2} (a^* \theta, b^* \bar{\theta}) + \frac{1}{2} (a^* \bar{\theta}, b^* \theta). \quad (2.12)$$

As a manifold, the internally fused double  $\mathbf{D}(G)$  is again the direct product  $G \times G$  equipped with the  $G$  action

$$g \triangleright (a, b) \equiv (g a g^{-1}, g b g^{-1}), \quad (2.13)$$

the moment map

$$\mu(a, b) \equiv a b a^{-1} b^{-1} \quad (2.14)$$

and the two-form

$$\Omega = \frac{1}{2} (a^* \theta, b^* \bar{\theta}) + \frac{1}{2} (a^* \bar{\theta}, b^* \theta) + \frac{1}{2} ((ab)^* \theta, (a^{-1} b^{-1})^* \bar{\theta}). \quad (2.15)$$

Let us now list some of the properties of the quasi-Hamiltonian spaces relevant for this paper (see [2] for more details):

- First of all, a quasi-Hamiltonian manifold  $M$  equipped with the same  $G$ -action, a form  $-\Omega$  and a moment map  $\mu^{-1}$  is again quasi-Hamiltonian, it is referred to as the inverse quasi-Hamiltonian space and denoted as  $M^-$ .
- Suppose that the unit element  $e \in G$  is the regular value of the moment map  $\mu$ . The axioms of the quasi-Hamiltonian geometry imply that  $\mathcal{G} \equiv \text{Lie}(G)$  acts on the unit-level submanifold  $\mu^{-1}(e)$  without fixed points and thus  $\mu^{-1}(e)/G$  is a symplectic orbifold (not necessarily manifold because there still may be points in  $\mu^{-1}(e)$  with a discrete isotropy subgroup). This orbifold is usually denoted as  $(M)_e$  and it is called the unit-level *quasi-Hamiltonian reduction* of  $M$ . By construction, the pull-back of the symplectic form  $\omega$  from  $(M)_e$  to  $\mu^{-1}(e)$  is equal to the restriction of  $\Omega$  to  $\mu^{-1}(e)$ , however, we stress that  $\omega$  is the symplectic form in the usual sense, whilst  $\Omega$  is neither closed nor globally non-degenerate in general.
- A direct product of two quasi-Hamiltonian manifolds  $M_1 \times M_2$  is again a quasi-Hamiltonian manifold if it is equipped with the diagonal  $G$ -action, a moment map being the Lie group product  $\mu_1 \mu_2$  of the respective moment maps  $\mu_1$  for  $M_1$  and  $\mu_2$  for  $M_2$  and with a two-form

$$\Omega_{12} = \Omega_1 + \Omega_2 + \frac{1}{2}(\mu_1^* \theta, \mu_2^* \bar{\theta}). \quad (2.16)$$

The quasi-Hamiltonian manifold  $(M_1 \times M_2, \mu_1 \mu_2, \Omega_{12})$  is called the *fusion product* and is denoted as  $M_1 \otimes M_2$ . In the case of a multiple fusion  $M_1 \otimes M_2 \otimes \dots \otimes M_n$ , the mixed term in (2.16) gives rise to a multitude of terms in the resulting reduced symplectic form on  $(M_1 \otimes M_2 \otimes \dots \otimes M_n)_e$  which look quite awkward without the conceptual quasi-Hamiltonian understanding of their origin. It is indeed the purpose of the present paper to go in the opposite direction and to give the quasi-Hamiltonian *raison d'être* for the multitude of cross-terms in the symplectic structures induced by the WZNW defects.

- Any  $G$ -invariant function  $H$  on a quasi-Hamiltonian manifold  $(M, \mu, \Omega)$  defines a "quasi-Hamiltonian dynamics", in the sense that there is a unique (evolution) vector field  $v_H$  satisfying the conditions

$$\iota(v_H)\Omega = \delta H, \quad \iota(v_H)\mu^* \theta = 0. \quad (2.17)$$

Here  $\delta$  stands for the de Rham differential. The Hamiltonian vector field  $v_H$  is  $G$ -invariant and preserves  $\omega$  and  $\mu$  [2].

- Perhaps the most remarkable property of the quasi-Hamiltonian spaces is the equivalence theorem of Ref. [2]. It states that every quasi-Hamiltonian space determines a standard Hamiltonian loop group space with proper moment map and vice versa. In this way many structural questions which can be asked about infinite-dimensional symplectic manifolds admitting the Hamiltonian actions of loop groups can be reformulated and solved in an analytically more friendly environment, in particular, if the corresponding quasi-Hamiltonian space turns out to be finite dimensional. Speaking more precisely, the Hamiltonian  $LG$  space  $N$  with an equivariant moment map  $\Phi : N \rightarrow LG^*$  and a symplectic form  $\omega$  gives rise to the quasi-Hamiltonian structure on the manifold  $\text{Hol}(N) \equiv N/\Omega G$  where  $\Omega G$  is the group of based loops (i.e. loops taking the value  $e$  at the distinguished point  $\sigma = 0$ ).

In order to make explicit the quasi-Hamiltonian form and the quasi-Hamiltonian moment map on  $\text{Hol}(N)$ , we need to introduce some technical tools, namely, the space of quasi-periodic maps  $W$  and a map  $\text{Hol}: LG^* \rightarrow W$ .

The space  $W$  consists of smooth maps  $l: \mathbb{R} \rightarrow G$  with the property

$$l(\sigma + 2\pi) = l(\sigma)M, \quad \forall \sigma \in \mathbb{R}. \quad (2.18)$$

The element  $M \in G$  does not depend on  $\sigma$  and it is called the monodromy of  $l \in W$ . For every  $A \in LG^*$  there is then a unique element  $w_A \in W$  such that

$$A = w_A(\sigma)^{-1} \partial_\sigma w_A(\sigma) d\sigma, \quad w_A(0) = e. \quad (2.19)$$

We have thus defined the map  $\text{Hol}: LG^* \rightarrow W$

$$\text{Hol}(A) := w_A. \quad (2.20)$$

The loop group  $LG$  acts on  $LG^*$  by gauge transformations (the Hamiltonian moment map  $\Phi$  is equivariant precisely with respect to this action!):

$$g \triangleright A = gAg^{-1} - g^* \bar{\theta}, \quad g \in LG. \quad (2.21)$$

The transformation (2.21) then induces the following transformation of the holonomy:

$$w_{g \triangleright A}(\sigma) = g(0)w_A(\sigma)g(\sigma)^{-1}. \quad (2.22)$$

In particular,  $w_A(2\pi)$  is gauge invariant with respect to the transformations from the based loop group  $\Omega G$  since in this case  $g(0) = g(2\pi) = e$ . It is this gauge invariance which permits to define the quasi-Hamiltonian moment map  $\mu: \text{Hol}(N) \rightarrow G$  as

$$\mu := w_\Phi(2\pi). \quad (2.23)$$

The quasi-Hamiltonian form  $\Omega$  on  $\text{Hol}(N)$  is constructed as follows. First of all, consider a two-form  $\Upsilon$  on  $LG^*$  defined by

$$\Upsilon = \frac{1}{2} \int_0^{2\pi} d\sigma (\text{Hol}_\sigma^* \bar{\theta}, \partial_\sigma \text{Hol}_\sigma^* \bar{\theta}). \quad (2.24)$$

Note that the definition (2.24) makes sense since, for a fixed value of  $\sigma$ ,  $\text{Hol}_\sigma$  is a map from  $LG^* \rightarrow G$ . The vector fields corresponding to the infinitesimal action of the group  $\Omega G$  on  $N$  turn out to be the degeneracy directions of the following two-form on  $N$ :

$$\omega + \Phi^* \Upsilon. \quad (2.25)$$

This two-form is therefore the pull-back of some form  $\Omega$  on  $\text{Hol}(N)$ , which is nothing but the quasi-Hamiltonian form on  $\text{Hol}(N)$ .



### 3 Quasi-Hamiltonian equivalent of the WZNW model

The full WZNW model [30] is the standard symplectic dynamical system, the phase space  $P_{WZ}$  of which admits two different Hamiltonian actions of the loop group  $LG$ . One of those actions has the equivariant moment map in the sense of Definition 8.2 of [2] (see also Eq. (3.30) of the present paper). Following the discussion at the end of Section 2, we can associate to the equivariant Hamiltonian  $LG$ -manifold  $P_{WZ}$  the equivalent quasi-Hamiltonian dynamical system on the space  $\text{Hol}(P_{WZ})$  equipped with the corresponding  $G$ -action induced by some quasi-Hamiltonian moment map. It is the goal of this section to show that this equivalent quasi-Hamiltonian system is nothing but the quasi-Hamiltonian version of the chiral WZNW model.

Ideologically, we shall describe here the WZNW model in the language of the twisted Heisenberg double [29, 21]. Thus the phase space  $P_{WZ}$  of the WZNW model is the cotangent bundle of the loop group  $LG$  parametrized by  $J_L(\sigma) \in L\mathcal{G}$  and  $g(\sigma) \in LG$ , however, the symplectic form is not the canonical one on the cotangent bundle since it contains the additional term (the twist):

$$\omega_{WZ} = -\delta \int_0^{2\pi} d\sigma (J_L, \delta g g^{-1}) - \frac{1}{2} \int_0^{2\pi} d\sigma (\delta g g^{-1}, \partial_\sigma (\delta g g^{-1})). \quad (3.26)$$

Here  $\delta$  is the de Rham differential on  $P_{WZ}$ .

There are two Hamiltonian actions of the loop group  $LG$  on the phase space  $P_{WZ}$ :

$$\begin{aligned} h \triangleright_L (J_L, g) &:= (h J_L h^{-1} + \partial_\sigma h h^{-1}, h g), & h \in LG; \\ h \triangleright_R (J_L, g) &:= (J_L, g h^{-1}), & h \in LG. \end{aligned} \quad (3.27)$$

The moment maps of these two actions are  $J_L$  and  $J_R$ , respectively, where

$$J_R := -g^{-1} J_L g + g^{-1} \partial_\sigma g. \quad (3.28)$$

Indeed, it is easy to check that it holds

$$\iota(v_\xi^L)\omega_{WZ} = \delta \int_0^{2\pi} (J_L, \xi) d\sigma, \quad \iota(v_\xi^R)\omega_{WZ} = \delta \int_0^{2\pi} (J_R, \xi) d\sigma, \quad (3.29)$$

where  $\xi \in L\mathcal{G}$  and  $v_\xi^{L,R}$  are the respective vector fields corresponding to the infinitesimal actions of  $\xi$  on  $P_{WZ}$ . Note the transformation of the right current  $J_R$  under the action  $\triangleright_R$  on  $P_{WZ}$  by an element  $h \in LG$ :

$$J_R \rightarrow h J_R h^{-1} - \partial h h^{-1}. \quad (3.30)$$

We observe that the moment map  $J_R$  is equivariant following the conventions of Sections 8.1 and 8.2 of [2]. However, the left current  $J_L$  transforms under the action  $\triangleright_L$  with the opposite sign of the inhomogeneous term:

$$J_L \rightarrow h J_L h^{-1} + \partial h h^{-1}. \quad (3.31)$$

We shall refer to the moment map  $J_L$  as 'anti-equivariant'. We finish the resuming of the WZNW model by defining its Hamiltonian:

$$H_{WZ} = -\frac{1}{2} \int_0^{2\pi} (J_L, J_L) d\sigma - \frac{1}{2} \int_0^{2\pi} (J_R, J_R) d\sigma. \quad (3.32)$$

Since the phase space  $P_{WZ}$  with the right action  $\triangleright_R$  of  $LG$  is the Hamiltonian  $LG$ -space in the sense of the definition 8.2 of [2], we can construct the corresponding quasi-Hamiltonian  $G$ -space  $\text{Hol}(P_{WZ})$  following the recipe described at the end of Section 2. This gives the statement of the following important Theorem:

**Theorem 1:** *The quasi-Hamiltonian space  $\text{Hol}(P_{WZ})$  is the space of quasi-periodic maps  $W$ , the corresponding quasi-Hamiltonian moment map  $\mu : W \rightarrow G$  is the inverse monodromy of the element  $l \in W$*

$$\mu(l) = l(2\pi)^{-1}l(0) \quad (3.33)$$

and the quasi-Hamiltonian form  $\Omega$  on  $W$  induced by  $\omega_{WZ}$  on  $P_{WZ}$  reads

$$\Omega(l) := \frac{1}{2} \left[ \int_0^{2\pi} (l^{-1}\delta l, \partial_\sigma(l^{-1}\delta l)) d\sigma + (\delta l l^{-1}|_0, \delta l l^{-1}|_{2\pi}) \right]. \quad (3.34)$$

*Proof.* Denote by  $g_R \in W$  the element  $w_{J_R}$  defined by (2.19), i.e.

$$J_R = g_R^{-1} \partial_\sigma g_R, \quad g_R(0) = e. \quad (3.35)$$

We can also conveniently parametrize the current  $J_L$  as

$$J_L = \partial_\sigma g_L g_L^{-1}, \quad g_L(0) = e \quad (3.36)$$

and the field  $g(\sigma)$  as

$$g(\sigma) = g_L(\sigma) b(\sigma) g_R(\sigma). \quad (3.37)$$

The relation (3.28) then implies that  $b(\sigma)$  in fact does not depend on  $\sigma$  and it is therefore equal to  $g(0)$ . In what follows, we set

$$l(\sigma) := g_L(\sigma) b(\sigma) = g_L(\sigma) g(0) \quad (3.38)$$

and express straightforwardly the symplectic form  $\omega_{WZ}$  in terms of the variables  $l(\sigma)$  and  $g_R(\sigma)$ :

$$\omega_{WZ} = \frac{1}{2} \int_0^{2\pi} (l^{-1}\delta l, \partial_\sigma(l^{-1}\delta l)) - \frac{1}{2} (l^{-1}\delta l, \delta g_R g_R^{-1}) \Big|_0^{2\pi} - \frac{1}{2} \int_0^{2\pi} (\delta g_R g_R^{-1}, \partial_\sigma(\delta g_R g_R^{-1})). \quad (3.39)$$

Because of the fact that  $g_R(0) = e$  and

$$g(2\pi) = l(2\pi) g_R(2\pi) = l(0) = g(0), \quad (3.40)$$

we conclude that the quasi-Hamiltonian form (2.25) becomes

$$\omega_{WZ} + J_R^* \Upsilon = \omega_{WZ} + \frac{1}{2} \int_0^{2\pi} (\delta g_R g_R^{-1}, \partial_\sigma (\delta g_R g_R^{-1})) = \frac{1}{2} \left[ \int_0^{2\pi} (l^{-1} \delta l, \partial_\sigma (l^{-1} \delta l)) d\sigma + (\delta l l^{-1}|_0, \delta l l^{-1}|_{2\pi}) \right]. \quad (3.41)$$

Now Eqs. (2.23) and (3.40) show that the quasi-Hamiltonian moment map is indeed the inverse monodromy of  $l \in W$

$$\mu(l) = g_R(2\pi) = l(2\pi)^{-1}l(0). \quad (3.42)$$

Finally, it remains to identify the quasi-Hamiltonian space  $\text{Hol}(P_{WZ})$  with  $W$ . Note that  $\text{Hol}(P_{WZ})$  is the space of cosets  $P_{WZ}/\Omega G$ , so starting from the parametrization  $(J_L, g)$  of  $P_{WZ}$  we see that  $\text{Hol}(P_{WZ})$  can be parametrized by means of  $g_L$  and  $g(0)$  as  $J_L = \partial_\sigma g_L g_L^{-1}$ ,  $g = g(0)$ . Following (3.38),  $\text{Hol}(P_{WZ})$  can be parametrized also by  $l \in W$  since  $g(0) = l(0)$ . From (3.27), we conclude that the  $G$ -action on  $W$  is given by

$$l(\sigma) \rightarrow l(\sigma)h^{-1}, \quad l(\sigma) \in W, \quad h \in G. \quad (3.43)$$

□

Although from the general theorems of Ref. [2] it follows that the triple  $(W, \Omega(l), \mu(l))$  given by Eqs. (3.34), (3.42) and (3.43) is the quasi-Hamiltonian  $G$ -space, we prefer to provide a direct proof of this fact in order to make the present paper more self-contained:

**Theorem 2:** *Define a function on  $W$  by the formula*

$$H(l) = -\frac{1}{2} \int_0^{2\pi} (\partial_\sigma l l^{-1}, \partial_\sigma l l^{-1}) d\sigma, \quad (3.44)$$

the  $G$ -action on  $W$  by

$$l(\sigma) \rightarrow l(\sigma)h^{-1}, \quad l(\sigma) \in W, \quad h \in G, \quad (3.45)$$

the moment map  $\mu : W \rightarrow G$  by

$$\mu(l) = l(2\pi)^{-1}l(0) \quad (3.46)$$

and the two-form  $\Omega(l)$  on  $W$  by

$$\Omega(l) := \frac{1}{2} \int_0^{2\pi} (l^{-1} \delta l, \partial_\sigma (l^{-1} \delta l)) d\sigma + \frac{1}{2} (\delta l l^{-1}|_0, \delta l l^{-1}|_{2\pi}) \quad (3.47)$$

The quadruple  $(W, \Omega(l), \mu, H)$  is then the quasi-Hamiltonian dynamical system.

*Proof.* We immediately observe from (2.18), (3.45) and (3.46) that

$$\mu(h \triangleright l) = h\mu(l)h^{-1}, \quad (3.48)$$

which means that the first defining quasi-Hamiltonian property (2.3) is verified.

A simple bookkeeping of boundary terms gives the second defining quasi-Hamiltonian property (2.4):

$$\delta\Omega(l) = \frac{1}{12} (l^{-1} \delta l|_{2\pi}, [l^{-1} \delta l|_{2\pi}, l^{-1} \delta l|_{2\pi}]) - \frac{1}{12} (l^{-1} \delta l|_0, [l^{-1} \delta l|_0, l^{-1} \delta l|_0]) + \frac{1}{2} \delta (\delta l l^{-1}|_0, \delta l l^{-1}|_{2\pi}) =$$

$$= -\frac{1}{12}(\mu_l^{-1}\delta\mu_l, [\mu_l^{-1}\delta\mu_l, \mu_l^{-1}\delta\mu_l]), \quad (3.49)$$

where we have set  $\mu(l) = \mu_l$ .

Let us now verify the third property (2.5). First of all, let  $\xi_W$  be a vector field on  $W$  induced by the infinitesimal action of an element  $\xi \in \mathcal{G}$ . We infer easily

$$\iota(\xi_W)l^{-1}\delta l = -\xi, \quad \iota(\xi_W)(\delta ll^{-1}|_0) = -l(0)\xi l(0)^{-1}, \quad \iota(\xi_W)(\delta ll^{-1}|_{2\pi}) = -l(2\pi)\xi l(2\pi)^{-1}, \quad (3.50)$$

hence we find indeed that

$$\iota(\xi_W)\Omega = \frac{1}{2}(\xi, \delta\mu_l\mu_l^{-1} + \mu_l^{-1}\delta\mu_l). \quad (3.51)$$

It remains to verify the last property (2.6). First of all we note that  $W$  is a submanifold of the group  $\mathbb{R}G$  consisting of all smooth maps from  $\mathbb{R}$  to  $G$ . Therefore any vector field  $v$  at a point  $l$  of  $W$  can be written as the left transport  $-L_{l*}\zeta$  of some  $-\zeta \in \text{Lie}(\mathbb{R}G)$ . From this information we find

$$\iota(v)(l^{-1}\delta l) = -\zeta \quad (3.52)$$

therefore

$$\iota(v)\Omega = \int_0^{2\pi} (l^{-1}\delta l, \partial_\sigma\zeta) d\sigma - \frac{1}{2}(\zeta, l^{-1}\delta l) \Big|_0^{2\pi} - \frac{1}{2}(l_0\zeta_0 l_0^{-1}, \delta ll^{-1}|_{2\pi}) + \frac{1}{2}(\delta ll^{-1}|_0, l_{2\pi}\zeta_{2\pi} l_{2\pi}^{-1}). \quad (3.53)$$

If  $v$  is to be in the kernel of  $\Omega$  then obviously  $\partial_\sigma\zeta = 0$  and

$$\iota(v)\Omega = \frac{1}{2}(\zeta, \delta\mu_l\mu_l^{-1} + \mu_l^{-1}\delta\mu_l) = \frac{1}{2}(\mu_l\zeta\mu_l^{-1} + \zeta, \delta\mu_l\mu_l^{-1}). \quad (3.54)$$

From the last formula, the wanted property (2.6) readily follows.

We conclude the demonstration by noting that the Hamiltonian (3.44) is evidently  $G$ -invariant, as it should be. □

**Definition:** We shall refer to the quasi-Hamiltonian dynamical system  $(W, \Omega(l), \mu(l), H(l))$  as to the quasi-Hamiltonian chiral WZNW model.

**Remark 1:** Historically, the origin of the concept of the chiral WZNW model lies in the attempt to equip the left and right movers of the WZNW model with independent dynamics. Recall that every solution of the WZNW model in the configuration space  $LG$  can be described as the product of left and right movers [15, 16]:

$$g(\sigma, \tau) = l(\sigma + \tau)r^{-1}(\sigma - \tau), \quad \sigma \in [0, 2\pi[, \quad \tau \in \mathbb{R}, \quad (3.55)$$

where both left and right movers  $l$  and  $r$  are the elements of  $W$  and can be viewed as almost independent coordinates on the infinite-dimensional phase space  $P_{WZ}$  of the theory. Indeed,  $l$  and  $r$  are tied only by the requirement that they must have *the same monodromies* in order to insure the periodicity of the WZNW field  $g(\sigma)$ . In [15], the symplectic form  $\omega_{WZ}$  was expressed in terms of the left and right movers as

$$\omega_{WZ} = \Omega(l) - \Omega(r), \quad (3.56)$$

where the two-form  $\Omega(l)$  is nothing but our quasi-Hamiltonian friend (3.34). The form of the WZNW symplectic form (3.56) suggests that it may be possible to separate completely the left and right movers by allowing the independent monodromies for them. However, the trouble in doing that was remarked already in [15]. The point is that the exterior derivatives of the forms  $\Omega(l)$  and  $\Omega(r)$  do not vanish separately as it can be seen from (3.49) (in fact, in calculating  $\delta\omega_{WZ}$ , they cancel with each other precisely when the left and right monodromies are the same). As the solution to the problem of non-closedness of  $\Omega(l)$ , it was proposed in [15] to add to  $\Omega(l)$  a two-form  $\rho(\mu(l))$  depending exclusively on the inverse monodromy  $\mu(l)$  and to define the chiral WZNW model as a theory on the phase space  $W$ , with the symplectic form  $\Omega(l) + \rho(\mu(l))$  and the quadratic current Hamiltonian (3.44). The problem with this definition is the ambiguity of the choice of the two-form  $\rho(\mu(l))$  as well as the fact that, strictly speaking, such  $\rho$  exists only on a dense open subset of the group manifold  $G$ . In this section, we did not attempt to define the chiral dynamics in the symplectic way, but we adopted the quasi-Hamiltonian point of view. Said in other words, we have defined the chiral WZNW model as the *quasi-Hamiltonian dynamical system*. For this, we did not need to add any term to the two-form  $\Omega(l)$  on  $W$ , but we let it as it stands. Of course, all this is just a shift of interpretation but it will turn out soon that our quasi-Hamiltonian version of the chiral WZNW has some good structural properties, namely it is useful for the compact description of the symplectic properties of the WZNW defects.

## 4 Loop group equivalent of a quasi-Hamiltonian space

We devote this section to the formulation and proof of a technical Theorem 3, which will be of big utility in Section 5. It gives a convenient description of the Hamiltonian  $LG$ -space equivalent to a given quasi-Hamiltonian space in the sense of the equivalence formulated in Section 2:

**Theorem 3:** *The Hamiltonian  $LG$ -manifold  $(N, \omega, \Phi)$  with equivariant proper moment map equivalent to a quasi-Hamiltonian  $G$ -manifold  $(M, \Omega, \mu)$  is given by  $N = (M \otimes W^-)_e$ , i.e. by the quasi-Hamiltonian fusion of  $M$  and  $W^-$  followed by the unit-level quasi-Hamiltonian reduction. The corresponding  $LG$ -action on  $(M \otimes W^-)_e$  is given by*

$$(x, l(\sigma)) \rightarrow (x, h(\sigma)l(\sigma)), \quad x \in M, \quad l(\sigma) \in W, \quad h(\sigma) \in LG \quad (4.57)$$

and the corresponding  $LG^*$ -valued moment map  $\Phi$  is given by

$$\Phi(x, l) = -\partial_\sigma l l^{-1} d\sigma, \quad x \in M, \quad l \in W. \quad (4.58)$$

*Proof.* We start by checking, that the formula (4.57) consistently defines the  $LG$ -action on the quasi-Hamiltonian quotient  $(M \otimes W^-)_e$ . First of all, the monodromy of the configuration  $h(\sigma)l(\sigma)$  is the same as that of  $l(\sigma)$  for every  $h(\sigma) \in LG$  therefore the action (4.57) survives the unit-level reduction constraint  $\mu\mu_l^{-1} = e$ . On the top of that, the action (4.57) obviously commutes with the quasi-Hamiltonian  $G$ -action (3.45) on  $W$ , it descends therefore to the  $G$ -quotient.

In what follows, we find more convenient to describe the space  $(M \otimes W^-)_e$  differently. For that, consider the quasi-Hamiltonian  $G \times G$  action on  $M \times W^-$ , that is the  $G$ -action on  $M$  and the action (3.45) on  $W^-$ . Now the diagonal subaction, the quotient with respect to which

we consider, permits a global slice given by the requirement  $l(0) = e$ . We shall denote by  $\tilde{l}$  the elements of  $W$  for which this requirement is respected, i.e.  $\tilde{l}(0) = 0$  and we parametrize  $(M \otimes W^-)_e$  as

$$(M \otimes W^-)_e = \{(x, \tilde{l}) \in M \times W, \quad \tilde{l}(0) = e, \quad \mu(x)\tilde{l}(2\pi) = e\}. \quad (4.59)$$

We infer from (3.34) that, in the parametrization (4.59), the symplectic form  $\omega$  on  $(M \otimes W^-)_e$  obtained from the quasi-Hamiltonian reduction reads

$$\omega = \Omega - \frac{1}{2} \int_0^{2\pi} (\tilde{l}^{-1} \delta \tilde{l}, \partial_\sigma(\tilde{l}^{-1} \delta \tilde{l})) d\sigma. \quad (4.60)$$

In order to verify that (4.58) gives the moment map of the  $LG$ -action (4.57), we have to characterize this action in the parametrization (4.59). We distinguish two cases: the action of the based loops from  $\Omega G$  and the action of the constant loops from  $G$ . We find

$$(x, \tilde{l}) \rightarrow (x, h\tilde{l}), \quad h \in \Omega G; \quad (4.61)$$

$$(x, \tilde{l}) \rightarrow (h \triangleright x, h\tilde{l}h^{-1}), \quad h \in G. \quad (4.62)$$

Here  $h \triangleright x$  stands for the  $G$ -action on  $M$ .

Denote by  $v_\xi$  the vector field corresponding to the infinitesimal action (4.61) of an element  $\xi \in \text{Lie}(\Omega G)$ . Then we find easily

$$\begin{aligned} \iota(v_\xi)\omega &= -\frac{1}{2} \int_0^{2\pi} (\tilde{l}^{-1} \xi \tilde{l}, \partial_\sigma(\tilde{l}^{-1} \delta \tilde{l})) d\sigma + \frac{1}{2} \int_0^{2\pi} (\tilde{l}^{-1} \delta \tilde{l}, \partial_\sigma(\tilde{l}^{-1} \xi \tilde{l})) d\sigma = \\ &= -\int_0^{2\pi} (\xi, \tilde{l} \partial_\sigma(\tilde{l}^{-1} \delta \tilde{l}) \tilde{l}^{-1}) d\sigma = -\delta \int_0^{2\pi} (\xi, \partial_\sigma \tilde{l} \tilde{l}^{-1}) d\sigma \end{aligned} \quad (4.63)$$

We note that all boundary terms in the computation (4.63) vanished because of  $\xi(0) = 0$ .

Denote by  $v_\xi$  the vector field corresponding to the infinitesimal action (4.62) of an element  $\xi \in \text{Lie}(G)$ . Then we find easily

$$\begin{aligned} \iota(v_\xi)\omega &= \iota(v_\xi)\Omega - \frac{1}{2} \int_0^{2\pi} (\tilde{l}^{-1} \xi \tilde{l} - \xi, \partial_\sigma(\tilde{l}^{-1} \delta \tilde{l})) + \frac{1}{2} \int_0^{2\pi} (\tilde{l}^{-1} \delta \tilde{l}, \partial_\sigma(\tilde{l}^{-1} \xi \tilde{l} - \xi)) = \\ &= \frac{1}{2} (\xi, \delta \mu \mu^{-1} + \mu^{-1} \delta \mu) + \frac{1}{2} (\xi, \tilde{l}^{-1} \delta \tilde{l})|_0^{2\pi} + \frac{1}{2} (\tilde{l}^{-1} \delta \tilde{l}, \tilde{l}^{-1} \xi \tilde{l})|_0^{2\pi} - \int_0^{2\pi} (\xi, \tilde{l} \partial_\sigma(\tilde{l}^{-1} \delta \tilde{l}) \tilde{l}^{-1}) = \\ &= \frac{1}{2} (\xi, \delta \mu \mu^{-1} + \mu^{-1} \delta \mu) + \frac{1}{2} (\xi, \delta \tilde{l}(2\pi) \tilde{l}(2\pi)^{-1} + \tilde{l}(2\pi)^{-1} \delta \tilde{l}(2\pi)) - \delta \int_0^{2\pi} (\xi, \partial_\sigma \tilde{l} \tilde{l}^{-1}) d\sigma. \end{aligned} \quad (4.64)$$

Following (4.59), the first two terms on the r.h.s. of (4.64) vanish because of the constraint  $\mu(x)\tilde{l}(2\pi) = e$ . Combining this fact with (4.63), we conclude that  $\Phi = -\partial_\sigma \tilde{l} \tilde{l}^{-1} d\sigma \in L\mathcal{G}^*$  is indeed the moment map of the  $LG$ -action (4.57) on the symplectic manifold  $(M \otimes W^-)_e$ .

It remains to prove that the Hamiltonian  $LG$ -space  $((M \otimes W^-)_e, \omega, \Phi)$  is equivalent to the quasi-Hamiltonian  $G$ -space  $(M, \Omega, \mu)$  in the sense of the equivalence discussed at the end of Section 2.

For that we shall determine the equivalent system  $(M', \Omega', \mu')$  to  $((M \otimes W^-)_e, \omega, \Phi)$  and then show that  $(M', \Omega', \mu')$  and  $(M, \Omega, \mu)$  are isomorphic as the quasi-Hamiltonian spaces. Let us first prove that the quotient  $M' \equiv (M \otimes W^-)_e / \Omega G$  indeed coincides with  $M$  as manifold. For that, we use the following well-known parametrization of the space  $W$  of quasi-periodic maps used in [15, 16]:

$$l(\sigma) \equiv h(\sigma)e^{i\tau\sigma}g_0^{-1}, \quad (4.65)$$

where  $h(\sigma) \in LG$ ,  $g_0 \in G$  and  $\tau$  is the element of the Weyl alcove. It follows from (4.65), in particular, that the elements  $\tilde{l}(\sigma)$  can be parametrized as

$$l(\sigma) \equiv k(\sigma)g_0e^{i\tau\sigma}g_0^{-1}, \quad (4.66)$$

where  $k(\sigma)$  is in the based loop group  $\Omega G$ . The quotient  $M' = (M \otimes W^-)_e / \Omega G$  can be therefore identified with the set of elements  $(x, g_0e^{i\tau\sigma}g_0^{-1}) \in M \times W$  such that  $g_0e^{i2\pi\tau}g_0^{-1} = \mu(x)^{-1}$  and this set, in turn, can be directly identified with  $M$ .

Following Eq. (2.25), the quasi-Hamiltonian form  $\Omega'$  on  $M$  which corresponds to the Hamiltonian  $LG$ -space  $((M \otimes W^-)_e, \omega, \Phi)$  is given by the formula

$$\Omega' = \omega - \frac{1}{2} \int_0^{2\pi} (\delta g_R g_R^{-1}, \partial_\sigma (\delta g_R g_R^{-1})) = \Omega - \frac{1}{2} \int_0^{2\pi} (\tilde{l}^{-1} \delta \tilde{l}, \partial_\sigma (\tilde{l}^{-1} \delta \tilde{l})) d\sigma - \frac{1}{2} \int_0^{2\pi} (\delta g_R g_R^{-1}, \partial_\sigma (\delta g_R g_R^{-1})), \quad (4.67)$$

where  $g_R \in W$ ,  $g_R(0) = e$  is defined by

$$g_R^{-1} \partial_\sigma g_R d\sigma = \Phi = -\partial_\sigma \tilde{l}^{-1} d\sigma. \quad (4.68)$$

We thus see that  $g_R = \tilde{l}^{-1}$  and

$$\Omega' = \Omega. \quad (4.69)$$

The fact that the induced  $G$ -action on  $M'$  is clearly given by the restriction of the action (4.62) on the first term, i.e. by the  $G$ -action on  $M$ . Finally, the moment map  $\mu'$  is given by  $g_R(2\pi) = \tilde{l}(2\pi)^{-1} = \mu(x)$  which finishes the proof.  $\square$

Repeating step by step the proof of Theorem 3, we obtain also

**Corollary:** *The manifold  $(M \otimes W)_e$  is the Hamiltonian  $LG$ -space with the  $LG$ -action given by (4.57) and the anti-equivariant moment map given by*

$$\Phi^- = \partial_\sigma l l^{-1} d\sigma. \quad (4.70)$$

**Remark 2:** Theorem 3 can be easily generalized to the case where the manifold  $M$  is the quasi-Hamiltonian  $G \times G$ -space and we transform to the loop group language only one copy of  $G$ . We find then that the resulting fusion/reduction  $(M \otimes W^-)_e$  does not give a symplectic manifold but it yields the quasi-Hamiltonian  $G$ -space with respect to the copy of  $G$  which we "did not touch". For example, if  $M$  is the quasi-Hamiltonian double  $D(G)$  then a one-line computation

shows that  $(D(G) \otimes W^-)_e = W^-$ , or, said in other words,  $D(G)$  acts as identity with respect to the partial fusion.

To a given Hamiltonian  $LG$ -space  $(N, \omega)$  with the equivariant moment map  $\Phi$ , one can canonically associate its "loop-reversal" Hamiltonian  $LG$ -space  $(N, \omega)$  with the anti-equivariant moment map  $\Phi^-$  (the term anti-equivariant was defined by means of Eq. (3.31)). To do that, we define the loop-reversal map  $I : S^1 \rightarrow S^1$  as

$$I(\sigma) = 2\pi - \sigma, \quad \sigma \in [0, 2\pi[. \quad (4.71)$$

Let now act the loop group  $LG$  on  $N$  as

$$h \triangleright_I y := (I^*h) \triangleright y, \quad h \in LG, \quad y \in N, \quad (4.72)$$

where  $\triangleright$  stands for the original  $LG$ -action with the equivariant moment map  $\Phi$  and  $\triangleright_I$  stands for the new action defined in terms of the original one and of the pull-back  $I^*$  of the map  $I$ . It is easy to see that the new action  $\triangleright_I$  has the anti-equivariant moment map  $\Phi^- = -I^*\Phi$ . Indeed, we have

$$\iota(I^*\xi)\omega = \delta \int (\Phi, I^*\xi) = \delta \int (-I^*\Phi, \xi). \quad (4.73)$$

We have now the following proposition

**Theorem 4:** *The anti-equivariant Hamiltonian  $LG$ -space  $(M \otimes W)_e$  is isomorphic to the loop reversal of the equivariant space  $(M \otimes W^-)_e$ .*

*Proof.* The quasi-Hamiltonian space  $(W, \Omega(l), \mu(l))$  corresponding to the chiral WZNW model has an interesting property that its quasi-Hamiltonian inverse  $(W, -\Omega(l), \mu(l)^{-1})$  is isomorphic to the original space  $(W, \Omega(l), \mu(l))$ . This isomorphism  $I^* : W \rightarrow W$  is simply the extension of the pull-back of the loop reversal map and, with a slight abuse of notation, we have denoted it again by  $I^*$ :

$$(I^*l)(\sigma) := l(2\pi - \sigma). \quad (4.74)$$

To see that this is isomorphism, we just check that the  $G$ -action (3.45) commutes with  $I^*$  and it holds  $I^*\Omega = -\Omega$  and  $\mu(I(l)) = \mu(l)^{-1}$ .

The existence of the isomorphism  $I^*$  obviously implies that the reduction/fusion  $(M \otimes W)_e$  is isomorphic to  $(M \otimes W^-)_e$  as symplectic manifold, but not necessarily as  $LG$ -space. Indeed, the symplectic form  $\Omega + \Omega(l)$  on  $(M \otimes W)_e$  can be rewritten in the coordinates  $\hat{l} \equiv I^*l$  on  $W$  as  $\Omega - \Omega(\hat{l})$  and in the same coordinates the action (4.57) of the loop group becomes the action

$$(x, \hat{l}(\sigma)) \rightarrow (x, h(2\pi - \sigma)\hat{l}(\sigma)), \quad x \in M, \quad \hat{l}(\sigma) \in W, \quad h(\sigma) \in LG. \quad (4.75)$$

Thus we see that the change of coordinates  $l \rightarrow \hat{l}$  gives the loop reversed action  $\triangleright_I$  of  $LG$  on  $(M \otimes W^-)_e$ .  $\square$



## 5 Flat connections and the proof of formula (1.2)

In this section, we wish to deal with the side  $AC$  of the triangle on Fig. 1 and to prove the formula (1.2).

Let  $G$  be a compact simple connected and simply connected Lie group,  $\mathcal{G}$  its Lie algebra and  $\Sigma$  be a Riemann surface with boundaries  $\partial\Sigma$ . Denote by  $G(\Sigma)$  the group of smooth maps from  $\Sigma$  to  $G$ . The group  $G(\Sigma)$  naturally acts on the space of connections on the trivial bundle  $\Sigma \times G$  which we denote as  $\Omega^1(\Sigma, \mathcal{G})$ :

$$A^g = gAg^{-1} - g^*(\bar{\theta}), \quad A \in \Omega^1(\Sigma, \mathcal{G}), \quad g \in G(\Sigma). \quad (5.76)$$

The space  $\Omega^1(\Sigma, \mathcal{G})$  is symplectic; its symplectic form  $\omega$  is defined by

$$\omega = \int_{\Sigma} (\delta A \frown \delta A), \quad (5.77)$$

where  $\delta$  stands for the de Rham differential on the infinite-dimensional manifold  $\Omega^1(\Sigma, \mathcal{G})$  and  $(\cdot, \cdot)$  is the Killing-Cartan form on  $\mathcal{G}$ . It turns out [1] that the action (5.76) is symplectic with the moment map  $\Psi$  given by

$$\langle \Psi(A), \xi \rangle \equiv \int_{\Sigma} (dA + A^2, \xi) + \int_{\partial\Sigma} (A, \xi), \quad (5.78)$$

where  $\xi \in \text{Lie}(G(\Sigma)) \equiv \Omega^0(\Sigma, \mathcal{G})$  and  $d$  is the de Rham differential on the surface  $\Sigma$ . Note that we take the orientation on  $\partial\Sigma$  opposite to the induced orientation on  $\Sigma$  as in [2, 24].

The object of central interest for us is obtained by a partial symplectic reduction of the full connection space  $\Omega^1(\Sigma, \mathcal{G})$  by the subgroup  $G_{\partial}(\Sigma)$  of  $G(\Sigma)$  consisting of map sending the boundaries to the unit element  $e$  of  $G$ . The moment map of this action is given just by the first term in (5.78) with  $\xi \in \text{Lie}(G_{\partial}(\Sigma)) \subset \Omega^0(\Sigma, \mathcal{G})$ . Setting the moment map to the zero value (flat connections!) and factoring the corresponding 0-level set by the partial gauge group  $G_{\partial}(\Sigma)$  we obtain the principal actor of our game:

$$M(\Sigma) \equiv \Omega^1(\Sigma, \mathcal{G}) // G_{\partial}(\Sigma) \quad (5.79)$$

It was proved in [6], that in the case of non-empty boundary the moduli space  $M(\Sigma)$  is a smooth symplectic manifold. Needless to say, for  $\Sigma$  being the annulus,  $M(\Sigma)$  is the phase space of the standard WZNW model.

Denote by  $G(\partial\Sigma)$  the factor group  $G(\Sigma)/G_{\partial}(\Sigma)$ . Obviously,  $G(\partial\Sigma)$  can be identified with the group of smooth maps from  $\partial\Sigma$  to  $G$  and it acts on the moduli space  $M(\Sigma)$  in the Hamiltonian way. The equivariant moment map of this residual action is given by the second term on the r.h.s. of (5.78) where  $A$  is the restriction of the representant of the class  $[A] \in M(\Sigma)$  to  $\Omega^1(\partial\Sigma, \mathcal{G})$ .

If the boundary  $\partial\Sigma$  has  $r+1$  connected components then to each one corresponds the equivariant Hamiltonian action of a copy of the loop group  $LG$  on  $M(\Sigma)$ . The explicit description of the

manifold  $M(\Sigma)$  with  $k$ -handles was given in [23] as

$$M(\Sigma) = \left\{ (a, c, \zeta) \in G^{2k} \times G^r \times (LG^*)^{r+1} \left| \prod_{i=1}^{2k} [a_{2i-1}, a_{2i}] = \prod_{i=1} \text{Hol}(\zeta_i) \right. \right\}. \quad (5.80)$$

where  $c_0 = e$  and  $[\cdot, \cdot]$  stands for the group commutator. In this description, the action of  $h = (h_0, \dots, h_r) \in (LG)^{r+1}$  is given by

$$h \triangleright a_i = \text{Ad}_{h_0(0)} a_i, \quad h \triangleright c_j = h_0(0) c_j h_j(0)^{-1}, \quad h \triangleright \zeta_j = \text{Ad}_{h_j} \zeta_j - dh_j h_j^{-1}. \quad (5.81)$$

The equivariant moment map is the projection to the  $(LG^*)^{r+1}$ -factor.

The expression for the symplectic form on  $M(\Sigma)$  in the parametrization (5.80) is complicated and it was not given in [23]. We shall find now an alternative description of the space  $M(\Sigma)$  in which the structure of the symplectic form becomes transparent and it is given in terms of the quasi-Hamiltonian fusion.

**Theorem 5:** *Let  $\Sigma$  be a Riemann surface with  $k$  handles and  $r + 1$  boundaries. Then*

$$M(\Sigma) = \left( \underbrace{W^- \otimes W^- \otimes \dots \otimes W^-}_{r+1 \text{ times}} \otimes \underbrace{\mathbf{D}(G) \otimes \mathbf{D}(G) \otimes \dots \otimes \mathbf{D}(G)}_{k \text{ times}} \right)_e, \quad (5.82)$$

where  $\mathbf{D}(G)$  is the internally fused quasi-Hamiltonian double of  $G$ .

*Proof.* We shall start with the quasi-Hamiltonian  $\underbrace{G \times G \times \dots \times G}_{r+1 \text{ times}}$ -equivalent of the Hamiltonian  $\underbrace{LG \times LG \times \dots \times LG}_{r+1 \text{ times}}$ -space  $M(\Sigma)$  as obtained in [2]:

$$\text{Hol}(M(\Sigma)) = \underbrace{D(G) \otimes D(G) \otimes \dots \otimes D(G)}_{r \text{ times}} \otimes \underbrace{\mathbf{D}(G) \otimes \mathbf{D}(G) \otimes \dots \otimes \mathbf{D}(G)}_{k \text{ times}}. \quad (5.83)$$

Here  $D(G)$  is the standard quasi-Hamiltonian double. Using Theorem 4 and Remark 2, the quasi-Hamiltonian representation (5.82) of  $M(\Sigma)$  follows directly.  $\square$

**Corollary:** *The moduli space of flat connections on the surface with  $n$  boundaries,  $m$  Wilson lines insertions and  $k$  handles reads:*

$$M_{nmk}(\Sigma) \equiv \left( \underbrace{W^- \otimes \dots \otimes W^-}_{n \text{ times}} \otimes C_1^- \otimes C_2^- \otimes \dots \otimes C_m^- \otimes \underbrace{\mathbf{D}(G) \otimes \dots \otimes \mathbf{D}(G)}_{k \text{ times}} \right)_e. \quad (5.84)$$

*Proof.* Suppose that  $r + 1 = n + m$ . First of all, we convert into  $W^-$  via Theorem 4 and Remark 2 only  $n - 1$  factors  $D(G)$  in the fusion product (5.83). Then we use the fact proved in [2], that the inclusion of the Wilson line with holonomy in a conjugacy class  $\mathcal{C}_i$  amounts to the reduction of  $D(G)$  at  $\mathcal{C}_i$  and it is equal to  $C_i^-$ . On the remaining  $m$  factors  $D(G)$  we thus perform the reduction at a tuple of the conjugacy classes  $\mathcal{C} = (\mathcal{C}_1, \dots, \mathcal{C}_m)$  to obtain the desired formula (5.84).  $\square$

## 6 Symplectic geometry of defects

So far we have been dealing with the side  $AC$  of the triangle on Figure 1 and we have proved the quasi-Hamiltonian formula (1.2) expressing the symplectic structure of the moduli space of flat connections on the surface with  $n$  boundaries,  $m$  Wilson lines insertions and  $k$  handles. We shall now turn to the side  $BC$  of the triangle and perform the explicit evaluation of the symplectic structures of several particular WZNW defects starting from the formula (1.2) and applying successively the formula (2.16). In all cases, we shall find the perfect agreement with the results obtained before from the detailed analysis of the WZNW dynamics [15, 16, 17, 27]. This fact confirms that the concept of the quasi-Hamiltonian fusion is the unique structural ingredient explaining the multitude of terms in the defect symplectic forms. Before doing the actual calculations, we should comment on two things: 1) We note that the fusion product introduced in Section 2 is commutative only on the *isomorphism classes* of quasi-Hamiltonian spaces. Although the isomorphism between  $M_1 \otimes M_2$  and  $M_2 \otimes M_1$  is described explicitly in [2], in practice it turns out to be more convenient to reshuffle the order of the fused manifolds to ensure a direct comparison of the symplectic forms issued from the formula (1.2) with the symplectic forms of the corresponding WZNW defects as obtained previously in [15, 16, 17, 27]. 2) In physical literature the loop group actions on symplectic manifolds related to the WZNW dynamics are often considered with the anti-equivariant moment map [15, 16, 17, 27]. We already know from Section 4, that this is a mere convention since the loop reversal map changes the anti-equivariant moment map into the equivariant. In order to match the same convention, sometimes we perform the transition from equivariant to anti-equivariant at the level of the formula (1.2). As it was proved in Theorem 4, this amounts simply to replacing  $W^-$  by  $W$ .

### 1. Bulk WZNW model with no defects.

This is an important warm up case to start with. It is well-known that the phase space of the standard WZNW model is the moduli space of flat connections on the surface with two boundaries [7]. Conventionally, the action of the loop group corresponding to one of the boundaries is taken to be anti-equivariant and the other one equivariant. Following our main formula (1.2), this phase space should therefore coincide with the symplectic manifold  $(W \otimes W^-)_e$ . Let us see that this is indeed true. Following the fusion formula (2.16) we obtain

$$\Omega_{W \otimes W^-} = \Omega(l) - \Omega(r) - \frac{1}{2}(\mu_l^{-1} \delta \mu_l, \mu_r^{-1} \delta \mu_r). \quad (6.85)$$

Note the opposite sign of  $\Omega(r)$  and the inverse of the moment map  $\mu_r$  related to the fact that the right sector correspond to the inverse quasi-Hamiltonian space  $W^-$  in the sense explained in Section 2. Now the quasi-Hamiltonian reduction of  $W \otimes W^-$  at the unit level of the fused moment map  $\mu_l \mu_r^{-1} = e$  makes the last term on the r.h.s. of (6.85) disappear and we are left with

$$\Omega_{W \otimes W^-} \Big|_{\mu_l = \mu_r} = \Omega(l) - \Omega(r), \quad (6.86)$$

where the configurations  $l, r \in W$  have the same monodromies. But this coincides with the expression of the standard WZNW symplectic form  $\omega_{WZ}$  in terms of the left and right movers as given by (3.56) (cf. also [15]).

2. *Bulk WZNW model with one defect.*

The defect in the bulk WZNW model means that the WZNW configuration field  $g(\sigma)$  is allowed to jump at some point  $\sigma_0$  of the loop; we choose  $\sigma_0 = 0$ . As shows the analysis of [13], the preservation of the full  $LG \times LG$  symmetry requires that the jump of the configuration field must lie in some conjugacy class  $\mathcal{C} \subset G$ . Following our general philosophy of relying on the formula (1.2), the WZNW symplectic form in the presence of the defect should be given by the unit-level reduction of the fusion product  $W \otimes W^- \otimes \mathcal{C}$ . Following the formulae (1.2), (2.7), (2.16) and (3.34), we find that the quasi-Hamiltonian form on  $W \otimes W^- \otimes \mathcal{C}$  restricted to the unit value of the product moment map  $\mu_l \mu_r^{-1} \mu$  reads:

$$\Omega_{W \otimes W^- \otimes \mathcal{C}} \Big|_{\mu = \mu_r \mu_l^{-1}} = \Omega(l) - \Omega(r) + \alpha_{\mu_r \mu_l^{-1}}^{\mathcal{C}} + \frac{1}{2}(\mu_l^{-1} \delta \mu_l, \delta \mu_r^{-1} \mu_r). \quad (6.87)$$

Here we recall that  $\alpha_{\mu_r \mu_l^{-1}}^{\mathcal{C}}$  is the quasi-Hamiltonian form (2.7),(2.9). Our expression (6.87) coincides with Eq. (106) of [27], where the symplectic structure of the WZNW model with one defect was described ( $\mu_{l,r} \equiv \gamma_{L,R}^{-1}$ ,  $\alpha^{\mathcal{C}} = -\frac{1}{2}\omega$  in the notation of [27]).

3. *Bulk WZNW model with two defects.*

Again from the formulae (1.2), (2.7), (2.16) and (3.34), we find the quasi-Hamiltonian form on the fusion product  $W \otimes W^- \otimes \mathcal{C}_1 \otimes \mathcal{C}_2$  restricted to the unit value of the product moment map  $\mu_l \mu_r^{-1} \mu_1 \mu_2$ :

$$\Omega_{W \otimes W^- \otimes \mathcal{C}_1 \otimes \mathcal{C}_2} \Big|_{\mu_r \mu_l^{-1} = \mu_1 \mu_2} = \Omega(l) - \Omega(r) + \alpha_{\mu_1}^{\mathcal{C}_1} + \alpha_{\mu_2}^{\mathcal{C}_2} + \frac{1}{2}(\mu_l^{-1} \delta \mu_l, \delta \mu_r^{-1} \mu_r) + \frac{1}{2}(\mu_1^{-1} \delta \mu_1, \delta \mu_2 \mu_2^{-1}). \quad (6.88)$$

This expression is equivalent to Eq. (121) of [27], where the symplectic structure of the WZNW model with two defects was derived. To see this, some more work is needed. First of all, we have to identify the notations here and in [27]:  $\mu_{l,r} \equiv \gamma_{L,R}^{-1}$ ,  $\mu_1 \equiv \tilde{d}_\beta$  and  $\mu_2 \equiv d_\alpha$ . Then we have to reexpress  $\Omega(l)$  in terms of the parametrization (4.65) of  $l \in W$ :

$$l(\sigma) \equiv h(\sigma) e^{i\tau\sigma} g_0^{-1}, \quad (6.89)$$

where  $h(\sigma)$  is strictly periodic (therefore it is an element of  $LG$ ),  $\tau$  is in the Weyl alcove of  $\mathcal{G}$  and  $g_0$  is in  $G$ . With this parametrization, we obtain

$$\begin{aligned} \Omega(l) = & \frac{1}{2} \int_0^{2\pi} \left[ (h^{-1} \delta h, \partial_\sigma (h^{-1} \delta h)) - 2i \delta(\tau, h^{-1} \delta h) \right] d\sigma + \\ & + 2\pi i (\delta\tau, g_0^{-1} \delta g_0) + \frac{1}{2} (g_0^{-1} \delta g_0, e^{2\pi i \tau} g_0^{-1} \delta g_0 e^{-2\pi i \tau}). \end{aligned} \quad (6.90)$$

Inserting (6.90) into (6.88), we obtain the formula which coincides with Eq. (121) of [27].

4. *Boundary WZNW model with open string ending on the conjugacy classes  $\mathcal{C}_1$  and  $\mathcal{C}_2$ .*

It was found in [17], that the symplectic structure of the boundary WZNW model with open string ending on two conjugacy classes  $\mathcal{C}_1$  and  $\mathcal{C}_2$  is the same as that of the moduli

space of flat connections on the disc with two Wilson lines insertions with the holonomies in  $\mathcal{C}_1$  and  $\mathcal{C}_2$ . Following our quasi-Hamiltonian dictionary, we shall evaluate the unit-level reduction of the fusion product  $\mathcal{C}_2 \otimes \mathcal{C}_1^- \otimes W$ . Thus, assembling the quasi-Hamiltonian form on the conjugacy classes (2.7), the expression of the quasi-Hamiltonian form issued from the fusion (2.16) and the chiral WZNW form (3.34), we find the following formula for the quasi-Hamiltonian form on the fusion product  $\mathcal{C}_2 \otimes \mathcal{C}_1^- \otimes W$  restricted to the unit value of the product moment map  $\mu_2 \mu_1^{-1} \mu_l$ :

$$\Omega_{\mathcal{C}_2 \otimes \mathcal{C}_1^- \otimes W} \Big|_{\mu_l^{-1} \mu_1 = \mu_2} = \Omega(l) + \frac{1}{2}(\mu_1 \delta \mu_1^{-1}, \delta \mu_l \mu_l^{-1}) - \alpha_{\mu_1}^{\mathcal{C}_1} + \alpha_{\mu_2}^{\mathcal{C}_2}. \quad (6.91)$$

This expression coincides with Eq. (53) of [17], where the symplectic structure of the boundary WZNW model was first determined (to see it, one must identify  $\mu_l^{-1} \equiv \gamma$ ,  $\mu_1 \equiv h_0$  and  $\mu_2 \equiv \gamma h_0$ ).

### 5. Boundary WZNW model with one defect.

From the formulae (1.2), (2.7), (2.16) and (3.34), we find the quasi-Hamiltonian form on the fusion product  $\mathcal{C}_1^- \otimes \mathcal{C}_2 \otimes W \otimes \mathcal{C}_3$  restricted to the unit value of the product moment map  $\mu_1^{-1} \mu_2 \mu_l \mu_3$ :

$$\Omega_{\mathcal{C}_1^- \otimes \mathcal{C}_2 \otimes W \otimes \mathcal{C}_3} \Big|_{\mu_1 = \mu_2 \mu_l \mu_3} = \Omega(l) + \frac{1}{2}(\mu_1 \delta \mu_1^{-1}, \delta \mu_2 \mu_2^{-1}) + \frac{1}{2}(\mu_l^{-1} \delta \mu_l, \delta \mu_3 \mu_3^{-1}) - \alpha_{\mu_1}^{\mathcal{C}_1} + \alpha_{\mu_2}^{\mathcal{C}_2} + \alpha_{\mu_3}^{\mathcal{C}_3}. \quad (6.92)$$

In order that this expression coincide with Eq. (139) of [27], where the symplectic structure of the boundary WZNW model with one defect was computed, we must insert (6.90) into (6.92) and identify  $\mu_1 \equiv h_0$ ,  $\mu_2 \equiv d_\alpha$ ,  $\mu_3 \equiv h_\pi$  and  $\mu_l \equiv \gamma^{-1}$ .

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