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# MONODROMY FILTRATION AND MOTIVES

ABHIJIT LASKAR

ABSTRACT. Given a (pure) motive  $M$  over a number field  $F$  and a non-archimedean valuation  $v$  on  $F$ , we state an analogue of the weight monodromy conjecture for  $M$ , via a local weight filtration  $\mathfrak{W}_\bullet$  on the  $\ell$ -adic realization  $H_\ell(M)$ . We prove special cases of this conjecture. In some of these cases, we are able to show that the local  $L$ -factor at  $v$ , of the  $L$ -function of  $M$ , is well-defined i.e., rational and independent of  $\ell$ .

## 0. INTRODUCTION AND NOTATION

Let  $F$  be a number field with an embedding  $\tau : F \hookrightarrow \mathbb{C}$ ;  $\bar{F}$  is a fixed algebraic closure of  $F$ ;  $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$  is an extension of  $\tau$ . Let  $v$  be a discrete valuation on  $F$ ;  $F_v$  is the completion of  $F$  at  $v$ ;  $\bar{v}$  is an extension of  $v$  to  $\bar{F}$ ;  $\bar{F}_v$  is the localization of  $\bar{F}$  at  $\bar{v}$ . The residue fields of  $F_v$  and  $\bar{F}_v$  are denoted by  $k_v$  and  $\bar{k}_v$ , respectively. The characteristic of  $k_v$  is  $p > 0$ . We write  $\Gamma_v := \text{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \text{Gal}(\bar{F}/F)$  and  $I_v \subset \Gamma_v$  is the inertia group. A geometric Frobenius  $\Psi_v \in \Gamma_v$  is an element which induces the inverse of the Frobenius automorphism  $\phi_v$  of  $\bar{k}_v$ . We denote by  $W_v$  the *Weil group* of  $F_v$ , i.e., the dense subgroup formed by elements  $w \in \Gamma_v$  which induce on  $\bar{k}_v$  an integral power  $\phi_v^{\alpha(w)}$ . The map  $\alpha : W_v \rightarrow \mathbb{Z}$  thus defined is a group homomorphism and  $\ker(\alpha) = I_v$ . We will suppose all our algebraic varieties to be geometrically irreducible.

Consider a (pure) motive  $M$  over  $F$ . For any prime number  $\ell$ , the  $\ell$ -adic realization  $H_\ell(M)$  gives us a representation  $\rho_\ell : \Gamma_F \rightarrow \text{GL}(H_\ell(M))$  of  $\Gamma_F$ . Let  $v$  be a non-archimedean valuation of  $F$ , by restricting  $\rho_\ell$  to  $\Gamma_v$  and using Grothendieck's  $\ell$ -adic monodromy theorem, we obtain a nilpotent morphism  $N_\ell : H_\ell(M)(1) \rightarrow H_\ell(M)$  of  $\Gamma_v$ -modules. This morphism induces the local monodromy filtration  $\mathfrak{M}_\bullet$  on  $H_\ell(M)$ . In case if  $H_\ell(M) \cong H_{\acute{e}t}^r(X_{\bar{F}}, \mathbb{Q}_\ell)$  ( as  $\Gamma_F$ -modules) for some smooth projective algebraic variety  $X$  over  $F$ , then Deligne's weight monodromy conjecture ( cf. [3]) predicts that *for any element  $w \in W_v$ , the eigenvalues of the induced automorphisms  $\bar{\rho}_{\ell,j}(w)$  on the  $j$ -th graded of  $\mathfrak{M}_\bullet$ , are algebraic integers with identical complex absolute value  $q_v^{(r+j)\alpha(w)/2}$* . We generalize (see Con.3.2) this conjecture to the case of any arbitrary motive  $M$ , via a local (w.r.t.  $v$ ) weight filtration  $\mathfrak{W}_\bullet$  on  $H_\ell(M)$ . We verify this conjecture (see Cor.3.7) for the category of motives which is  $\otimes$ -generated by Artin motives; abelian varieties; surfaces; and smooth complete intersections in any projective space, as a consequence of some more general results.

Next we prove (see Thm.4.4) that for any motive  $M$  and  $w \in W_v$ , the characteristic polynomial of  $\rho_{M,\ell}(w)$  has coefficients in  $\mathbb{Q}$  and is independent of  $\ell$ . This fact combined with the above mentioned results, allows us to verify in a large number of new cases ( see Thm.4.2 and Cor.4.5), that the local  $L$ -factors in the expression for the  $L$ -function of motives, are well-defined, i.e., rational and independent of  $\ell$ . In particular, we deduce that if  $X$  is finite product of hyperkähler varieties of  $K3^{[n]}$  type; unirational varieties of  $\dim \leq 4$ ; uniruled varieties of  $\dim \leq 3$ ; cubic 4-folds; moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves and Fermat hypersurfaces; then  $H_{\acute{e}t}^r(X_{\bar{F}}, \mathbb{Q}_\ell)$  has a well-defined  $L$ -function for  $0 \leq r \leq 2 \dim X$ , which is independent of the choice of  $\ell$ .

## 1. MONODROMY FILTRATION

1.1. **Generalities.** Let  $\mathcal{A}$  be an abelian category and  $A \in \mathbf{Ob}(\mathcal{A})$ . Let  $N \in \text{End}(A)$  be a nilpotent endomorphism. Then, we define the *kernel filtration* associated to  $N$ , as the increasing filtration  $\mathfrak{F}_\bullet$  on  $A$  satisfying

- $\mathfrak{F}_p A = \text{Ker}(N^{p+1} : A \rightarrow A)$  for  $p \geq 0$  and
- $\mathfrak{F}_p A = 0$  for  $p < 0$ .

We define the image filtration  $\mathfrak{G}^\bullet$  associated to  $N$ , as the decreasing filtration on  $A$ , satisfying

- $\mathfrak{G}^q A = \text{Im}(N^q : A \rightarrow A)$  for  $q > 0$  and
- $\mathfrak{G}^q A = A$  for  $q \leq 0$ .

The convolution  $\mathfrak{M} = \mathfrak{F} * \mathfrak{G}$  is the increasing filtration on  $A$  satisfying

$$\mathfrak{M}_r A := \sum_{p-q=r} \mathfrak{F}_p A \cap \mathfrak{G}^q A \text{ for } r \in \mathbb{Z},$$

where  $+$  stands for product in  $\mathcal{A}$ . Now, by the universal property of coproducts, the natural ‘inclusion’ morphisms  $\mathfrak{F}_p A \cap \mathfrak{G}^q A \xrightarrow{(0, \dots, 1, \dots, 0)} \mathfrak{M}_r A$  induces a morphism

$$\Pi : \bigoplus_{p-q=r} \mathfrak{F}_p A \cap \mathfrak{G}^q A \rightarrow \mathfrak{M}_r A,$$

where  $\bigoplus$  stands for coproduct in  $\mathcal{A}$ .

Now, note that the filtration  $\mathfrak{G}$  also induces a filtration

$$\mathfrak{G}^q Gr_p^{\mathfrak{F}} A := \text{Im}(\mathfrak{G}^q A \cap \mathfrak{F}_p A \rightarrow Gr_p^{\mathfrak{F}} A)$$

on the graded pieces  $Gr_p^{\mathfrak{F}} A := \mathfrak{F}_p A / \mathfrak{F}_{p-1} A$ . The following is an elementary, but useful characterization of graded pieces of  $\mathfrak{M}_\bullet$ , in terms of the above filtration.

**Lemma 1.2.** *There is a natural isomorphism  $\bigoplus_{p-q=r} Gr_{\mathfrak{G}}^q Gr_p^{\mathfrak{F}} A \rightarrow Gr_r^{\mathfrak{M}} A$*

*Proof.* First note that the morphism  $\Pi : \bigoplus_{p-q=r} \mathfrak{F}_p A \cap \mathfrak{G}^q A \rightarrow \mathfrak{M}_r A$  defined above is surjective, i.e.,  $\text{coker } \Pi = 0$ . Now, as  $(\mathfrak{G}^q A \cap \mathfrak{F}_p A) \cap (\mathfrak{G}^{q+1} A \cap \mathfrak{F}_{p-1} A) = (\mathfrak{G}^q A \cap \mathfrak{F}_{p-1} A) + (\mathfrak{G}^{q+1} A \cap \mathfrak{F}_p A)$ , it follows that

$$\text{ker } \Pi = \bigoplus_{p-q=r} ((\mathfrak{G}^q A \cap \mathfrak{F}_{p-1} A) + (\mathfrak{G}^{q+1} A \cap \mathfrak{F}_p A)).$$

Now, as

$$Gr_{\mathfrak{G}}^q Gr_p^{\mathfrak{F}} A = \mathfrak{G}^q A \cap \mathfrak{F}_{p-1} A / ((\mathfrak{G}^q A \cap \mathfrak{F}_{p-1} A) + (\mathfrak{G}^{q+1} A \cap \mathfrak{F}_p A)),$$

the assertion follows.  $\square$

The next lemma is well-known

**Lemma 1.3.** *The convolution  $\mathfrak{M} = \mathfrak{F} * \mathfrak{G}$  is the unique filtration on  $A$  satisfying the following properties :*

- (i) *is an increasing filtration  $\dots \mathfrak{M}_{i-1} A \subset \mathfrak{M}_i A \subset \mathfrak{M}_{i+1} A \dots$  of  $\Gamma_v$  representations, such that  $\mathfrak{M}_i A = 0$  for sufficiently small  $i$  and  $\mathfrak{M}_i A = A$  for sufficiently large  $i$ .*
- (ii)  *$N_\ell(\mathfrak{M}_i A) \subseteq \mathfrak{M}_{i-2} A$  for all  $i$ .*
- (iii) *Using the second condition we can define an induced map  $\bar{N} : Gr_i^{\mathfrak{M}} A \rightarrow Gr_{i-2}^{\mathfrak{M}} A$ , where  $Gr_i^{\mathfrak{M}} A = \mathfrak{M}_i A / \mathfrak{M}_{i-1} A$ . Then  $\bar{N}^r : Gr_r^{\mathfrak{M}} A \rightarrow Gr_{-r}^{\mathfrak{M}} A$  is an isomorphism for each  $r \geq 0$ .*

**1.4. Local monodromy filtration for motives.** Let the notation be as §0. We denote by  $\mathcal{V}_F$  the category of smooth projective varieties over  $F$ . In this article, we will be interested in the following three different categories  $\mathcal{M}_\alpha(F)$  of pure motives over  $F$  :

- the category of (Grothendieck) motives,  $\mathcal{M}_{hom}(F)$  defined in terms of algebraic cycles modulo homological equivalence.
- the category of motives  $\mathcal{M}_{mot}(F)$ , defined in terms of motivated correspondences;
- the category of motives  $\mathcal{M}_{AH}(F)$  defined in terms of absolute Hodge correspondences.

In each of these categories, there is a functor  $h_\alpha : \mathcal{V}_F \rightarrow \mathcal{M}_\alpha(F)$ , such that classical Weil-cohomology theories (with Tate twists)  $H_B$  (Betti cohomology),  $H_\ell$  ( $\ell$ -adic cohomology, for any prime number  $\ell$ ) and  $H_{dR}$  (De Rham cohomology) on  $\mathcal{V}_F$ , factors through  $\mathcal{M}_\alpha(F)$ . This allows to define various *realization functors* on  $\mathcal{M}_\alpha(F)$ , corresponding to these cohomology theories. In particular, if  $M := (X, p, n) \in \text{Ob}\mathcal{M}_\alpha(F)$ , then its  $\ell$ -adic realization is defined as

$$H_\ell(M) = \bigoplus_{w=0}^{2 \dim X} p^* H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n),$$

where  $p^*$  denotes the image of  $p \in \text{Corr}_\alpha^0(X \times X)$  under the  $\Gamma_F$ -invariant isomorphism

$$H_{et}^{2 \dim X}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(\dim X) \cong \bigoplus_{w=0}^{2 \dim X} \text{End}(H_{et}^w(X_{\bar{F}}, \mathbb{Q}_\ell)).$$

It is also clear that the  $\mathbb{Q}_\ell$ -vector space  $V_\ell := H_\ell(M)$  has a natural action of  $\Gamma_F$ . This gives rise to the  $\ell$ -adic representations  $\rho_\ell : \Gamma_F \rightarrow \text{GL}(V_\ell)$ , which in turn gives us the local representations

$$\rho_{\ell,v} : \Gamma_v \rightarrow \text{GL}(V_\ell).$$

For any prime number  $\ell$ , we denote by  $\mu_{\ell^n}$  the group of  $\ell^n$ -th roots of unity in  $\bar{k}_v$  and  $\mathbb{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}$ . For any  $\mathbb{Q}_\ell$ -vector space  $U$ , we write  $U(1) := U \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$ , where  $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$ .

The inertia group  $I_v \subset \Gamma_v$  fits into the following exact sequence

$$1 \rightarrow P \rightarrow I_v \xrightarrow{t} \mathbb{Z}_{(p')}(1) \rightarrow 1,$$

where  $P$  is a pro- $p$ -group and  $\mathbb{Z}_{(p')}(1) = \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$ . Let  $\ell \neq p$  be a prime number. We

denote by  $t_\ell : I_K \rightarrow \mathbb{Z}_\ell(1)$ , the  $\ell$ -component of  $t$ . Explicitly, the surjective map  $t_\ell$  is given as  $x \mapsto \left( x(\pi^{\frac{1}{\ell^n}})/\pi^{\frac{1}{\ell^n}} \right)_n$ , where  $\pi \in \mathcal{O}_v$  is a uniformizer. The map  $t_\ell$  is unique upto multiplication by an element of  $\mathbb{Z}_\ell^\times$ .

By Grothendieck's monodromy theorem there exists a unique nilpotent morphism  $N_\ell : V_\ell(1) \rightarrow V_\ell$  of  $\Gamma_v$ -representations such that for a sufficiently small open subgroup  $J$  of  $I_v$  we have

$$(1) \quad \rho_{\ell,v}(x) = \exp(t_\ell(x)N_\ell), \quad \text{for all } x \in J.$$

The morphism  $N_\ell$  is called the *monodromy operator* associated to  $\rho_{\ell,v}$ . Let  $\mathfrak{M}_\bullet$  be the unique filtration on  $V_\ell$  induced by  $N_\ell$  and satisfying the properties of Lemma 1.3.

We leave it to the reader to verify the validity of the next lemma.

**Lemma 1.5.** *Let  $M$  and  $M'$  be motives in  $\mathcal{M}_\alpha(F)$ .*

(a) *If  $M$  is a sub-motive of  $M'$ , then  $N_{M,\ell} = N_{M',\ell}|_{H_\ell(M)}$  and  $\mathfrak{M}_i H_\ell(M) = \mathfrak{M}_i H_\ell(M')|_{H_\ell(M)}$  for all  $i$ .*

- (b) The monodromy operator associated to the direct sum of motives  $M \oplus M'$ , is given by  $N_{M \oplus M', \ell} = N_{M, \ell} \oplus N_{M', \ell}$  and  $\mathfrak{M}_i H_\ell(M \oplus M') = \mathfrak{M}_i H_\ell(M) \oplus \mathfrak{M}_i H_\ell(M')$ .
- (c) The monodromy operator associated to the tensor product of motives  $M \otimes M'$  is given by  $N_{M \otimes M', \ell} = N_{M, \ell} \otimes \text{Id} + \text{Id} \otimes N_{M', \ell}$  and

$$\mathfrak{M}_i H_\ell(M \otimes M') = \sum_{f+g=i} \mathfrak{M}_f H_\ell(M) \otimes \mathfrak{M}_g H_\ell(M').$$

- (d) The monodromy operator associated to the dual  $M^\vee$  is given by  $N_{M^\vee, \ell} = -{}^t N_{M, \ell}$  and  $\mathfrak{M}_i H_\ell(M^\vee) = (\mathfrak{M}_{-i-1} H_\ell(M))^\perp$ .

**Theorem 1.6.** *Let  $M$  be any motive over  $F$  and  $v$  a non-archimedean valuation on  $F$ . Then there exists an open subgroup  $J$  of  $I_v$  such that  $\rho_{M, \ell}|_J$  is unipotent for every  $\ell \neq p$ .*

*Proof.* Let  $M$  be a direct summand of a motive  $h(X)(n)$  for some smooth projective algebraic variety  $X$  and  $n \in \mathbb{Z}$ . Then it follows that the  $\ell$ -adic realization  $H_\ell(M)$  of  $M$ , is a  $\Gamma_F$ -submodule of the  $\ell$ -adic realization  $H_\ell(h(X)(n))$  of  $h(X)(n)$ . Let us denote by  $\rho_{X, \ell} : \Gamma_F \rightarrow \text{GL}(H_\ell(h(X)(n)))$  the  $\ell$ -adic representation associated to the motive  $h(X)$ . Let  $\xi_{n, \ell} : \Gamma_F \rightarrow \text{GL}(\mathbb{Q}_\ell(n))$  be the  $\ell$ -adic representation associated to the motive  $\mathbb{Q}(n)$ . Write  $\rho_{X, \ell}(n) := \rho_{X, \ell} \otimes \xi_{n, \ell}$ , then it is clear that  $\rho_{X, \ell}(n)$  is the  $\ell$ -representation associated to the motive  $h(X)(n)$ . Assume that there exists a  $J_X \subset I_v$  such that  $\rho_{X, \ell}|_{J_X}$  is unipotent for all  $\ell \neq p$ , i.e., there exists a  $r \in \mathbb{N}$  such that  $(\rho_{X, \ell}(\sigma) - \text{Id})^r = 0$  for all  $\sigma \in J_X$ . As  $\xi_{1, \ell}$  is unramified at all non-archimedean places  $v$  of  $F$ , the same holds for  $\xi_{n, \ell} = (\xi_{1, \ell})^{\otimes n}$ . Therefore,

$$(\rho_{X, \ell}(n)(\sigma) - \text{Id})^r = (\rho_{X, \ell}(\sigma) - \text{Id})^r \otimes \text{Id} = 0, \text{ for all } \sigma \in J_X.$$

This implies that  $\rho_{M, \ell}|_{J_X}$  is unipotent for all  $\ell \neq p$ .

It remains to show that there exists a  $J_X \subset I_v$  such that  $\rho_{X, \ell}|_{J_X}$  is unipotent for all  $\ell \neq p$ . In order to see this, first note that  $h(X)$  has a direct sum decomposition as  $h(X) = \bigoplus_{i=0}^{2 \dim X} h^i(X)$ . Let  $\rho_{X, \ell}^i : \Gamma_F \rightarrow \text{GL}(H_\ell(h^i(X)))$  be the  $\ell$ -adic representation associated to the motive  $h^i(X)$ . Then it follows that

$$\rho_{X, \ell} = \bigoplus_{i=0}^{2 \dim X} \rho_{X, \ell}^i.$$

Assume that for every  $i$ , there exists a  $J_X^i \subset I_v$  such that  $\rho_{X, \ell}^i|_{J_X^i}$  is unipotent for all  $\ell \neq p$ . By setting  $J_X = \bigcap_{i=0}^{2 \dim X} J_X^i$ , we see that  $\rho_{X, \ell}|_{J_X}$  is unipotent for all  $\ell \neq p$ .

So we are reduced to showing that for every  $0 \leq i \leq 2 \dim X$ , there exists a  $J_X^i \subset I_v$  such that  $\rho_{X, \ell}^i|_{J_X^i}$  is unipotent for all  $\ell \neq p$ .

First assume that  $X_{F_v}$  has a proper strictly semi-stable model  $\mathcal{X}$  over the ring of integers  $\mathcal{O}_v$  of  $F_v$ . Let  $\mathcal{X}_{\bar{k}_v} = \mathcal{X} \otimes_{\mathcal{O}_v} \bar{k}_v$  be the geometric special fiber of  $\mathcal{X}$ . Now it is a well-known fact that  $I_v$  acts trivially on the sheaf of vanishing cycles  $R^q \Psi(\mathbb{Q}_\ell)$  on  $\mathcal{X}_{\bar{k}_v}$ . It now follows from the  $\Gamma_v$ -invariant spectral sequence of vanishing cycles

$$E_2^{p, q} = H_{et}^p(\mathcal{X}_{\bar{k}_v}, R^q \Psi(\mathbb{Q}_\ell)) \Rightarrow H_{et}^n(X_{\bar{F}_v}, \mathbb{Q}_\ell),$$

that for all  $\sigma \in I_v$ ,  $(\sigma - \text{Id})^{i+1}$  acts trivially on  $H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  for every  $\ell \neq p$ .

Now in general the existence of a semi-stable model for  $X_{F_v}$  is unknown. But we know from [2, Thm. 6.5] that, there is a finite extension  $F_{v'}$  of  $F_v$  and a strictly semi-stable scheme  $Y'$  over the ring of integers  $\mathcal{O}_{v'}$  of  $F_{v'}$ , with generic fiber  $X'$  and a  $F_v$ -alteration  $f : X' \rightarrow X_{F_v}$ . The induced morphism  $f' : X' \rightarrow X_{F_{v'}}$  is again an alteration. Since  $f'$  is finite of degree  $\deg f'$  on some open dense subscheme of  $X'$ , the pull back  $f'^* : H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell) \rightarrow H_{et}^i(X'_{\bar{F}_v}, \mathbb{Q}_\ell)$  and the pushforward  $f'_* : H_{et}^i(X'_{\bar{F}_v}, \mathbb{Q}_\ell) \rightarrow H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  satisfy  $f'_* \circ f'^* = \deg f' \cdot \text{id}$ . As  $f'^*$  and  $f'_*$  commute with the action of  $\Gamma_{v'}$ , it follows that  $H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  is a direct summand

of  $H_{et}^i(X'_{\bar{F}_v}, \mathbb{Q}_\ell)$ , as  $\Gamma_{v'}$ -module and hence as  $I_{v'}$  (inertia subgroup of  $\Gamma_{v'}$ ) module. As  $X'$  has a semi-stable model, it follows from our previous discussion that  $I_{v'}$  acts trivially on  $H_{et}^i(X'_{\bar{F}_v}, \mathbb{Q}_\ell)$  and hence on  $H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ , for every  $\ell \neq p$ .

Finally, as the  $\ell$ -adic realization  $H_\ell(h^i(X))$  of the motive  $h^i(X)$  is isomorphic to  $H_{et}^i(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  as  $\Gamma_F$ -modules, the proposition follows.  $\square$

## 2. WEIGHT FILTRATION

**Definition 2.1.** A  $\Gamma_v$ -module  $U$  is said to have a (unique) weight filtration  $\mathfrak{W}_\bullet$  if it verifies the following properties :

- (i)  $\mathfrak{W}_\bullet$  is an increasing filtration  $\cdots \mathfrak{W}_{i-1}U \subset \mathfrak{W}_iU \subset \mathfrak{W}_{i+1}U \cdots$  of  $\Gamma_v$ -representations such that  $\mathfrak{W}_iU = 0$  for sufficiently small  $i$  and  $\mathfrak{W}_{i+1}U = U$  for sufficiently large  $i$ ;
- (ii) the action of  $I_v$  on  $\text{Gr}_i^{\mathfrak{W}} = \mathfrak{W}_iU/\mathfrak{W}_{i-1}U$  is through a finite quotient;
- (iii) the eigenvalues of the action of  $\Psi_v$  on  $\text{Gr}_i^{\mathfrak{W}} = \mathfrak{W}_iU/\mathfrak{W}_{i-1}U$  are all algebraic integers of complex absolute value  $q^{i/2}$ , where  $q = |k_v|$ .

**Example 2.2.**  $\mathbb{Q}_\ell(1)$  is a one-dimensional  $\Gamma_v$ -representation which is unramified for every  $v$  (such that  $p \neq \ell$ ). Moreover, we know that the action of any arithmetic Frobenius on  $\mathbb{Q}_\ell(1)$  is through multiplication by  $q$ , hence the action of the geometric Frobenius is through multiplication by  $q^{-1}$ . This allows, us to define the weight filtration as following

$$\mathfrak{W}_j\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell(1) \text{ for } j \geq -2 \text{ and } \mathfrak{W}_j\mathbb{Q}_\ell(1) = 0 \text{ for } j < -2.$$

**Lemma 2.3.** Let  $U, U'$  be  $\Gamma_v$ -modules.

- (a) If  $U$  has weight filtration and  $U'$  is a sub-module of  $U$ , then  $U'$  has weight filtration.
- (b) If both  $U$  and  $U'$  has weight filtrations, then the  $\Gamma_v$ -modules  $U \oplus U'$  and  $U \otimes U'$  has weight filtrations.
- (c) If  $U$  has weight filtration then the dual  $U_1^\vee$  has weight filtration.

*Proof.* (a) We verify that  $\mathfrak{W}_iU' := \mathfrak{W}_iU \cap U'$  defines the weight filtration on  $U'$ .

(b) We verify that  $\mathfrak{W}_i(U \oplus U') := \mathfrak{W}_iU \oplus \mathfrak{W}_iU'$  defines the weight filtration on  $U \oplus U'$  and

$$\mathfrak{W}_i(U \otimes U') = \sum_{f+g=i} \mathfrak{W}_fU \otimes \mathfrak{W}_gU',$$

defines the weight filtration on  $U \otimes U'$ .

(c) we verify that the dual filtration  $\mathfrak{W}_iU^\vee := (\mathfrak{W}_{-i-1}U)^\perp$  defines the weight filtration on  $U^\vee$ .  $\square$

**Proposition 2.4.** Let  $M$  be any motive over  $F$ . Then for every prime number  $\ell \neq p$ , there exists a weight filtration on the  $\Gamma_v$ -module  $H_\ell(M)$ .

*Proof.* Let  $M = (X, p, n)$  be a motive over  $F$ , where  $X$  is a smooth projective algebraic variety,  $p \in \text{Corr}^0(X \times X)$  and  $n \in \mathbb{Z}$ . By definition, the  $\ell$ -adic realization of  $M$  is

$$H_\ell(M) = \bigoplus_{w=0}^{2 \dim X} p^*H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n).$$

Now,  $p^*H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)$  is a  $\Gamma_v$ -sub-representation of  $H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell) \cong H_{et}^{w+2n}(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ . Now suppose that  $H_{et}^w(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  has a weight filtration. Then by 2.3(a) we get a weight filtration on  $p^*H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)$ . Now, it follows from 2.2(i) and 2.3(b), that there exists a weight filtration on  $p^*H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(1)^{\otimes n} = p^*H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n)$ . Finally, using

2.3[(b)] we conclude that there exists a weight filtration on  $\bigoplus_{w=0}^{2\dim X} p^* H_{et}^{w+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n) = H_\ell(M)$ .

It remains to exhibit that  $H_{et}^w(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  has a weight filtration. To start with let us assume that  $X_{F_v}$  has a proper strictly semi-stable model  $\mathcal{X}$  over the ring of integers  $\mathcal{O}_v$  of  $F_v$ . Let  $X_1, \dots, X_m$  be the irreducible components of the special fiber of  $\mathcal{X}$ , and set

$$X^{(k)} := \coprod_{1 \leq i_1 < \dots < i_k \leq m} X_{i_1} \cap \dots \cap X_{i_k}.$$

Then  $X^{(k)}$  is a proper smooth variety of dimension  $n - k + 1$  over the residue field  $k_v$  of  $F_v$ . Now, by [6], there is a  $\Gamma_v$ -invariant spectral sequence

$$(2) \quad E_1^{-r, w+r} = \bigoplus_{k \geq \max\{0, -r\}} H_{et}^{w-r-2k}(X_{\bar{k}_v}^{(2k+r+1)}, \mathbb{Q}_\ell(-r-k)) \Rightarrow H_{et}^w(X_{\bar{F}_v}, \mathbb{Q}_\ell).$$

Now the inertia group  $I_v$  acts trivially on each  $E_1^{i,j}$  and hence there is an action of  $\text{Gal}(\bar{k}_v/k_v)$ . By the Weil conjectures, we know that  $H_{et}^{w-r-2k}(X_{\bar{k}_v}^{(2k+r+1)}, \mathbb{Q}_\ell(-r-k))$  has weight  $(w-r-2k) - 2(r-k) = w+r$ . Thus,  $E_1^{i,j}$  has weight  $j$  and the filtration induced by the spectral sequence (2), is the required weight filtration defined in 2.1.

We now treat the general case. As in the proof of Thm.1.6, there is a finite extension  $F'_{v'}$  of  $F_v$ , such that there is a strictly semi-stable scheme  $Y'$  over the ring of integers  $\mathcal{O}_{v'}$  of  $F'_{v'}$ , with generic fiber  $X'$  and a  $F_v$ -alteration  $f: X' \rightarrow X_{F_v}$ . It follows that  $H_{et}^w(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  is a direct summand of  $H_{et}^w(X'_{\bar{F}_v}, \mathbb{Q}_\ell)$ , as  $\Gamma_{v'}$ -modules. As  $X'$  has a semi-stable model, hence by the above observation, there is a weight filtration on  $H_{et}^w(X'_{\bar{F}_v}, \mathbb{Q}_\ell)$ . It now follows from Lemma 2.3, that  $H_{et}^w(X_{\bar{F}_v}, \mathbb{Q}_\ell)$  has a weight filtration.  $\square$

### 3. ON THE PURITY OF LOCAL MONODROMY FILTRATION

**Definition 3.1.** Let  $M$  be a motive over  $F$ . The *weight*  $wt(M)$  of  $M$  is defined to be the weight of the underlying Hodge structure of the Betti realization  $H_\tau(M)$ . It is independent of the choice of the embedding  $\tau$ .

The weight monodromy conjecture (WMC) predicts that the monodromy filtration on the absolute  $\ell$ -adic cohomology of any proper smooth variety over  $F_v$ , coincides with the weight filtration but upto a shift. This conjecture has the following analogue for motives.

**Conjecture 3.2.** *Let  $M$  be any motive over  $F$  of weight  $w$ . Then,*

$$\mathfrak{M}_i H_\ell(M) = \mathfrak{M}_{i+wt(M)} H_\ell(M) \quad \forall i.$$

**Proposition 3.3.** *Let  $M$  be an Artin motive over  $F$ . Then, conjecture 3.2 holds for  $M$ .*

*Proof.* We know that the category of Artin motives  $\mathcal{M}_F^0 \cong \text{Rep}_{\mathbb{Q}}(\Gamma_F)$ . Let  $\rho: \Gamma_F \rightarrow \text{GL}(V)$  be the image of  $M$  under this isomorphism. Then we have,

$$H_\tau(M) = V \quad \text{and} \quad H_\ell(M) = V \otimes \mathbb{Q}_\ell, \quad \text{as a } \Gamma_F \text{-module.}$$

Thus, the  $\ell$ -adic representation  $\rho_\ell: \Gamma_F \rightarrow \text{GL}(H_\ell(M))$  associated to  $M$ , is simply  $\rho \otimes \mathbb{Q}_\ell$ . Now, as  $\rho$  is continuous for the Krull topology on  $\Gamma_F$  and the discrete topology on  $V$ , there exists a finite extension  $F'/F$ , such that  $\text{Gal}(\bar{F}/F')$  acts trivially on  $V$ . In particular, if  $v'$  is an extension of  $v$  to  $F'$ , then  $\text{Gal}(\bar{F}_v/F'_{v'})$  and hence the inertia group corresponding to this extension, acts trivially on  $V$ . Thus, the  $\ell$ -adic representation  $\rho_\ell$  is potentially unramified at every  $v$  such that  $p \neq \ell$ . This implies that the monodromy operators corresponding to

the local representations  $\rho_{\ell,v}$  are all trivial. In particular, the monodromy filtration  $\mathfrak{M}_\bullet$  of  $H_\ell(M)$  is trivial, i.e.

$$\mathfrak{M}_j H_\ell(M) = H_\ell(M) \text{ for } j \geq 0 \text{ and } \mathfrak{M}_j H_\ell(M) = 0 \text{ for } j < 0.$$

Now, as the weight of an Artin motive is 0, in order to prove the proposition, we only need to show that the weight filtration on  $H_\ell(M)$  coincides with the monodromy filtration on  $H_\ell(M)$ . It is obvious that  $\mathfrak{M}_\bullet H_\ell(M)$ , verifies property (i) of 2.1. As,  $\rho_\ell$  is potentially unramified at every  $v$ , so  $\mathfrak{M}_\bullet H_\ell(M)$  also verifies property (ii) of 2.1. Finally, note that as  $\rho_\ell = \rho \otimes \mathbb{Q}_\ell$ , the eigenvalues of the action of  $\Psi_v$  on  $\text{Gr}_0^{\text{mf}} H_\ell(M) = H_\ell(M) = V \otimes \mathbb{Q}_\ell$  are algebraic integers ( independent of  $\ell$  ). Now, as  $\text{Gal}(\bar{F}/F')$  acts trivially on  $V$ , the eigenvalues of the action of  $\Psi^{[k'_v:k_v]}$  on  $V$  are all 1, where  $k'_v$  denotes the residue field of  $F'_{v'}$ . This implies that the eigenvalue of the action of  $\Psi_v$  on  $V$  is a root of unity and hence of complex absolute value  $q^{0/2} = 1$ . This establishes property (iii) of 2.1. By the uniqueness of the weight filtration, we conclude that on  $H_\ell(M)$ , the monodromy filtration coincides with the weight filtration.  $\square$

**Definition 3.4.** For  $\beta = \text{mot}$  or  $AH$ , we denote by  $\mathcal{M}_\beta^{\text{pmf}}(F)$  the Tannakian subcategory of  $\mathcal{M}_\beta(F)$ , which is  $\otimes$ -generated by the family  $\mathcal{C} := (M_i | i \in I)$  of motives, such that each  $M_i$  verifies Conjecture 3.2.

**Proposition 3.5.** *Conjecture 3.2 holds for every  $M \in \text{Ob}(\mathcal{M}_\beta^{\text{pmf}}(F))$ .*

*Proof.* Any motive  $M \in \text{Ob}(\mathcal{M}_\beta^{\text{pmf}}(F))$  is isomorphic to a sub-quotient of an object of the form  $P(M'_j)$  ( a polynomial expression), where  $\{M'_j\}_{j \in J \subset I}$  consists of objects of  $\mathcal{C}$  or their duals and  $P(t_j) \in \mathbb{N}[t_j]_{j \in J}$ ; multiplication in  $P(M'_j)$  is interpreted as  $\otimes$  and addition as  $\oplus$ . As  $\mathcal{M}_\beta(F)$  is a semi-simple Tannakian category, it follows that every quotient object of  $P(M'_j)$ , can be identified with a subobject. Thus, in view of 3.5, it suffices to verify the conjecture for the motive  $P(M'_j)$ . Now, by combining Lemma1.5 and Lemma2.3, we see that Conjecture 3.2 is stable under sub-objects,  $\oplus$ ,  $\otimes$  and taking duals of motives. A repeated application of this fact and the definition of  $\mathcal{C}$ , implies the proposition.  $\square$

**Definition 3.6.** Define  $\mathcal{M}_{AH}^{\text{acs}}(F)$  ( resp.  $\mathcal{M}_{\text{mot}}^{\text{acs}}(F)$ ) to be the Tannakian subcategory of  $\mathcal{M}_{AH}(F)$  ( resp.  $\mathcal{M}_{\text{mot}}(F)$  ), which is  $\otimes$ -generated by the following motives

- (i) Artin motives;
- (ii)  $h^1(A)$ 's of abelian varieties;
- (iii)  $h^2(S)$ 's of surfaces;
- (iv) for any  $m \in \mathbb{N}$ ,  $h^m(Y)$ 's of  $m$ -dimensional (smooth) complete intersections in any projective space.

We also define  $\mathcal{M}_{\text{hom}}^{\text{acs}}(F)$  to be the abelian  $\otimes$ -subcategory of  $\mathcal{M}_{\text{hom}}(F)$ , generated by (i) the Tate motive  $\mathbb{Q}(1)$  (ii) abelian varieties (iii) 0 dimensional varieties (iv) surfaces and (v) complete intersections in projective spaces.

**Corollary 3.7.** *For  $\alpha = \text{hom}$ ,  $\text{mot}$  or  $AH$ , Conjecture 3.2 holds for every  $M \in \text{Ob}(\mathcal{M}_\alpha^{\text{acs}}(F))$ .*

*Proof.* First note that it suffices to show the corollary for  $\mathcal{M}_{AH}^{\text{acs}}(F)$ , as  $\mathcal{M}_{\text{hom}}(F)$  and  $\mathcal{M}_{\text{mot}}(F)$  are subcategories ( a priori not full) of  $\mathcal{M}_{AH}(F)$  and the the  $\ell$ -adic realization of a motive is unchanged by passing from one category to the other.

Now, by Prop. 3.3 the Artin motives verifies Conjecture 3.2, by [1] we know that  $h^1(A)$ 's for  $A$  any abelian variety, verifies the conjecture, by [6] the conjecture is true for  $h^2(S)$  for any surface  $S$ , and finally by [8], the conjecture holds for  $h^m(Y)$ 's (for any  $m \in \mathbb{N}$  ) of  $m$ -dimensional (smooth) complete intersections in any projective space. It follows that  $\mathcal{M}_{AH}^{\text{acs}}(F)$  is a subcategory of  $\mathcal{M}_{AH}^{\text{pmf}}(F)$  and hence verifies Conjecture 3.2.  $\square$

**Remark 3.8.** Denote by  $\mathcal{M}_{num}(F)$  the category of (Grothendieck) motives for numerical equivalence on algebraic cycles. Let  $\mathcal{M}_{num}^{av}(F)$  be the abelian  $\otimes$ -subcategory of  $\mathcal{M}_{num}(F)$ , generated by (i) the Tate motive  $\mathbb{Q}(1)$  (ii) abelian varieties and (iii) 0 dimensional varieties. By [4], we know that  $\mathcal{M}_{num}^{av}(F)$  is a  $F$ -linear semi-simple Tannakian category. As homological equivalence coincides with numerical equivalence for zero dimensional varieties and abelian varieties [5], we can identify  $\mathcal{M}_{num}^{av}(F)$  to a subcategory of  $\mathcal{M}_{AH}^{acs}(F)$ . Hence it makes sense to speak of  $\ell$ -adic realizations of motives in  $\mathcal{M}_{num}^{av}(F)$ . The previous corollary then implies that Conjecture 3.2 holds for objects in  $\mathcal{M}_{num}^{av}(F)$ .

**Corollary 3.9.** *Let  $X$  be a smooth projective variety over  $F$  which is a finite product of*

- (i) hyperkähler varieties of  $K3^{[n]}$  type;
- (ii) unirational varieties of  $\dim \leq 4$ ;
- (iii) uniruled varieties of  $\dim \leq 3$ ;
- (iv) cubic 4-folds;
- (v) moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves;
- (vi) Fermat hypersurfaces;
- (vii) curves;
- (viii) surfaces;
- (ix) abelian varieties;

then the weight monodromy conjecture holds for  $H^i(X_{\bar{F}}, \mathbb{Q}_\ell)$  for any  $i \geq 0$ .

*Proof.* In view of 3.5, it suffices to establish the result for each of the varieties in the list. If  $X$  is either a hyperkähler variety of  $K3^{[n]}$  type, a cubic 4-fold, a unirational variety of  $\dim \leq 3$ , a Fermat hypersurface, moduli space of stable vector bundles of coprime rank and degree over a smooth projective curve, then the motive  $h(X) \in \text{Ob}(\mathcal{M}_F^{av})$ . So, in these cases the weight monodromy conjecture follows from 3.7.

The case of curves [1] and surfaces [6], are well known. The case of unirational varieties of dimension 4, follows from the fact that the motive of any such variety is an object of the Tannakian category generated by the motive of variety of dimension at most 2. Similarly, the motive of a uniruled 3-fold belongs to the Tannakian category of motives generated by a surface, and hence the WMC holds in this case as well.

In the case of any abelian variety  $A$  the only new thing that we obtain is that WMC holds for the higher cohomology groups  $H^i(A_{/\bar{F}_v}, \mathbb{Q}_\ell)$  ( $i \geq 2$ ) as well, since the motive  $h^i(A) = \wedge^i h^1(A)$ .  $\square$

#### 4. L-FUNCTIONS

Let  $M$  be any motive in  $\mathcal{M}_\alpha(F)$ , for  $\alpha = hom, mot$  or  $AH$ . For any prime number  $\ell$ , we denote by  $V_\ell := H_\ell(M)$ . For any non-archimedean valuation  $v$  on  $F$  of residual characteristic  $p$ , consider the characteristic polynomial

$$Z(V_\ell, T) := \det_{\mathbb{Q}_\ell}(1 - T\rho_{M,\ell}(\Psi_v)|V_\ell^{I_v}), \quad (\ell \neq p)$$

for the action of a geometric Frobenius element  $\Psi_v \in \Gamma_v$  on the inertia invariant part  $V_\ell^{I_v}$  part of  $V_\ell$ . By definition,  $Z(V_\ell, T) \in \mathbb{Q}_\ell[T]$ .

**Definition 4.1.** A motive  $M$  as above is said to have a well-defined  $L$ -function, if  $Z(V_\ell, T) \in \mathbb{Q}[T]$  and is independent of  $\ell$ . In this case, the  $L$ -function of  $M$  is defined as the Euler product

$$L(M, s) = \prod_v L_v(M, s),$$

where the local  $L$ -factors  $L_v(M, s)$  is defined as

$$L_v(M, s) = \frac{1}{Z(V_\ell, q_v^{-s})}, \text{ for } \ell \neq p.$$

**Theorem 4.2.** *Let  $M \in \text{Ob } \mathcal{M}_{AH}^{\text{pmf}}(F)$  be a motive, such that for every non-archemdean valuation  $v$  on  $F$  and  $w \in W_v$ , the characteristic polynomial  $P_\ell(w, T)$  of  $\rho_{M, \ell}(w)$  has coefficients in  $\mathbb{Q}$  and is independent of  $\ell$ . Then  $M$  has a well-defined  $L$ -function.*

*Proof.* We need to verify that  $Z(V_\ell, T)$  has coefficients in  $\mathbb{Q}$  and is independent of  $\ell$ .

First consider the non-archemdean valuations  $v$ , where  $M$  has good reduction, i.e.,  $V_\ell$  is unramified at  $v$  for every  $\ell \neq p$ . In this case

$$Z(V_\ell, T) = P_\ell(\Psi_v, T),$$

hence by hypothesis,  $Z(V_\ell, T)$  has coefficients in  $\mathbb{Q}$  and is independent of  $\ell$ .

Now consider the non-archemdean valuations  $v$ , where  $M$  has bad reduction. As  $V_\ell^{I_v}$  is a subspace of  $V_\ell$ , it follows from the hypothesis, that for every  $w \in W_v$ , the coefficients of the characteristic polynomial  $\det_{\mathbb{Q}_\ell}(1 - T\rho_{M, \ell}(w)|V_\ell^{I_v})$  are algebraic numbers independent of  $\ell$ . Thus, in order to prove our claim, it suffices to show that  $\text{Tr}(w^* : V_\ell^{I_v}) \in \mathbb{Q}$  and independent of  $\ell$  and then use Newton's lemma.

Let  $\bar{\rho}_{\ell, j} : \Gamma_v \rightarrow \text{GL}(\text{Gr}_j^{\mathfrak{M}} V_\ell)$  denote the representation induced by  $\rho_{M, \ell}$  on the graded parts of the local monodromy filtration  $\mathfrak{M}_\bullet$  on  $V_\ell$ . For every  $w \in W_v$ , let  $\bar{P}_{\ell, j}(w, T)$  be the characteristic polynomial of  $\bar{\rho}_{\ell, j}(w)$ .

**Lemma 4.3.** *Fix a  $w \in W_v$ , then for every  $j$ ,  $\bar{P}_{\ell, j}(w, T) \in \mathbb{Q}[T]$  and is independent of  $\ell$ .*

*Proof.* Let  $L$  be the sub-field of an algebraic closure  $\bar{\mathbb{Q}}_\ell$  of  $\mathbb{Q}_\ell$ , generated by the roots of  $P_\ell(w, T)$ . By hypothesis,  $P_\ell(w, T) \in \mathbb{Q}[T]$  and has coefficients independent of  $\ell$ . This implies that if  $P_\ell(w, \beta) = 0$ , then  $P_\ell(w, \sigma(\beta)) = 0$ , for every  $\sigma \in \text{Gal}(L/\mathbb{Q})$ . Now, from linear algebra, we know that  $P_\ell(w, T) = \prod_j \bar{P}_{\ell, j}(w, T)$ . Let  $\eta$  be an eigenvalue  $\bar{\rho}_{\ell, j}(w)$  and hence of  $\rho_\ell(w)$ . By Prop. 3.5, the complex absolute value

$$(3) \quad |\eta|_{\mathbb{C}} = q_v^{(\text{wt}(M)+j)\alpha(w)/2} \text{ for every } j.$$

As  $|\beta|_{\mathbb{C}} = |\sigma(\beta)|_{\mathbb{C}}$ , it follows from (3), that  $\beta$  and  $\sigma(\beta)$  occurs as the roots of a same factor (say)  $\bar{P}_{\ell, j_0}(w, T)$  of  $P_\ell(w, T)$ . Now suppose  $\bar{P}_{\ell, j_0}(w, T) = T^m + \sum_{r=1}^m a_{m-r} T^{m-r}$ . As the coefficients  $a_r$ 's are symmetric polynomials in the roots of  $\bar{P}_{\ell, j_0}(w, T)$ , it follows from the previous observation that  $\sigma(a_r) = a_r$ , for every  $\sigma \in \text{Gal}(L/\mathbb{Q})$ . In other words,  $a_r \in \mathbb{Q}$ , i.e.,  $\bar{P}_{\ell, j_0}(w, T) \in \mathbb{Q}[T]$  and is independent of  $\ell$ . By varying  $\beta$  over all roots of  $P_\ell(w, T)$ , we conclude the  $\bar{P}_{\ell, j}(w, T) \in \mathbb{Q}[T]$  and is independent of  $\ell$ , for every  $j$ .  $\square$

Now, let us denote by  $\mathfrak{F}_\bullet$  and  $\mathfrak{G}^\bullet$  the kernel filtration and image filtration on  $V_\ell$ . By lemma 1.2, we have an equality of  $\Gamma_v$ -modules

$$\text{Gr}_{-j+1}^{\mathfrak{M}} V_\ell = \text{Gr}_{\mathfrak{G}}^j \text{Gr}_1^{\mathfrak{F}} V_\ell \oplus \text{Gr}_{-j-1}^{\mathfrak{M}} V_\ell(1), \text{ for every } j \geq 0.$$

As,  $\text{Gr}_{\mathfrak{G}}^j \text{Gr}_1^{\mathfrak{F}} V_\ell = \text{Gr}_{\mathfrak{G}}^j \mathfrak{F}_0 V_\ell$  (this is the graded for the filtration induced by  $\mathfrak{G}^\bullet$  on  $\mathfrak{F}_0 V_\ell$ ), we get that

$$\text{Gr}_{-j+1}^{\mathfrak{M}} V_\ell = \text{Gr}_{\mathfrak{G}}^j \mathfrak{F}_0 V_\ell \oplus \text{Gr}_{-j-1}^{\mathfrak{M}} V_\ell(1)$$

for every  $j \geq 0$ . This implies that the traces  $\text{Tr}(w^* : \text{Gr}_{\mathfrak{G}}^j \mathfrak{F}_0 V_\ell) \in \mathbb{Q}$  and is independent of  $\ell$ . It now follows from linear algebra that  $\text{Tr}(w^* : \mathfrak{F}_0 V_\ell) \in \mathbb{Q}$  and is independent of  $\ell$ .

Now, the action of  $I_v$  on  $\mathfrak{F}_0 V_\ell$  factors through a finite quotient  $I_v/J$  and  $V_\ell^{I_v} = (\mathfrak{F}_0 V_\ell)^{I_v/J}$ .

Hence for a complete set of  $T \subset I_v$  of representatives of  $I_v/J$ , we have

$$[I_v : J] \cdot \text{Tr}(w^* : V_\ell^{I_v}) = \sum_{\tau \in T} \text{Tr}((w \circ \tau)^* : \mathfrak{F}_0 V_\ell).$$

Now by our previous discussion,  $\text{Tr}((w \circ \tau)^* : \mathfrak{F}_0 V_\ell) \in \mathbb{Q}$  and is independent of  $\ell$  and by Thm.1.6,  $[I_v : J]$  is independent of  $\ell$ . It follows that  $\text{Tr}(w^* : V_\ell^{I_v}) \in \mathbb{Q}$  and independent of  $\ell$ . This completes the proof of the theorem.  $\square$

**Theorem 4.4.** *Assume that the Künneth Standard conjecture holds for (absolute)  $\ell$ -adic cohomology (with Tate twists) of varieties defined over  $F$ . Then, for every motive  $M \in \mathcal{M}_{\text{hom}}(F)$ , and non-archimedean valuation  $v$  on  $F$ , the characteristic polynomial  $P_{M,\ell}(w, T)$  for the action of any element  $w \in W_v$ , has coefficients in  $\mathbb{Q}$  and is independent of  $\ell$ .*

*Proof.* First note that it suffices to show that for every  $w \in W_v$ , the traces  $\text{Tr}(\rho_{M,\ell}(w))$  are in  $\mathbb{Q}$  and is independent of  $\ell$ , for then one can resolve the Newton's identities relating symmetric polynomials and power sums, to conclude that the polynomial  $P_{M,\ell}(w, T) \in \mathbb{Q}[T]$  and is independent of  $\ell$ .

Let  $M := (X, p, n)$  and  $M' := (X, p, 0)$ . Then it is easy to see that

$$\text{Tr}(\rho_{M,\ell}(w)) = \chi_\ell^n(w) \cdot \text{Tr}(\rho_{M',\ell}(w)),$$

where  $\chi_\ell : \Gamma_F \rightarrow \mathbb{Q}_\ell^\times$  is the  $\ell$ -adic cyclotomic character of  $\Gamma_F$ . As  $\chi_\ell(\lambda) = 1$ , for every  $\lambda$  in the inertia subgroup  $I_v$ , it follows that if  $w = \Psi_v^m \cdot \lambda$  (for some  $m \in \mathbb{Z}$ ), then

$$\chi_\ell^n(w) = q_v^{mn}.$$

As  $q_v^{mn} \in \mathbb{Q}$  and is independent of  $\ell$ , thus it suffices to show the proposition for motives of the form  $M' = (X, p, 0)$ .

First, we make an observation. Let  $e, e' \in CH^d(X \times X)$  be any two algebraic correspondences. Then, by definition

$$e' \circ e = pr_{13*}(pr_{12}^* e \cdot pr_{23}^* e'),$$

where  $pr_{\alpha\beta}$  denotes the projection maps from  $X \times X \times X$  to  $X \times X$ , for  $\alpha, \beta \in \{1, 2, 3\}$ . Write  $d := \dim X$  and  $\gamma_{X \times X}^d : CH^d(X \times X) \rightarrow H_{\text{et}}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d)$  be the cycle class map to  $\ell$ -adic cohomology of  $X \times X$ . As cycle class maps are compatible with push-forward, we get  $\gamma_{X \times X}^d(e' \circ e) = pr_{13*}(\gamma_{X \times X \times X}^d(pr_{12}^* e \cdot pr_{23}^* e'))$ . Since cycle class maps are compatible with intersection products in  $CH^*(-)$  and cup products in cohomology, we have  $\gamma_{X \times X}^d(e' \circ e) = pr_{13*}(\gamma_{X \times X \times X}^d(pr_{12}^* e) \cdot \gamma_{X \times X \times X}^d(pr_{23}^* e'))$ . Finally, as cycle class maps are compatible with pull-backs we get  $\gamma_{X \times X}^d(e' \circ e) = pr_{13*}(pr_{12}^*(\gamma_{X \times X}^d(e)) \cdot pr_{23}^*(\gamma_{X \times X}^d(e')))$ . But this is by definition the composition of correspondences  $\gamma_{X \times X}^d(e') \circ \gamma_{X \times X}^d(e)$  under the identification

$$(4) \quad H_{\text{et}}^{2d}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(d) \cong \bigoplus_{r \geq 0} \text{End}_{\mathbb{Q}_\ell}(H_{\text{et}}^r(X_{\bar{F}}, \mathbb{Q}_\ell)).$$

Hence, writing  $\underline{e} := \gamma_{X \times X}^d(e)$ , when viewing it as a  $\mathbb{Q}_\ell$ -linear endomorphism of the (graded) vector space  $V := \bigoplus_{r \geq 0} H_{\text{et}}^r(X_{\bar{F}}, \mathbb{Q}_\ell)$  (under the identification (4)), we conclude that

$$\underline{e}' \circ \underline{e} = \underline{e}' \circ \underline{e}.$$

Now consider the motive  $M' = (X, p, 0)$ , by definition  $p$  is an idempotent in  $CH^d(X \times X)$ , i.e.,  $p \circ p = p$ . It follows from the above discussion that  $\underline{p}$  is an idempotent linear map. This implies that

$$V = \text{Ker}(\underline{p}) \oplus \text{Im}(\underline{p}),$$

as  $\Gamma_F$ -modules. Let  $w_*$  be the automorphism of  $V$ , induced by the action of  $w$ . It now follows by linear algebra that

$$\text{Tr}(\rho_{M',\ell}(w)) = \text{Tr}(w_*|_{\text{Im}(\underline{p})}) = \text{Tr}(w_* \circ \underline{p}).$$

Thus in order to prove the proposition it suffices to show that  $\text{Tr}(w_* \circ \underline{p}) \in \mathbb{Q}$  and is independent of  $\ell$ .

For brevity, let us denote by  $\text{Tr}^{\text{alt}}(w_* \circ \underline{p})$  the alternating sum

$$\sum_{r=0}^{2d} (-1)^r \text{Tr}(w_* \circ \underline{p} : H_{\text{et}}^r(X_{\bar{F}}, \mathbb{Q}_\ell)).$$

By hypothesis the Künneth standard conjecture holds for  $X$ , i.e, for every  $0 \leq r \leq 2d$ , there exists an algebraic cycle  $\pi_r$  such that  $\underline{\pi}_r$  is the canonical projection

$$V \rightarrow H_{\text{et}}^r(X_{\bar{F}}, \mathbb{Q}_\ell) \hookrightarrow V,$$

followed by inclusion. It now follows from the above discussion that

$$\text{Tr}(w_* \circ (\underline{p} \circ \pi_r)) = \text{Tr}(w_* \circ \underline{p} : H_{\text{et}}^r(X_{\bar{F}}, \mathbb{Q}_\ell)) = \text{Tr}^{\text{alt}}(w_* \circ (\underline{p} \circ \pi_r)).$$

By [7, Thm 0.1],  $\text{Tr}^{\text{alt}}(w_* \circ (\underline{p} \circ \pi_r)) \in \mathbb{Q}$  and is independent of  $\ell$ . It follows that

$$\sum_{r=0}^{2d} \text{Tr}(w_* \circ \underline{p} : H_{\text{et}}^r(X_{\bar{F}}, \mathbb{Q}_\ell)) = \sum_{r=0}^{2d} \text{Tr}(w_* \circ (\underline{p} \circ \pi_r)) = \text{Tr}(w_* \circ (\underline{p} \circ \Delta_{X \times X})) = \text{Tr}(w_* \circ \underline{p}),$$

is in  $\mathbb{Q}$  and is independent of  $\ell$ . □

**Corollary 4.5.** *Every motive  $M \in \mathcal{M}_{\text{hom}}^{\text{acs}}(F)$  has a well-defined  $L$ -function.*

*Proof.* As  $\mathcal{M}_{\text{hom}}^{\text{acs}}(F)$  is a subcategory of  $\mathcal{M}_{\text{AH}}^{\text{pmf}}(F)$ , thus in view of Thm.4.2, we only need to verify that for any non-archimedean valuation  $v$  on  $F$  and  $w \in W_v$ , the characteristic polynomial  $P_{M,\ell}(w, T)$  of  $\rho_{M,\ell}(w)$  has coefficients in  $\mathbb{Q}$  and is independent of  $\ell$ .

Write  $M = (X, p, n)$ , then  $X$  is a product of abelian varieties, zero dimensional varieties, surfaces or complete intersection in projective spaces. As each of these varieties satisfies Künneth standard conjecture, it follows that  $X$  satisfies Künneth standard conjecture. It now follows from the proof of Thm.4.4, that  $P_{M,\ell}(w, T) \in \mathbb{Q}[T]$  and is independent of  $\ell$ . □

**Corollary 4.6.** *Let  $X$  be as in Cor. 3.9. Then, for every  $i \in \mathbb{N}$ , the cohomology groups  $H_{\text{et}}^i(X_{\bar{F}}, \mathbb{Q}_\ell)$  has well defined  $L$ -function, which is independent of the choice of  $\ell$ .*

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