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Locality and Continuity in Constitutive Theory

REUVEN SEGEV

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1. Introduction

In the theory of constitutive relations in continuum mechanics, various axioms of locality are postulated. These axioms restrict the amount of information concerning the motion of the body that is needed in order to determine the stress at any material point.

Following NOLL [1], TRUESDELL & NOLL [2] set up the following hierarchy of locality axioms (see also [3]). The most general locality assumption, the principle of determinism, states that the stress in a body is determined by the history of the configuration of that body. A stronger locality assumption is provided by the principle of local action which states that the motion outside an arbitrary neighborhood of a material point X may be disregarded in determining the stress at X . Next, materials of grade n are those for which the stress at any point depends only on the history of the values of the first n derivatives of the deformation at X . Clearly, materials of grade n satisfy the principle of local action and materials of grade one, which are called simple materials, satisfy the strongest locality assumption.

In this paper I make some observations regarding relations between the three types of locality mentioned above. The presentation differs from the traditional approach in that I assume that the force acting on a body, rather than the stress, is determined by the motion. With this approach, Cauchy's postulate, which is a traditional consistency assumption needed for the proof of the existence of the stress, implies the principle of local action. Finally, I prove that if all bodies have a finite and bounded memory, and if the stress on a body depends continuously on the motion, the following results hold. (a) The body satisfies the locality condition for materials of grade n . (b) The mapping ψ_X that assigns the stress at X to the first n derivatives of the deformation at X is continuous. (c) The mappings ψ_X varies continuously with X .

2. Preliminaries

In this section I present the framework in which the aforementioned constitutive theory is formulated, and I review the basic results of continuum mechanics that will be used in subsequent sections.

A *body* is modeled mathematically as the closure of an open connected subset of \mathbf{R}^3 having a smooth boundary, and the *physical space* is modeled by \mathbf{R}^3 . A *configuration of class n* of the body B in space is a mapping $\kappa: B \rightarrow \mathbf{R}^3$ having the following properties: it is one to one; it is n times continuously differentiable; if $D\kappa$ is the derivative of κ , then $\det(D\kappa) \neq 0$ at all points in the body. Here, $D\kappa_{iI} = \kappa_{i,I}$, where κ_i is the i^{th} component of κ , and a comma denotes partial differentiation with respect to the body coordinate X_I . The *configuration space* Q_B is the set of all configurations of class n , where n is some fixed positive integer. Let $[t_0, t]$ be an interval in \mathbf{R} . A *motion* of B in the time interval $[t_0, t]$ is a continuous mapping $H: [t_0, t] \rightarrow Q_B$. The collection of all motions on $[t_0, t]$ will be denoted by M_B .

A triplet $\mu = (\mu^1, \mu^2, \mu^3)$ of nonnegative integers will be referred to as a multi-index and the notation

$$|\mu| = \sum_p \mu^p, \quad \mu! = \mu^1! \mu^2! \mu^3!, \quad X^\mu = (X_1)^{\mu^1} (X_2)^{\mu^2} (X_3)^{\mu^3},$$

$$D^\mu \kappa_i = \frac{\partial^{|\mu|} \kappa_i}{\partial X_1^{\mu^1} \partial X_2^{\mu^2} \partial X_3^{\mu^3}}.$$

will be used subsequently.

Let $g: B \rightarrow \mathbf{R}^3$ be an n times differentiable mapping. For an integer m such that $0 \leq m \leq n$, the m^{th} *jet*, $j^m g(X)$, of g at the point $X \in B$ is the collection of all partial derivatives $\{D^\mu g_i; |\mu| \leq m\}$. Clearly, for any n times differentiable mapping g and any point X in B , $j^m g(X)$ belongs to the vector space $\bigoplus_{0 \leq p \leq n} L^p(\mathbf{R}^3, \mathbf{R}^3)_S$, where $L^p(\mathbf{R}^3, \mathbf{R}^3)_S$ denotes the vector space of p -multilinear symmetric mappings. In addition, for any element $X \in B$ and any element $\xi \in \bigoplus_{0 \leq p \leq n} L^p(\mathbf{R}^3, \mathbf{R}^3)_S$, it is possible to construct a mapping $g: B \rightarrow \mathbf{R}^3$ such that $j^m g(X) = \xi$. Thus, the m^{th} *jet space*, i.e., the collection of m -jets of functions, will be identified with $\bigoplus_{0 \leq p \leq n} L^p(\mathbf{R}^3, \mathbf{R}^3)_S$ and it will be denoted by J^m . For an element $\xi \in J^m$, $\xi^{i\mu}$ will denote the component of ξ corresponding to $D^\mu \kappa_i$; this can be regarded as the component of an array in $L^{|\mu|}(\mathbf{R}^3, \mathbf{R}^3)_S$. The m^{th} jet space will be endowed with the norm $\|\cdot\|_J$, such that for every $\xi \in J^m$, $\|\xi\|_J = \max_{|\mu| \leq m, i} \{|\xi^{i\mu}|\}$. Given a motion $H: [t_0, t] \rightarrow Q_B$ and a material point X , $j^m H_X: [t_0, t] \rightarrow J^m$, is the continuous mapping that assigns the jet $j^m(H(\tau))(X)$ to any time τ .

In terms of the notion of a jet the assumption of n^{th} grade locality can be formulated simply by stating that the stress at a point X is determined by the history of the n^{th} jet of the configuration at X , i.e., by the mapping $j^m H_X: [t_0, t] \rightarrow J^m$. Henceforth, I will use the term n^{th} *jet locality* as a synonym of n^{th} grade locality.

Let $C^n(B, \mathbf{R}^3)$ denote the Banach space of n -times continuously differentiable mappings $u: B \rightarrow \mathbf{R}^3$ equipped with the norm $\|u\|_{C^n} = \sup_{X \in B} \{\max_{|\mu| \leq n} \{|D^\mu u_i(X)|\}\}$. It can be shown (see [4]) that Q_B is an open subset of $C^n(B, \mathbf{R}^3)$, and it will be endowed with the induced topology given by the metric

$$d_Q: Q_B \times Q_B \rightarrow \mathbf{R} \quad \text{with} \quad d_Q(\kappa, \kappa') = \sup_{X \in B} \{\max_{|\mu| \leq n} \{|D^\mu \kappa_i(X) - D^\mu \kappa'_i(X)|\}\}.$$

Similarly, M_B will be endowed with the topology of uniform convergence, *i.e.*, a metric d_M on M_B is used such that for two motions H and H'

$$\begin{aligned} d_M(H, H') &= \sup_{\tau \in [t_0, t]} \{d_Q(H(\tau), H'(\tau))\} \\ &= \sup_{\tau \in [t_0, t]} \{ \sup_{X \in B} \{ \max_{|\mu| \leq n} \{ |D^\mu H(\tau)_i(X) - D^\mu H'(\tau)_i(X)| \} \} \}. \end{aligned}$$

Clearly, M_B can be identified with the set

$$\{G: [t_0, t] \times B \rightarrow \mathbf{R}^3; G|_{\tau} \in Q_B, \tau \in [t_0, t]\},$$

endowed with the metric d_M as defined above. For any material point X , $j^m H_X$ belongs to $C([t_0, t], J^m)$, the space of continuous mappings $[t_0, t] \rightarrow J^m$ on which the metric d_C defined by $d_C(\zeta, \zeta') = \sup_{\tau \in [t_0, t]} \|\zeta(\tau) - \zeta'(\tau)\|_J$ is used.

Let X be a point in B and let $u, v \in C^n(B, \mathbf{R}^3)$. The mappings u and v are *germ equivalent* at X if there is an open subset U of B containing X such that for every $Y \in U$, $u(Y) = v(Y)$. Clearly, germ equivalence of functions is an equivalence relation. The equivalence class of the mapping u will be denoted by $\text{germ}_X(u)$ and will be termed the *germ* of u at X . The quotient space, *i.e.*, the collection of all germs at X , will be denoted by G_X . In the language of germs the principle of local action described in the introduction states that the stress at a point X is determined by the history of the germ of the motion at X , *i.e.*, the stress at X is determined by the mapping $[t_0, t] \rightarrow G_X$ given by $\tau \rightarrow \text{germ}_X(H(\tau))$. Henceforth, I refer to this as *germ locality*.

Recalling that forces in continuum mechanics are given in terms of body force fields and surface force fields, by a *force* f I mean a pair of vector fields: the body force field \mathbf{b} defined in B and the surface force field \mathbf{t} defined on the boundary ∂B of B . The collection of forces acting on B will be denoted by Φ_B . Thus, Φ_B can be identified with the collection $\{(\mathbf{b}, \mathbf{t})\}$ of pairs of vector fields, where the first is defined on the body and the second is defined on its boundary.

A body P is a *subbody* of the body B if P is a subset of B . A *force system* on B is a mapping that assigns to each subbody P of B a force F_P , and $(\mathbf{b}_P, \mathbf{t}_P)$ will denote the corresponding body force field and surface force field on P . A force system satisfies Cauchy's postulate if the following conditions hold: the total force of each subbody vanishes; $\mathbf{t}_P(X)$ depends on the subbody P only through the unit normal \mathbf{n} to the boundary of P at X , *i.e.*, $\mathbf{t}_P(X) = \mathbf{t}(X, \mathbf{n})$; $\mathbf{t}(X, \mathbf{n})$ is a continuous function of its arguments. It is noted that a given force on B cannot be restricted uniquely to subbodies of B .

The basic result concerning stresses and forces in continuum mechanics states: if a force system satisfies Cauchy's postulate, there exists a unique continuous (two point) tensor field σ_{Ti} defined on B such that $\mathbf{t}_P(X)_i = \sigma_{Ti}(X) \mathbf{n}_T$. Denote by Σ_B the vector space of continuous stress fields (two point tensor fields) on B , and for any stress field σ_{Ti} define the norm

$$\|\sigma\| = \sup_{X \in B} \{ \max_{T,i} \{ |\sigma_{Ti}(X)| \} \}.$$

This makes Σ_B into a Banach space.

3. The Basic Postulates

In this section I postulate the basic principles of constitutive theory and present their immediate consequences. Since continuity plays no role in this section, no use is made of the topological structures defined in the previous section. In addition, there is no need to specify the time interval on which a motion is defined. Thus, in this section I extend the definition of a motion to include those defined on $(-\infty, t]$. M_B denotes the collection of such motions, and no topology is introduced on M_B .

The Principle of Body Self-Determinism: The force acting on a body at the time t is determined by the motion $H: (-\infty, t] \rightarrow Q_B$.

From this principle it follows that for any body B there is a mapping

$$A_B: M_B \rightarrow \Phi_B,$$

called the *loading* of B , that assigns to any motion H of B the force $f = A_B(H)$ acting on B at the time t . Let P be a subbody of B . For any motion H of B , let $H|P: (-\infty, t] \rightarrow Q_P$ be the motion defined for all $X \in P$ by $H|P(\tau)(X) = H(\tau)(X)$. Since the restriction $H|P$ of the motion H to any subbody P determines, by the principle of body self determinism, a force $A_P(H|P)$ on P , it follows that a motion H of the body B determines a force system on B at the time t . The second basic postulate is concerned with this force system.

The Principle of Consistency: The force system $\{A_P(H|P); P \subset B\}$ generated by the motion H of the body B satisfies Cauchy's Postulate.

As a consequence of this principle it follows immediately that any motion of the body determines a stress field on the body at time t . The corresponding mapping $\Psi_B: M_B \rightarrow \Sigma_B$ is called the *constitutive relation* for B .

Proposition 3.1. *Let $\Psi_B: M_B \rightarrow \Sigma_B$ and $\Psi_P: M_P \rightarrow \Sigma_P$ be constitutive relations for B and P , where $P \subset B$. Then, for any motion $H: (-\infty, t] \rightarrow Q_B$ and any $X \in P$,*

$$\Psi_P(H|P)(X) = \Psi_B(H)(X).$$

Proof. For any motion $H: (-\infty, t] \rightarrow Q_B$, $\Psi_P(H|P)$ is a stress field on P . Assume that $\Psi_B(H)(X) \neq \Psi_P(H|P)(X)$ for some $X \in P$. Then, for some subbody P' of P whose boundary contains X ,

$$\Psi_B(H)(X)_{li} n_I \neq \Psi_P(H|P)(X)_{li} n_I.$$

Here n is the unit normal to the boundary of P' at X . However, since the restriction of H to P' is a configuration of P' that induces, by the principle of body self determinism, a unique force on P' , and since each of the two unequal terms above represents the traction acting on P' at X , one obtains a contradiction.

It follows from this proposition that the stress at a point is determined by the motion of any subbody containing that point. In other words, given a point X in B and two motions H and H' , the stresses at X due to H and H' will be equal if there exists a subbody P of B such that $H|P = H'|P$. Since any subbody P of B contains by definition some open subset of B , and any open subset of B contains some subbody of B , the last statement is equivalent to germ locality.

Corollary 3.2. *The principle of body self-determinism and the principle of consistency imply germ locality.*

Remark. The basic principle of body self-determinism as postulated here is different from the principle of determinism stated in TRUESDELL & NOLL [2] and TRUESDELL [3]: it is forces rather than stresses that are determined by the history motion of the body. Together with the principle of consistency, the principle of body self-determinism is equivalent to the principle of determinism. My reason for deviating from tradition is that thus I can avoid the unnecessary repetition of Cauchy's postulate. As suggested here, Cauchy's postulate is a constitutive hypothesis, and once it is stated in the context of constitutive theory there is no need to restate it. In addition, as can be seen in TRUESDELL [3], Cauchy's postulate can be proved on the basis of some mild assumptions.

4. The Consequences of Continuity

In this section I make the following additional assumption:

For any time t there is a time $t_0 < t$ such that the force acting on any body B at the time t is determined by the motion $H: [t_0, t] \rightarrow Q_B$.

Remark. This principle implies that bodies have a limited memory, and that the motion outside the interval $[t_0, t]$ can be disregarded. Henceforth, it is assumed that a motion is defined on the interval $[t_0, t]$, and the topology defined on M_B in Section I is used.

Proposition 4.1. *If a constitutive relation $\Psi_B: M_B \rightarrow \Sigma_B$ on B is continuous, then Ψ_B is jet local, i.e., for any point $X \in B$ there is a function $\psi_X: C([t_0, t], J^n) \rightarrow L(\mathbb{R}^3, \mathbb{R}^3)$, which will be referred to as the local constitutive relation, such that*

$$\Psi_B(H)(X) = \psi_X(j^n H_X).$$

Remark. Since ψ_X is defined only on the collection of jets of configurations, i.e., jets of embeddings, the notation here is somewhat inaccurate. However, since this set is open in J^n (see [4]), this abuse of notation will not affect the arguments.

Lemma 4.2. *Let $X \in B$ and $\varkappa \in Q_B$. Then, given any $\delta > 0$, there exists a $\varrho > 0$ such that $d_Q(\varkappa|P, j^n(\varkappa, X)|P) < \delta$ for every subbody P contained in a closed ball of radius ϱ centered at X . Here $j^n(\varkappa, X) \in C^n(B, \mathbb{R}^3)$ denotes the n^{th} order*

Taylor expansion of κ about X , i.e., $j^n(\kappa, X)_i = j^n(\kappa_i, X)$, where

$$j^n(v, X)(Y) = \sum_{|\mu| \leq n} \frac{(Y - X)^\mu}{\mu!} D^\mu v(X)$$

is the n^{th} order Taylor expansion of the mapping $v: B \rightarrow \mathbf{R}$.

Proof of Lemma 4.2. Taylor's theorem states that for a p -times continuously differentiable mapping $u: B \rightarrow \mathbf{R}^3$, if the line segment between X and Y is contained in B , one has

$$u_i(Y) = \sum_{|\mu| \leq p} \frac{(Y - X)^\mu}{\mu!} D^\mu u_i + \theta_i(Y - X) = j^p(u_i, X)(Y) + \theta_i(Y - X),$$

where θ_i is a continuous function such that

$$\lim_{Y \rightarrow X} \frac{|\theta_i(Y - X)|}{|Y - X|^p} = 0.$$

Thus, for the configuration κ , a given point $X \in B$ and a $\delta > 0$, one can find a $\varrho_0 > 0$ such that for any Y in B with $|Y - X| \leq \varrho_0$, $|z_i(Y) - j^n(\kappa_i, X)(Y)| < \delta$. Similarly, for any μ such that $|\mu| \leq n$, $D^\mu \kappa_i$ is an $n - |\mu|$ times continuously differentiable mapping, and it follows that for any $\delta > 0$ there exists an $r_\mu > 0$ such that for any Y in B with $|Y - X| \leq r_\mu$, $|D^\mu \kappa_i(Y) - j^n(D^\mu \kappa_i, X)(Y)| < \delta/2$.

Writing $\nu > \mu$ for the multi-indices ν and μ if $\nu_p \geq \mu_p$, $p = 1, 2, 3$, and $|\nu| > |\mu|$ one has

$$D^\mu (X - Y)^\nu = \frac{\nu!}{(\nu - \mu)!} (X - Y)^{\nu - \mu}$$

for $\nu \geq \mu$ and $D^\mu (X - Y)^\nu = 0$ for $\nu < \mu$. Hence,

$$D^\mu j^n(\kappa_i, X)(Y) = \sum_{\substack{|\nu| \leq n \\ \nu \geq \mu}} \frac{(Y - X)^{\nu - \mu}}{(\nu - \mu)!} D^\nu \kappa_i(X)$$

and letting $\eta = \nu - \mu \geq 0$ one has

$$D^\mu j^n(\kappa_i, X)(Y) = \sum_{|\mu| + |\eta| \leq n} \frac{(Y - X)^\eta}{\eta!} D^{\mu + \eta} \kappa_i(X).$$

On the other hand,

$$\begin{aligned} j^n(D^\mu \kappa_i, X)(Y) &= \sum_{|\eta| \leq n} \frac{(Y - X)^\eta}{\eta!} D^\eta (D^\mu \kappa_i)(X) \\ &= \sum_{|\eta| \leq n} \frac{(Y - X)^\eta}{\eta!} D^{\eta + \mu} \kappa_i(X) \\ &= D^\mu j^n(\kappa_i, X)(Y) + \sum_{n - |\mu| < |\eta| \leq n} \frac{(Y - X)^\eta}{\eta!} D^{\eta + \mu} \kappa_i(X), \end{aligned}$$

so that $D^\mu j^n(\kappa_i, X)$ is the Taylor expansion of $j^n(D^\mu \kappa_i, X)$ of order $(n - |\mu|)$. Thus, by Taylor's theorem,

$$\lim_{Y \rightarrow X} \frac{|j^n(D^\mu \kappa_i, X)(Y) - D^\mu j^n(\kappa_i, X)(Y)|}{|Y - X|^{n-|\mu|}} = 0.$$

It follows that, for any $\delta > 0$ and every multi-index μ , there is an $r'_\mu > 0$ such that $|j^n(D^\mu \kappa_i, X)(Y) - D^\mu j^n(\kappa_i, X)(Y)| < \delta/2$ for any Y with $|Y - X| \leq r'_\mu$. By the triangle inequality one obtains

$$\begin{aligned} \delta &> |j^n(D^\mu \kappa_i, X)(Y) - D^\mu j^n(\kappa_i, X)(Y)| + |D^\mu \kappa_i(Y) - j^n(D^\mu \kappa_i, X)(Y)| \\ &\geq |D^\mu \kappa_i(Y) - D^\mu j^n(\kappa_i, X)(Y)| \end{aligned}$$

for all Y such that $|Y - X| \leq \varrho_\mu = \min\{r'_\mu, r_\mu\}$. Set $\varrho = \min_{0 \leq |\mu| \leq n} \{\varrho_\mu\}$ to conclude that for a configuration κ , a point $X \in B$ and any $\delta > 0$, one can find a $\varrho > 0$ such that $d_Q(\kappa|P, j^n(\kappa, X)|P) < \delta$ for any subbody P contained in a ball of radius ϱ centered at X .

Lemma 4.3. *Given $\kappa \in Q_B$, $X \in B$, and $\delta > 0$, let $r = \sup\{\varrho\}$ for all ϱ such that $d_Q(\kappa|P, j^n(\kappa, X)|P) < \delta$ for subbodies P contained in a closed ball of radius ϱ centered at X . Similarly given another configuration $\kappa' \in Q_B$, let $r' = \sup\{\varrho'\}$ for all ϱ' such that $d_Q(\kappa'|P', j^n(\kappa', X)|P') < \delta$ for subbodies P' contained in a ball of radius ϱ' centered at X . Then.*

- (i) for each $e > 0$ there is a $d > 0$ such that $r' > r - e$ if $d_Q(\kappa, \kappa') < d$;
- (ii) r depends continuously on κ .

Proof of Lemma 4.3. (i) In order to prove this part of the lemma it is sufficient to show that for each $\delta > 0$ and each $0 < \varrho < r$ there is a $d > 0$ such that, for all subbodies P contained in a closed ball of radius ϱ centered at X , $d_Q(\kappa, \kappa') < d$ implies that $d_Q(\kappa'|P, j^n(\kappa', X)|P) < \delta$.

Given any $\varrho < r$, let

$$m = \max_{\substack{|Y-X| \leq \varrho \\ \nu \leq n}} \{|D^\nu \kappa_i(Y) - D^\nu j^n(\kappa_i, X)(Y)|\}.$$

Since the closed ball is compact, m exists, and moreover $m < \delta$. Let

$$m' = \max_{\substack{|Y-X| \leq \varrho \\ \nu \leq n}} \{|D^\nu \kappa'_i(Y) - D^\nu j^n(\kappa'_i, X)(Y)|\}.$$

One has

$$\begin{aligned} m' &= \max_{\substack{|Y-X| \leq \varrho \\ \nu \leq n}} \{|D^\nu \kappa'_i(Y) - D^\nu \kappa_i(Y) + D^\nu \kappa_i(Y) - D^\nu j^n(\kappa_i, X)(Y) \\ &\quad + D^\nu j^n(\kappa_i, X)(Y) - D^\nu j^n(\kappa'_i, X)(Y)|\} \\ &\leq \max_{\substack{|Y-X| \leq \varrho \\ \nu \leq n}} \{|D^\nu \kappa'_i(Y) - D^\nu \kappa_i(Y)|\} + \max_{\substack{|Y-X| \leq \varrho \\ \nu \leq n}} \{|D^\nu \kappa_i(Y) - D^\nu j^n(\kappa_i, X)(Y)|\} \\ &\quad + \max_{\substack{|Y-X| \leq \varrho \\ \nu \leq n}} \{|D^\nu j^n(\kappa_i, X)(Y) - D^\nu j^n(\kappa'_i, X)(Y)|\}. \end{aligned}$$

In addition,

$$\begin{aligned} & \max_{\substack{|Y-X| \leq \varrho \\ v \leq n}} \{ |D^v j^n(\kappa_i, X)(Y) - D^v j^n(\kappa'_i, X)(Y)| \} \\ &= \max_{\substack{|Y-X| \leq \varrho \\ v \leq n}} \left\{ \left| \sum_{|\mu|+|\eta| \leq n} \frac{(Y-X)^\eta}{\eta!} D^{\mu+\eta}(\kappa_i - \kappa'_i)(X) \right| \right\} \\ &\leq d_Q(\kappa, \kappa') \sum_{|\eta| \leq n} \frac{\varrho^\eta}{\eta!}. \end{aligned}$$

Thus,

$$m' \leq d_Q(\kappa, \kappa') + m + d_Q(\kappa, \kappa') \sum_{|\eta| \leq n} \frac{\varrho^\eta}{\eta!}.$$

Let d be any positive number such that

$$\delta - m > d + d \sum_{|\eta| \leq n} \frac{\varrho^\eta}{\eta!}.$$

Then, for any κ' such that $d_Q(\kappa, \kappa') < d$, $m' < \delta$ as required.

(ii) Reverse the roles of κ and κ' in part (i) of the lemma to conclude that for each $e > 0$ there is a $d > 0$ such that $d_Q(\kappa, \kappa') < d$ implies that $r > r' - e$. Hence, for each $e > 0$ there is a $d > 0$ such that $|r - r'| < e$ if $d_Q(\kappa, \kappa') < d$.

Lemma 4.4. *Given a motion $H: [t_0, t] \rightarrow Q_B$ and a point $X \in B$, let $j^n(H, X): [t_0, t] \rightarrow C^n(B, \mathbf{R}^3)$ be the motion such that $j^n(H, X)(\tau) = j^n(H(\tau), X)$. Then, for any $\delta > 0$, there is a $\varrho > 0$ such that $d_M(H|P, j^n(H, X)|P) < \delta$ for each P contained in a closed ball of radius ϱ centered at X .*

Proof of Lemma 4.4. Given H and $\delta > 0$, let $r(\tau)$ be the r defined in Lemma 4.3 corresponding to the configuration $H(\tau)$. Since H is a continuous function, it follows from Lemma 4.3 that $r(\tau)$ depends continuously on τ . As $[t_0, t]$ is compact, $r(\tau)$ has a minimum and any $\varrho < \min_{\tau} \{r(\tau)\}$ will satisfy the required condition.

Proof of Proposition 4.1. The continuity of Ψ_B implies that for any $\varepsilon > 0$ there exists a $\delta > 0$ such that $\|\Psi_B(H') - \Psi_B(H)\| < \varepsilon$ if $d(H', H) < \delta$. By the definition of the norm on Σ_B it follows that if $d_M(H', H) < \delta$, then

$$|\sigma'_{ii}(Y) - \sigma_{ii}(Y)| < \varepsilon \quad \text{for all } Y \in B, \quad I, i = 1, 2, 3,$$

where $\sigma'_{ii} = \Psi_B(H')_{ii}$ and $\sigma_{ii} = \Psi_B(H)_{ii}$. In addition, given any $X \in B$ and $\varepsilon > 0$, it follows from WHITNEY'S extension theorem [5] that there is a closed ball P_0 centered at X such that if $d_M(H'|P_0, H|P_0) < \delta$ implies $\|\Psi_{P_0}(H') - \Psi_{P_0}(H)\| < \varepsilon$, then $d_M(H'|P, H|P) < \delta$ implies $\|\Psi_P(H') - \Psi_P(H)\| < \varepsilon$ for all closed balls $P \subset P_0$ centered at X . In other words, it is sufficient that two configurations be close to one another in a neighborhood of X in order for the resulting stresses to be close to one another in that neighborhood. Let H be any motion of B and let (ε_s) , $s = 1, 2, \dots$, be a sequence of positive real numbers converging to zero. For $X \in B$, let $\varrho_s > 0$ be the real number satis-

fying $d_M(H|P, j^n(H, X)|P) < \delta_s$ for any subbody P contained in a ball of radius ϱ_s , centered at X , where $\delta_s > 0$ is a positive real number such that $d_M(H'|P, H|P) < \delta_s$ implies $\|\Psi_P(H'|P) - \Psi_P(H|P)\| < \varepsilon_s$. Finally, for each $s = 1, 2, \dots$, let P_s be a subbody containing the point X such that P_s is contained in a ball of radius ϱ_s , centered at X . It follows that

$$|\Psi_{P_s}(j^n(H|P_s, X))_{II}(Y) - \Psi_{P_s}(H|P_s)_{II}(Y)| < \varepsilon_s$$

for all $Y \in P_s$, $i, I = 1, 2, 3$ and all s . By Proposition 3.1

$$\Psi_{P_s}(H|P_s)_{II}(Y) = \Psi_B(H)_{II}(Y), \quad Y \in P_s,$$

$$\Psi_{P_s}(j^n(H|P_s, X))_{II}(Y) = \Psi_{P_s}(j^n(H, X)|P_s)_{II}(Y) = \Psi_B(j^n(H, X))_{II}(Y), \quad Y \in P_s,$$

and one has $|\Psi_B(j^n(H, X))_{II}(Y) - \Psi_B(H)_{II}(Y)| < \varepsilon_s$, for all $i, I, Y \in P_s$ and all s . However, $X \in \bigcap_s P_s$ so that $|\Psi_B(j^n(H, X))_{II}(X) - \Psi_B(H)_{II}(X)| < \varepsilon_s$ for all s , and one concludes that

$$\Psi_B(j^n(H, X))_{II}(X) = \Psi_B(H)_{II}(X).$$

Since $j^n(H, X)$ depends only on $j^n H_X$, there is a mapping $\psi_X: C([t_0, t], J^n) \rightarrow L(\mathbb{R}^3, \mathbb{R}^3)$ such that $\Psi_B(H)(X) = \psi_X(j^n H_X)$. Thus the proposition follows.

Let m be an integer such that $1 \leq m \leq n$, and denote by Q'_B the set of configurations $\lambda: B \rightarrow \mathbb{R}^3$ of class m . Clearly, $Q_B \subset Q'_B$, and by the definitions of the topologies of these sets, the inclusion mapping $i_Q: Q_B \rightarrow Q'_B$ is continuous. Similarly, let M'_B denote the corresponding set of motions $\{H: [t_0, t] \rightarrow Q'_B\}$. Again, $M_B \subset M'_B$ and the inclusion mapping $i_M: M_B \rightarrow M'_B$ is continuous. Hence, if $\Psi'_B: M'_B \rightarrow \Sigma_B$ is a continuous constitutive relation, $\Psi'_B \circ i_M: M_B \rightarrow \Sigma_B$, which is the restriction of Ψ'_B to M_B , is a continuous constitutive relation. Thus, if a material is of grade m with $1 \leq m < n$, it is also of grade n as expected.

5. Some Properties of Local Constitutive Relations

Proposition 5.1. *The mapping $\psi_X: C([t_0, t], J^n) \rightarrow L(\mathbb{R}^3, \mathbb{R}^3)$ is continuous, i.e., the stress at a point X depends continuously on the history of the jet of the configurations at X .*

Proof. By Propositions 3.1 and 4.1 one has

$$\psi_X(j^n H_X) = \Psi_B(j^n(H, X))(X) = \Psi_P(j^n(H, X)|P)(X)$$

for all motions H and subbodies P containing X where Ψ_P is continuous. Thus, it is sufficient to prove that for all $\varepsilon > 0$ there is a $\delta > 0$ such that for any two motions H and H' of B , $d_C(j^n H_X, j^n H'_X) < \delta$ implies that

$$|\Psi_P(j^n(H, X)|P)(X) - \Psi_P(j^n(H', X)|P)(X)| < \varepsilon$$

for some subbody P containing X . By the continuity of Ψ_P , it is sufficient to prove that for all $\lambda > 0$, there is a $\delta > 0$ and a subbody P containing X , such

that $d_C(j^n H_X, j^n H'_X) < \delta$ implies that $d_M(j^n(H, X)|P, j^n(H', X)|P) < \lambda$. One has

$$\begin{aligned} & d_M(j^n(H, X)|P, j^n(H', X)|P) \\ &= \sup_{\tau \in [t_0, t]} \left\{ \sup_{Y \in B} \left\{ \max_{|\mu| \leq n} \left\{ |D^{\mu} j^n H(\tau)_i(Y) - D^{\mu} j^n H'(\tau)_i(Y)| \right\} \right\} \right\} \\ &= \sup_{Y \in P} \left\{ \sup_{\tau \in [t_0, t]} \left\{ \max_{|\mu| \leq n} \left\{ \sum_{\substack{|\nu| \leq n \\ \mu \leq \nu}} \frac{(Y - X)^{\nu - \mu}}{(\nu - \mu)!} (D^{\nu} H(\tau)_i(X) - D^{\nu} H'(\tau)_i(X)) \right\} \right\} \right\}. \end{aligned}$$

Thus,

$$d_M(j^n(H, X)|P, j^n(H', X)|P) \leq d_C(j^n H_X, j^n H'_X) \left\{ \sum_{|\nu| \leq n} \varrho^{\nu} \right\},$$

where ϱ is the diameter of P . This completes the proof.

Proposition 5.2. The mapping $\psi: B \times C([t_0, t], J^n) \rightarrow L(\mathbf{R}^3, \mathbf{R}^3)$ given by

$$\psi(X, j^n H_X) = \psi_X(j^n H_X)$$

is continuous in its first argument, i.e., the “jet local constitutive relation” varies continuously with X .

Proof. One has to show that for every $\zeta \in C([t_0, t], J^n)$ and every $\varepsilon > 0$, there is a $\delta > 0$ such that $\|\psi_Y(\zeta) - \psi_X(\zeta)\| < \varepsilon$ if $\|Y - X\| < \delta$. Let H be a motion of B such that $j^n H_Y = \zeta$, $Y \in B$. Given any $\varepsilon > 0$, let $\delta = \min\{\delta_1, \delta_2\}$, where $\delta_1 > 0$ and $\delta_2 > 0$ satisfy

$$\|\Psi_B(H)(Y) - \Psi_B(H)(X)\| < \varepsilon/2 \quad \text{for } \|Y - X\| < \delta_1,$$

$$\|\psi_X(j^n H_X) - \psi_X(j^n H_Y)\| < \varepsilon/2 \quad \text{for } \|Y - X\| < \delta_2.$$

The existence of δ_1 is guaranteed by the continuity of the stress field corresponding to H . The existence of δ_2 is guaranteed by the continuity (for each τ) of $j^n H(\tau)(X)$ with respect to X (since $H(\tau)$ is n -times continuously differentiable) and by the continuity of ψ_X proved in Proposition 5.1. To show that δ satisfies the required condition, I write

$$\begin{aligned} \|\Psi_B(H)(Y) - \Psi_B(H)(X)\| &= \|\psi_Y(j^n H_Y) - \psi_X(j^n H_X)\| \\ &= \|\psi_Y(j^n H_Y) - \psi_X(j^n H_Y) - (\psi_X(j^n H_X) - \psi_X(j^n H_Y))\| \\ &\geq \|\psi_Y(j^n H_Y) - \psi_X(j^n H_Y)\| - \|\psi_X(j^n H_X) - \psi_X(j^n H_Y)\|, \end{aligned}$$

where in the first line Proposition 4.1 is used. Hence,

$$\|\Psi_B(H)(Y) - \Psi_B(H)(X)\| + \|\psi_X(j^n H_X) - \psi_X(j^n H_Y)\| \geq \|\psi_Y(\zeta) - \psi_X(\zeta)\|$$

and by hypothesis, for $\|Y - X\| < \delta$, $\varepsilon > \|\psi_Y(\zeta) - \psi_X(\zeta)\|$.

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