



HAL
open science

Monodromy filtration and motives

Abhijit Laskar

► **To cite this version:**

| Abhijit Laskar. Monodromy filtration and motives. 2014. hal-01068065v1

HAL Id: hal-01068065

<https://hal.science/hal-01068065v1>

Preprint submitted on 24 Sep 2014 (v1), last revised 28 Mar 2015 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Monodromy Filtration and Motives

Abhijit Laskar ¹

Abstract

For a given number field F and a non-archimedean valuation on F , we establish the existence of weight filtration on the ℓ -adic realizations of any pure motive M defined over F . Using this filtration we state a motivic analogue of the Weight monodromy conjecture of Deligne. We prove this conjecture in special cases and from this we deduce some new instances of the original Weight monodromy conjecture of Deligne.

1 Monodromy

Let F be a number field with an embedding $\tau : F \hookrightarrow \mathbb{C}$; \bar{F} is a fixed algebraic closure of F ; $\bar{\tau} : \bar{F} \hookrightarrow \mathbb{C}$ is an extension of τ . Let v be a discrete valuation on F ; F_v is the completion of F at v ; \bar{v} is an extension of v to \bar{F} ; \bar{F}_v is the localization of \bar{F} at \bar{v} . The residue fields of F_v and \bar{F}_v are denoted by k_v and \bar{k}_v , respectively. The characteristic of k_v is $p > 0$. We write $\Gamma_v := \text{Gal}(\bar{F}_v/F_v) \subset \Gamma_F := \text{Gal}(\bar{F}/F)$ and $I_v \subset \Gamma_v$ is the inertia group. A geometric Frobenius $\Psi_v \in \Gamma_v$ is an element which induces the inverse of the Frobenius automorphism of \bar{k}_v .

In this article, the category (pure) motives \mathcal{M}_F over F , would mean any one of the following categories :

- the category of Grothendieck motives, defined in terms of algebraic cycles modulo homological equivalence;
- the category of André motives, defined in terms of motivated correspondences;
- the category of motives defined in terms of absolute Hodge correspondences.

It would be clear from the context, which category we're using. Now, let M be a (pure) motive over F and ℓ any prime number. The ℓ -adic realization $V := H_\ell(M)$ has a natural action of Γ_F , that gives rise to the ℓ -adic representations $\rho_\ell : \Gamma_F \rightarrow \text{GL}(V)$, which in turn gives us the local representations

$$\rho_{\ell,v} : \Gamma_v \rightarrow \text{GL}(V).$$

For any prime number ℓ , we denote by μ_{ℓ^n} the group of ℓ^n -th roots of unity in \bar{k}_v and $\mathbb{Z}_\ell(1) := \varprojlim_n \mu_{\ell^n}$. For any \mathbb{Q}_ℓ -vector space U , we write $U(1) := U \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(1)$, where $\mathbb{Q}_\ell(1) = \mathbb{Q}_\ell \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell(1)$.

The inertia group $I_v \subset \Gamma_v$ fits into the following exact sequence

$$1 \rightarrow P \rightarrow I_v \xrightarrow{t} \mathbb{Z}_{(\ell)}(1) \rightarrow 1,$$

¹Abhijit Laskar: Fakultät für Mathematik, Universität Regensburg, 93040 Regensburg, Germany. E-mail: abhijit.laskar@gmail.com.

where P is a pro- p -group and $\mathbb{Z}_{(p')}(1) = \prod_{\ell \neq p} \mathbb{Z}_\ell(1)$. Let $\ell \neq p$ be a prime number. We

denote by $t_\ell : I_K \rightarrow \mathbb{Z}_\ell(1)$, the ℓ -component of t . Explicitly, the surjective map t_ℓ is given as $x \mapsto \left(x(\pi^{\frac{1}{\ell^n}}) / \pi^{\frac{1}{\ell^n}} \right)_n$, where $\pi \in \mathcal{O}_v$ is a uniformizer. The map t_ℓ is unique upto multiplication by an element of \mathbb{Z}_ℓ^\times .

By Grothendieck's monodromy theorem there exists a unique nilpotent morphism $N : V(1) \rightarrow V$ of Γ_v -representations such that for a sufficiently small open subgroup J of I_v we have

$$\rho_{\ell,v}(x) = \exp(t_\ell(x)N), \quad \text{for all } x \in J. \quad (1)$$

The morphism N is called the *monodromy operator* associated to $\rho_{\ell,v}$.

Definition 1.1. There exists a unique filtration \mathfrak{M}_\bullet on V called the monodromy filtration, characterized by the following properties :

- (i) \mathfrak{M}_\bullet is an increasing filtration $\cdots \mathfrak{M}_{i-1}V \subset \mathfrak{M}_iV \subset \mathfrak{M}_{i+1}V \cdots$ of Γ_v representations, such that $\mathfrak{M}_iV = 0$ for sufficiently small i and $\mathfrak{M}_iV = V$ for sufficiently large i .
- (ii) $N_\ell(\mathfrak{M}_iV(1)) \subseteq \mathfrak{M}_{i-2}V$ for all i .
- (iii) Using the second condition we can define an induced map $N : \text{Gr}_i^{\mathfrak{M}}V(1) \rightarrow \text{Gr}_{i-2}^{\mathfrak{M}}V$, where $\text{Gr}_i^{\mathfrak{M}}V = \mathfrak{M}_iV / \mathfrak{M}_{i-1}V$. Then $N^r : \text{Gr}_r^{\mathfrak{M}}V(r) \rightarrow \text{Gr}_{-r}^{\mathfrak{M}}V$ is an isomorphism for each $r \geq 0$.

We leave it to the reader to verify the validity of the next lemma.

Lemma 1.2. Let M and M' be two motives over F .

- (a) If M is a sub-motive of M' , then $N_{M,\ell} = N_{M',\ell}|_{H_\ell(M)}$ and $\mathfrak{M}_iH_\ell(M) = \mathfrak{M}_iH_\ell(M')|_{H_\ell(M)}$ for all i .
- (b) The monodromy operator associated to the direct sum of motives $M \oplus M'$, is given by $N_{M \oplus M',\ell} = N_{M,\ell} \oplus N_{M',\ell}$ and $\mathfrak{M}_iH_\ell(M \oplus M') = \mathfrak{M}_iH_\ell(M) \oplus \mathfrak{M}_iH_\ell(M')$.
- (c) The monodromy operator associated to the tensor product of motives $M \otimes M'$ is given by $N_{M \otimes M',\ell} = N_{M,\ell} \otimes \text{Id} + \text{Id} \otimes N_{M',\ell}$ and

$$\mathfrak{M}_iH_\ell(M \otimes M') = \sum_{f+g=i} \mathfrak{M}_fH_\ell(M) \otimes \mathfrak{M}_gH_\ell(M).$$

- (d) The monodromy operator associated to the dual M^\vee is given by $N_{M^\vee,\ell} = -{}^tN_{M,\ell}$ and $\mathfrak{M}_iH_\ell(M^\vee) = (\mathfrak{M}_{-i-1}H_\ell(M))^\perp$.

2 Weight Filtration

Definition 2.1. A Γ_v -mdoule U is said to have a (unique) weight filtration \mathfrak{W}_\bullet if it verifies the following properties :

- (i) \mathfrak{W}_\bullet is an increasing filtration $\cdots \mathfrak{W}_{i-1}U \subset \mathfrak{W}_iU \subset \mathfrak{W}_{i+1}U \cdots$ of Γ_v -representations such that $\mathfrak{W}_iU = 0$ for sufficiently small i and $\mathfrak{W}_{i+1}U = U$ for sufficiently large i ;

- (ii) the action of I_v on $\mathrm{Gr}_i^{\mathfrak{w}} = \mathfrak{W}_i U / \mathfrak{W}_{i-1} U$ is through a finite quotient;
- (iii) the eigenvalues of the action of Ψ_v on $\mathrm{Gr}_i^{\mathfrak{w}} = \mathfrak{W}_i U / \mathfrak{W}_{i-1} U$ are all algebraic integers of complex absolute value $q^{i/2}$, where $q = |k_v|$.

Example 2.2. (i) $\mathbb{Q}_\ell(1)$ is a one-dimensional Γ_v -representation which is unramified for every v (such that $p \neq \ell$). Moreover, we know that the action of any arithmetic Frobenius on $\mathbb{Q}_\ell(1)$ is through multiplication by q , hence the action of the geometric Frobenius is through multiplication by q^{-1} . This allows, us to define the weight filtration as following

$$\mathfrak{W}_j \mathbb{Q}_\ell(1) = \mathbb{Q}_\ell(1) \text{ for } j \geq -2 \text{ and } \mathfrak{W}_j \mathbb{Q}_\ell(1) = 0 \text{ for } j < -2.$$

- (ii) If Y is any smooth proper variety over F_v and $U := \mathrm{H}_{\acute{e}t}^r(Y_{\bar{F}_v}, \mathbb{Q}_\ell)$, then U has a weight filtration. Indeed, the case where Y has a proper strictly semistable model over \mathcal{O}_v , the existence of this filtration follows from the Weil conjectures and the Rappoport-Zink weight spectral sequence [4]. The general case can be reduced to this case using de jong's alterations [2].

Lemma 2.3. *Let U, U' be Γ_v -modules.*

- (a) *If U has weight filtration and U' is a sub-module of U , then U' has weight filtration.*
- (b) *If both U and U' has weight filtrations, then the Γ_v -modules $U \oplus U'$ and $U \otimes U'$ has weight filtrations.*
- (c) *If U has weight filtration then the dual U_1^\vee has weight filtration.*

Proof. (a) We verify that $\mathfrak{W}_i U' := \mathfrak{W}_i U \cap U'$ defines the weight filtration on U' .

- (b) We verify that $\mathfrak{W}_i(U \oplus U') := \mathfrak{W}_i U \oplus \mathfrak{W}_i U'$ defines the weight filtration on $U \oplus U'$ and

$$\mathfrak{W}_i(U \otimes U') = \sum_{f+g=i} \mathfrak{W}_f U \otimes \mathfrak{W}_g U',$$

defines the weight filtration on $U \otimes U'$.

- (c) we verify that the dual filtration $\mathfrak{W}_i U^\vee := (\mathfrak{W}_{-i-1} U)^\perp$ defines the weight filtration on U^\vee .

□

Proposition 2.4. *Let M be any motive over F . Then for every prime number $\ell \neq p$, there exists a weight filtration on the Γ_v -module $H_\ell(M)$.*

Proof. Let $M = (X, p, n)$ be a motive over F , where X is a smooth projective algebraic variety, $p \in \mathrm{Corr}^0(X \times X)$ and $n \in \mathbb{Z}$. The ℓ -adic realization of M is given by

$$H_\ell(M) = \bigoplus_{r=0}^{2 \dim X} p^* \mathrm{H}_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n),$$

where p^* denotes the image of p under the Γ_F -invariant isomorphism

$$\mathrm{H}_{\acute{e}t}^{2 \dim X}(X_{\bar{F}} \times X_{\bar{F}}, \mathbb{Q}_\ell)(\dim X) \cong \bigoplus_{r=0}^{2 \dim X} \mathrm{End}(\mathrm{H}_{\acute{e}t}^r(X_{\bar{F}}, \mathbb{Q}_\ell)).$$

Now, $p^*H_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)$ is a Γ_v -sub-representation of $H_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell) \cong H_{\acute{e}t}^{r+2n}(X_{\bar{F}_v}, \mathbb{Q}_\ell)$. Therefore, by 2.3(a) the weight filtration on $H_{\acute{e}t}^r(X_{\bar{F}_v}, \mathbb{Q}_\ell)$ (2.2(ii)), induces weight filtration on $p^*H_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)$. Now, it follows from 2.2(i) and 2.3(b), that there exists a weight filtration on $p^*H_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell) \otimes \mathbb{Q}_\ell(1)^{\otimes n} = p^*H_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n)$. Finally, using 2.3[(b)] we conclude that there exists a weight filtration on $\bigoplus_{r=0}^{2 \dim X} p^*H_{\acute{e}t}^{r+2n}(X_{\bar{F}}, \mathbb{Q}_\ell)(n) = H_\ell(M)$. \square

3 On the purity of Monodromy filtration

Definition 3.1. Let M be a motive over F . The weight w of M is defined to be the weight of the underlying Hodge structure of the Betti realization $H_\tau(M)$. It is independent of the choice of the embedding τ .

The weight monodromy conjecture [3] predicts that the monodromy filtration on the absolute ℓ -adic cohomology of any proper smooth variety over F_v , coincides with the weight filtration but upto a shift. This conjecture has the following analogue for motives.

Conjecture 3.2. *Let M be any motive over F of weight w . Then,*

$$\mathfrak{M}_i H_\ell(M) = \mathfrak{M}_{i+w} H_\ell(M) \quad \forall i.$$

Proposition 3.3. *Let M be an Artin motive over F . Then, conjecture 3.2 holds for M .*

Proof. We know that the category of Artin motives $\mathcal{M}_F^0 \cong \text{Rep}_{\mathbb{Q}}(\Gamma_F)$. Let $\rho : \Gamma_F \rightarrow \text{GL}(V)$ be the image of M under this isomorphism. Then we have,

$$H_\tau(M) = V \quad \text{and} \quad H_\ell(M) = V \otimes \mathbb{Q}_\ell, \quad \text{as a } \Gamma_F \text{ - module.}$$

Thus, the ℓ -adic representation $\rho_\ell : \Gamma_F \rightarrow \text{GL}(H_\ell(M))$ associated to M , is simply $\rho \otimes \mathbb{Q}_\ell$. Now, as ρ is continuous for the Krull topology on Γ_F and the discrete topology on V , there exists a finite extension F'/F , such that $\text{Gal}(\bar{F}/F')$ acts trivially on V . In particular, if v' is an extension of v to F' , then $\text{Gal}(\bar{F}_v/F'_{v'})$ and hence the inertia group corresponding to this extension, acts trivially on V . Thus, the ℓ -adic representation ρ_ℓ is potentially unramified at every v such that $p \neq \ell$. This implies that the monodromy operators corresponding to the local representations $\rho_{\ell,v}$ are all trivial. In particular, the monodromy filtration \mathfrak{M}_\bullet of $H_\ell(M)$ is trivial, i.e.

$$\mathfrak{M}_j H_\ell(M) = H_\ell(M) \quad \text{for } j \geq 0 \quad \text{and} \quad \mathfrak{M}_j H_\ell(M) = 0 \quad \text{for } j < 0.$$

Now, as the weight of an Artin motive is 0, in order to prove the proposition, we only need to show that the weight filtration on $H_\ell(M)$ coincides with the monodromy filtration on $H_\ell(M)$. It is obvious that $\mathfrak{M}_\bullet H_\ell(M)$, verifies property (i) of 2.1. As, ρ_ℓ is potentially unramified at every v , so $\mathfrak{M}_\bullet H_\ell(M)$ also verifies property (ii) of 2.1. Finally, note that as $\rho_\ell = \rho \otimes \mathbb{Q}_\ell$, the eigenvalues of the action of Ψ_v on $\text{Gr}_0^{\mathfrak{M}} H_\ell(M) = H_\ell(M) = V \otimes \mathbb{Q}_\ell$ are algebraic integers (independent of ℓ). Now, as $\text{Gal}(\bar{F}/F')$ acts trivially on V , the eigenvalues of the action of $\Psi^{[k'_v:k_v]}$ on V are all 1, where k'_v denotes the residue field of $F'_{v'}$. This implies that the eigenvalue of the action of Ψ_v on V is a root of unity and hence of complex absolute value $q^{0/2} = 1$. This establishes property (iii) of 2.1. By the uniqueness of the weight filtration, we conclude that on $H_\ell(M)$, the monodromy filtration coincides with the weight filtration. \square

Lemma 3.4. *The conjecture 3.2 is stable under sub-objects, direct sums, tensor products and taking duals of motives.*

Proof. This follows by combining 1.2 and 2.3. \square

Theorem 3.5. *Let \mathcal{M}_F^{av} be the Tannakian category of motives \otimes -generated by $h^1(A)$'s of abelian varieties and the Artin motives. Then conjecture 3.2 holds for every $M \in \text{Ob}(\mathcal{M}_F^{av})$.*

Proof. Let $\mathcal{C} = \{ Y \in \text{Ob}(\mathcal{M}_F) \mid Y = h^1(A) \text{ of an abelian variety } A \text{ or is an Artin motive} \}$. As \mathcal{M}_F^{av} is generated by \mathcal{C} , every object $M \in \mathcal{M}_F^{av}$ is isomorphic to a sub-quotient of an object of the form $P(X_j)$ (a polynomial expression), where $\{X_j\}_{j \in J}$ consists of objects of \mathcal{C} or their duals and $P(t_j) \in \mathbb{N}[t_j]_{j \in J}$; multiplication in $P(X_j)$ is interpreted as \otimes and addition as \oplus . Thus, in view of 3.4, it suffices to verify the conjecture for an expression $P(X_j)$ as above. Now, for $h^1(A)$ of an abelian variety A , it is well known [1] that the conjecture holds true and for Artin motives we've 3.3. The theorem now follows from 3.4. \square

Corollary 3.6. *Let X be a smooth projective variety over F which is a finite product of*

- (i) *hyperkähler varieties of $K3^{[n]}$ type;*
- (ii) *unirational varieties of $\dim \leq 4$;*
- (iii) *uniruled varieties of $\dim \leq 3$;*
- (iv) *cubic 4-folds;*
- (v) *moduli spaces of stable vector bundles of coprime rank and degree over smooth projective curves;*
- (vi) *Fermat hypersurfaces;*
- (vii) *curves;*
- (viii) *surfaces;*
- (ix) *abelian varieties;*

then the Weight monodromy conjecture holds for $H^i(X_{/\bar{F}_v}, \mathbb{Q}_\ell)$ for any $i \geq 0$.

Proof. In view of 3.4, it suffices to establish the result for each of the varieties in the list. If X is either a hyperkähler variety of $K3^{[n]}$ type, a cubic 4-fold, a unirational variety of $\dim \leq 3$, a Fermat hypersurface, moduli space of stable vector bundles of coprime rank and degree over a smooth projective curve, then the motive $h(X) \in \text{Ob}(\mathcal{M}_F^{av})$. So, in these cases the weight monodromy conjecture follows from 3.5.

The case of curves [1] and surfaces [4],[5], are well known. The case of unirational varieties of dimension 4, follows from the fact that the motive of any such variety is an object of the Tannakian category generated by the motive of variety of dimension at most 2 (for which the WMC is known). Similarly, the motive of a uniruled 3-fold belongs to the Tannakian category of motives generated by a surface, and hence the WMC holds in this case as well.

In the case of any abelian variety A the only new thing that we obtain is that WMC holds for the higher cohomology groups $H^i(A_{/\bar{F}_v}, \mathbb{Q}_\ell)$ ($i \geq 2$) as well, since the motive $h^i(A) = \wedge^i h^1(A)$. \square

References

- [1] *Groupes de monodromie en géométrie algébrique. I.* Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, Berlin, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I), Dirigé par A. Grothendieck. Avec la collaboration de M. Raynaud et D. S. Rim.
- [2] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.
- [3] Pierre Deligne. Théorie de Hodge. I. In *Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1*, pages 425–430. Gauthier-Villars, Paris, 1971.
- [4] M. Rapoport and Th. Zink. Über die lokale Zetafunktion von Shimuravarietäten. Monodromiefiltration und verschwindende Zyklen in ungleicher Charakteristik. *Invent. Math.*, 68(1):21–101, 1982.
- [5] Takeshi Saito. Weight spectral sequences and independence of l . *J. Inst. Math. Jussieu*, 2(4):583–634, 2003.