A Note on Limit Analysis

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Abstract: We present a mathematical framework for the theory of limit analysis of rigid, perfectly plastic bodies where the equality of the static multiplier and kinematic multiplier for incompressible fields is formulated and proved in a compact form. Assuming that the failure criterion is a norm on the space of deviatoric stress fields, we use standard properties of linear operators on Banach spaces.

Key Words: limit analysis, plasticity, reduced kinematic multiplier, static multiplier, Banach's closed range theorem.

1. INTRODUCTION

The equality of the static multiplier and the kinematic multiplier is a basic result of the theory of limit analysis of rigid, perfectly plastic bodies. This result may be described as follows.

Let f be a given load on an isotropic homogeneous rigid perfectly plastic body Ω for which the stress $\sigma(x)$ at $x \in \Omega$ is considered admissible if $\sigma(x) \in C_Y$, where C_Y is a convex subset of the space of stress matrices. Let Φ_f be the collection of stress fields that are in equilibrium with f and set

$$\lambda_f = \sup\left\{\lambda > 0 \mid \exists \sigma \in \Phi_{\lambda f}, \quad \text{such that} \quad \sigma(x) \in C_Y,\ x \in \Omega\right\}. \tag{1}$$

Thus, λ_f, the *static multiplier*, is the largest multiplier of the load such that $\lambda_f f$ can be equilibrated by an admissible stress field. For this reason λ_f is sometimes referred to as the *factor of safety*.

For a strain matrix e, let the dissipation function $d(e)$ be defined by

$$d(e) = \sup_{\tau \in C_Y} \tau_{ij} e_{ij} \tag{2}$$

and set

$$\mu_f = \inf \frac{\int_\Omega d(\varepsilon(w)(x))\,\mathrm{d}V}{f(w)}, \tag{3}$$

where the infimum is taken over all virtual displacement fields w, $\varepsilon(w)$ denotes the virtual strain field corresponding to w, and $f(w)$ denotes the virtual work performed by the load f for the virtual displacement field w. The number μ_f is referred to as the *kinematic multiplier*. Thus, it is the objective of the theory of limit analysis to prove that $\lambda_f = \mu_f$, under appropriate conditions.

A particular complication arises because the prevalent yield conditions are given in terms of sets C_Y which are not bounded in the space of matrices. The yield conditions bound only the deviatoric component $\tau_{Dij} = \tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij}$ of the stress. In such cases, the equality of the multipliers still holds under appropriate technical conditions if in the definition of the kinematic multiplier, the infimum is taken over incompressible (isochoric) virtual displacements only. Following [1], we refer to this quantity as the *reduced kinematic multiplier*. The prevalent proof (see [2, pp. 91–98] and [1, Chapter VI]) that the reduced kinematic multiplier is equal to the static multiplier uses the pressure restoration theorem asserting, within the framework of L^2-stress fields, that a force that performs no work on incompressible vector fields is the gradient of a hydrostatic pressure field[1].

In this paper, we limit ourselves to the case where the yield function is given as a norm on the space of deviatoric stress matrices. While this property holds for the prevalent yield criteria, it implies that the yield function should give the same value to the matrices τ and $-\tau$. This assumption enables us to study the problem using operator theory on Banach spaces rather than convex analysis. Consequently, one obtains a compact formulation and proof of this basic result of limit analysis and related issues described below. In particular, the methods we use enable strength analysis of structures under *a priori* unknown loadings (see [5]).

As an introduction to the mathematical framework, in Section 2 we consider the simpler situation where the failure criterion, or equivalently cost function, is given in terms of a norm on the space of stress fields. In Section 3 we present the main result and construction for the case where the failure criterion is given in terms of a norm on the space of deviatoric stress fields. In fact, the decomposition into spherical and deviatoric fields is just an example of a Whitney sum structure needed for this formulation.

Forces considered in Sections 2 and 3 are modeled as elements of the dual space of the vector spaces of virtual displacements. In the vast majority of applications, only subspaces of all forces are considered. In addition, the norm used for forces was restricted by a technical condition that we postulate and cannot be chosen according to some physical guidelines. A framework for stress analysis of the body for an arbitrary loading in such a space of forces, where the earlier restriction on the norm is removed, is presented in Section 4.

The mathematical framework of Sections 2–4 is abstract in the sense that no particular choices are made for the classes of virtual displacements, forces, stresses, etc. To provide an example of a situation where the assumptions made in earlier chapters indeed hold, in Section 5 we give a natural such choice for the case of continuous bodies (rather than discrete structures) in which stress fields are essentially bounded tensor fields over the body and

virtual displacements are assumed to be integrable vector fields for which the corresponding strain fields are integrable.

2. FAILURE CRITERIA GIVEN BY NORMS

We consider first the case where the failure criterion is given by a norm on the space of stress fields.

2.1. Abstract Mathematical Framework

Our basic object is a Banach space $\mathcal{W}$ whose elements are interpreted physically as virtual velocity fields or virtual displacement fields (depending on one's favorite interpretation) for a given configuration of the body in space. The dual space $\mathcal{W}^*$ represents the collection of generalized forces acting on the body at that a given configuration, so that for each linear functional $F \in \mathcal{W}^*$ and $w \in \mathcal{W}$, $F(w)$ is interpreted as the virtual power or virtual work performed by the generalized force F on the field represented by w.

The space of virtual strain fields is denoted by $\mathcal{S}$ and is also assumed to be a Banach space. The elements of its dual space $\mathcal{S}^*$ are the stress fields and for $\sigma \in \mathcal{S}^*$, $\chi \in \mathcal{S}$, $\sigma(\chi)$ is the virtual work performed by the stress object. The norms on the various Banach spaces introduced so far are induced by a failure criterion for stress fields. Thus, we first assume that the failure criterion, or alternatively, a yield function or a cost function, is given in terms of a given norm $\| \cdot \|$ on the space of stresses such that the body does not fail if and only if $\|\sigma\| \leq s_Y$, where s_Y may interpreted as the yield stress. Thus, the norm on $\mathcal{S}$ is related to the failure criterion by

$$\|\chi\| = \sup_{\sigma \in \mathcal{S}^*} \left\{ \frac{\sigma(\chi)}{\|\sigma\|} \mid \|\sigma\| \neq 0 \right\}. \tag{4}$$

It is assumed that there is a linear mapping

$$\varepsilon : \mathcal{W} \to \mathcal{S} \tag{5}$$

that gives the virtual strain field $\chi = \varepsilon(w)$ corresponding to the virtual displacement field w. The mapping ε is assumed to have the following properties.

(i) It is assumed that ε is injective. This assumption can be motivated as follows. Assuming that the body is supported in such a way that rigid body motion is prevented, the only virtual displacement field that corresponds to the zero strain field is the zero field. Thus, Kernel $\varepsilon = 0$ and the mapping ε is one to one. It is noted that we do not assume that ε is surjective because it is natural to include in $\mathcal{S}$ strain fields that are incompatible, i.e. strain fields that cannot be obtained from virtual velocity fields.

(ii) It is assumed that ε is norm preserving, i.e.

$$\|\varepsilon(w)\| = \|w\|. \tag{6}$$

In fact, this requirement will determine the norm on $\mathcal{W}$. As ε is assumed to be injective, given the norm on $\mathcal{S}$ as induced by the failure criterion on $\mathcal{S}^*$, Equation (6) defines a unique norm on $\mathcal{W}$.

Let $F \in \mathcal{W}^*$ be given. The principle of virtual work in this abstract setting is

$$F(w) = \sigma(\varepsilon(w)) = \varepsilon^*(\sigma)(w), \quad \text{for all} \quad w \in \mathcal{W}, \tag{7}$$

where $\varepsilon^* : \mathcal{S}^* \to \mathcal{W}^*$ is the mapping dual to ε. Thus, we refer to the equation

$$F = \varepsilon^*(\sigma) \tag{8}$$

as the equilibrium equation. It is noted that ε^* is not injective as ε was not surjective. This is the abstract manifestation of statical indeterminacy. Thus, for a given force $F \in \mathcal{W}^*$ there is an affine subset $\Phi_F \subset \mathcal{S}^*$ containing all of the solutions σ to the equilibrium equation.

In Section 5 we present natural candidates for these spaces and norms for the case of continuous bodies.

2.2. The Optimal Stress-limit Analysis Problem

For a given force $F \in \mathcal{W}^*$, the optimal stress problem seeks the infimum of the norms of all stresses that are in equilibrium with F. This infimum, s_F^{opt}, is referred to as the *optimal stress*. Thus, for $\Phi_F = \{\sigma \in \mathcal{S}^* \mid F = \varepsilon^*(\sigma)\}$,

$$s_F^{\text{opt}} = \inf_{\sigma \in \Phi_F} \|\sigma\|. \tag{9}$$

If there exists a stress field $\sigma^{\text{opt}} \in \mathcal{S}^*$ equilibrating F such that $\|\sigma^{\text{opt}}\| = s_F^{\text{opt}}$ (so that s^{opt} is attained for σ^{opt} and not merely obtained as a limit), then we refer to σ^{opt} as the *optimal stress field*. While it is natural to use a supremum norm for stress fields so that the optimal stress field is an equilibrating stress field whose maximum is the least, it is possible to interpret the norm of a stress field as a cost function as in traditional problems of structural optimization (e.g. [6, 7]).

Given the abstract yield stress s_Y, we say that the force F is a *collapse force* if

$$s_F^{\text{opt}} = s_Y \tag{10}$$

and we refer to such a state of loading as a *limit state*.

Given a force F the *static multiplier*, *limit design factor*, or the *factor of safety* is defined as

$$\lambda_F = \sup_{\lambda \in \mathbb{R}^+} \left\{ \lambda \in \mathbb{R}^+ \mid \exists \sigma \in \Phi_{\lambda F}, \quad \|\sigma\| \leq s_Y \right\}. \tag{11}$$

One can verify that

$$\lambda_F = \frac{s_Y}{s_F^{\text{opt}}}. \tag{12}$$

It is a result of plasticity theory, that for rigid perfectly plastic materials, bodies can support the loads λF for all $\lambda < \lambda_F$ and will collapse for all $\lambda > \lambda_F$. In other words, optimal stress fields occur in rigid perfectly plastic bodies in the case where $s_F^{\text{opt}} = s_Y$.

2.3. Optimal Stresses for Norm-failure Criteria

For $F \in \mathcal{W}^*$, the fact that ε is injective and norm preserving implies that

$$\widehat{\sigma} = F \circ \varepsilon^{-1} : \text{Image}\, \varepsilon \subset \mathcal{S} \to \mathbb{R} \tag{13}$$

is a continuous linear functional on Image ε. As

$$F(w) = \widehat{\sigma} \circ \varepsilon(w), \tag{14}$$

we conclude that every stress σ that equilibrates F is an extension of $\widehat{\sigma}$ to $\mathcal{S}$. It follows from the Hahn–Banach theorem that there is an extension $\sigma^{\text{hb}} \in \mathcal{S}^*$ such that

$$\|\sigma^{\text{hb}}\| = \sup_{\chi \in \mathcal{S}} \frac{\sigma^{\text{hb}}(\chi)}{\|\chi\|} = \sup_{\chi_0 \in \text{Image } A} \frac{\widehat{\sigma}(\chi_0)}{\|\chi_0\|} = \sup_{w \in \mathcal{W}} \frac{F(w)}{\|w\|} = \|F\|. \tag{15}$$

Since the norm of any extension $\sigma \in \Phi_F$ of $\widehat{\sigma}$ is greater than or equal to $\|\widehat{\sigma}\| = \|F\|$, one has

$$s_F^{\text{opt}} = \|F\| = \sup_{w \in \mathcal{W}} \frac{F(w)}{\|w\|} \tag{16}$$

and the optimum is actually attained for the Hahn–Banach extension σ^{hb} of $\widehat{\sigma}$. We observe that the norm for forces was induced by the failure criterion and cannot be defined independently following some physical guidelines. This restriction is overcome using the mathematical settings introduced in Section 4.

It is noted that the normalized optimal stress,

$$K_F = \frac{s_F^{\text{opt}}}{\|F\|}, \tag{17}$$

(to which we referred in [8–10] as the optimal stress concentration factor) is independent of the force and is always equal to one.

3. FAILURE CRITERIA GIVEN BY SEMINORMS

3.1. Mathematical Framework

Traditional yield conditions for plasticity do not satisfy the foregoing conditions as they are semi-norms on the space of stresses rather than norms. For such yield conditions we have the following structure. The space $\mathcal{S}$ has the structure of a Whitney sum

$$\mathcal{S} = D \oplus P \tag{18}$$

with projections

$$\Pi_D : \mathcal{S} \to D, \quad \Pi_P : \mathcal{S} \to P, \tag{19}$$

onto the complete subspaces D and P, interpreted as the deviatoric and spherical components of the strain, respectively. The projections are the left inverses of the formal inclusions

$$I_D : D \to \mathcal{S}, \quad I_P : P \to \mathcal{S}. \tag{20}$$

In particular, we note for further reference that

$$\Pi_P \circ I_D = 0. \tag{21}$$

The Whitney sum structure on $\mathcal{S}$ induces a Whitney sum structure

$$\mathcal{S}^* = D^* \oplus P^* \tag{22}$$

on $\mathcal{S}^*$ such that every stress field σ may be written uniquely as

$$\sigma = \sigma_D + \sigma_P, \tag{23}$$

where we naturally refer to σ_D as the deviatoric component of the stress and to σ_P as the pressure component of the stress. Thus, one has

$$\sigma(\chi) = \sigma_D(\chi_D) + \sigma_P(\chi_P) \tag{24}$$

where $\chi_D = \Pi_D(\chi) \in D$ and $\chi_P = \Pi_P(\chi) \in P$ are the deviatoric and pressure components of the virtual strain. The components of a stress are given using the projections provided by the dual mappings

$$I_D^* : \mathcal{S}^* \to D^*, \quad I_P^* : \mathcal{S}^* \to P^* \tag{25}$$

and are included in the space of stresses by

$$\Pi_D^* : D^* \to \mathcal{S}^*, \quad \Pi_P^* : P^* \to \mathcal{S}^*. \tag{26}$$

The failure criterion is now assumed to be of the form

$$\|\sigma\|_D = \|I_D^*(\sigma)\|, \tag{27}$$

where $\|\cdot\|$ is a given norm on $\mathcal{S}^*$. It is clear that $\|\cdot\|_D$ is a seminorm because its value will vanish for all stresses σ with $\sigma_D = I_D^*(\sigma) = 0$, i.e. for pure hydrostatic pressures.

3.2. Optimal Stresses for Seminorm-failure Criteria

In this section we develop the analog of Equation (16) to the case where the failure criterion is given by Equation (27), i.e. we look for

$$s_F^{\text{opt}} = \inf\left\{\|I_D^*(\sigma)\| \mid \sigma \in \Phi_F\right\}. \tag{28}$$

Assumption 3.1. The mapping $\Pi_P \circ \varepsilon : \mathcal{W} \to P$ has a closed image.

We consider the following two closed subspaces of $\mathcal{W}$:

$$\mathcal{W}_D = \text{Kernel}\,(\Pi_P \circ \varepsilon), \quad \mathcal{W}_P = \text{Kernel}\,(\Pi_D \circ \varepsilon) \tag{29}$$

so that $\mathcal{W}_D$ contains incompressible displacement fields and $\mathcal{W}_P$ contains purely expansive fields, that is, vector fields whose corresponding strain fields are deviatoric and spherical, respectively. One has natural inclusions

$$\imath_D : \mathcal{W}_D \to \mathcal{W}, \quad \imath_P : \mathcal{W}_P \to \mathcal{W}, \tag{30}$$

and the mappings

$$\varepsilon_D = \Pi_D \circ \varepsilon \circ \imath_D, \quad \varepsilon_P = \Pi_P \circ \varepsilon \circ \imath_P \tag{31}$$

as shown in the commutative diagrams below. Let $u \in \mathcal{W}_D$, with $\varepsilon_D(u) = 0$. Then, since by definition $\Pi_P(\varepsilon(\imath_D(u))) = 0$, and $\Pi_D(\varepsilon(\imath_D(u))) = 0$, as both $\imath_D$ and ε are injections, one concludes that $u = 0$ so that ε_D is also injective. In addition, since ε, $\imath_D$ and I_D are all injective and norm preserving, one has

$$\|u\| = \|\varepsilon \circ \imath_D(u)\| = \|I_D \circ \varepsilon_D(u)\| = \|I_D(\varepsilon_D(u)\| = \|\varepsilon_D(u)\|, \tag{32}$$

and it follows that ε_D is norm preserving. Obviously, the analogous observations hold for ε_P.

$$\begin{array}{ccc} \mathcal{W} & \xrightarrow{\ \varepsilon\ } & \mathcal{S} \\ \uparrow{\scriptstyle \imath_D} & & {\scriptstyle I_D}\uparrow\downarrow{\scriptstyle \Pi_D} \\ \mathcal{W}_D & \xrightarrow{\ \varepsilon_D\ } & \mathcal{S}_D \end{array} \qquad \begin{array}{ccc} \mathcal{W} & \xrightarrow{\ \varepsilon\ } & \mathcal{S} \\ \uparrow{\scriptstyle \imath_P} & & {\scriptstyle I_P}\uparrow\downarrow{\scriptstyle \Pi_P} \\ \mathcal{W}_P & \xrightarrow{\ \varepsilon_P\ } & \mathcal{S}_P \end{array}. \tag{33}$$

Similarly to the framework of Section 2.1, one has the dual spaces $\mathcal{W}_D^*$ and $\mathcal{W}_P^*$. Dual to the natural inclusions $\imath_D, \imath_P$, one has the natural projections

$$\imath_D^* : \mathcal{W}^* \to \mathcal{W}_D^*, \quad \imath_P^* : \mathcal{W}^* \to \mathcal{W}_P^* \tag{34}$$

and the resulting dual commutative diagrams:

$$\begin{array}{ccc} \mathcal{W}^* & \xleftarrow{\varepsilon^*} & \mathcal{S}^* \\ \downarrow{\scriptstyle \imath_D^*} & & I_D^* \uparrow\downarrow \Pi_D^* \\ \mathcal{W}_D^* & \xleftarrow{\varepsilon_D^*} & \mathcal{S}_D^* \end{array} \qquad \begin{array}{ccc} \mathcal{W}^* & \xleftarrow{\varepsilon^*} & \mathcal{S}^* \\ \downarrow{\scriptstyle \imath_P^*} & & I_P^* \uparrow\downarrow \Pi_P^* \\ \mathcal{W}_P^* & \xleftarrow{\varepsilon_P^*} & \mathcal{S}_P^* \end{array} . \tag{35}$$

Consider $F_D \in \mathcal{W}_D^*$ and the corresponding *reduced equilibrium equation*

$$F_D = \varepsilon_D^*(\sigma_D), \quad \sigma_D \in \mathcal{S}_D^*. \tag{36}$$

We use the notation

$$\Phi_{F_D}^D = \left\{ \sigma_D \in \mathcal{S}_D^* \mid F_D = \varepsilon_D^*(\sigma_D) \right\} \tag{37}$$

for the collection of solutions of the restricted equilibrium equation. Thus, one may consider the *reduced (restricted) stress optimization problem*

$$s_{F_D}^{\text{opt}} = \inf_{\sigma_D \in \Phi_{F_D}^D} \|\sigma_D\|. \tag{38}$$

Since the mapping ε_D has the same properties as the mapping ε, the results of Section 2.3 hold for the reduced problem and we conclude that

$$s_{F_D}^{\text{opt}} = \|F_D\| = \sup_{u \in \mathcal{W}_D} \frac{F_D(u)}{\|u\|}. \tag{39}$$

We now prove a useful lemma for the restricted equilibrium problem.

Lemma 3.2. *Given $F \in \mathcal{W}^*$, then*

$$I_D^* \{\Phi_F\} = \Phi_{\imath_D^*(F)}^D. \tag{40}$$

Proof. We first show that $I_D^* \{\Phi_F\} \subset \Phi_{\imath_D^*(F)}^D$. Let $\sigma \in \Phi_F$, so $F(w) = \sigma(\varepsilon(w))$, for any $w \in \mathcal{W}$. Thus, for any incompressible velocity field $v \in \mathcal{W}_D$ one has

$$\begin{aligned} \imath_D^*(F)(v) &= F(\imath_D(v)) = \sigma(\varepsilon(\imath_D(v))) \\ &= (I_D^*(\sigma) + I_P^*(\sigma)) [\Pi_D(\varepsilon(\imath_D(v))) + \Pi_P(\varepsilon(\imath_D(v)))] \\ &= I_D^*(\sigma) [\Pi_D(\varepsilon(\imath_D(v)))] = \imath_D^* \circ \varepsilon^* \circ \Pi_D^*(I_D^*(\sigma))(v) = \varepsilon_D^*(I_D^*(\sigma))(v). \end{aligned} \tag{41}$$

We conclude that if F is in equilibrium with σ, then

$$\imath_D^*(F) = \varepsilon_D^*(I_D^*(\sigma)), \tag{42}$$

so $\imath_D^*(F)$ is in equilibrium with $I_D^*(\sigma)$.

To show that $I_D^*\{\Phi_F\} \supset \Phi^D_{\imath_D^*(F)}$, we consider an arbitrary $\tau \in \Phi^D_{\imath_D^*(F)}$ so that

$$\imath_D^*(F) = \varepsilon_D^*(\tau). \tag{43}$$

One has to show that there is a stress $\sigma \in \Phi_F^*$, i.e. $F = \varepsilon^*(\sigma)$, such that $I_D^*(\sigma) = \tau$. Set $\sigma_0 = \Pi_D^*(\tau) \in \mathcal{S}^*$, $F_0 = \varepsilon^*(\sigma_0) = \varepsilon^* \circ \Pi_D^*(\tau) \in \mathcal{W}^*$, and $F_1 = F - F_0$. For any $v \in \mathcal{W}_D$ one has

$$\begin{aligned} F_1(\imath_D(v)) &= F(\imath_D(v)) - F_0(\imath_D(v)) = \imath_D^*(F)(v) - \varepsilon^* \circ \Pi_D^*(\tau)(\imath_D(v)) \\ &= \varepsilon_D^*(\tau)(v) - \imath_D^* \circ \varepsilon^* \circ \Pi_D^*(\tau)(v) = 0. \end{aligned} \tag{44}$$

It follows that $\imath_D^*(F_0) = \imath_D^*(F)$. In addition, let $\sigma_1 \in \mathcal{S}^*$ be a stress in equilibrium with F_1, i.e. $F_1 = \varepsilon^*(\sigma_1)$, then

$$F = F_1 + F_0 = \varepsilon^*(\sigma_1) + \varepsilon^*(\sigma_0), \tag{45}$$

and it follows that $F = \varepsilon^*(\sigma_1 + \sigma_0)$. It remains to show that $I_D^*(\sigma_1) = 0$ so that $I_D^*(\sigma_0 + \sigma_1) = \tau$. (Note that so far we have $\imath_D^* \circ \varepsilon^*(\sigma_1) = 0$, however, this is not enough because ε^* is not injective.) Thus, we want to prove that σ_1 may be chosen from $\mathcal{S}_P^*$.

It is noted that Equation (44) implies that

$$F_1 \in (\mathcal{W}_D)^\perp = (\text{Kernel}\,(\Pi_P \circ \varepsilon))^\perp. \tag{46}$$

Now, in Assumption 3.1 we postulated that $\Pi_P \circ \varepsilon$ has a closed image. We recall that Banach's closed range theorem asserts that for any Banach space X and a bounded linear mapping $A : X \to Y$ having a closed image in the Banach space Y, $\text{Image}\,(A^*) = (\text{Kernel}\,A)^\perp$ (e.g. [11, p. 70]). We conclude that

$$(\mathcal{W}_D)^\perp = (\text{Kernel}\,(\Pi_P \circ \varepsilon))^\perp = \text{Image}\,((\Pi_P \circ \varepsilon)^*)\,. \tag{47}$$

Thus, for some $\sigma_P \in \mathcal{S}_P^*$, F_1 is of the form

$$F_1 = (\Pi_P \circ \varepsilon)^*(\sigma_P) = \varepsilon^*(\Pi_P^*(\sigma_P)), \tag{48}$$

and σ_1 is of the form

$$\sigma_1 = \Pi_P^*(\sigma_P), \quad \sigma_P \in \mathcal{S}_P^*. \tag{49}$$

It follows that

$$I_D^*(\sigma_1) = I_D^*(\Pi_P^*(\sigma_P)) = (\Pi_P \circ I_D)^*(\sigma_P) = 0. \tag{50}$$

□

It is noted that unlike stresses and strain spaces, we do not use a Whitney sum structure on $\mathcal{W}$ (e.g. the Helmholtz–Weyl decomposition as in [3, Section 1.2] and [12, p. 98]).

The following theorem provides the main result for the stress optimization or limit analysis problem for the seminorm failure criterion $\|\sigma\|_D = \|I_D^*(\sigma)\|$ (Equation (27)). It asserts that for a seminorm failure criterion, we will obtain the same result if we either compute the optimal stress for the given force F using the definition in Equation (28) or we solve the restricted stress optimization problem for the restriction of the given force to 'incompressible' virtual displacements, i.e. we compute the optimal deviatoric stress for F_D as in Equation (39).

Theorem 3.3. *For $F \in \mathcal{W}^*$, let*

$$s_F^{\text{opt}} = \inf\{\|\sigma\|_D \mid \sigma \in \Phi_F\} \tag{51}$$

and for $G \in \mathcal{W}_D^*$, let

$$s_G^{\text{opt},D} = \inf\left\{\|\sigma_D\| \mid \sigma_D \in \Phi_G^D \subset \mathcal{S}_D^*\right\}. \tag{52}$$

Then,

$$s_F^{\text{opt}} = s_{\iota_D^*(F)}^{\text{opt},D}. \tag{53}$$

Proof. Using Lemma 3.2, one has

$$\begin{aligned} s_F^{\text{opt}} &= \inf\left\{\|I_D^*(\sigma)\| \mid \sigma \in \Phi_F\right\} = \inf\left\{\|\sigma_D\| \mid \sigma_D \in I_D^*\{\Phi_F\}\right\} \\ &= \inf\left\{\|\sigma_D\| \mid \sigma_D \in \Phi_{\iota_D^*(F)}^D\right\} = s_{\iota_D^*(F)}^{\text{opt},D}. \end{aligned} \tag{54}$$

□

Using the definition of the dual of the inclusion mapping one obtains immediately the following result.

Corollary 3.4. For the case of a seminorm-failure criterion, one has,

$$s_F^{\text{opt}} = \|\iota_D^*(F)\| = \sup_{u \in \mathcal{W}_D} \frac{F(u)}{\|u\|}. \tag{55}$$

4. APPLIED LOADINGS, STRESS SENSITIVITY AND LOAD CAPACITY

The following analysis allows one to consider forces in subspaces of $\mathcal{W}^*$ and norms other than the one that makes ε an isometry. Thus, it is assumed that there is a Banach space $\mathcal{M}$ and a continuous linear mapping

$$\beta : \mathcal{W} \to \mathcal{M}. \tag{56}$$

Considering forces in $\mathcal{W}^*$ of the form $F = \beta^*(f)$ for $f \in \mathcal{M}^*$, we set

$$s^{\text{opt}}_{\beta^*(f)} = \|\imath_D^*(\beta^*(f))\| = \sup_{v \in \mathcal{W}_D} \frac{\beta^*(f)(v)}{\|v\|}. \tag{57}$$

The notions of stress sensitivity (or generalized stress concentration factors in [8–10]) and load capacity ratios of perfectly plastic bodies (see [8–10]) are used when, rather than one particular load case, an entire class of loadings is considered. Consider a homogeneous isotropic rigid perfectly plastic body or a structure whose yield stress is s_Y. Then, the load capacity ratio is the largest number C that depends only on the geometry of the body such that the body will not collapse plastically under any loading whose norm $\|f\|$ satisfies

$$\|f\| \leq s_Y C \tag{58}$$

independently of the distribution of the load.

Using the terminology introduced above, and considering all loadings $f \in \mathcal{M}^*$, the stress sensitivity, $K^{\mathcal{M}}$, is defined as largest ratio of the optimal stress $s^{\text{opt}}_{\beta^*(f)}$ to norm of f, i.e.

$$K^{\mathcal{M}} = \sup_{f \in \mathcal{M}^*} \frac{s^{\text{opt}}_{\beta^*(f)}}{\|f\|}. \tag{59}$$

It can be shown (see [8–10] for details) that the load capacity is given by

$$C = \frac{1}{K^{\mathcal{M}}}. \tag{60}$$

The following results are concerned with stress sensitivity and load capacity ratios. Examples for the computations of C and corresponding worst case loadings for various structures are given in [5].

Theorem 4.1. *Let*

$$K = \sup_{F \in \mathcal{W}^*} \frac{s^{\text{opt}}_F}{\|F\|}. \tag{61}$$

Then, $K = 1$.

Proof. Using Corollary 3.4, one has

$$K = \sup_{F \in \mathcal{W}^*} \frac{\|\imath_D^*(F)\|}{\|F\|} = \|\imath_D^*\| = \|\imath_D\| = \sup_{v \in \mathcal{W}_D} \frac{\|\imath_D(v)\|}{\|v\|} = 1, \tag{62}$$

where we used the fact that the norm of a dual map is equal to the norm of the original map and the fact that $\imath_D$ is just the inclusion mapping so it preserves norms. □

It is noted that unlike the case of a norm-failure criterion, for a generic $F \in \mathcal{W}^*$,

$$\frac{s_F^{\text{opt}}}{\|F\|} = \frac{\|\imath_D^*(F)\|}{\|F\|} \neq 1. \tag{63}$$

In particular, for a stress $\sigma_1 = \Pi_P^*(p)$, $p \in \mathcal{S}_P^*$, $F_1 = \varepsilon^*(\sigma_1)$, one has $\imath_D^*(F_1) = 0$, $\|\sigma_1\|_D = 0$. It follows that $s_{F_1}^{\text{opt}} = \inf\left\{\|I_D^*(\sigma)\| \mid \sigma \in \Phi_{F_1}\right\} = 0$ while $\|F_1\| \neq 0$.

Theorem 4.2. *The stress sensitivity is given by*

$$K^{\mathcal{M}} = \|\beta \circ \imath_D\|. \tag{64}$$

Proof. Using Equation (57), the expression for $K^{\mathcal{M}}$ may be written as

$$K^{\mathcal{M}} = \sup_{f \in \mathcal{M}^*} \frac{\|\imath_D^*(\beta^*(f))\|}{\|f\|} = \|\imath_D^* \circ \beta^*\| = \|\beta \circ \imath_D\|. \tag{65}$$

□

5. PERFECTLY PLASTIC CONTINUOUS BODIES

Consider a continuous body that in a given configuration in the physical space $\mathbb{R}^3$ occupies the region Ω. It is assumed that Ω is an open subset of $\mathbb{R}^3$ and that its boundary $\partial\Omega$ is Lipschitz. Furthermore, there are two open subsets $\Gamma_0 \subset \partial\Omega$ and $\Gamma_t \subset \partial\Omega$ such that Γ_0 is the region where the body is supported and Γ_t is the region where the body is not supported so that a surface traction field t may be exerted on the body on Γ_t. Thus, it is natural to assume that Γ_0 and Γ_t are non-empty and disjoint, $\overline{\Gamma}_0 \cup \overline{\Gamma}_t = \partial\Omega$, and $\Lambda = \partial\Gamma_0 = \partial\Gamma_t$ is a curve on $\partial\Omega$.

Identifying a symmetric 3×3 stress matrix with an element of $\mathbb{R}^6$, a stress field is a symmetric tensor field $\sigma : \Omega \to \mathbb{R}^6$. The spherical projection and deviatoric projection are defined for a symmetric matrix τ by

$$\pi_P(\tau)_{ij} = \frac{1}{3}\tau_{kk}\delta_{ij}, \quad \pi_D(\tau) = \tau - \pi_P(\tau), \tag{66}$$

and the condition that the body does not collapse is of the form

$$\sup_{x\in\Omega} |\pi_D(\sigma(x))| \le s_Y, \tag{67}$$

where $|\cdot|$ is a norm on the space of matrices. For example, the von Mises yield criterion uses the 2-norm $|\tau|^2 = \tau_{ij}\tau_{ij}$. For technical reasons one usually allows higher stresses on subsets of the body of zero volume and sets

$$\|\sigma\|_D = \|\pi_D \circ \sigma\|^\infty := \operatorname{ess}\sup_{x\in\Omega} |\pi_D(\sigma(x))|\,. \tag{68}$$

Thus, it is natural to set

$$\mathcal{S} = L^1(\Omega, \mathbb{R}^6), \quad \Pi_P = \pi_P \circ \chi, \quad \Pi_D = \pi_D \circ \chi, \tag{69}$$

with D and P being, respectively, the spaces of deviatoric (incompressible) and spherical L^1-strain fields. The L^1-norm

$$\|\chi\|^1 := \int_\Omega |\chi(x)|\, \mathrm{d}V \tag{70}$$

should be used on $\mathcal{S}$, where for the values of a strain field, $|\cdot|$ is the dual norm on the space of matrices to the one used for stress matrices. It follows that $\mathcal{S}^* = L^\infty(\Omega, \mathbb{R}^6)$ with L^∞-norm

$$\|\sigma\|^\infty = \operatorname{ess}\sup_{x\in\Omega} |\sigma(x)|\,. \tag{71}$$

In order for the strain mapping ε, with

$$\varepsilon(w) = \frac{1}{2}\left(\nabla w + (\nabla w)^{\mathrm{T}}\right), \tag{72}$$

to be an isometry, the norm of a virtual displacement field w should be given by

$$\|w\| = \|\varepsilon(w)\|^1. \tag{73}$$

The preceding natural setting motivates the use of the Banach space $LD(\Omega)$, see [2, 13–14], for the space $\mathcal{W}$. Indeed, let $LD(\Omega)$ be the Banach space of L^1-vector fields on Ω, having integrable strain fields, and equipped with the norm

$$\|w\|^{LD} = \|w\|^1 + \|\varepsilon(w)\|^1. \tag{74}$$

Then, $LD(\Omega)$ is a Banach space having a well-defined bounded trace mapping

$$\gamma : LD(\Omega) \to L^1(\partial\Omega, \mathbb{R}^3) \tag{75}$$

that agrees with the restrictions of continuous functions defined on $\overline{\Omega}$. Let $LD(\Omega))_0$ be the subspace of $LD(\Omega)$ containing virtual displacements whose traces (boundary values) vanish on Γ_0 so they satisfy the kinematic boundary conditions. Then, it is implied by the

results of [2] that $LD(\Omega)_0$ is indeed a Banach space with the norm $\|w\| = \|\varepsilon(w)\|^1$. Thus, $\mathcal{W} = LD(\Omega)_0$ satisfies all of the requirements we specified in Section 2.1. It is shown next that Assumption 1 is also satisfied.

Lemma 5.1. *The mappings*

$$\Pi_D \circ \varepsilon : \mathcal{W} \to D, \quad \textit{and} \quad \Pi_p \circ \varepsilon : \mathcal{W} \to P, \tag{76}$$

have closed images.

Proof. By a standard result on linear operators (*e.g.*, [7, pp. 67–68]) the image of a bounded linear $A : X \to Y$ is closed if there is a number $C > 0$ such that for all $x \in X$

$$\inf_{y \in \text{Kernel } A} \|x - y\| \leqslant C\|A(x)\|. \tag{77}$$

This property holds for ε since Kernel $\varepsilon = \{0\}$ and $\|\varepsilon(\chi)\| = \|w\|$. Thus, Image ε is closed in $\mathcal{S}$.

Consider the restriction $\widehat{\Pi}_P$ of the deviatoric projection to Image ε

$$\widehat{\Pi}_P = \Pi_P|_{\text{Image } \varepsilon} : \text{Image } \varepsilon \to P. \tag{78}$$

To show that Image $\widehat{\Pi}_P$ = Image $(\Pi_P \circ \varepsilon)$ is closed, we observe that $P \subset$ Image ε, that is all spherical tensor fields are compatible. If $\xi = \rho I \in P$ is smooth (so the real function ρ is smooth), then, there is a vector field u that solves the equation

$$\nabla \cdot u = u_{i,i} = 3\rho. \tag{79}$$

(See, for example, [2, Section III.3] for the existence theorem.) In other words, $\xi = \rho I = \pi_P \circ \varepsilon(u)$, so $\xi \in$ Image ε. Since the subset of smooth functions is dense (by regularization) in $L^1(\Omega)$, for any $\chi_P \in P$, there is a sequence $\xi_n \to \chi_P$ such that $\xi_n \in P$ are smooth and hence, ξ_n may be also regarded as elements $\widehat{\xi}_n$ of Image ε. As Image ε is closed, the sequence $\widehat{\xi}_n$ has a limit $\widehat{\xi} \in$ Image ε. Since for each n, $\Pi_D(\widehat{\xi}_n) = 0$ and Π_D is continuous, it follows that $\Pi_D(\widehat{\xi}) = 0$ so that $\widehat{\xi} \in P$. Furthermore, $\widehat{\Pi}_P(\widehat{\xi}) = \chi_P$. We conclude that $\widehat{\Pi}_P$ is surjective and its image is simply the Banach space P.

Next, consider the restriction $\widehat{\Pi}_D$ of the deviatoric projection to Image ε

$$\widehat{\Pi}_D = \Pi_D|_{\text{Image } \varepsilon} : \text{Image } \varepsilon \to D. \tag{80}$$

We want to show that $\widehat{\Pi}_D$ has a closed image. To obtain that, we show that for each $\chi_0 = \chi_{0D} + \chi_{0P} \in$ Image ε,

$$\inf_{\chi_P \in \text{Kernel } \widehat{\Pi}_D} \|\chi_0 - \chi_P\| = \inf_{\chi_P \in \text{Kernel } \widehat{\Pi}_D} \|\chi_{0D} + \chi_{0P} - \chi_P\| \leqslant \|\widehat{\Pi}_D(\chi_0)\|. \tag{81}$$

Now

$$\inf_{\chi_P \in \text{Kernel}\,\widehat{\Pi}_D} \|\chi_{0D} + \chi_{0P} - \chi_P\| \leqslant \|\chi_{0D}\| + \inf_{\chi_P \in \text{Kernel}\,\widehat{\Pi}_D} \|\chi_{0P} - \chi_P\| \tag{82}$$

and since $\|\widehat{\Pi}_D(\chi_0)\| = \|\chi_{0D}\|$, it is sufficient to show that

$$\inf_{\chi_P \in \text{Kernel}\,\widehat{\Pi}_D} \|\chi_{0P} - \chi_P\| = 0 \tag{83}$$

for the assertion to hold. Note that χ_{P0} need not be compatible a-priori, while χ_P is required to be compatible by definition. However, since $\widehat{\Pi}_P$ is surjective, χ_{0P} is compatible also, that is $\chi_{0P} \in \text{Kernel}\,\widehat{\Pi}_D$, and we may simply choose $\chi_{0P} = \chi_P$, to show that Equation (83) holds. □

As it is customary in continuum mechanics to consider loadings of continuous bodies that consist of body force fields on Ω and surface force fields on Γ_t, we use the loading space

$$\mathcal{M}^* = L^\infty(\Omega, \mathbb{R}^3) \times L^\infty(\Gamma_t, \mathbb{R}^3). \tag{84}$$

Thus, naturally,

$$\mathcal{M} = L^1(\Omega, \mathbb{R}^3) \times L^1(\Gamma_t, \mathbb{R}^3), \tag{85}$$

and the mapping

$$\beta : LD(\Omega)^1_0 \to L(\Omega, \mathbb{R}^3) \times L^1(\Gamma_t, \mathbb{R}^3) \tag{86}$$

is given by

$$\beta(w) = \left(w, \gamma(w)\,|_{\Gamma_t}\right). \tag{87}$$

Thus, for a load $f = (b, t) \in \mathcal{M}^*$,

$$\|f\| = \max\left\{\operatorname{ess\,sup}_{x\in\Omega} |b(x)|\,,\ \operatorname{ess\,sup}_{y\in\Gamma_t} |t(y)|\right\}. \tag{88}$$

In particular, for the case of vanishing body forces, so $\mathcal{M} = L^1(\Gamma_t, \mathbb{R}^3)$, the parameter $C = 1/K^{\mathcal{M}}$, will be the largest bound such that the body will not collapse for any boundary loading t as long as

$$\operatorname{ess\,sup}_{y\in\Gamma_t} |t(y)| \leq s_Y C. \tag{89}$$

Evidently, one may consider cases where loads on other parts of the body or its boundary are considered. A number of examples for the computations of C for structures, i.e. cases in which the dimension of $\mathcal{M}$ is finite, are presented in [5].

As a final remark it is noted that using the terminology of optimal stresses, the only constitutive information one uses is the failure criterion. On the other hand, we cannot claim that virtual displacement are related in any way to actual displacement fields. It is the theory of rigid, perfectly plastic bodies as in [1, 2] that relates, using a flow rule, extremizing sequences of virtual displacements with the collapse velocity fields.

NOTE

1. Temam [2] refers the readers to [3] and his [4] for the proof and [1] considers the pressure field restoration in Section (VI.2).

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