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Abstract. This paper aims at justifying the low Mach number convergence to the incompressible Navier-Stokes equations for viscous compressible flows in the *ill-prepared data* case. The fluid domain is either the whole space, or the torus.

A number of works have been dedicated to this classical issue, all of them being, to our knowledge, related to L^2 spaces and to energy type arguments. In the present paper, we investigate the low Mach number convergence *in the L^p type critical regularity framework*. More precisely, in the barotropic case, the divergence-free part of the initial velocity field just has to be bounded in the critical Besov space $\dot{B}^{d/p-1}_{p,r}\cap\dot{B}^{-1}_{\infty,1}$ for some suitable $(p,r)\in[2,4]\times[1,+\infty]$. We still require L^2 type bounds on the low frequencies of the potential part of the velocity and on the density, though, an assumption which seems to be unavoidable in the ill-prepared data framework, because of acoustic waves.

In the last part of the paper, our results are extended to the full Navier-Stokes system for heat conducting fluids.

We are concerned with the study of the convergence of the solutions to the compressible Navier-Stokes equations when the Mach number ε goes to 0. In the barotropic case, the system under consideration reads

$$(NSC_\varepsilon)\quad\begin{cases}\partial_t\rho^\varepsilon+\operatorname{div}(\rho^\varepsilon u^\varepsilon)=0,\\ \partial_t(\rho^\varepsilon u^\varepsilon)+\operatorname{div}(\rho^\varepsilon u^\varepsilon\otimes u^\varepsilon)-\operatorname{div}\big(2\mu(\rho^\varepsilon)D(u^\varepsilon)+\lambda(\rho^\varepsilon)\operatorname{div}u^\varepsilon\,\mathrm{Id}\big)+\dfrac{\nabla P^\varepsilon}{\varepsilon^2}=0,\end{cases}$$

where $\rho^\varepsilon=\rho^\varepsilon(t,x)\in\mathbb{R}_+$ stands for the density, $u^\varepsilon=u^\varepsilon(t,x)\in\mathbb{R}^d$, for the velocity field, $P^\varepsilon=P(\rho^\varepsilon)\in\mathbb{R}$ is the pressure, $\lambda=\lambda(\rho^\varepsilon)$ and $\mu=\mu(\rho^\varepsilon)$ are the (given) viscosity functions that are assumed to satisfy $\mu>0$ and $\lambda+2\mu>0$. Finally, $D(u^\varepsilon)$ stands for the deformation tensor, that is $(D(u^\varepsilon))_{ij}:=\frac{1}{2}(\partial_iu^{\varepsilon,j}+\partial_ju^{\varepsilon,i})$. We assume that the functions P, λ and μ are smooth, and we restrict our attention to the case where the fluid domain is either the whole space $\mathbb{R}^d$ or the periodic box $\mathbb{T}^d$ (combinations such as $\mathbb{T}\times\mathbb{R}^{d-1}$ and so on may be considered as well).

At the formal level, in the low Mach number asymptotic, we expect ρ^ε to tend to some constant positive density ρ^* (say $\rho^*=1$ for simplicity) and u^ε to tend to some vector field v satisfying the (homogeneous) *incompressible* Navier-Stokes equations:

$$(NS)\qquad\begin{cases}\partial_tv+v\cdot\nabla v-\mu(1)\Delta v+\nabla\Pi=0,\\ \operatorname{div}v=0.\end{cases}$$

This heuristics has been justified rigorously in different contexts (see e.g. [11, 12, 15, 16, 17, 18, 20, 24, 25, 26, 28, 29, 30]). In the present paper, we want to consider *ill-prepared data* of the form $\rho_0^\varepsilon=\rho^*+\varepsilon a_0^\varepsilon$ and u_0^ε where $(a_0^\varepsilon,u_0^\varepsilon)$ are bounded in a sense that will be specified later on. Assuming (with no loss of generality) that $P'(\rho^*)=\rho^*=1$ and setting $\rho^\varepsilon=1+\varepsilon a^\varepsilon$,

we get the following system for $(a^\varepsilon, u^\varepsilon)$:

$$\left\{\begin{aligned} &\partial_t a^\varepsilon + \frac{\operatorname{div} u^\varepsilon}{\varepsilon} = -\operatorname{div}(a^\varepsilon u^\varepsilon),\\ &\partial_t u^\varepsilon + u^\varepsilon\cdot\nabla u^\varepsilon - \frac{\mathcal{A}u^\varepsilon}{1+\varepsilon a^\varepsilon} + \frac{\nabla a^\varepsilon}{\varepsilon} = \frac{k(\varepsilon a^\varepsilon)}{\varepsilon}\nabla a^\varepsilon\\ &\qquad\qquad + \frac{1}{1+\varepsilon a^\varepsilon}\operatorname{div}\bigl(2\widetilde{\mu}(\varepsilon a^\varepsilon)D(u^\varepsilon) + \widetilde{\lambda}(\varepsilon a^\varepsilon)\operatorname{div} u^\varepsilon\,\mathrm{Id}\bigr), \end{aligned}\right. \tag{0.1}$$

where $\mathcal{A} := \mu\Delta + (\lambda+\mu)\nabla\operatorname{div}$ with $\lambda := \lambda(1)$ and $\mu := \mu(1)$,

$$k(z) := -\frac{P'(1+z)}{1+z} + P'(1), \quad \widetilde{\mu}(z) := \mu(1+z) - \mu(1) \ \text{ and } \ \widetilde{\lambda}(z) := \lambda(1+z) - \lambda(1).$$

In what follows, the exact value of functions k, $\widetilde{\lambda}$ and $\widetilde{\mu}$ will not matter. We shall only use that those functions are smooth and vanish at 0.

We strive for *critical regularity assumptions* consistent with those of the well-posedness issue for the limit system (NS). At this stage, let us recall that, by definition, critical spaces for (NS) are norm invariant for all $\ell > 0$ by the scaling transformations $T_\ell : v(t,x) \longmapsto \ell v(\ell^2 t, \ell x)$, in accordance with the fact that v is a solution to (NS) if and only if so does $T_\ell v$ (provided the initial data has been changed accordingly of course).

As first observed in [9], in the context of the barotropic Navier-Stokes equations (0.1), the relevant scaling transformations read

$$(a,u)(t,x) \longmapsto (a, \ell u)(\ell^2 t, \ell x), \qquad \ell > 0, \tag{0.2}$$

which suggest our taking initial data (a_0, u_0) in spaces invariant by $(a_0,u_0)(x) \mapsto (a_0, \ell u_0)(\ell x)$.

In order to be more specific, let us introduce now the notations and function spaces that will be used throughout the paper. For simplicity, we focus on the $\mathbb{R}^d$ case. Similar notations and definitions may be given in the $\mathbb{T}^d$ case.

We are given an homogeneous Littlewood-Paley decomposition $(\dot{\Delta}_j)_{j\in\mathbb{Z}}$ that is a dyadic decomposition in the Fourier space for $\mathbb{R}^d$. One may for instance set $\dot{\Delta}_j := \varphi(2^{-j}D)$ with $\varphi(\xi) := \chi(\xi/2) - \chi(\xi)$, and χ a non-increasing nonnegative smooth function supported in $B(0,4/3)$, and with value 1 on $B(0,3/4)$ (see [2], Chap. 2 for more details).

We then define, for $1 \leq p,r \leq \infty$ and $s \in \mathbb{R}$, the semi-norms

$$\|z\|_{\dot{B}^s_{p,r}} := \bigl\|2^{js}\|\dot{\Delta}_j z\|_{L^p(\mathbb{R}^d)}\bigr\|_{\ell^r(\mathbb{Z})}.$$

Like in [2], we adopt the following definition of homogeneous Besov spaces, which turns out to be well adapted to the study of nonlinear PDEs:

$$\dot{B}^s_{p,r} = \Bigl\{z \in \mathcal{S}'(\mathbb{R}^d) : \|z\|_{\dot{B}^s_{p,r}} < \infty \ \text{ and } \ \lim_{j\to-\infty}\|\dot{S}_j z\|_{L^\infty} = 0\Bigr\} \quad \text{with} \quad \dot{S}_j := \chi(2^{-j}D).$$

As we shall work with *time-dependent functions* valued in Besov spaces, we introduce the norms:

$$\|u\|_{L^q_T(\dot{B}^s_{p,r})} := \bigl\|\|u(t,\cdot)\|_{\dot{B}^s_{p,r}}\bigr\|_{L^q(0,T)}.$$

As pointed out in [6], when using parabolic estimates in Besov spaces, it is somehow natural to take the time-Lebesgue norm *before* performing the summation for computing the Besov norm. This motivates our introducing the following quantities:

$$\|u\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} := \bigl\|(2^{js}\|\dot{\Delta}_j u\|_{L^q_T(L^p)})\bigr\|_{\ell^r(\mathbb{Z})}.$$

The index T will be omitted if $T = +\infty$ and we shall denote by $\widetilde{\mathcal{C}}_b(\dot{B}^s_{p,r})$ the subset of functions of $\widetilde{L}^\infty(\dot{B}^s_{p,r})$ which are also continuous from $\mathbb{R}_+$ to $\dot{B}^s_{p,r}$.

Let us emphasize that, owing to Minkowski inequality, we have if $r \leq q$

$$\|z\|_{L^q_T(\dot{B}^s_{p,r})} \leq \|z\|_{\widetilde{L}^q_T(\dot{B}^s_{p,r})}$$

with equality if and only if $q = r$. Of course, the opposite inequality occurs if $r \geq q$.

An important example where those nonclassical norms are suitable is the heat equation

$$\partial_t z - \mu\Delta z = f, \qquad z|_{t=0} = z_0 \tag{0.3}$$

for which the following family of inequalities holds true (see [2, 6]):

$$\|z\|_{\widetilde{L}^m_T(\dot{B}^{s+2/m}_{p,r})} \leq C\big(\|z_0\|_{\dot{B}^s_{p,r}} + \|f\|_{\widetilde{L}^1_T(\dot{B}^s_{p,r})}\big) \tag{0.4}$$

for any $T > 0$, $1 \leq m, p, r \leq \infty$ and $s \in \mathbb{R}$.

Restricting ourselves to the case of *small and global-in-time* solutions (just for simplicity), the reference global well-posedness result for (NS) that we have in mind reads as follows[1]:

Theorem 0.1. *Let $u_0 \in \dot{B}^{d/p-1}_{p,r}$ with $\operatorname{div} u_0 = 0$ and $p < \infty$, and $r \in [1, +\infty]$. There exists $c > 0$ such that if*

$$\|u_0\|_{\dot{B}^{d/p-1}_{p,r}} \leq c\mu$$

then (NS) has a unique global solution u in the space

$$\widetilde{L}^\infty(\mathbb{R}_+; \dot{B}^{d/p-1}_{p,r}) \cap \widetilde{L}^1(\mathbb{R}_+; \dot{B}^{d/p+1}_{p,r}),$$

which is also in $\mathcal{C}(\mathbb{R}_+; \dot{B}^{d/p-1}_{p,r})$ if $r < \infty$. Besides, we have

$$\|u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r})} + \mu\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})} \leq C\|u_0\|_{\dot{B}^{d/p-1}_{p,r}}, \tag{0.5}$$

for some constant C depending only on d and p.

Although Theorem 0.1 is not related to energy arguments, to our knowledge, all the mathematical results proving the convergence of (NSC_ε) to (NS), strongly rely on the use of L^2 type norms in order to get estimates independent of ε. This is due to the presence of singular first order skew symmetric terms (which disappear when performing L^2 or H^s estimates) in the following linearized equations of (0.1):

$$\begin{cases} \partial_t a^\varepsilon + \dfrac{\operatorname{div} u^\varepsilon}{\varepsilon} = f^\varepsilon, \\ \partial_t u^\varepsilon - \mathcal{A}u^\varepsilon + \dfrac{\nabla a^\varepsilon}{\varepsilon} = g^\varepsilon. \end{cases} \tag{0.6}$$

However, it is clear that those singular terms do not affect the divergence-free part $\mathcal{P}u^\varepsilon$ of the velocity, which just satisfies the heat equation (0.3). We thus expect handling $\mathcal{P}u^\varepsilon$ to be doable by means of a L^p type approach similar to that of Theorem 0.1. At the same time, for low frequencies ('low' meaning small with respect to $(\varepsilon\nu)^{-1}$), the singular terms tend to dominate the evolution of a^ε and $\mathcal{Q}u^\varepsilon$, which precludes a L^p-type approach with $p \neq 2$, as the wave equation is ill-posed in such spaces. Finally, for very high frequencies (that is greater than $(\varepsilon\nu)^{-1}$), it is well known that a^ε and $\mathcal{Q}u^\varepsilon$ tend to behave as the solutions of

[1]The statement in the Sobolev framework is due to H. Fujita and T. Kato in [19]. Data in general critical Besov spaces, with a slightly different solution space, have been considered by H. Kozono and M. Yamazaki in [27], and by M. Cannone, Y. Meyer and F. Planchon in [4]. The above statement has been proved exactly under this shape by J.-Y. Chemin in [6].

a damped equation and of a heat equation, respectively, and are thus tractable in L^p type spaces. Besides, keeping in mind the notion of critical space introduced in (0.2), it is natural to work at the same level of regularity for ∇a^ε and $\mathcal{Q}u^\varepsilon$ (see e.g. [2], Chap. 10, or [5] for more explanations). The rest of the paper is devoted to clarifying this heuristics, first in the barotropic case (Sections 1 to 4), and next for the full Navier-Stokes-Fourier system (Section 5).

1. Main results

Before stating our main results, let us introduce some notation. From now on, we agree that for $z \in \mathcal{S}'(\mathbb{R}^d)$,

$$z^{\ell,\alpha} := \sum_{2^j\alpha\leq 2^{j_0}} \dot\Delta_j z \quad\text{and}\quad z^{h,\alpha} := \sum_{2^j\alpha> 2^{j_0}} \dot\Delta_j z, \tag{1.7}$$

for some large enough nonnegative integer j_0 depending only on p, d, and on the functions k, λ/ν, μ/ν with $\nu := \lambda+2\mu$. The corresponding "truncated" semi-norms are defined as follows:

$$\|z\|^{\ell,\alpha}_{\dot B^\sigma_{p,r}} := \|z^{\ell,\alpha}\|_{\dot B^\sigma_{p,r}} \quad\text{and}\quad \|z\|^{h,\alpha}_{\dot B^\sigma_{p,r}} := \|z^{h,\alpha}\|_{\dot B^\sigma_{p,r}}.$$

Let $\widetilde\varepsilon := \varepsilon\nu$. Based on the heuristics of the introduction, it is natural to consider families of data $(a_0^\varepsilon, u_0^\varepsilon)$ so that

- $(a_0^\varepsilon, \mathcal{Q}u_0^\varepsilon)^{\ell,\widetilde\varepsilon} \in \dot B^{d/2-1}_{2,1}$,
- $(a_0^\varepsilon)^{h,\widetilde\varepsilon} \in \dot B^{d/p}_{p,1}$, $\quad(\mathcal{Q}u_0^\varepsilon)^{h,\widetilde\varepsilon} \in \dot B^{d/p-1}_{p,1}$,
- $\mathcal{P}u_0^\varepsilon \in \dot B^{d/p-1}_{p,r} \cap \dot B^{-1}_{\infty,1}$.

Recall that $\dot B^{d/p-1}_{p,r}$ is only embedded in $\dot B^{-1}_{\infty,r}$. The reason why we prescribe the slightly stronger assumption $\dot B^{-1}_{\infty,1}$ for $\mathcal{P}u_0^\varepsilon$ is that we need the constructed velocity to have gradient in $L^1(\mathbb{R}_+;L^\infty)$ in order to preserve the Besov regularity of a^ε through the mass equation. Indeed, it is well known that for a solution z to the free heat equation, the norm of ∇z in $L^1(\mathbb{R}_+;L^\infty)$ is equivalent to that of z_0 in $\dot B^{-1}_{\infty,1}$ (see e.g. [2], Chap. 2).

Our assumptions on the data induce us to look for a solution to (0.1) in the space $X^{p,r}_{\varepsilon,\nu}$ of functions (a,u) such that

- $(a^{\ell,\widetilde\varepsilon}, \mathcal{Q}u^{\ell,\widetilde\varepsilon}) \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{d/2-1}_{2,1}) \cap L^1(\mathbb{R}_+;\dot B^{d/2+1}_{2,1})$,
- $a^{h,\widetilde\varepsilon} \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{d/p}_{p,1}) \cap L^1(\mathbb{R}_+;\dot B^{d/p}_{p,1})$,
- $\mathcal{Q}u^{h,\widetilde\varepsilon} \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{d/p-1}_{p,1}) \cap L^1(\mathbb{R}_+;\dot B^{d/p+1}_{p,1})$,
- $\mathcal{P}u \in \widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1}) \cap \widetilde L^1(\mathbb{R}_+;\dot B^{d/p+1}_{p,r}\cap\dot B^{1}_{\infty,1})$ (only weak continuity in $\dot B^{d/p-1}_{p,r}$ if $r=\infty$).

We shall endow that space with the norm:

$$\begin{aligned}\|(a,u)\|_{X^{p,r}_{\varepsilon,\nu}} :=& \|(a,\mathcal{Q}u)\|^{\ell,\widetilde\varepsilon}_{\widetilde L^\infty(\dot B^{d/2-1}_{2,1})} + \|\mathcal{Q}u\|^{h,\widetilde\varepsilon}_{\widetilde L^\infty(\dot B^{d/p-1}_{p,1})} + \|\mathcal{P}u\|_{\widetilde L^\infty(\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1})} + \widetilde\varepsilon\|a\|^{h,\widetilde\varepsilon}_{\widetilde L^\infty(\dot B^{d/p}_{p,1})}\\ &+\nu\|(a,\mathcal{Q}u)\|^{\ell,\widetilde\varepsilon}_{L^1(\dot B^{d/2+1}_{2,1})} + \nu\|\mathcal{Q}u\|^{h,\widetilde\varepsilon}_{L^1(\dot B^{d/p+1}_{p,1})} + \nu\|\mathcal{P}u\|_{\widetilde L^1(\dot B^{d/p+1}_{p,r}\cap\dot B^{1}_{\infty,1})} + \varepsilon^{-1}\|a\|^{h,\widetilde\varepsilon}_{L^1(\dot B^{d/p}_{p,1})}.\end{aligned}$$

Our main result reads as follows:

Theorem 1.1. *Assume that the fluid domain is either $\mathbb{R}^d$ or $\mathbb{T}^d$, that the initial data $(a_0^\varepsilon, u_0^\varepsilon)$ are as above with $1\leq r\leq p/(p-2)$ and that, in addition,*

- *Case $d=2$: $2\leq p<4$,*

- *Case $d=3$: $2\leq p\leq 4$,*
- *Case $d\geq 4$: $2\leq p<2d/(d-2)$, or $p=2d/(d-2)$ and $r=1$.*

Let $\widetilde{\varepsilon}:=\varepsilon\nu$. There exists a constant η independent of ε and of ν such that if

$$C_0^{\varepsilon,\nu}:=\|(a_0^\varepsilon,\mathcal{Q}u_0^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\dot B^{d/2-1}_{2,1}}+\|\mathcal{Q}u_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{d/p-1}_{p,1}}+\|\mathcal{P}u_0^\varepsilon\|_{\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1}}+\widetilde{\varepsilon}\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{d/p}_{p,1}}\leq\eta\nu, \tag{1.8}$$

then System (0.1) with initial data $(a_0^\varepsilon,u_0^\varepsilon)$ has a global solution $(a^\varepsilon,u^\varepsilon)$ in the space $X^{p,r}_{\varepsilon,\nu}$ with, for some constant C independent of ε and ν,

$$\|(a^\varepsilon,u^\varepsilon)\|_{X^{p,r}_{\varepsilon,\nu}}\leq CC_0^{\varepsilon,\nu}. \tag{1.9}$$

In addition, $\mathcal{Q}u^\varepsilon$ converges weakly to 0 when ε goes to 0, and, if $\mathcal{P}u_0^\varepsilon\rightharpoonup v_0$ then $\mathcal{P}u^\varepsilon$ converges in the sense of distributions to the solution of

$$\partial_t u+\mathcal{P}(u\cdot\nabla u)-\mu\Delta u=0,\qquad u|_{t=0}=v_0. \tag{1.10}$$

Finally, if the fluid domain is $\mathbb{R}^d$ and $d\geq 3$ then we have

$$\nu^{1/2}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}\leq CC_0^{\varepsilon,\nu}\widetilde{\varepsilon}^{1/2-1/p}\quad\text{and}$$

$$\|\mathcal{P}u^\varepsilon-v\|_{\widetilde{L}^\infty(\dot B^{(d+1)/p-3/2}_{p,r})}+\mu\|\mathcal{P}u^\varepsilon-v\|_{\widetilde{L}^1(\dot B^{(d+1)/p+1/2}_{p,r})}\leq C\big(\|\mathcal{P}u_0^\varepsilon-v_0\|_{\dot B^{(d+1)/p-3/2}_{p,r}}+C_0^{\varepsilon,\nu}\widetilde{\varepsilon}^{1/2-1/p}\big).$$

In the $\mathbb{R}^2$ case, we have,

$$\nu^{1/2}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\widetilde{L}^2(\dot B^{(c+2)/p-c/2}_{p,1})}\leq CC_0^{\varepsilon,\nu}\widetilde{\varepsilon}^{c(1/2-1/p)}\quad\text{and}$$

$$\|\mathcal{P}u^\varepsilon-v\|_{\widetilde{L}^\infty(\dot B^{(c+2)/p-c/2-1}_{p,r})}+\mu\|\mathcal{P}u^\varepsilon-v\|_{\widetilde{L}^1(\dot B^{(c+2)/p-c/2+1}_{p,r})}\leq C\big(\|\mathcal{P}u_0^\varepsilon-v_0\|_{\dot B^{(c+2)/p-c/2-1}_{p,r}}+C_0^{\varepsilon,\nu}\widetilde{\varepsilon}^{c(1/2-1/p)}\big)$$

where the constant c verifies the conditions $0\leq c\leq 1/2$ and $c<(8-2p)/(p-2)$.

Some remarks are in order:

(1) According to [14], uniqueness holds true if $r=1$. We conjecture that it also holds in the other cases but, to the best of our knowledge, the question has not been addressed.

(2) The first part of the theorem (the global existence issue) may be extended to $2d/(d+2)\leq p<2$ and all $r\in[1,\infty]$ provided the following smallness condition is fulfilled:

$$\|(a_0^\varepsilon,\mathcal{Q}u_0^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\dot B^{d/2-1}_{2,1}}+\|\mathcal{Q}u_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{d/2-1}_{2,1}}+\|\mathcal{P}u_0^\varepsilon\|_{\dot B^{d/2-1}_{2,r}\cap\dot B^{-1}_{\infty,1}}+\widetilde{\varepsilon}\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot B^{d/2}_{2,1}}\leq\eta\nu.$$

Indeed, Theorem 1.1 provides a global small solution in $X^{2,r}_{\varepsilon,\nu}$. Therefore it is only a matter of checking that the constructed solution has additional regularity $X^{p,r}_{\varepsilon,\nu}$. This may be checked by following steps 3 and 4 of the proof below, knowing already that the solution is in $X^{2,r}_{\varepsilon,\nu}$. The condition that $2d/(d+2)\leq p$ comes from the part $u^{\ell,\widetilde{\varepsilon}}\cdot\nabla a$ of the convection term in the mass equation, as $\nabla u^{\ell,\widetilde{\varepsilon}}$ is only in $L^1(\mathbb{R}_+;\dot B^{d/2}_{2,1})$, and the regularity to be transported is $\dot B^{d/p}_{p,1}$. Hence we need to have $d/p\leq d/2+1$ (see e.g. Chap. 3 of [2]). The same condition appears when handling $k(\varepsilon a^\varepsilon)\nabla a^\varepsilon$.

As we believe the case $p<2$ to be somewhat anecdotic (it is just a regularity result), we decided to concentrate on the case $p\geq 2$ in the rest of the paper.

(3) We can afford source terms in the mass and velocity equations, the regularity of which is given by scaling considerations.

(4) We also expect results in the same spirit (but only local-in-time) to be provable for large data, as in [11, 12].

(5) To keep the paper a reasonable size, we also refrained to establish more accurate convergence results in the case of periodic boundary conditions, based on Schochet's filtering method (see [12] for more details on that issue if $p = 2$).

2. The proof of global existence for fixed ε and ν

Recall that $\nu := \lambda + 2\mu$. Performing the change of unknowns

$$(a,u)(t,x) := \varepsilon(a^\varepsilon, u^\varepsilon)(\varepsilon^2\nu t, \varepsilon\nu x) \tag{2.11}$$

and the change of data

$$(a_0,u_0)(x) := \varepsilon(a_0^\varepsilon, u_0^\varepsilon)(\varepsilon\nu x) \tag{2.12}$$

reduces the proof of the global existence to the case $\nu = 1$ and $\varepsilon = 1$. So in the rest of this section, we assume that $\varepsilon = \nu = 1$, and simply denote

$$z^\ell := z^{\ell,1} \quad \text{and} \quad z^h := z^{h,1}, \tag{2.13}$$

$$\|z\|^\ell_{\dot B^\sigma_{p,r}} := \|z^{\ell,1}\|_{\dot B^\sigma_{p,r}} \quad \text{and} \quad \|z\|^h_{\dot B^\sigma_{p,r}} := \|z^{h,1}\|_{\dot B^\sigma_{p,r}}. \tag{2.14}$$

The threshold between low and high frequencies will be set at 2^{j_0} for some large enough nonnegative integer j_0 depending only on d, k, $\widetilde\mu/\nu$ and $\widetilde\lambda/\nu$.

Resuming to the original variables will yield the desired uniform estimate (1.9) under Condition (1.8). Indeed, we notice that we have up to some harmless constant:

$$\begin{aligned}\|(a_0^\varepsilon, \mathcal{Q}u_0^\varepsilon)\|^{\ell,\widetilde\varepsilon}_{\dot B^{d/2-1}_{2,1}} + \|\mathcal{Q}u_0^\varepsilon\|^{h,\widetilde\varepsilon}_{\dot B^{d/p-1}_{p,1}} + \|\mathcal{P}u_0^\varepsilon\|_{\dot B^{d/p-1}_{p,r}\cap \dot B^{-1}_{\infty,1}} + \widetilde\varepsilon\|a_0^\varepsilon\|_{\dot B^{d/p}_{p,1}}\\ = \nu\big(\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot B^{d/2-1}_{2,1}} + \|\mathcal{Q}u_0\|^h_{\dot B^{d/p-1}_{p,1}} + \|\mathcal{P}u_0\|_{\dot B^{d/p-1}_{p,r}\cap \dot B^{-1}_{\infty,1}} + \|a_0\|_{\dot B^{d/p}_{p,1}}\big)\end{aligned}$$

and

$$\|(a^\varepsilon,u^\varepsilon)\|_{X^{p,r}_{\varepsilon,\nu}} = \nu\|(a,u)\|_{X^{p,r}_{1,1}}.$$

2.1. **A priori estimates of the solutions to system** (2.15)**.** In this paragraph, we concentrate on the proof of global estimates for a global smooth solution (a,u) to the following system:

$$\begin{cases}\partial_t a + \operatorname{div} u = -\operatorname{div}(au),\\ \partial_t u + u\cdot\nabla u - \widetilde{\mathcal{A}}u + \nabla a = k(a)\nabla a - J(a)\widetilde{\mathcal{A}}u\\ \qquad\qquad + \dfrac{1}{1+a}\operatorname{div}\left(2\dfrac{\widetilde\mu(a)}{\nu}D(u) + \dfrac{\widetilde\lambda(a)}{\nu}\operatorname{div} u\,\mathrm{Id}\right),\end{cases} \tag{2.15}$$

with k, $\widetilde\lambda$, $\widetilde\mu$ as above, $J(a) := a/(1+a)$ and $\widetilde{\mathcal{A}} := \mathcal{A}/\nu$.

To simplify the presentation we assume the viscosity coefficients λ and μ to be constant (i.e. the last line of the velocity equation in (2.15), is zero). The general case will be discussed at the end of the subsection.

Throughout we make the assumption that

$$\sup_{t\in\mathbb{R}_+,\,x\in\mathbb{R}^d}|a(t,x)| \leq 1/2 \tag{2.16}$$

which will enable us to use freely the composition estimate stated in Proposition 6.1. Note that as $\dot B^{d/p}_{p,1} \hookrightarrow L^\infty$, Condition (2.16) will be ensured by the fact that the constructed solution has small norm in $X^{p,r}_{1,1}$.

Step 1: the incompressible part of the velocity. Projecting the velocity equation onto the set of divergence free vector fields yields

$$\partial_t \mathcal{P}u - \widetilde{\mu}\Delta\mathcal{P}u = -\mathcal{P}\big(J(a)\widetilde{\mathcal{A}}u\big) - \mathcal{P}(u\cdot\nabla u) \quad \text{with} \quad \widetilde{\mu} := \mu/\nu.$$

Hence, using the estimates (0.4) for the heat equation, we get

$$\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r})\cap\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^{d/p-1}_{p,r}} + \|\mathcal{P}(J(a)\nabla^2 u) + \mathcal{P}(u\cdot\nabla u)\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})} \tag{2.17}$$

$$\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})\cap L^1(\dot{B}^{1}_{\infty,1})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^{-1}_{\infty,1}} + \|\mathcal{P}(J(a)\nabla^2 u) + \mathcal{P}(u\cdot\nabla u)\|_{L^1(\dot{B}^{-1}_{\infty,1})}. \tag{2.18}$$

In order to bound the right-hand sides, we use the fact that the 0-th order Fourier multiplier $\mathcal{P}$ maps $\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})$ (or $L^1(\dot{B}^{-1}_{\infty,1})$) into itself. In addition, classical product laws and Proposition 6.1 give (if $p<2d$):

$$\|J(a)\nabla^2 u\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})} \lesssim \|a\|_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})}\|\nabla^2 u\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})},$$
$$\|u\cdot\nabla u\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})} \lesssim \|u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r})}\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}.$$

Because, by Bernstein inequality,

$$\|a\|_{\dot{B}^{d/p}_{p,1}} \lesssim \|a^\ell\|_{\dot{B}^{d/p}_{p,1}} + \|a^h\|_{\dot{B}^{d/p}_{p,1}} \lesssim 2^{j_0}\|a^\ell\|_{\dot{B}^{d/2-1}_{2,1}} + \|a^h\|_{\dot{B}^{d/p}_{p,1}}, \tag{2.19}$$

we deduce from (2.17) that

$$\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r})\cap\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^{d/p-1}_{p,r}} + 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.20}$$

Next, in order to bound the r.h.s. of (2.18), we use Bony's decomposition (see [3] and the definition in appendix):

$$J(a)\nabla^2 u = T_{J(a)}\nabla^2 u + T_{\nabla^2 u}J(a) + R(J(a),\nabla^2 u)$$

and, with the summation convention over repeated indices,

$$(u\cdot\nabla u)^i = T_{u^j}\partial_j u^i + T_{\partial_j u^i}u^j + \partial_j R((\mathcal{P}u)^j, u^i) + R((\mathcal{Q}u)^j,\partial_j u^i) \quad \text{with} \quad i=1,\cdots,d. \tag{2.21}$$

On the one hand, T maps $L^\infty\times\dot{B}^{-1}_{\infty,1}$ and $\dot{B}^{-1}_{\infty,1}\times L^\infty$ in $\dot{B}^{-1}_{\infty,1}$ while R maps $\dot{B}^{d/p}_{p,1}\times\dot{B}^{d/p-1}_{p,\infty}$ in $\dot{B}^{d/p-1}_{p,1}$, if $2\le p<2d$. Hence, taking advantage of functional embeddings (adapted to $\widetilde{L}^m(\dot{B}^s_{p,r})$ spaces),

$$\|J(a)\nabla^2 u\|_{L^1(\dot{B}^{-1}_{\infty,1})} \lesssim \|a\|_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})}\|\nabla^2 u\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})}. \tag{2.22}$$

On the other hand, thanks to the fact that, if $2\le p<2d$,

$$\begin{aligned}
\|T_{u^j}\partial_j u^i\|_{L^1(\dot{B}^{-1}_{\infty,1})} &\lesssim \|u^j\|_{L^\infty(\dot{B}^{-1}_{\infty,1})}\|\partial_j u^i\|_{L^1(\dot{B}^{0}_{\infty,\infty})}\\
\|T_{\partial_j u^i}u^j\|_{L^1(\dot{B}^{-1}_{\infty,1})} &\lesssim \|\partial_j u^i\|_{L^\infty(\dot{B}^{-2}_{\infty,1})}\|u^j\|_{L^1(\dot{B}^{1}_{\infty,\infty})}\\
\|\partial_j R((\mathcal{P}u)^j,u^i)\|_{L^1(\dot{B}^{-1}_{\infty,1})} &\lesssim \|(\mathcal{P}u)^j\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})}\|u^i\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}\\
\|R((\mathcal{Q}u)^j,\partial_j u^i)\|_{L^1(\dot{B}^{-1}_{\infty,1})} &\lesssim \|(\mathcal{Q}u)^j\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})}\|\partial_j u^i\|_{\widetilde{L}^1(\dot{B}^{d/p}_{p,r})},
\end{aligned}$$

we get

$$\|u\cdot\nabla u\|_{L^1(\dot{B}^{-1}_{\infty,1})} \lesssim \big(\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})} + \|\mathcal{Q}u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})}\big)\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}.$$

Plugging this latter inequality and (2.22) in (2.18) and using Bernstein inequality, we end up with

$$\|\mathcal{P}u\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})\cap L^1(\dot{B}^{1}_{\infty,1})} \lesssim \|\mathcal{P}u_0\|_{\dot{B}^{-1}_{\infty,1}} + 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.23}$$

Step 2: the low frequencies of $(a, \mathcal{Q}u)$. Throughout, we set $p^* = 2p/(p-2)$ (that is $1/p + 1/p^* = 1/2$) and $1/r^* = \theta/r + 1 - \theta$ with $\theta = p/2 - 1$. Because $2 \leq p \leq \min(4, 2d/(d-2))$, we have $\max(p, d) \leq p^*$, and $r^* \in [1, r]$. We shall use repeatedly the following facts, based on straightforward interpolation inequalities :

- The space $\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r}) \cap \widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})$ is continuously embedded in $\widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,r^*})$,
- The space $\widetilde{L}^2(\dot{B}^{d/p}_{p,r}) \cap \widetilde{L}^2(\dot{B}^{0}_{\infty,1})$ is continuously embedded in $\widetilde{L}^2(\dot{B}^{d/4}_{4,2})$ (here comes that $r \leq p/(p-2)$),
- We have $1/r^* + 1/r \geq 1$ (again, we use that $r \leq p/(p-2)$),
- If $p = d^*$ (that is $p = 2d/(d-2)$) then $r = 1$ by assumption, and thus $r^* = 1$, too.

Now, to estimate the low frequencies of $(a, \mathcal{Q}u)$, we write that

$$\begin{cases} \partial_t a + \operatorname{div} \mathcal{Q}u = -\operatorname{div}(au), \\ \partial_t \mathcal{Q}u - \Delta \mathcal{Q}u + \nabla a = -\mathcal{Q}(u \cdot \nabla u) - \mathcal{Q}(J(a)\widetilde{\mathcal{A}}u) + k(a)\nabla a, \end{cases} \tag{2.24}$$

and the energy estimates for the barotropic linearized equations (see [2], Prop. 10.23, or [9]) thus give

$$\begin{aligned} \|(a, \mathcal{Q}u)\|^\ell_{\widetilde{L}^\infty(\dot{B}^{d/2-1}_{2,1}) \cap L^1(\dot{B}^{d/2+1}_{2,1})} \lesssim \|(a_0, \mathcal{Q}u_0)\|^\ell_{\dot{B}^{d/2-1}_{2,1}} + \|\operatorname{div}(au)\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})} \\ + \|\mathcal{Q}(u \cdot \nabla u)\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})} + \|\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})} + \|k(a)\nabla a\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})}. \end{aligned}$$

Let us first bound[2] $u \cdot \nabla u$ in $L^1(\dot{B}^{d/2-1}_{2,1})$. For that, we use again decomposition (2.21). To handle the first term of (2.21), we just use that (see [2], Chap. 2)

$$T : \widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,r^*}) \times \widetilde{L}^1(\dot{B}^{d/p}_{p,r}) \longrightarrow L^1(\dot{B}^{d/2-1}_{2,1}).$$

This is due to the fact that $1/p + 1/p^* = 1/2$, and that either $d/p^* - 1 < 0$ and $1/r + 1/r^* \geq 1$, or $d/p^* - 1 = 0$ and $r^* = 1$. As $T : \widetilde{L}^\infty(\dot{B}^{d/p^*-2}_{p^*,r^*}) \times \widetilde{L}^1(\dot{B}^{d/p+1}_{p,r}) \longrightarrow L^1(\dot{B}^{d/2-1}_{2,1})$, the second term of (2.21) also satisfies quadratic estimates with respect to the norm of the solution in $X^{p,r}_{1,1}$.

Next, because

$$R : \dot{B}^{d/4}_{4,2} \times \dot{B}^{d/4}_{4,2} \longrightarrow \dot{B}^{d/2}_{2,1},$$

we have

$$\|\partial_j R((\mathcal{P}u)^j, u^i)\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim \|\mathcal{P}u\|_{L^2(\dot{B}^{d/4}_{4,2})} \|u\|_{L^2(\dot{B}^{d/4}_{4,2})}.$$

For the last term of (2.21), we just have to use that

$$R : \widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1}) \times \widetilde{L}^1(\dot{B}^{d/p}_{p,r}) \longrightarrow L^1(\dot{B}^{d/2-1}_{2,1}) \quad \text{for} \quad p \in [2, 4] \cap [2, 2d).$$

Putting all the above informations together, we conclude that

$$\|u \cdot \nabla u\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim \|(a, u)\|^2_{X^{p,r}_{1,1}}. \tag{2.25}$$

In order to bound $(\operatorname{div}(au))^\ell$, we notice that

$$\big(\operatorname{div}(au)\big)^\ell = \big(\operatorname{div}\big(R(a, u) + T_a u\big)\big)^\ell + \operatorname{div} T_{u^\ell} a^\ell + \big(\operatorname{div}\big(\dot{S}_{j_0} u\, \dot{\Delta}_{j_0+1} a\big)\big)^\ell. \tag{2.26}$$

[2]We do not get anything better by just considering the low frequencies of $\mathcal{Q}(u \cdot \nabla u)$.

Now, the remainder R and the paraproduct T map $\widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,1}) \times \widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})$ in $L^1(\dot{B}^{d/2}_{2,1})$ and we have $\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1}) \hookrightarrow \widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,1})$ because $p^* \geq p$. Hence

$$\|\mathrm{div}\,(R(a,u)+T_a u)\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim \|a\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})}\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}.$$

To handle the third term of (2.26), it suffices to use the fact that

$$T : L^2(L^\infty) \times L^2(\dot{B}^{d/2}_{2,1}) \longrightarrow L^1(\dot{B}^{d/2}_{2,1}).$$

Finally,

$$\begin{aligned}\|\dot{S}_{j_0}u\,\dot{\Delta}_{j_0+1}a\|_{L^1(L^2)} &\leq \|\dot{S}_{j_0}u\|_{L^\infty(L^{p^*})}\|\dot{\Delta}_{j_0+1}a\|_{L^1(L^p)}\\ &\lesssim 2^{j_0(1-d/p^*)}\|u\|_{\widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,r})}\big(2^{j_0d/p}\|\dot{\Delta}_{j_0+1}a\|_{L^1(L^p)}\big)2^{-j_0d/p}.\end{aligned}$$

Hence

$$2^{j_0d/2}\|\dot{S}_{j_0}u\,\dot{\Delta}_{j_0+1}a\|_{L^1(L^2)} \lesssim 2^{j_0}\|u\|_{\widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,r})}\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})}. \tag{2.27}$$

We can thus conclude that

$$\|\mathrm{div}\,(au)\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.28}$$

Next, denoting $\mathcal{Q}^\ell := \dot{S}_{j_0+1}\mathcal{Q}$, we write that

$$\mathcal{Q}^\ell(J(a)\widetilde{\mathcal{A}}u) = \mathcal{Q}^\ell\big(T_{\widetilde{\mathcal{A}}u}J(a) + R(\widetilde{\mathcal{A}}u, J(a))\big) + T_{J(a)}\Delta\mathcal{Q}^\ell u + [\mathcal{Q}^\ell, T_{J(a)}]\widetilde{\mathcal{A}}u. \tag{2.29}$$

To handle the first two terms, it suffices to notice that

$$R \text{ and } T \text{ map } \widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,r^*}) \times \widetilde{L}^1(\dot{B}^{d/p}_{p,1}) \text{ to } L^1(\dot{B}^{d/2-1}_{2,1}), \tag{2.30}$$

and to use Proposition 6.1. Therefore, by virtue of (2.19),

$$\|T_{\widetilde{\mathcal{A}}u}J(a) + R(\widetilde{\mathcal{A}}u, J(a))\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}.$$

For the third term, we just have to use that $T : L^\infty \times \dot{B}^{d/2-1}_{2,1} \to \dot{B}^{d/2-1}_{2,1}$. Finally the commutator term may be handled according to Lemma 6.1, which ensures that[3]

$$\|[\mathcal{Q}^\ell, T_{J(a)}]\widetilde{\mathcal{A}}u\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim \|\nabla J(a)\|_{\widetilde{L}^\infty(\dot{B}^{d/p^*-1}_{p^*,1})}\|\nabla^2 u\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})}.$$

Hence using embeddings and composition estimates, we end up with

$$\|\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.31}$$

Finally, we decompose $k(a)\nabla a$ as follows:

$$\big(k(a)\nabla a\big)^\ell = \big(T_{\nabla a}k(a) + R(\nabla a, k(a))\big)^\ell + T_{(k(a))^\ell}\nabla a^\ell + \big(\dot{S}_{j_0}k(a)\dot{\Delta}_{j_0+1}\nabla a\big)^\ell.$$

To bound the first two terms, we use again (2.30) and composition estimates. For the third term, we use that $T : L^2(L^\infty) \times L^2(\dot{B}^{d/2-1}_{2,1}) \to L^1(\dot{B}^{d/2-1}_{2,1})$. For the last term, we proceed as in (2.27) and get

$$2^{j_0(d/2-1)}\|\dot{S}_{j_0}k(a)\,\dot{\Delta}_{j_0+1}\nabla a\|_{L^2} \lesssim 2^{j_0}\|k(a)\|_{\dot{B}^{d/p^*-1}_{p^*,1}}\|a\|^h_{\dot{B}^{d/p}_{p,1}}.$$

Therefore, by embedding,

$$2^{j_0(d/2-1)}\|\dot{S}_{j_0}k(a)\,\dot{\Delta}_{j_0+1}\nabla a\|_{L^1(L^2)} \lesssim 2^{j_0}\|k(a)\|_{L^\infty(\dot{B}^{d/p-1}_{p,1})}\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})}.$$

[3]Recall that $r = 1$ if $p^* = d$.

For bounding $k(a)$, one cannot use directly Proposition 6.1 as it may happen that $d/p-1<0$. So we write

$$k(a)=k'(0)\,a+a\widetilde{k}(a)\quad\text{with}\quad\widetilde{k}(0)=0.$$

Now, combining Proposition 6.1 and product laws in Besov spaces, we get for $2\leq p<2d$,

$$\|k(a)\|_{\dot{B}^{d/p-1}_{p,1}}\lesssim(|k'(0)|+\|a\|_{\dot{B}^{d/p}_{p,1}})\|a\|_{\dot{B}^{d/p-1}_{p,1}}. \tag{2.32}$$

So finally,

$$\|k(a)\nabla a\|_{L^1(\dot{B}^{d/2-1}_{2,1})}\lesssim 2^{j_0}(1+\|a\|_{L^\infty(\dot{B}^{d/p}_{p,1})})\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.33}$$

Putting together Inequalities (2.25), (2.28), (2.31) and (2.33), we conclude that

$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde{L}^\infty(\dot{B}^{d/2-1}_{2,1})\cap L^1(\dot{B}^{d/2+1}_{2,1})}\lesssim\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot{B}^{d/2-1}_{2,1}}+2^{j_0}(1+\|a\|_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})})\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.34}$$

Step 3: Effective velocity. To estimate the high frequencies of $\mathcal{Q}u$, we follow the approach of [21, 22, 23], and introduce the following "effective" velocity field[4]:

$$w:=\mathcal{Q}u+(-\Delta)^{-1}\nabla a.$$

We find out that

$$\partial_t w-\Delta w=-\mathcal{Q}(u\cdot\nabla u)-\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)+k(a)\nabla a+\mathcal{Q}(au)+w-(-\Delta)^{-1}\nabla a.$$

Applying the heat estimates (0.4) for the high frequencies of w only, we get

$$\begin{aligned}\|w\|^h_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})\cap L^1(\dot{B}^{d/p+1}_{p,1})}\lesssim\|w_0\|^h_{\dot{B}^{d/p-1}_{p,1}}+\|u\cdot\nabla u\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}+\|\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}\\+\|k(a)\nabla a\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}+\|\mathcal{Q}(au)\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}+\|w\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}+\|a\|^h_{L^1(\dot{B}^{d/p-2}_{p,1})}.\end{aligned}$$

The important point is that, owing to the high frequency cut-off at $|\xi|\sim 2^{j_0}$,

$$\|w\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}\lesssim 2^{-2j_0}\|w\|^h_{L^1(\dot{B}^{d/p+1}_{p,1})}\quad\text{and}\quad\|a\|^h_{L^1(\dot{B}^{d/p-2}_{p,1})}\lesssim 2^{-2j_0}\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})}.$$

Hence, if j_0 is large enough then the term $\|w\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}$ may be absorbed by the l.h.s. The other terms satisfy quadratic estimates. Indeed, it is clearly the case of $u\cdot\nabla u$ according to (2.25), for $\dot{B}^{d/2-1}_{2,1}$ embeds in $\dot{B}^{d/p-1}_{p,1}$. Next, because the product maps $\dot{B}^{d/p}_{p,1}\times\dot{B}^{d/p-1}_{p,1}$ in $\dot{B}^{d/p-1}_{p,1}$, we have if $p<2d$,

$$\|k(a)\nabla a\|_{L^1(\dot{B}^{d/p-1}_{p,1})}\lesssim\|a\|^2_{L^2(\dot{B}^{d/p}_{p,1})}.$$

To handle $\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)$, we decompose it into

$$\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)=T_{J(a)}\Delta\mathcal{Q}u+\mathcal{Q}R(J(a),\widetilde{A}u)+\mathcal{Q}T_{\widetilde{\mathcal{A}}u}J(a)+[\mathcal{Q},T_{J(a)}]\widetilde{\mathcal{A}}u.$$

Arguing as from proving (2.31), we readily get

$$\|\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)\|_{L^1(\dot{B}^{d/p-1}_{p,1})}\lesssim\|a\|_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})}(\|\mathcal{Q}u\|_{L^1(\dot{B}^{d/p+1}_{p,1})}+\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}).$$

Finally, using Bony's decomposition, we see that

$$\|au\|_{\widetilde{L}^1(\dot{B}^{d/p}_{p,r})}\lesssim\|a\|_{\widetilde{L}^2(\dot{B}^{d/p}_{p,1})}\|u\|_{\widetilde{L}^2(\dot{B}^{d/p}_{p,r}\cap\dot{B}^0_{\infty,1})}.$$

Because

$$\|\mathcal{Q}(au)\|^h_{L^1(\dot{B}^{d/p-1}_{p,1})}\lesssim\|au\|_{\widetilde{L}^1(\dot{B}^{d/p}_{p,r})},$$

[4]The idea is to write the term $\Delta\mathcal{Q}u-\nabla a$ in (2.24) as the Laplacian of some gradient-like vector-field.

we conclude that

$$\|w\|^h_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})\cap L^1(\dot{B}^{d/p+1}_{p,1})} \lesssim \|w_0\|^h_{\dot{B}^{d/p-1}_{p,1}} + 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}} + 2^{-2j_0}\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})}. \tag{2.35}$$

Step 4: High frequencies of the density. We notice that

$$\partial_t a + u\cdot\nabla a + a = -a\,\mathrm{div}\,u - \mathrm{div}\,w.$$

To bound the high frequencies of a, we write that for all $j \geq j_0$,

$$\partial_t\dot{\Delta}_j a + \dot{S}_{j-1}u\cdot\nabla\dot{\Delta}_j a + \dot{\Delta}_j a = -\dot{\Delta}_j(T_{\nabla a}\cdot u + R(\nabla a,u) + a\,\mathrm{div}\,u + \mathrm{div}\,w) + R_j$$

with $R_j := \dot{S}_{j-1}u\cdot\nabla\dot{\Delta}_j a - \dot{\Delta}_j(T_u\cdot\nabla a)$.

Arguing as in [13], we thus get for all $t\geq 0$,

$$\begin{aligned}\|\dot{\Delta}_j a(t)\|_{L^p} + \int_0^t \|\dot{\Delta}_j a\|_{L^p}\,d\tau &\leq \|\dot{\Delta}_j a_0\|_{L^p} + \frac{1}{p}\int_0^t \|\mathrm{div}\,\dot{S}_{j-1}u\|_{L^\infty}\|\dot{\Delta}_j a\|_{L^p}\,d\tau\\ &\quad + \int_0^t \|\dot{\Delta}_j(T_{\nabla a}\cdot u + R(\nabla a,u) + a\,\mathrm{div}\,u + \mathrm{div}\,w)\|_{L^p}\,d\tau + \int_0^t \|R_j\|_{L^p}\,d\tau.\end{aligned} \tag{2.36}$$

Now, because $\dot{B}^{d/p}_{p,1}$ is an algebra, we may write

$$\|a\,\mathrm{div}\,u\|_{\dot{B}^{d/p}_{p,1}} \lesssim \|a\|_{\dot{B}^{d/p}_{p,1}}\|\mathrm{div}\,u\|_{\dot{B}^{d/p}_{p,1}},$$

and continuity results for the paraproduct, and remainder yield

$$\begin{aligned}\|T_{\nabla a}\cdot u\|_{L^1(\dot{B}^{d/p}_{p,1})} &\lesssim \|\nabla a\|_{\widetilde{L}^\infty(\dot{B}^{-1}_{\infty,1})}\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})},\\ \|R(\nabla a,u)\|_{L^1(\dot{B}^{d/p}_{p,1})} &\lesssim \|\nabla a\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})}\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}.\end{aligned}$$

Finally, because

$$R_j = \dot{\Delta}_j\sum_{|j'-j|\leq 4}(\dot{S}_{j-1}-\dot{S}_{j'-1})u\cdot\nabla\dot{\Delta}_{j'}a + \sum_{|j'-j|\leq 4}[\dot{S}_{j-1}u,\dot{\Delta}_j]\cdot\nabla\dot{\Delta}_{j'}a,$$

commutator estimates from [2] lead to

$$\sum_{j\in\mathbb{Z}} 2^{jd/p}\|R_j\|_{L^p} \leq C\|\nabla u\|_{L^\infty}\|a\|_{\dot{B}^{d/p}_{p,1}}.$$

Multiplying (2.36) by $2^{jd/p}$, using the above inequalities, and summing up over $j\geq j_0$ thus leads to

$$\begin{aligned}\|a\|^h_{\widetilde{L}^\infty_t(\dot{B}^{d/p}_{p,1})} + \int_0^t \|a\|^h_{\dot{B}^{d/p}_{p,1}}\,d\tau &\leq \|a_0\|^h_{\dot{B}^{d/p}_{p,1}} + C\int_0^t\big(\|\nabla u\|_{L^\infty} + \|\mathrm{div}\,u\|_{\dot{B}^{d/p}_{p,1}}\big)\|a\|_{\dot{B}^{d/p}_{p,1}}\,d\tau\\ &\quad + C\|\nabla a\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,1})}\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})} + C\|w\|^h_{L^1(\dot{B}^{d/p+1}_{p,1})}.\end{aligned}$$

Therefore,

$$\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})\cap\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})} \lesssim \|a_0\|^h_{\dot{B}^{d/p}_{p,1}} + 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}} + \|w\|^h_{L^1(\dot{B}^{d/p+1}_{p,1})}. \tag{2.37}$$

Plugging (2.35) in (2.37) and taking j_0 large enough, we thus get

$$\|a\|^h_{L^1(\dot{B}^{d/p}_{p,1})\cap\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})} \lesssim \|a_0\|^h_{\dot{B}^{d/p}_{p,1}} + \|\mathcal{Q}u_0\|^h_{\dot{B}^{d/p-1}_{p,1}} + 2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.38}$$

Step 5: Closing the a priori estimates. Resuming to (2.35) yields

$$\|w\|^h_{\widetilde{L}^\infty(\dot B^{d/p-1}_{p,1})\cap L^1(\dot B^{d/p+1}_{p,1})}\lesssim \|a_0\|^h_{\dot B^{d/p}_{p,1}}+\|\mathcal{Q}u_0\|^h_{\dot B^{d/p-1}_{p,1}}+2^{j_0}\|(a,u)\|^2_{X^{p,r}_{1,1}}. \tag{2.39}$$

As $\mathcal{Q}u^h=w^h-(-\Delta)^{-1}\nabla a^h$, the same inequality holds true for $\mathcal{Q}u^h$. Finally, putting together (2.20), (2.23), (2.34) and (2.38), we conclude that

$$\begin{aligned}\|(a,u)\|_{X^{p,r}_{1,1}}\leq C\big(\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot B^{d/2-1}_{2,1}}+\|\mathcal{P}u_0\|_{\dot B^{d/p-1}_{p,r}\cap\dot B^{-1}_{\infty,1}}+\|a_0\|^h_{\dot B^{d/p}_{p,1}}\\ +\|\mathcal{Q}u_0\|^h_{\dot B^{d/p-1}_{p,1}}+2^{j_0}(1+\|(a,u)\|_{X^{p,r}_{1,1}})\|(a,u)\|^2_{X^{p,r}_{1,1}}\big)\cdot\end{aligned} \tag{2.40}$$

It is now easy to close the estimates if the data are small enough; we end up with (1.9).

Step 6: The case of nonconstant viscosity coefficients. It is only a matter of checking that the last line of (2.15) satisfies quadratic estimates. To this end, we write that

$$\frac{1}{1+a}\mathrm{div}\,\big(\widetilde{\mu}(a)D(u)\big)=\frac{\widetilde{\mu}(a)}{1+a}\mathrm{div}\,D(u)+\frac{\widetilde{\mu}'(a)}{1+a}D(u)\cdot\nabla a, \tag{2.41}$$

and a similar relation for the term pertaining to $\widetilde{\lambda}$.

The first term of the r.h.s. of (2.41) may be handled exactly as $J(a)\widetilde{\mathcal{A}}u$. As for the second term, it suffices to estimate it in $L^1(\dot B^{d/p-1}_{p,1})$ and to show that applying $\mathcal{Q}^\ell$ to it leads to estimates in $L^1(\dot B^{d/2-1}_{2,1})$.

Throughout, we use the fact that $\frac{\widetilde{\mu}'(a)}{1+a}\nabla a=\nabla(L(a))$ for some smooth function L vanishing at 0. Now, continuity properties of R and T imply that

$$\begin{aligned}\|T_{\nabla(L(a))}\nabla u\|_{L^1(\dot B^{d/2-1}_{2,1})}&\lesssim\|\nabla(L(a))\|_{\widetilde{L}^\infty(\dot B^{d/p^*-1}_{p^*,1})}\|\nabla u\|_{\widetilde{L}^1(\dot B^{d/p}_{p,r})},\\ \|R(\nabla(L(a)),\nabla u)\|_{L^1(\dot B^{d/2-1}_{2,1})}&\lesssim\|\nabla(L(a))\|_{\widetilde{L}^\infty(\dot B^{d/p-1}_{p,1})}\|\nabla u\|_{\widetilde{L}^1(\dot B^{d/p}_{p,r})},\\ \|T_{\nabla u}\nabla(L(a))\|_{L^1(\dot B^{d/p-1}_{p,1})}&\lesssim\|\nabla u\|_{L^\infty(L^\infty)}\|\nabla(L(a))\|_{L^\infty(\dot B^{d/p-1}_{p,1})},\end{aligned}$$

which in particular yields quadratic estimates for the $L^1(\dot B^{d/p-1}_{p,1})$ norm, after using suitable embedding and the composition estimate (2.33).

To complete the proof, it is only a matter of getting quadratic estimates for $\mathcal{Q}^\ell(T_{\nabla u}\nabla L(a))$ in $L^1(\dot B^{d/2-1}_{2,1})$. To this end, we observe that

$$\|\mathcal{Q}^\ell(T_{\nabla u}\nabla L(a))\|_{L^1(\dot B^{d/2-1}_{2,1})}\lesssim 2^{j_0}\|\mathcal{Q}^\ell(T_{\nabla u}\nabla L(a))\|_{L^1(\dot B^{d/2-2}_{2,1})},$$

and thus

$$\begin{aligned}\|\mathcal{Q}^\ell(T_{\nabla u}\nabla L(a))\|_{L^1(\dot B^{d/2-1}_{2,1})}&\lesssim 2^{j_0}\|\nabla u\|_{\widetilde{L}^2(\dot B^{d/p^*-1}_{p^*,r})}\|\nabla(L(a))\|_{\widetilde{L}^2(\dot B^{d/p-1}_{p,1})}\\ &\lesssim 2^{j_0}\|u\|_{\widetilde{L}^2(\dot B^{d/p^*}_{p^*,r})}\|a\|_{\widetilde{L}^2(\dot B^{d/p}_{p,1})}.\end{aligned}$$

Therefore we end up with

$$\Big\|\frac{\widetilde{\mu}'(a)}{1+a}D(u)\cdot\nabla a\Big\|_{L^1(\dot B^{d/p-1}_{p,1})}\lesssim 2^{j_0}(1+\|a\|_{L^\infty(\dot B^{d/p}_{p,1})})\|(a,u)\|^2_{X^{p,r}_{1,1}},$$

and one may conclude that (2.40) is still fulfilled in this more general situation.

2.2. **Existence of a global solution to System** (2.15)**.** Let us now give a few words on the existence issue. The simplest way is to smooth out the initial velocity u_0 into a sequence of initial velocities $(u_0^n)_{n\in\mathbb{N}}$ with $(u_0^n)^\ell$ in $\dot{B}^{d/2-1}_{2,1}$ uniformly, and $(u_0^n)^h$ in $\dot{B}^{d/p-1}_{p,1}$. Then using the results of [8, 14] yields a unique local-in-time solution (a^n,u^n) to (2.15) with data (a_0,u_0^n). From the above estimates, we know in addition that (with obvious notation)

$$\|(a^n,u^n)\|_{X^{p,r}_{1,1}(0,t)}\leq C\big(\|(a_0^n,\mathcal{Q}u_0^n)\|^\ell_{\dot{B}^{d/2-1}_{2,1}}+\|\mathcal{P}u_0^n\|_{\dot{B}^{d/p-1}_{p,r}\cap\dot{B}^{-1}_{\infty,1}}+\|a_0^n\|^h_{\dot{B}^{d/p}_{p,1}}+\|\mathcal{Q}u_0^n\|^h_{\dot{B}^{d/p-1}_{p,1}}\big)$$

is fulfilled whenever t is smaller than the lifespan T_*^n of (a^n,u^n). As the above inequality implies that

$$\|a^n\|_{\widetilde{L}^\infty_{T_*^n}(\dot{B}^{d/p}_{p,1})}+\int_0^{T_*^n}\|\nabla u^n\|_{L^\infty}\,dt<\infty,$$

a straightforward adaptation of Prop. 10.10 of [2] to $p\neq 2$ implies that $T_*^n=+\infty$. We thus have for all $n\in\mathbb{N}$,

$$\|(a^n,u^n)\|_{X^{p,r}_{1,1}}\leq C\big(\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot{B}^{d/2-1}_{2,1}}+\|\mathcal{P}u_0\|_{\dot{B}^{d/p-1}_{p,r}\cap\dot{B}^{-1}_{\infty,1}}+\|a_0\|^h_{\dot{B}^{d/p}_{p,1}}+\|\mathcal{Q}u_0\|^h_{\dot{B}^{d/p-1}_{p,1}}\big).$$

Next, compactness arguments similar to those of e.g. [2] or [8] allow to conclude that $(a^n,u^n)_{n\in\mathbb{N}}$ weakly converges (up to extraction) to some global solution of (2.15) with the desired regularity properties, and satisfying (1.9) (with $\varepsilon=\nu=1$ of course). Resuming to the original unknowns completes the proof of the first part of Theorem 1.1.

3. The incompressible limit: weak convergence

Granted with the uniform estimates established in the previous section, it is now possible to pass to the limit in the system in the sense of distributions. As in the work by P.-L. Lions and N. Masmoudi [30] dedicated to the finite energy weak solutions of (0.1), the proof relies on compactness arguments, and works the same in the $\mathbb{R}^d$ and $\mathbb{T}^d$ cases. To simplify the presentation, we assume that the viscosity functions λ and μ are constant.

So we consider a family $(a_0^\varepsilon,u_0^\varepsilon)$ of data satisfying (1.8) and $\mathcal{P}u_0^\varepsilon\rightharpoonup v_0$ when ε goes to 0. We denote by $(a^\varepsilon,u^\varepsilon)$ the corresponding solution of (0.1) given by Theorem 1.1. Because

$$\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot{B}^{d/p-1}_{p,1}}\lesssim\widetilde{\varepsilon}\|a_0^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\dot{B}^{d/p}_{p,1}},\tag{3.42}$$

the data $(a_0^\varepsilon,u_0^\varepsilon)$ are uniformly bounded in $\dot{B}^{d/p-1}_{p,1}\times(\dot{B}^{d/p}_{p,r}\cap\dot{B}^{-1}_{\infty,1})$, and thus in $\dot{B}^{d/4-1}_{4,2}$. Likewise, (1.9) ensures that $(a^\varepsilon,u^\varepsilon)$ is bounded in the space $\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot{B}^{d/4-1}_{4,2})$, given our assumptions on p and r. Therefore there exists a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ decaying to 0 so that

$$(a_0^{\varepsilon_n},u_0^{\varepsilon_n})\rightharpoonup(a_0,u_0)\ \text{ in }\ \dot{B}^{d/4-1}_{4,2}\quad\text{and}\quad(a^{\varepsilon_n},u^{\varepsilon_n})\rightharpoonup(a,u)\ \text{ in }\ L^\infty(\mathbb{R}_+;\dot{B}^{d/4-1}_{4,2})\quad\text{weakly }*.$$

Of course, we have $\mathcal{P}u_0=v_0$.

The strong convergence of the density to 1 is obvious: we have $\rho^{\varepsilon_n}=1+\varepsilon_n a^{\varepsilon_n}$, and $(a^{\varepsilon_n})_{n\in\mathbb{N}}$ is bounded (in $L^2(\mathbb{R}_+;\dot{B}^{d/p}_{p,1})$ for instance).

In order to justify that $\operatorname{div}u=0$, we rewrite the mass equation as follows:

$$\operatorname{div}u^{\varepsilon_n}=-\varepsilon_n\operatorname{div}(a^{\varepsilon_n}u^{\varepsilon_n})-\varepsilon_n\partial_t a^{\varepsilon_n}.$$

Given that a^{ε_n} and u^{ε_n} are bounded in $L^2(\mathbb{R}_+;\dot{B}^{d/4}_{4,2}\cap L^\infty)$ (use the definition of $X^{p,r}_{\varepsilon,\nu}$ and interpolation), the first term in the right-hand side is $\mathcal{O}(\varepsilon_n)$ in $L^1(\mathbb{R}_+;\dot{B}^{d/4-1}_{4,2})$. As for the

last term, it tends to 0 in the sense of distributions, for $a^{\varepsilon_n} \rightharpoonup a$ in $L^\infty(\mathbb{R}_+;\dot B^{d/4-1}_{4,2})$ weakly $*$. We thus have $\operatorname{div} u^{\varepsilon_n} \rightharpoonup 0$, whence $\operatorname{div} u = 0$.

To complete the proof of the weak convergence, it is only a matter of justifying that u^{ε_n} converges in the sense of distributions to the solution u of (1.10). To achieve it, we project the velocity equation onto divergence-free vector fields, and get

$$\partial_t \mathcal{P}u^{\varepsilon_n} - \mu\Delta\mathcal{P}u^{\varepsilon_n} = -\mathcal{P}(u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}) - \mathcal{P}\big(J(\varepsilon_n a^{\varepsilon_n})\mathcal{A}u^{\varepsilon_n}\big). \tag{3.43}$$

Because $\mathcal{Q}u = 0$, the left-hand side weakly converges to $\partial_t u - \mu\Delta u$.

To prove that the last term tends to 0, we use the fact that having $\widetilde{\varepsilon}(a^\varepsilon)^{h,\widetilde{\varepsilon}}$ and $(a^\varepsilon)^{\ell,\widetilde{\varepsilon}}$ bounded in $\widetilde L^\infty(\dot B^{d/p}_{p,1})$ and $\widetilde L^\infty(\dot B^{d/p-1}_{p,1})$, respectively, implies that, for all $\alpha\in[0,1]$,

$$\widetilde{\varepsilon}^\alpha a^\varepsilon \quad \text{is bounded in} \quad \widetilde L^\infty(\dot B^{d/p-1+\alpha}_{p,1}). \tag{3.44}$$

Now $\mathcal{A}u^\varepsilon$ is bounded in $\widetilde L^1(\dot B^{d/p-1}_{p,r})$ and $p<2d$. Hence, according to product laws in Besov spaces, composition inequality and (3.44), we get $J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon = \mathcal{O}(\widetilde{\varepsilon}^{1-\alpha})$ in $\widetilde L^1(\dot B^{d/p-2+\alpha}_{p,r})$, whenever $2\max(0,1-d/p)<\alpha\leq 1$.

In order to prove that $\mathcal{P}(u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}) \rightharpoonup \mathcal{P}(u\cdot\nabla u)$, we note that

$$u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n} = \frac12\nabla|\mathcal{Q}u^{\varepsilon_n}|^2 + \mathcal{P}u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n} + \mathcal{Q}u^{\varepsilon_n}\cdot\nabla\mathcal{P}u^{\varepsilon_n}.$$

Projecting the first term onto divergence free vector fields gives 0, and we also know that $\mathcal{P}u = u$. Hence we just have to prove that

$$\mathcal{P}(\mathcal{P}u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n}) \rightharpoonup \mathcal{P}(\mathcal{P}u\cdot\nabla u) \quad\text{and}\quad \mathcal{P}(\mathcal{Q}u^{\varepsilon_n}\cdot\nabla\mathcal{P}u^{\varepsilon_n}) \rightharpoonup 0. \tag{3.45}$$

This requires our proving results of strong convergence for $\mathcal{P}u^{\varepsilon_n}$. To this end, we shall exhibit uniform bounds for $\partial_t\mathcal{P}u^{\varepsilon_n}$ in a suitable space. First, arguing by interpolation, we see that $(\nabla^2 u^{\varepsilon_n})$ is bounded in $\widetilde L^m(\dot B^{d/p+2/m-3}_{p,r})$ for any $m\geq 1$. Choosing $m>1$ so that $2d/p+2/m-3>0$ (this is possible as $p<2d$) and remembering that $(\varepsilon^n a^{\varepsilon_n})$ is bounded in $\widetilde L^\infty(\dot B^{d/p}_{p,1})$, we thus get $(J(\varepsilon_n a^{\varepsilon_n})\mathcal{A}u^{\varepsilon_n})$ bounded in $\widetilde L^m(\dot B^{d/p+2/m-3}_{p,r})$. Similarly, combining the facts that (u^{ε_n}) and $(\nabla u^{\varepsilon_n})$ are bounded in $\widetilde L^\infty(\dot B^{d/p-1}_{p,r})$ and $\widetilde L^m(\dot B^{d/p+2/m-2}_{p,r})$, respectively, we see that $(u^{\varepsilon_n}\cdot\nabla u^{\varepsilon_n})$ is bounded in $\widetilde L^m(\dot B^{d/p+2/m-3}_{p,r})$, too. Computing $\partial_t\mathcal{P}u^{\varepsilon_n}$ from (3.43), it is now clear that $(\partial_t\mathcal{P}u^{\varepsilon_n})$ is bounded in $\widetilde L^m(\dot B^{d/p+2/m-3}_{p,r})$. Hence $(\mathcal{P}u^{\varepsilon_n}-\mathcal{P}u_0^{\varepsilon_n})$ is bounded in $\mathcal{C}^{1-1/m}(\mathbb{R}_+;\dot B^{d/p+2/m-3}_{p,r})$. As $\mathcal{P}u^{\varepsilon_n}$ is also bounded in $\widetilde{\mathcal{C}}_b(\mathbb{R}_+;\dot B^{d/p-1}_{p,r})$, and as the embedding of $\dot B^{d/p-1}_{p,1}$ in $\dot B^{d/p+2/m-3}_{p,1}$ is locally compact (see e.g. [2], page 108), we conclude by means of Ascoli theorem that, up to a new extraction, for all $\phi\in\mathcal{S}(\mathbb{R}^d)$ and $T>0$,

$$\phi\mathcal{P}u^{\varepsilon_n}\longrightarrow\phi\mathcal{P}u \quad\text{in}\quad \mathcal{C}([0,T];\dot B^{d/p+2/m-3}_{p,1}). \tag{3.46}$$

Interpolating with the uniform in $\mathcal{C}_b(\mathbb{R}_+;\dot B^{d/p-1}_{p,r})$, we can upgrade the strong convergence in (3.46) to the space $\mathcal{C}([0,T];\dot B^{d/p-1-\alpha}_{p,1})$ for all small enough $\alpha>0$, and all $T>0$. Combining with the properties of weak convergence for ∇u^{ε_n} to ∇u, and $\mathcal{Q}u^{\varepsilon_n}$ to 0 that may be deduced from the uniform bounds on u^{ε_n}, it is now easy to conclude to (3.45).

4. The incompressible limit: strong convergence in the whole space case

In this section, we combine Strichartz estimates for the following acoustic wave equations

$$\left\{\begin{array}{l} \partial_t a^\varepsilon + \dfrac{\operatorname{div} u^\varepsilon}{\varepsilon} = F^\varepsilon, \\ \partial_t u^\varepsilon + \dfrac{\nabla a^\varepsilon}{\varepsilon} = G^\varepsilon \end{array}\right. \qquad (t,x)\in\mathbb{R}_+\times\mathbb{R}^d \tag{4.47}$$

associated to (0.1), with the uniform bounds (1.9) for the constructed solution $(a^\varepsilon,u^\varepsilon)$ so as to establish the strong convergence for u^ε to the solution v of (1.10) in a proper function space. Recall that in a different context (that of global weak solutions), the idea of taking advantage of Strichartz estimates for investigating the incompressible limit goes back to the work of B. Desjardins and E. Grenier in [16].

Throughout the proof, we assume the viscosity coefficients to be constant, for simplicity. Recall that $C_0^{\varepsilon,\nu}$ denotes the l.h.s. of (1.8).

We first consider the case $d\geq 3$ which is slightly easier than the two-dimensional case, owing to more available Strichartz estimates.

The case $d\geq 3$. Let us assume that $\varepsilon=\nu=1$ for a while. Then the solution (a,u) to (0.1) satisfies (4.47) with $F=-\operatorname{div}(au)$ and $G=\Delta\mathcal{Q}u-\mathcal{Q}(u\cdot\nabla u)-\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)+k(a)\nabla a$, and Proposition 2.2 in [11] ensures that for all $q\in[2,\infty)$, we have

$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde{L}^{2q/(q-2)}(\dot B^{(d-1)/q-1/2}_{q,1})}\lesssim\|(a_0,\mathcal{Q}u_0)\|^\ell_{\dot B^{d/2-1}_{2,1}}+\|(F,G)\|^\ell_{L^1(\dot B^{d/2-1}_{2,1})}.$$

Following the proof of (2.34) to bound F and G, we eventually get

$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde{L}^{2q/(q-2)}(\dot B^{(d-1)/q-1/2}_{q,1})}\lesssim C_0^{1,1}.$$

As we also have

$$\|(a,\mathcal{Q}u)\|^\ell_{L^1(\dot B^{d/2+1}_{2,1})}\lesssim C_0^{1,1},$$

we conclude by using the following complex interpolation result

$$[L^1(\dot B^{d/2+1}_{2,1}),\widetilde{L}^{2q/(q-2)}(\dot B^{(d-1)/q-1/2}_{q,1})]_{q/(q+2)}=\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})\quad\text{with}\quad p=(q+2)/2,$$

that

$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}\lesssim C_0^{1,1}\quad\text{for all}\quad p\in[2,+\infty).$$

Back to the original variables in (2.11), we deduce that for all positive ε and ν,

$$\nu^{1/2}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}\lesssim\widetilde{\varepsilon}^{1/2-1/p}\,C_0^{\varepsilon,\nu}.$$

Of course, for the above inequality to be true, we need in addition that the index p fulfills the assumptions in Theorem 1.1. Now, taking advantage of the high-frequency cut-off (second line below) and (1.9) (third line), we get

$$\begin{aligned}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}&\lesssim\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}+\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{h,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}\\&\lesssim\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})}+\widetilde{\varepsilon}^{1/2-1/p}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{h,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot B^{d/p}_{p,1})}\\&\lesssim\nu^{-1/2}\,\widetilde{\varepsilon}^{1/2-1/p}\,C_0^{\varepsilon,\nu},\end{aligned}$$

which yields the strong convergence of $(a^\varepsilon,\mathcal{Q}u^\varepsilon)$ to 0 in $\widetilde{L}^2(\dot B^{(d+1)/p-1/2}_{p,1})$, with an explicit rate.

Let us now go to the proof of the convergence of $\mathcal{P}u^\varepsilon$. Setting $\delta u^\varepsilon := \mathcal{P}u^\varepsilon - u$, we see that

$$\partial_t\delta u^\varepsilon - \mu\Delta\delta u^\varepsilon + \mathcal{P}(\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon + \delta u^\varepsilon\cdot\nabla u) = -\mathcal{P}\Big(u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon + \mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon + J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\Big)\cdot$$

In what follows, we aim at estimating δu^ε in the space $\widetilde{L}^\infty(\dot{B}^{(d+1)/p-3/2}_{p,r})\cap\widetilde{L}^1(\dot{B}^{(d+1)/p+1/2}_{p,r})$. First, applying (0.4) and the fact that $\mathcal{P}$ is a self-map in any homogeneous Besov space gives

$$\begin{aligned}\delta U^\varepsilon &:= \|\delta u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-3/2}_{p,r})} + \mu\|\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p+1/2}_{p,r})}\\ &\lesssim \|\delta u_0^\varepsilon\|_{\dot{B}^{(d+1)/p-3/2}_{p,r}} + \|\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon + \delta u^\varepsilon\cdot\nabla u\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})}\\ &\quad + \|u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon + \mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon + J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})}.\end{aligned}$$

Next, product and composition estimates in the spirit of those of the previous sections (where we use repeatedly that $(d+1)/p-1/2\leq d/p$ and $(d+1)/p-3/2+d/p>0$) yield:

$$\begin{aligned}\|\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})} &\lesssim \|\mathcal{P}u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r})}\|\nabla\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-1/2}_{p,r})}\\ &\quad + \|\mathcal{P}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}\|\nabla\delta u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-5/2}_{p,r})},\\ \|\delta u^\varepsilon\cdot\nabla u\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})} &\lesssim \|\nabla u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-2}_{p,r})}\|\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p+1/2}_{p,r})}\\ &\quad + \|\nabla u\|_{\widetilde{L}^1(\dot{B}^{d/p}_{p,r})}\|\delta u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-3/2}_{p,r})},\end{aligned}$$

and also

$$\begin{aligned}\|u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})} &\lesssim \|\nabla\mathcal{Q}u^\varepsilon\|_{\widetilde{L}^2(\dot{B}^{(d+1)/p-3/2}_{p,1})}\|u^\varepsilon\|_{\widetilde{L}^2(\dot{B}^{d/p}_{p,r}\cap\dot{B}^0_{\infty,1})},\\ \|\mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})} &\lesssim \|\mathcal{Q}u^\varepsilon\|_{\widetilde{L}^2(\dot{B}^{(d+1)/p-1/2}_{p,1})}\|\nabla\mathcal{P}u^\varepsilon\|_{\widetilde{L}^2(\dot{B}^{d/p-1}_{p,r})},\\ \|J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(d+1)/p-3/2}_{p,r})} &\lesssim \|J(\varepsilon a^\varepsilon)\|_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})}\|\mathcal{A}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{d/p-1}_{p,r})}\\ &\lesssim (1+\|\varepsilon a^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})})\|\varepsilon a^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})}\|u^\varepsilon\|_{L^1(\dot{B}^{d/p+1}_{p,r})}.\end{aligned}$$

Let us observe that

$$\begin{aligned}\|\varepsilon a^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})} + \|\varepsilon a^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\widetilde{L}^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})}\\ &\lesssim \nu^{-1}\widetilde{\varepsilon}^{1/2-1/p}\|a^\varepsilon\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^\infty(\dot{B}^{d/2-1}_{2,1})} + \widetilde{\varepsilon}^{1/2-1/p}\|\varepsilon a^\varepsilon\|^{h,\widetilde{\varepsilon}}_{\widetilde{L}^\infty(\dot{B}^{d/p}_{p,1})}\\ &\lesssim \nu^{-1}\widetilde{\varepsilon}^{1/2-1/p}C_0^{\varepsilon,\nu}. \qquad (4.48)\end{aligned}$$

Therefore, putting together all the above estimates and using (1.9), we get

$$\begin{aligned}\delta U^\varepsilon &\lesssim \|\delta u_0^\varepsilon\|_{\dot{B}^{(d+1)/p-3/2}_{p,r}}\\ &\quad + \mu^{-1}\big(\|u\|_{\widetilde{L}^\infty(\dot{B}^{d/p-1}_{p,r})} + \mu\|u\|_{\widetilde{L}^1(\dot{B}^{d/p+1}_{p,r})}\big)\delta U^\varepsilon + \nu^{-1}\widetilde{\varepsilon}^{1/2-1/p}(1+\nu^{-1}C_0^{\varepsilon,\nu})(C_0^{\varepsilon,\nu})^2.\end{aligned}$$

Note that Theorem 0.1 implies that as v_0 is small compared to μ (a consequence of smallness condition (1.8)) then the solution u to (1.10) with data v_0 exists globally and satisfies (0.5). We thus get

$$\delta U^\varepsilon \lesssim \|\delta u_0^\varepsilon\|_{\dot{B}^{(d+1)/p-3/2}_{p,r}} + \widetilde{\varepsilon}^{1/2-1/p}C_0^{\varepsilon,\nu},$$

which completes the proof of convergence in $\mathbb{R}^d$ if $d\geq 3$.

The case $d=2$. Applying Proposition 2.2 in [11] to (4.47) in the case $d=2$, and using the bounds of the previous section to bound the r.h.s. in $L^1(\dot{B}^0_{2,1})$, we now get if $\varepsilon=\nu=1$,

$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde{L}^r(\dot{B}^{2/q-1+1/r}_{q,1})}\lesssim C_0^{1,1}\quad\text{whenever}\quad 2/r\leq 1/2-1/q.$$

Let us emphasize that in contrast with the high-dimensional case, we cannot have r smaller than 4. In what follows, we set $1/r=c(1/2-1/q)$ with $c\in[0,1/2]$ to be fixed later on. Observing that (1.9) implies that

$$\|(a,\mathcal{Q}u)\|^\ell_{L^1(\dot{B}^2_{2,1})}\lesssim C_0^{1,1}$$

and adapting the interpolation argument used in the previous paragraph, we get

$$\|(a,\mathcal{Q}u)\|^\ell_{\widetilde{L}^2(\dot{B}^{(c+2)/p-c/2}_{p,1})}\lesssim C_0^{1,1},$$

where p, c and q are interrelated through

$$p=\frac{4q+(4-2q)c}{q+2+(2-q)c}\cdot$$

Note that as $c\in[0,1/2]$ and $q\in[2,+\infty]$, one can achieve any $p\in[2,6]$, which is a weaker condition than that which is imposed for p in the statement of Theorem 1.1.

For general ε and ν, the above inequality recasts in

$$\nu^{1/2}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot{B}^{(c+2)/p-c/2}_{p,1})}\lesssim\widetilde{\varepsilon}^{c(1/2-1/p)}C_0^{\varepsilon,\nu}.$$

Arguing as in the high-dimensional case, one can get a similar inequality for the high frequencies of $(a^\varepsilon,\mathcal{Q}u^\varepsilon)$, namely

$$\begin{aligned}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|_{\widetilde{L}^2(\dot{B}^{(c+2)/p-c/2}_{p,1})}&\lesssim\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot{B}^{(c+2)/p-c/2}_{p,1})}+\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{h,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot{B}^{(c+2)/p-c/2}_{p,1})}\\&\lesssim\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot{B}^{(c+2)/p-c/2}_{p,1})}+\widetilde{\varepsilon}^{c(1/2-1/p)}\|(a^\varepsilon,\mathcal{Q}u^\varepsilon)\|^{h,\widetilde{\varepsilon}}_{\widetilde{L}^2(\dot{B}^{2/p}_{p,1})}\\&\lesssim\nu^{-1/2}\widetilde{\varepsilon}^{c(1/2-1/p)}C_0^{\varepsilon,\nu}.\end{aligned}$$

Let us finally prove the convergence of $\mathcal{P}u^\varepsilon$ to u in $\widetilde{L}^\infty(\dot{B}^{(c+2)/p-c/2-1}_{p,r})\cap\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2+1}_{p,r})$. Again, we apply Inequality (0.4) to the equation fulfilled by δu^ε, and get

$$\begin{aligned}\delta\!\!\!/U^\varepsilon:=\|\delta u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(c+2)/p-c/2-1}_{p,r})}+\|\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2+1}_{p,r})}&\lesssim\|\delta u_0^\varepsilon\|_{\dot{B}^{(c+2)/p-c/2-1}_{p,r}}\\&+\|\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon+\delta u^\varepsilon\cdot\nabla u\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2-1}_{p,r})}\\&+\|u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon+\mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon+J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2-1}_{p,r})}.\end{aligned}$$

In order to bound the nonlinear terms, we use standard continuity results for the product or paraproduct, and also (repeatedly) the fact that the condition on c in Theorem 1.1 is equivalent to $(c+2)/p-c/2-1+2/p>0$. Then we get

$$\begin{aligned}\|\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2-1}_{p,r})}&\lesssim\|\mathcal{P}u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{2/p-1}_{p,r})}\|\nabla\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2}_{p,r})}\\&\quad+\|\mathcal{P}u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{2/p+1}_{p,r})}\|\nabla\delta u^\varepsilon\|_{L^\infty(\dot{B}^{(c+2)/p-c/2-2}_{p,r})},\\\|\delta u^\varepsilon\cdot\nabla u\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2-1}_{p,r})}&\lesssim\|\nabla u\|_{\widetilde{L}^\infty(\dot{B}^{2/p-2}_{p,r})}\|\delta u^\varepsilon\|_{\widetilde{L}^1(\dot{B}^{(c+2)/p-c/2+1}_{p,r})}\\&\quad+\|\nabla u\|_{\widetilde{L}^1(\dot{B}^{2/p}_{p,r})}\|\delta u^\varepsilon\|_{\widetilde{L}^\infty(\dot{B}^{(c+2)/p-c/2-1}_{p,r})},\end{aligned}$$

and also

$$\begin{aligned}
\|u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon\|_{\widetilde{L}^1(\dot B^{(c+2)/p-c/2-1}_{p,r})} &\lesssim \|\nabla\mathcal{Q}u^\varepsilon\|_{\widetilde{L}^2(\dot B^{(c+2)/p-c/2-1}_{p,1})}\|u^\varepsilon\|_{\widetilde{L}^2(\dot B^{2/p}_{p,r}\cap\dot B^0_{\infty,1})},\\
\|\mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\|_{\widetilde{L}^1(\dot B^{(c+2)/p-c/2-1}_{p,r})} &\lesssim \|\mathcal{Q}u^\varepsilon\|_{\widetilde{L}^2(\dot B^{(c+2)/p-c/2}_{p,1})}\|\nabla\mathcal{P}u^\varepsilon\|_{\widetilde{L}^2(\dot B^{2/p-1}_{p,r})},\\
\|J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\|_{\widetilde{L}^1(\dot B^{(c+2)/p-c/2-1}_{p,r})} &\lesssim (1+\|\varepsilon a^\varepsilon\|_{\widetilde{L}^\infty(\dot B^{2/p}_{p,1})})\|\varepsilon a^\varepsilon\|_{\widetilde{L}^\infty(\dot B^{(c+2)/p-c/2}_{p,1})}\|u^\varepsilon\|_{\widetilde{L}^1(\dot B^{2/p+1}_{p,r})}.
\end{aligned}$$

In order to bound $\varepsilon a^\varepsilon$ in $\widetilde{L}^\infty(\dot B^{(c+2)/p-c/2}_{p,1})$, one may argue exactly as in the case $d\geq 3$:

$$\begin{aligned}
\|\varepsilon a^\varepsilon\|_{\widetilde{L}^\infty(\dot B^{(c+2)/p-c/2}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|^{\ell,\widetilde\varepsilon}_{\widetilde{L}^\infty(\dot B^{(c+2)/p-c/2}_{p,1})}+\|\varepsilon a^\varepsilon\|^{h,\widetilde\varepsilon}_{\widetilde{L}^\infty(\dot B^{(c+2)/p-c/2}_{p,1})}\\
&\lesssim \nu^{-1}\widetilde\varepsilon^{c(1/2-1/p)}\|a^\varepsilon\|^{\ell,\widetilde\varepsilon}_{\widetilde{L}^\infty(\dot B^0_{2,1})}+\widetilde\varepsilon^{c(1/2-1/p)}\|\varepsilon a^\varepsilon\|^{h,\widetilde\varepsilon}_{\widetilde{L}^\infty(\dot B^{2/p}_{p,1})}\\
&\lesssim \nu^{-1}\widetilde\varepsilon^{c(1/2-1/p)}C_0^{\varepsilon,\nu}.
\end{aligned}$$

So using Theorem 0.1 to bound the terms pertaining to u, it is now easy to conclude to the last inequality of Theorem 1.1. ∎

5. The full Navier-Stokes-Fourier system

In this final section, we aim at extending the previous results to the more physically relevant case of non-isothermal polytropic fluids. The corresponding governing equations, the so-called Navier-Stokes-Fourier system, involves the density of the fluid ρ^ε and its velocity u^ε. To fully describe the fluid, we need to consider a *third* (real valued) unknown, for instance the temperature θ^ε.

For simplicity, we only consider the case of perfect heat conducting and viscous gases. We set the reference density and temperature to be 1, and focus on *ill-prepared data* of the form $\rho_0^\varepsilon=1+\varepsilon a_0^\varepsilon$, u_0^ε and $\theta_0^\varepsilon=1+\varepsilon\vartheta_0^\varepsilon$ where $(a_0^\varepsilon,u_0^\varepsilon,\vartheta_0^\varepsilon)$ are bounded in a sense that will be specified later on[5]. Setting $\rho^\varepsilon=1+\varepsilon a^\varepsilon$ and $\theta^\varepsilon=1+\varepsilon\vartheta^\varepsilon$, we get the following system for $(a^\varepsilon,u^\varepsilon,\vartheta^\varepsilon)$:

$$\left\{\begin{aligned}
&\partial_t a^\varepsilon+\frac{\operatorname{div}u^\varepsilon}{\varepsilon}=-\operatorname{div}(a^\varepsilon u^\varepsilon),\\
&\partial_t u^\varepsilon+u^\varepsilon\cdot\nabla u^\varepsilon-\frac{\mathcal{A}u^\varepsilon}{1+\varepsilon a^\varepsilon}+\frac{\nabla(a^\varepsilon+\vartheta^\varepsilon+\varepsilon a^\varepsilon\vartheta^\varepsilon)}{\varepsilon(1+\varepsilon a^\varepsilon)}=0,\\
&\partial_t\vartheta^\varepsilon+\frac{\operatorname{div}u^\varepsilon}{\varepsilon}+\operatorname{div}(\vartheta^\varepsilon u^\varepsilon)-\kappa\frac{\Delta\vartheta^\varepsilon}{1+\varepsilon a^\varepsilon}=\frac{\varepsilon}{1+\varepsilon a^\varepsilon}\Bigl(2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2\Bigr)\cdot
\end{aligned}\right.\tag{5.49}$$

We assume that the fluid is genuinely viscous and heat-conductive, that is to say

$$\mu>0,\quad \nu:=\lambda+2\mu>0\ \text{ and }\ \kappa>0.$$

Even though our results should hold for coefficients λ, μ and κ depending smoothly on the density, we only consider the constant case, for simplicity.

Keeping in mind our results on the barotropic case, we want to consider families of small data $(a_0^\varepsilon,u_0^\varepsilon,\vartheta_0^\varepsilon)$ in the space $Y^p_{0,\varepsilon,\nu}$ defined by (still setting $\widetilde\varepsilon:=\varepsilon\nu$):

- $(a_0^\varepsilon,\mathcal{Q}u_0^\varepsilon,\vartheta_0^\varepsilon)^{\ell,\widetilde\varepsilon}\in\dot B^{d/2-1}_{2,1}$,
- $(a_0^\varepsilon)^{h,\widetilde\varepsilon}\in\dot B^{d/p}_{p,1}$, $\quad(\mathcal{Q}u_0^\varepsilon)^{h,\widetilde\varepsilon}\in\dot B^{d/p-1}_{p,1}$, $\quad(\vartheta_0^\varepsilon)^{h,\widetilde\varepsilon}\in\dot B^{d/p-2}_{p,1}$,
- $\mathcal{P}u_0^\varepsilon\in\dot B^{d/p-1}_{p,1}$.

[5]The reader may refer to [18] for the construction and the low Mach asymptotic of the weak solutions to the Navier-Stokes-Fourier equations, and to [1] for the case of smoother data with large entropy variations.

The existence space $Y^p_{\varepsilon,\nu}$ is the set of triplets (a,u,ϑ) so that

- $(a^{\ell,\widetilde{\varepsilon}}, \mathcal{Q}u^{\ell,\widetilde{\varepsilon}}, \vartheta^{\ell,\widetilde{\varepsilon}}) \in \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{d/2-1}_{2,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{d/2+1}_{2,1})$,
- $a^{h,\widetilde{\varepsilon}} \in \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{d/p}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{d/p}_{p,1})$,
- $\vartheta^{h,\widetilde{\varepsilon}} \in \mathcal{C}_b(\mathbb{R}_+;\dot{B}^{d/p-2}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{d/p}_{p,1})$,
- $\mathcal{Q}u^{h,\widetilde{\varepsilon}}$ and $\mathcal{P}u^{\varepsilon}$ are in $\mathcal{C}_b(\mathbb{R}_+;\dot{B}^{d/p-1}_{p,1}) \cap L^1(\mathbb{R}_+;\dot{B}^{d/p+1}_{p,1})$,

endowed with the norm:

$$\|(a,u,\vartheta)\|_{Y^p_{\varepsilon,\nu}} := \|(a,\mathcal{Q}u,\vartheta)\|^{\ell,\widetilde{\varepsilon}}_{L^\infty(\dot{B}^{d/2-1}_{2,1})} + \|(\mathcal{P}u,\mathcal{Q}u^{h,\widetilde{\varepsilon}})\|_{L^\infty(\dot{B}^{d/p-1}_{p,1})} + \widetilde{\varepsilon}\|a\|^{h,\widetilde{\varepsilon}}_{L^\infty(\dot{B}^{d/p}_{p,1})}$$
$$+\widetilde{\varepsilon}^{-1}\|\vartheta\|^{h,\widetilde{\varepsilon}}_{L^\infty(\dot{B}^{d/p-2}_{p,1})} + \nu\|(a,\mathcal{Q}u,\vartheta)\|^{\ell,\widetilde{\varepsilon}}_{L^1(\dot{B}^{d/2+1}_{2,1})} + \nu\|(\mathcal{P}u,\mathcal{Q}u^{h,\widetilde{\varepsilon}})\|_{L^1(\dot{B}^{d/p+1}_{p,1})} + \varepsilon^{-1}\|(a,\vartheta)\|^{h,\widetilde{\varepsilon}}_{L^1(\dot{B}^{d/p}_{p,1})}.$$

We also set

$$\|(a_0,u_0,\vartheta_0)\|_{Y^p_{0,\varepsilon,\nu}} := \|(a_0,\mathcal{Q}u_0,\vartheta_0)\|^{\ell,\widetilde{\varepsilon}}_{\dot{B}^{d/2-1}_{2,1}}$$
$$+\|(\mathcal{P}u_0,\mathcal{Q}u_0^{h,\widetilde{\varepsilon}})\|_{\dot{B}^{d/p-1}_{p,1}} + \widetilde{\varepsilon}\|a_0\|^{h,\widetilde{\varepsilon}}_{\dot{B}^{d/p}_{p,1}} + \widetilde{\varepsilon}^{-1}\|\vartheta_0\|^{h,\widetilde{\varepsilon}}_{\dot{B}^{d/p-2}_{p,1}}.$$

Here the integer j_0 appearing in the threshold between low and high frequencies depends only on $\widetilde{\kappa} := \kappa/\nu$, $\widetilde{\mu} := \mu/\nu$ and $\widetilde{\lambda} := \lambda/\nu$ with $\nu := \lambda + 2\mu$.

In the case $p = 2$ and $\varepsilon = 1$, global existence for (5.49) in the above space and for small data has been established in [10]. The main goal of this section is to extend the statement to more general p's, and to get estimates independent of ε and ν for the constructed solution. Furthermore, in the $\mathbb{R}^d$ case, we establish a strong convergence result in the low Mach number asymptotics, in the spirit of our recent work [15]. Here is the main result of this section:

Theorem 5.1. *Assume that the fluid domain is either $\mathbb{R}^d$ or $\mathbb{T}^d$ with $d \geq 3$, and that the initial data $(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)$ are as above with $2 \leq p < d$ and $p \leq 2d/(d-2)$. There exists a constant η independent of ε and of ν (but depending on κ/ν) such that if*

$$\|(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}} \leq \eta\nu, \tag{5.50}$$

then System (5.49) *with initial data $(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)$ has a unique global solution $(a^\varepsilon, u^\varepsilon, \vartheta^\varepsilon)$ in the space $Y^p_{\varepsilon,\nu}$ with, for some constant C independent of ε and ν,*

$$\|(a^\varepsilon, u^\varepsilon, \vartheta^\varepsilon)\|_{Y^p_{\varepsilon,\nu}} \leq C\|(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}}. \tag{5.51}$$

Furthermore, in the $\mathbb{R}^d$ case, if $(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)$ is a family of data fulfilling (5.50) *with $\mathcal{P}u_0^\varepsilon \to v_0$ and $\vartheta_0^\varepsilon - a_0^\varepsilon \to \Theta_0$ for suitable norms, then we have*

- *$(q^\varepsilon, \mathcal{Q}u^\varepsilon) \to 0$ with $q^\varepsilon := \vartheta^\varepsilon + a^\varepsilon$,*
- *$\mathcal{P}u^\varepsilon \to u$ with u solution to* (1.10)*,*
- *$\Theta^\varepsilon \to \Theta$ with $\Theta^\varepsilon := \vartheta^\varepsilon - a^\varepsilon$ and Θ satisfying*

$$\partial_t\Theta - \frac{\kappa}{2}\Delta\Theta + u\cdot\nabla\Theta = 0, \qquad \Theta|_{t=0} = \Theta_0. \tag{5.52}$$

More precisely, we have

$$\|(q^\varepsilon, \mathcal{Q}u^\varepsilon)\|^{\ell,\widetilde{\varepsilon}}_{L^2(\dot{B}^{(d+1)/p-1/2}_{p,1})} \lesssim \nu^{-1/2}\widetilde{\varepsilon}^{1/2-1/p}\|(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}}, \tag{5.53}$$

$$\|\mathcal{P}u^\varepsilon - u\|_{L^\infty(\dot{B}^{(d+1)/p-3/2}_{p,1})} + \mu\|\mathcal{P}u^\varepsilon - u\|_{L^1(\dot{B}^{(d+1)/p+1/2}_{p,1}} \lesssim \|\mathcal{P}u_0^\varepsilon - v_0\|_{\dot{B}^{(d+1)/p-3/2}_{p,1}} + \widetilde{\varepsilon}^{1/2-1/p}\|(a_0^\varepsilon, u_0^\varepsilon, \vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}}, \tag{5.54}$$

and

$$\|\delta\Theta^\varepsilon\|_{L^\infty(\dot B^{(d+1)/p-3/2}_{p,1}+\dot B^{d/p-2}_{p,1})}+\|\delta\Theta^\varepsilon\|_{L^2(\dot B^{(d+1)/p-1/2}_{p,1})+L^1(\dot B^{d/p}_{p,1})}\\ \lesssim \|\Theta_0^\varepsilon-\Theta_0\|_{\dot B^{(d+1)/p-3/2}_{2,1}+\dot B^{d/p-2}_{p,1}}+\widetilde\varepsilon^{1/2-1/p}\|(a_0^\varepsilon,u_0^\varepsilon,\vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}}. \tag{5.55}$$

Remark 5.1. *Regarding the global existence and convergence issues, we expect similar results for slightly larger Besov spaces, as in the barotropic case. Here we only considered Besov spaces* with last index 1 *for simplicity, in order to benefit from uniqueness (see* [7]*), an open question otherwise, and also because it allows us to avoid resorting to* $\widetilde L^m(\dot B^s_{p,r})$ *spaces and complicated product estimates.*

Proof. As in the barotropic case, performing a suitable change of unknowns reduces the proof to the case $\varepsilon=\nu=1$, and coefficients $\tilde\mu$, $\tilde\lambda$ and $\tilde\kappa$. More precisely, we set

$$(a,u,\vartheta)(t,x)=\varepsilon(a^\varepsilon,u^\varepsilon,\vartheta^\varepsilon)(\varepsilon^2\nu t,\varepsilon\nu x). \tag{5.56}$$

Thanks to (2.14), we notice that

$$\nu\|(a,u,\vartheta)\|_{Y^p_{1,1}}=\|(a^\varepsilon,u^\varepsilon,\vartheta^\varepsilon)\|_{Y^p_{\varepsilon,\nu}}\quad\text{and}\quad \nu\|(a_0,u_0,\vartheta_0)\|_{Y^p_{0,1,1}}=\|(a_0^\varepsilon,u_0^\varepsilon,\vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}}. \tag{5.57}$$

So we may assume from now on that $\nu=\varepsilon=1$, and thus omit the exponent ε.

Let us give the outline of the proof. The first six steps are dedicated to proving global-in-time a priori estimates (namely (5.51)) for smooth solutions to (5.49), which is a rather easy adaptation of what we did in the barotropic case. In Step 7, we sketch the proof of existence. The last step is dedicated to the low Mach number asymptotics in the $\mathbb{R}^d$ case. Throughout, we assume that (2.16) is satisfied, so that one may freely apply Proposition 6.1.

Step 1. Incompressible part of the velocity. Let $\widetilde{\mathcal{A}}:=\mathcal{A}/\nu$. we have

$$\partial_t\mathcal{P}u-\widetilde\mu\Delta\mathcal{P}u=-\mathcal{P}(u\cdot\nabla u)-\mathcal{P}(J(a)\widetilde{\mathcal{A}}u)-\mathcal{P}(\vartheta\nabla K(a))\quad\text{with}\quad J(0)=K(0)=0.$$

Hence heat estimates (0.4) yield

$$\|\mathcal{P}u\|_{L^\infty(\dot B^{d/p-1}_{p,1})\cap L^1(\dot B^{d/p+1}_{p,1})}\lesssim\|\mathcal{P}u_0\|_{\dot B^{d/p-1}_{p,1}}+\|\mathcal{P}(u\cdot\nabla u)+\mathcal{P}(J(a)\widetilde{\mathcal{A}}u)+\mathcal{P}(\vartheta\nabla(K(a)))\|_{L^1(\dot B^{d/p-1}_{p,1})}.$$

Only the last term is new compared to the barotropic case. Decomposing it into

$$\vartheta\nabla(K(a))=\vartheta^\ell\nabla(K(a))+\vartheta^h\nabla(K(a)),$$

we may write

$$\|\mathcal{P}(\vartheta\nabla(K(a)))\|_{L^1(\dot B^{d/p-1}_{p,1})}\lesssim\|\nabla(K(a))\|_{L^2(\dot B^{d/p-1}_{p,1})}\|\vartheta^\ell\|_{L^2(\dot B^{d/p}_{p,1})}\\ +\|\nabla(K(a))\|_{L^\infty(\dot B^{d/p-1}_{p,1})}\|\vartheta^h\|_{L^1(\dot B^{d/p}_{p,1})}. \tag{5.58}$$

So arguing as in the barotropic case and using (2.19), we eventually get

$$\|\mathcal{P}u\|_{L^\infty(\dot B^{d/p-1}_{p,1})\cap L^1(\dot B^{d/p+1}_{p,1})}\lesssim\|\mathcal{P}u_0\|_{\dot B^{d/p-1}_{p,1}}+2^{j_0}(1+\|a\|_{L^\infty(\dot B^{d/p}_{p,1})})\|(a,u,\vartheta)\|^2_{Y^p_{1,1}}. \tag{5.59}$$

Step 2. Low frequencies. Applying Projector $\mathcal{Q}$ to the velocity equation, we see that $(a, \mathcal{Q}u, \vartheta)$ fulfills

$$\begin{cases} \partial_t a + \operatorname{div} \mathcal{Q}u = -\operatorname{div}(au), \\ \partial_t \mathcal{Q}u - \Delta \mathcal{Q}u + \nabla(a+\vartheta) = \mathcal{Q}\big(-u\cdot\nabla u - J(a)\widetilde{\mathcal{A}}u + (a-\vartheta)\nabla(K(a))\big), \\ \partial_t \vartheta - \widetilde{\kappa}\Delta\vartheta + \operatorname{div}\mathcal{Q}u = -\operatorname{div}(\vartheta u) - \widetilde{\kappa}J(a)\Delta\vartheta + \dfrac{1}{1+a}\Big(2\widetilde{\mu}|Du|^2 + \widetilde{\lambda}(\operatorname{div} u)^2\Big). \end{cases}$$

The results of [10] guarantee that

$$\|(a,\mathcal{Q}u,\vartheta)\|^\ell_{L^\infty(\dot{B}^{d/2-1}_{2,1})\cap L^1(\dot{B}^{d/2+1}_{2,1})} \lesssim \|(a_0,\mathcal{Q}u_0,\vartheta_0)\|^\ell_{\dot{B}^{d/2-1}_{2,1}} + \|r.h.s.\|^\ell_{L^1(\dot{B}^{d/2-1}_{2,1})}. \tag{5.60}$$

Compared to the barotropic case, we have to bound in $L^1(\dot{B}^{d/2-1}_{2,1})$ the low frequencies of the following additional terms:

$$\vartheta\nabla(K(a)), \quad \operatorname{div}(\vartheta u), \quad \widetilde{\kappa}J(a)\Delta\vartheta \quad \text{and} \quad \frac{1}{1+a}\Big(2\widetilde{\mu}|Du|^2 + \widetilde{\lambda}(\operatorname{div} u)^2\Big). \tag{5.61}$$

To handle the first term, we start with the observation that

$$\|\nabla(K(a))\|^\ell_{\dot{B}^{d/2-1}_{2,1}} \lesssim \big(1+\|a\|_{\dot{B}^{d/p}_{p,1}}\big)\big(\|a\|^\ell_{\dot{B}^{d/2}_{2,1}} + \|a\|^h_{\dot{B}^{d/p}_{p,1}}\big) + 2^{j_0}\|a\|_{\dot{B}^{d/p-1}_{p,1}}\|a\|^h_{\dot{B}^{d/p}_{p,1}}. \tag{5.62}$$

Indeed, because $\nabla((K(a)) = K'(0)\nabla a + \widetilde{K}(a)\nabla a$ for some smooth function $\widetilde{K}$ vanishing at zero, it suffices to prove that[6]

$$\|\widetilde{K}(a)\nabla a\|^\ell_{\dot{B}^{d/2-1}_{2,1}} \lesssim \|a\|_{\dot{B}^{d/p}_{p,1}}\big(\|\nabla a\|^\ell_{\dot{B}^{d/2-1}_{2,1}} + \|\nabla a\|_{\dot{B}^{d/p^*-1}_{p^*,1}}\big) + 2^{j_0}\|a\|_{\dot{B}^{d/p-1}_{p,1}}\|\nabla a\|^h_{\dot{B}^{d/p-1}_{p,1}}.$$

To this end, we use Bony's decomposition restricted to low frequencies:

$$(\widetilde{K}(a)\nabla a)^\ell = (T_{\nabla a}\widetilde{K}(a))^\ell + (R(\nabla a,\widetilde{K}(a)))^\ell + T_{(\widetilde{K}(a))^\ell}\nabla a^\ell + (\dot{S}_{j_0}\widetilde{K}(a)\dot{\Delta}_{j_0+1}\nabla a)^\ell.$$

To deal with the first two terms, we just use (2.30). For the third one, we use that $T : L^\infty \times \dot{B}^{d/2-1}_{2,1} \to \dot{B}^{d/2-1}_{2,1}$ and the embedding $\dot{B}^{d/p}_{p,1} \hookrightarrow L^\infty$. For the last one, we argue as follows:

$$2^{j_0(d/2-1)}\|\dot{S}_{j_0}\widetilde{K}(a)\dot{\Delta}_{j_0+1}\nabla a\|_{L^2} \le 2^{j_0}\,2^{j_0(d/p^*-1)}\|\dot{S}_{j_0}\widetilde{K}(a)\|_{L^{p^*}}\,2^{j_0(d/p-1)}\|\dot{\Delta}_{j_0+1}\nabla a\|_{L^p}.$$

Putting all those inequalities together, and using also composition estimates and the fact that $d/p^* - 1 \le 0$ eventually leads to the desired inequality.

Let us now bound $(\vartheta\nabla(K(a)))^\ell$ in $L^1(\dot{B}^{d/2-1}_{2,1})$. We start again from Bony's decomposition:

$$\begin{aligned}(\vartheta\nabla(K(a)))^\ell = (T_{\nabla(K(a))}\vartheta)^\ell + (R(\nabla(K(a)),\vartheta))^\ell \\ + T_{\vartheta^\ell}\nabla(K(a))^\ell + (\dot{S}_{j_0}\vartheta\dot{\Delta}_{j_0+1}\nabla K(a))^\ell.\end{aligned} \tag{5.63}$$

The first two terms may be bounded by splitting ϑ into $\vartheta^\ell + \vartheta^h$, using the continuity of R and T from $\dot{B}^{d/p-1}_{p,1} \times \dot{B}^{d/p}_{p,1}$ to $\dot{B}^{d/2-1}_{2,1}$. We end up with

$$\|(T_{\nabla(K(a))}\vartheta)^\ell\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim \|\nabla(K(a))\|_{L^2(\dot{B}^{d/p-1}_{p,1})}\big(\|\vartheta^\ell\|_{L^2(\dot{B}^{d/p}_{p,1})} + 2^{j_0}\|\vartheta^h\|_{L^2(\dot{B}^{d/p-1}_{p,1})}\big).$$

and

$$\begin{aligned}\|(R(\nabla(K(a)),\vartheta))^\ell\|_{L^1(\dot{B}^{d/2-1}_{2,1})} \lesssim\ & \|\nabla(K(a))\|_{L^\infty(\dot{B}^{d/p-1}_{p,1})}\|\vartheta^h\|_{L^1(\dot{B}^{d/p}_{p,1})} \\ & + \|\nabla(K(a))\|_{L^2(\dot{B}^{d/p-1}_{p,1})}\|\vartheta^\ell\|_{L^2(\dot{B}^{d/p}_{p,1})}.\end{aligned}$$

[6]For p^*, we keep the definition $1/p + 1/p^* = 1/2$.

For the third term in (5.63), by virtue of (5.62), we write

$$\begin{aligned}\|T_{\vartheta^\ell}\nabla(K(a))^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})} &\lesssim \|\vartheta^\ell\|_{L^2(L^\infty)}\|\nabla(K(a))^\ell\|_{\dot B^{d/2-1}_{2,1}}\\ &\lesssim \|\vartheta^\ell\|_{L^2(\dot B^{d/2}_{2,1})}\big(1+2^{j_0}\|a^\ell\|_{L^\infty(\dot B^{d/2-1}_{2,1})}\\ &\quad+\|a^h\|_{L^\infty(\dot B^{d/p}_{p,1})}\big)\big(\|a\|^\ell_{L^2(\dot B^{d/2}_{2,1})}+\|a\|^h_{L^2(\dot B^{d/p}_{p,1})}\big).\end{aligned}$$

Finally,

$$2^{j_0(d/2-1)}\|\dot S_{j_0}\vartheta\dot\Delta_{j_0+1}\nabla K(a))\|_{L^2}\le 2^{j_0(d/p^*-1)}\|\dot S_{j_0}\vartheta\|_{L^{p^*}}\,2^{j_0d/p}\|\dot\Delta_{j_0+1}\nabla(K(a))\|_{L^p}.$$

Hence

$$\|\dot S_{j_0}\vartheta\dot\Delta_{j_0+1}\nabla(K(a))\|_{L^1(\dot B^{d/2-1}_{2,1})}\lesssim 2^{j_0}\|\vartheta^\ell\|_{L^\infty(\dot B^{d/p^*-1}_{p^*,1})}\|K(a)\|^h_{L^1(\dot B^{d/p}_{p,1})}.$$

That the last term does belong to $L^1(\dot B^{d/p}_{p,1})$ may be seen by writing

$$K(a)=K'(0)\,a+\widetilde K(a)\,a\quad\text{with}\quad\widetilde K(0)=0,$$

which ensures, using composition estimates in $\dot B^{d/p}_{p,1}$,

$$\|K(a)\|^h_{L^1(\dot B^{d/p}_{p,1}))}\lesssim\|a\|^h_{L^1(\dot B^{d/p}_{p,1})}+\|a\|^2_{L^2(\dot B^{d/p}_{p,1})}. \tag{5.64}$$

Resuming to (5.63), we conclude that

$$\begin{aligned}\|(\vartheta\nabla(K(a)))^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})}&\lesssim 2^{j_0}\|\vartheta^\ell\|_{L^\infty(\dot B^{d/2-1}_{2,1})}\big(\|a^h\|_{L^1(\dot B^{d/p}_{p,1})}+\|a\|^2_{L^2(\dot B^{d/p}_{p,1})}\big)\\ &\quad+\big(1+2^{j_0}\|a^\ell\|_{L^\infty(\dot B^{d/2-1}_{2,1})}+\|a^h\|_{L^\infty(\dot B^{d/p-1}_{p,1})}\big)\big(\|\vartheta^\ell\|_{L^2(\dot B^{d/2}_{2,1})}\\ &\quad+2^{j_0}\|\vartheta^h\|_{L^2(\dot B^{d/p-1}_{p,1})}+\|\vartheta^h\|_{L^1(\dot B^{d/p}_{p,1})}\big)\big(\|a\|^\ell_{L^2(\dot B^{d/2}_{2,1})}+\|a\|^h_{L^2(\dot B^{d/p}_{p,1})}\big).\end{aligned} \tag{5.65}$$

To handle $\operatorname{div}(\vartheta u)$, we decompose ϑ into low and high frequencies. To deal with both parts, we resort again to Bony's decomposition and continuity results for R and T. We end up with

$$\|\vartheta^\ell u\|_{\dot B^{d/2}_{2,1}}\lesssim\|\vartheta^\ell\|_{\dot B^{d/p^*-1}_{p^*,1}}\|u\|_{\dot B^{d/p+1}_{p,1}}+\|u\|_{L^\infty}\|\vartheta^\ell\|_{\dot B^{d/2}_{2,1}}, \tag{5.66}$$

$$\|\vartheta^h u\|_{\dot B^{d/2-1}_{2,1}}\lesssim\|\vartheta^h\|_{\dot B^{d/p-1}_{p,1}}\|u\|_{\dot B^{d/p}_{p,1}}+\|u\|_{\dot B^{d/p^*-1}_{p^*,1}}\|\vartheta^h\|_{\dot B^{d/p}_{p,1}}. \tag{5.67}$$

Therefore, taking advantage of the low frequency cut-off and of Bernstein inequality yields

$$\begin{aligned}\|(\operatorname{div}(\vartheta u))^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})}&\lesssim\|\vartheta^\ell\|_{L^\infty(\dot B^{d/2-1}_{2,1})}\|u\|_{L^1(\dot B^{d/p+1}_{p,1})}\\ &\quad+\big(\|\vartheta^\ell\|_{L^2(\dot B^{d/2}_{2,1})}+2^{j_0}\|\vartheta^h\|_{L^2(\dot B^{d/p-1}_{p,1})}\big)\|u\|_{L^2(\dot B^{d/p}_{p,1})}+2^{j_0}\|u\|_{L^\infty(\dot B^{d/p-1}_{p,1})}\|\vartheta^h\|_{L^1(\dot B^{d/p}_{p,1})}.\end{aligned} \tag{5.68}$$

For the next term, we use

$$J(a)\Delta\vartheta=J(a)\Delta\vartheta^\ell+J(a)\Delta\vartheta^h$$

and Bony's decomposition. For the first term, we easily get

$$\|J(a)\Delta\vartheta^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})}\lesssim\|\Delta\vartheta^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})}\|a\|_{L^\infty(\dot B^{d/p}_{p,1})}.$$

For the second one, we use that R and T map $\dot B^{d/p-2}_{p,1}\times\dot B^{d/p}_{p,1}$ to $\dot B^{d/2-2}_{2,1}$, if $p<d$ and $d\ge3$, and that

$$\|(T_{J(a)}\Delta\vartheta^h+R(J(a),\Delta\vartheta^h))^\ell\|_{\dot B^{d/2-2}_{2,1}}\lesssim\|J(a)\|_{\dot B^{d/p}_{p,1}}\|\Delta\vartheta^h\|_{\dot B^{d/p-2}_{p,1}}.$$

Hence, combining with Bernstein inequality,

$$\|(J(a)\Delta\vartheta)^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})} \lesssim 2^{j_0}\big(\|a\|_{L^\infty(\dot B^{d/p}_{p,1})}+2^{j_0}\|a\|_{L^\infty(\dot B^{d/p-1}_{p,1})}\big)\big(\|\Delta\vartheta^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})}+\|\Delta\vartheta^h\|_{L^1(\dot B^{d/p-2}_{p,1})}\big).$$

To handle the last term in (5.61), we use the fact that

$$\begin{aligned}\|T_{J(a)}Du\otimes Du\|_{\dot B^{d/2-2}_{2,1}} &\lesssim \|J(a)\|_{L^\infty}\|Du\otimes Du\|_{\dot B^{d/2-2}_{2,1}}\\ \|R(J(a),Du\otimes Du)\|_{\dot B^{d/2-2}_{2,1}} &\lesssim \|J(a)\|_{\dot B^{d/p}_{p,1}}\|Du\otimes Du\|_{\dot B^{d/2-2}_{2,1}}\\ \|T_{Du\otimes Du}J(a)\|_{\dot B^{d/2-2}_{2,1}} &\lesssim \|Du\otimes Du\|_{\dot B^{d/p^*-2}_{p^*,1}}\|J(a)\|_{\dot B^{d/p}_{p,1}}.\end{aligned}$$

At this point, we notice that, under assumption $2\leq p\leq 2d/(d-2)$, $p<d$ and $d\geq 3$, the usual product maps $\dot B^{d/p-1}_{p,1}\times\dot B^{d/p-1}_{p,1}$ to $\dot B^{d/2-2}_{2,1}$. Therefore

$$\Big\|\frac{1}{1+a}Du\otimes Du\Big\|_{L^1(\dot B^{d/2-2}_{2,1})}\lesssim\big(1+\|a\|_{L^\infty(\dot B^{d/p}_{p,1})}\big)\|Du\|^2_{L^2(\dot B^{d/p-1}_{p,1})}. \tag{5.69}$$

Inserting all the above inequalities in (5.60) and using (2.19), we thus end up with

$$\begin{aligned}\|(a,\mathcal{Q}u,\vartheta)\|^\ell_{L^\infty(\dot B^{d/2-1}_{2,1})\cap L^1(\dot B^{d/2+1}_{2,1})}&\\ \lesssim \|(a_0,\mathcal{Q}u_0,\vartheta_0)\|^\ell_{\dot B^{d/2-1}_{2,1}}&+2^{2j_0}(1+\|(a,u,\vartheta)\|_{Y^p_{1,1}})\|(a,u,\vartheta)\|^2_{Y^p_{1,1}}.\end{aligned} \tag{5.70}$$

Step 3. High frequencies: the effective velocity. Let $w:=\mathcal{Q}u+(-\Delta)^{-1}\nabla a$. We have

$$\partial_t w-\Delta w=-\mathcal{Q}(u\cdot\nabla u)+\mathcal{Q}(J(a)\widetilde{\mathcal{A}}u)-\mathcal{Q}(\vartheta\nabla K(a))+a\nabla K(a)+\mathcal{Q}(au)-\nabla\vartheta+w-(-\Delta)^{-1}\nabla a.$$

By virtue of (0.4), we have

$$\|w\|^h_{L^\infty(\dot B^{d/p-1}_{p,1})\cap L^1(\dot B^{d/p+1}_{p,1})}\lesssim\|w_0\|^h_{\dot B^{d/p-1}_{p,1}}+\|r.h.s.\|^h_{L^1(\dot B^{d/p-1}_{p,1})}.$$

Compared to the barotropic case, two new terms have to be handled : $\mathcal{Q}(\vartheta\nabla(K(a)))$ and $\nabla\vartheta$. The first one has been estimated in (5.58), and the second one is just linear. We eventually get if j_0 is large enough:

$$\begin{aligned}\|w\|^h_{L^\infty(\dot B^{d/p-1}_{p,1})\cap L^1(\dot B^{d/p+1}_{p,1})}\lesssim\|w_0\|^h_{\dot B^{d/p-1}_{p,1}}&\\ +2^{j_0}\|(a,u,\vartheta)\|^2_{Y^p_{1,1}}+\|\nabla\vartheta\|^h_{L^1(\dot B^{d/p-1}_{p,1})}&+2^{-2j_0}\|a\|^h_{L^1(\dot B^{d/p}_{p,1})}.\end{aligned} \tag{5.71}$$

Step 4. High frequencies: the temperature. Applying (0.4) to the heat equation

$$\partial_t\vartheta-\widetilde\kappa\Delta\vartheta=-a-\operatorname{div}w-\operatorname{div}(\vartheta u)-\widetilde\kappa J(a)\Delta\vartheta+\frac{1}{1+a}\Big(2\widetilde\mu|Du|^2+\widetilde\lambda(\operatorname{div}u)^2\Big)$$

yields

$$\|\vartheta\|^h_{L^\infty(\dot B^{d/p-2}_{p,1})\cap L^1(\dot B^{d/p}_{p,1})}\lesssim\|\vartheta_0\|^h_{\dot B^{d/p-2}_{p,1}}+\|r.h.s.\|^h_{L^1(\dot B^{d/p-2}_{p,1})}.$$

The term $\operatorname{div}(\vartheta u)$ can be bounded according to (5.66) and (5.67), using obvious embedding. For the other nonlinear terms, we observe that under condition $p<d$, we have

$$\begin{aligned}\|J(a)\Delta\vartheta^h\|_{L^1(\dot B^{d/p-2}_{p,1})}&\lesssim\|a\|_{L^\infty(\dot B^{d/p}_{p,1})}\|\Delta\vartheta^h\|_{L^1(\dot B^{d/p-2}_{p,1})},\\ \|J(a)\Delta\vartheta^\ell\|^h_{L^1(\dot B^{d/p-2}_{p,1})}&\lesssim 2^{-j_0}\|J(a)\Delta\vartheta^\ell\|^h_{L^1(\dot B^{d/p-1}_{p,1})}\lesssim 2^{-j_0}\|a\|_{L^\infty(\dot B^{d/p}_{p,1})}\|\Delta\vartheta^\ell\|_{L^1(\dot B^{d/2-1}_{2,1})},\end{aligned}$$

$$\Big\|\frac{1}{1+a}\Big(2\mu|Du|^2+\lambda(\operatorname{div}u)^2\Big)\Big\|_{L^1(\dot B^{d/p-2}_{p,1})}\lesssim (1+\|a\|_{L^\infty(\dot B^{d/p}_{p,1})})\|\nabla u\|^2_{L^2(\dot B^{d/p-1}_{p,1})},$$

whence

$$\|\vartheta\|^h_{L^\infty(\dot B^{d/p-2}_{p,1})\cap L^1(\dot B^{d/p}_{p,1})}\lesssim \|\vartheta_0\|^h_{\dot B^{d/p-2}_{p,1}}+2^{-2j_0}\|a+\operatorname{div}w\|^h_{L^1(\dot B^{d/p}_{p,1})}+2^{j_0}(1+\|(a,u,\vartheta)\|_{Y^p_{1,1}})\|(a,u,\vartheta)\|^2_{Y^p_{1,1}}. \tag{5.72}$$

Step 5. High frequencies: the density. Exactly as in the barotropic case, Inequality (2.37) is fulfilled.

Step 6. Closure of the estimates. Inserting (5.72) in (5.71), we get for large enough j_0

$$\|w\|^h_{L^\infty(\dot B^{d/p-1}_{p,1})\cap L^1(\dot B^{d/p+1}_{p,1})}\lesssim \|w_0\|^h_{\dot B^{d/p-1}_{p,1}}+\|\vartheta_0\|^h_{\dot B^{d/p-2}_{p,1}}+2^{j_0}(1+\|(a,u,\vartheta)\|_{Y^p_{1,1}})\|(a,u,\vartheta)\|^2_{Y^p_{1,1}}+2^{-2j_0}\|a\|^h_{L^1(\dot B^{d/p}_{p,1})}.$$

Next, plugging that latter inequality in (2.37), we get for large enough j_0,

$$\|a\|^h_{L^1\cap L^\infty(\dot B^{d/p}_{p,1})}\lesssim \|a_0\|^h_{\dot B^{d/p}_{p,1}}+\|w_0\|^h_{\dot B^{d/p-1}_{p,1}}+\|\vartheta_0\|^h_{\dot B^{d/p-2}_{p,1}}+2^{j_0}(1+\|(a,u,\vartheta)\|_{Y^p_{1,1}})\|(a,u,\vartheta)\|^2_{Y^p_{1,1}}.$$

Resuming to (5.59) and (5.70), it is now easy to conclude that

$$\|(a,u,\vartheta)\|_{Y^p_{1,1}}\lesssim \|(a_0,u_0,\vartheta_0)\|_{Y^p_{0,1,1}}+2^{2j_0}(1+\|(a,u,\vartheta)\|_{Y^p_{1,1}})\|(a,u,\vartheta)\|^2_{Y^p_{1,1}},$$

from which it is clear that we may get (5.51) if $\|(a_0,u_0,\vartheta_0)\|_{Y^p_{0,1,1}}$ is small enough.

Step 7. The proof of global existence and uniqueness. Uniqueness up to $p<d$ is just a consequence of the recent paper [7]. Local-in-time existence of a solution (a,u,ϑ) to (5.49) with $a\in\mathcal{C}([0,T];\dot B^{d/p}_{p,1})$, $u\in\mathcal{C}([0,T];\dot B^{d/p-1}_{p,1})\cap L^1(0,T;\dot B^{d/p+1}_{p,1})$ and $\vartheta\in\mathcal{C}([0,T];\dot B^{d/p-2}_{p,1})\cap L^1(0,T;\dot B^{d/p}_{p,1})$ has been established in [8]. That the additional low frequency L^2 type regularity is preserved during the evolution is a consequence of the computations that have been carried out in Step 2.

Finally, by slight modifications of the blow-up criterion of Prop. 10.10 of [2], one can show that if

$$\|\nabla u\|_{L^1_T(L^\infty)}+\|a\|_{L^\infty_T(\dot B^{d/p}_{p,1})}+\|\vartheta\|_{L^1_T(\dot B^{d/p}_{p,1})}<\infty$$

then the solution may be continued beyond T. As the norm in the space $Y^p_{1,1}$ (restricted to $[0,T)$) clearly controls the above l.h.s., Inequality (5.51) implies the global existence.

Step 8. Low Mach number limit : strong convergence in the whole space case. As in our recent work [15] dedicated to the Oberbeck-Boussinesq approximation, the proof of strong convergence relies on the dispersive properties of the system fulfilled by $q^\varepsilon:=\vartheta^\varepsilon+a^\varepsilon$ and $\mathcal{Q}u^\varepsilon$, namely

$$\begin{cases}\partial_t q^\varepsilon+\dfrac{2}{\varepsilon}\operatorname{div}\mathcal{Q}u^\varepsilon=-\operatorname{div}(u^\varepsilon q^\varepsilon)+\kappa\Delta\vartheta^\varepsilon+\kappa J(\varepsilon a^\varepsilon)\Delta\vartheta^\varepsilon+\dfrac{\varepsilon}{1+\varepsilon a^\varepsilon}\big(2\mu|Du^\varepsilon|^2+\lambda(\operatorname{div}u^\varepsilon)^2\big),\\ \partial_t\mathcal{Q}u^\varepsilon+\dfrac{1}{\varepsilon}\nabla q^\varepsilon=\nu\Delta\mathcal{Q}u^\varepsilon-\mathcal{Q}(u^\varepsilon\cdot\nabla u^\varepsilon)-\mathcal{Q}(J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon)+\mathcal{Q}\bigg((a^\varepsilon-\vartheta^\varepsilon)\dfrac{\nabla a^\varepsilon}{1+\varepsilon a^\varepsilon}\bigg).\end{cases}$$

Remembering that the low frequencies of the r.h.s. have been bounded in $L^1(\mathbb{R}_+;\dot B^{d/2-1}_{2,1})$ by $C_0^{\varepsilon,\nu}:=\|(a_0^\varepsilon,u_0^\varepsilon,\vartheta_0^\varepsilon)\|_{Y^p_{0,\varepsilon,\nu}}$ (see Step 2), one can mimic the proof of the strong convergence

for the barotropic case in the case $d \geq 3$ (see the beginning of Section 4) and easily conclude that (5.53) is satisfied.

The high frequencies of a^ε and $\mathcal{Q}u^\varepsilon$ may be bounded as in (5.53) (argue as in the barotropic case) but not $(\vartheta^\varepsilon)^{h,\widetilde{\varepsilon}}$ which is one derivative less regular than $(a^\varepsilon)^{h,\widetilde{\varepsilon}}$.

Let us now study the strong convergence of $\mathcal{P}u^\varepsilon$ to u. To this end, we observe that $\delta u^\varepsilon := \mathcal{P}u^\varepsilon - u$ fulfills

$$\begin{aligned}(5.73)\quad \partial_t \delta u^\varepsilon - \mu\Delta\delta u^\varepsilon + \mathcal{P}(\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon + \delta u^\varepsilon\cdot\nabla u) + \mathcal{P}\big(u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon + \mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\big) + J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\big)\\ = \mathcal{P}\bigg(\frac{1}{1+\varepsilon a^\varepsilon}\Big(\nabla(q^\varepsilon)^{\ell,\widetilde{\varepsilon}}a^\varepsilon + \nabla(a^\varepsilon)^{h,\widetilde{\varepsilon}}a^\varepsilon + \nabla(\vartheta^\varepsilon)^{h,\widetilde{\varepsilon}}a^\varepsilon\Big) + J(\varepsilon a^\varepsilon)\nabla(a^\varepsilon\vartheta^\varepsilon)\bigg)\cdot\end{aligned}$$

The first line may be handled as in the barotropic case : we get

$$\begin{aligned}\|\mathcal{P}(\mathcal{P}u^\varepsilon\cdot\nabla\delta u^\varepsilon + \delta u^\varepsilon\cdot\nabla u)\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})} &\lesssim \|\mathcal{P}u^\varepsilon\|_{L^\infty(\dot{B}^{d/p-1}_{p,1})}\|\nabla\delta u^\varepsilon\|_{L^1(\dot{B}^{(d+1)/p-1/2}_{p,1})}\\ &\quad + \|\delta u^\varepsilon\|_{L^\infty(\dot{B}^{(d+1)/p-3/2}_{p,1})}\|\nabla u\|_{L^1(\dot{B}^{d/p}_{p,1})},\\ \|\mathcal{P}\big(u^\varepsilon\cdot\nabla\mathcal{Q}u^\varepsilon + \mathcal{Q}u^\varepsilon\cdot\nabla\mathcal{P}u^\varepsilon\big) + J(\varepsilon a^\varepsilon)\mathcal{A}u^\varepsilon\big)\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})} &\lesssim \nu^{-1}\varepsilon^{1/2-1/p}(1+\nu^{-1}C_0^{\varepsilon,\nu})(C_0^{\varepsilon,\nu})^2.\end{aligned}$$

In order to bound the terms of the second line of (5.73), we shall use repeatedly the fact that for any smooth function K vanishing at 0, we have, by virtue of Proposition 6.1,

$$(5.74)\qquad \|K(\varepsilon a^\varepsilon)\|_{L^\infty(\dot{B}^{d/p}_{p,1})} \lesssim \nu^{-1}\big(\|a^\varepsilon\|^{\ell,\widetilde{\varepsilon}}_{L^\infty(\dot{B}^{d/p-1}_{p,1})} + \widetilde{\varepsilon}\|a^\varepsilon\|^{h,\widetilde{\varepsilon}}_{L^\infty(\dot{B}^{d/p}_{p,1})}\big) \lesssim \nu^{-1}C_0^{\varepsilon,\nu}.$$

On the one hand, using product laws in Besov spaces yields

$$\begin{aligned}\|\nabla(q^\varepsilon)^{\ell,\widetilde{\varepsilon}}a^\varepsilon\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})} &\lesssim \|(q^\varepsilon)^{\ell,\widetilde{\varepsilon}}\|_{L^2(\dot{B}^{(d+1)/p-1/2}_{p,1})}\|a^\varepsilon\|_{L^2(\dot{B}^{d/p}_{p,1})},\\ \|\nabla(a^\varepsilon)^{h,\widetilde{\varepsilon}}a^\varepsilon\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})} &\lesssim \|(a^\varepsilon)^{h,\widetilde{\varepsilon}}\|_{L^2(\dot{B}^{(d+1)/p-1/2}_{p,1})}\|a^\varepsilon\|_{L^2(\dot{B}^{d/p}_{p,1})},\\ \|\nabla(\vartheta^\varepsilon)^{h,\widetilde{\varepsilon}}a^\varepsilon\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})} &\lesssim \|(\vartheta^\varepsilon)^{h,\widetilde{\varepsilon}}\|_{L^1(\dot{B}^{d/p}_{p,1})}\|a^\varepsilon\|_{L^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})}\\ \|J(\varepsilon a^\varepsilon)\nabla(a^\varepsilon\vartheta^\varepsilon)\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|_{L^\infty(\dot{B}^{(d+1)/p-1/2}_{p,1})}\big(\|(\vartheta^\varepsilon)^{h,\widetilde{\varepsilon}}\|_{L^1(\dot{B}^{d/p}_{p,1})}\|a^\varepsilon\|_{L^\infty(\dot{B}^{d/p}_{p,1})}\\ &\quad + \|(\vartheta^\varepsilon)^{\ell,\widetilde{\varepsilon}}\|_{L^2(\dot{B}^{d/p}_{p,1})}\|a^\varepsilon\|_{L^2(\dot{B}^{d/p}_{p,1})}\big).\end{aligned}$$

Hence using (4.48), (5.53), (5.51) and (5.74),

$$\begin{aligned}\Big\|\mathcal{P}\Big(\frac{1}{1+\varepsilon a}\Big(\nabla(q^\varepsilon)^{\ell,\widetilde{\varepsilon}}a^\varepsilon + \nabla(a^\varepsilon)^{h,\widetilde{\varepsilon}}a^\varepsilon + \nabla(\vartheta^\varepsilon)^{h,\widetilde{\varepsilon}}a^\varepsilon\Big) + J(\varepsilon a^\varepsilon)\nabla(a^\varepsilon\vartheta^\varepsilon)\Big)\Big\|_{L^1(\dot{B}^{(d+1)/p-3/2}_{p,1})}\\ \lesssim \nu^{-1}(1+\nu^{-1}C_0^{\varepsilon,\nu})(C_0^{\varepsilon,\nu})^2.\end{aligned}$$

Putting together all the above inequalities and the uniform estimate (5.51), we end up with

$$\begin{aligned}\delta U^\varepsilon &:= \|\delta u^\varepsilon\|_{L^\infty(\dot{B}^{(d+1)/p-3/2}_{p,1})} + \mu\|\delta u^\varepsilon\|_{L^1(\dot{B}^{(d+1)/p+1/2}_{p,1})}\\ &\lesssim \|\mathcal{P}u_0^\varepsilon - v_0\|_{\dot{B}^{(d+1)/p-3/2}_{p,1}} + \nu^{-1}C_0^{\varepsilon,\nu}\delta U^\varepsilon + \nu^{-1}\widetilde{\varepsilon}^{1/2-1/p}(1+\nu^{-1}C_0^{\varepsilon,\nu})(C_0^{\varepsilon,\nu})^2,\end{aligned}$$

which obviously implies (5.54), owing to the smallness condition satisfied by $C_0^{\varepsilon,\nu}$.

Let us finally study the strong convergence of $\Theta^\varepsilon := \vartheta^\varepsilon - a^\varepsilon$ to the solution Θ of (5.52). Given the uniform bounds for $(\vartheta_0^\varepsilon)$ and for (a_0^ε), it is natural to assume that the limit Θ_0 belongs to $\dot{B}^{d/2-1}_{2,1}$ (as a matter of fact $\dot{B}^{d/p-1}_{p,1}$ is enough for what follows). Likewise, as $(\mathcal{P}u_0^\varepsilon)$ is bounded in $\dot{B}^{d/p-1}_{p,1}$, one may assume that its weak limit v_0 belongs to $\dot{B}^{d/p-1}_{p,1}$. Hence the corresponding solution u to (1.10) is in $\mathcal{C}(\mathbb{R}_+;\dot{B}^{d/p-1}_{p,1})\cap L^1(\mathbb{R}_+;\dot{B}^{d/p+1}_{p,1})$, and using the fact

that $\mathrm{div}\,(u\Theta) = u\cdot\nabla\Theta$, it is easy to prove that the linear equation (5.52) admits a unique solution $\Theta\in\mathcal{C}_b(\mathbb{R}_+;\dot B^{d/2-1}_{2,1})\cap L^1(\mathbb{R}_+;\dot B^{d/2+1}_{2,1})$.

Next, from (5.52), observing that $\Delta\vartheta^\varepsilon = \frac12\Delta\Theta^\varepsilon + \frac12\Delta q^\varepsilon$, we readily get that $\delta\Theta^\varepsilon := \Theta^\varepsilon - \Theta$ satisfies

$$\begin{aligned}(5.75)\quad \partial_t\delta\Theta^\varepsilon - \frac{\kappa}{2}\Delta\delta\Theta^\varepsilon = &-\mathcal{P}u^\varepsilon\cdot\nabla\delta\Theta^\varepsilon - \delta u^\varepsilon\cdot\nabla\Theta - \mathrm{div}\,(\mathcal{Q}u^\varepsilon\Theta^\varepsilon)\\ &-\frac{\kappa}{2}\Delta q^\varepsilon - \kappa J(\varepsilon a^\varepsilon)\Delta\vartheta^\varepsilon + \frac{\varepsilon}{1+\varepsilon a^\varepsilon}\big(2\mu|Du^\varepsilon|^2+\lambda(\mathrm{div}\,u^\varepsilon)^2\big).\end{aligned}$$

The level of regularity on which estimates for $\delta\Theta^\varepsilon$ may be proved, is essentially given by the available estimates for δu^ε, through the term $\delta u^\varepsilon\cdot\nabla\Theta = \mathrm{div}\,(\delta u^\varepsilon\Theta)$, by the fact that decay estimates are available for the low frequencies of the term Δq^ε in the space $L^2(\mathbb{R}_+;\dot B^{(d+1)/p-5/2}_{p,1})$ only through (5.53), and by observing that the high frequencies of Δq^ε (and more precisely of $\Delta\vartheta^\varepsilon$) are at most in the space $L^1(\dot B^{d/p-2}_{p,1})$, but have decay ε.

As regards $\delta u^\varepsilon\cdot\nabla\Theta$, product laws in Besov spaces give the following bound:

$$\|\delta u^\varepsilon\cdot\nabla\Theta\|_{L^1(\dot B^{(d+1)/p-3/2}_{p,1})}\lesssim\|\delta u^\varepsilon\|_{L^\infty(\dot B^{(d+1)/p-3/2}_{p,1})}\|\Theta\|_{L^1(\dot B^{d/2+1}_{2,1})}.$$

Note that only an L^2-in-time estimate is available for $(\Delta q^\varepsilon)^{\ell,\widetilde\varepsilon}$, through (5.53). However, a small variation on (0.4) (see e.g. [2]) ensures that the solution to

$$\partial_t z - \kappa\Delta z = -\frac{\kappa}{2}(\Delta q^\varepsilon)^{\ell,\widetilde\varepsilon},\qquad z|_{t=0}=0$$

belongs to $\mathcal{C}_b(\mathbb{R}_+;\dot B^{(d+1)/p-3/2}_{p,1})\cap L^2(\mathbb{R}_+;\dot B^{(d+1)/p-1/2}_{p,1})$ and satisfies

$$\|z\|_{L^2(\dot B^{(d+1)/p-1/2}_{p,1})}+\|z\|_{L^\infty(\dot B^{(d+1)/p-3/2}_{p,1})}\lesssim\|\Delta q^\varepsilon\|^{\ell,\widetilde\varepsilon}_{L^2(\dot B^{(d+1)/p-5/2}_{p,1})}.$$

So in short we expect to be able to bound $\delta\Theta^\varepsilon$ in

$$L^\infty(\mathbb{R}_+;\dot B^{(d+1)/p-3/2}_{p,1}+\dot B^{d/p-2}_{p,1})\cap\big(L^2(\mathbb{R}_+;\dot B^{(d+1)/p-1/2}_{p,1})+L^1(\mathbb{R}_+;\dot B^{d/p}_{p,1})\big).$$

Let us now look at the other terms in the r.h.s. of (5.75). It is clear that $(\Delta a^\varepsilon)^{h,\widetilde\varepsilon}$ may be bounded exactly as $(\Delta q^\varepsilon)^{\ell,\widetilde\varepsilon}$. Next, product laws easily give that

$$\begin{aligned}\|\mathcal{P}u^\varepsilon\cdot\nabla\delta\Theta^\varepsilon\|_{L^1(\dot B^{(d+1)/p-3/2}_{p,1}+\dot B^{d/p-2}_{p,1})} &\lesssim \|\mathcal{P}u^\varepsilon\|_{L^2(\dot B^{d/p}_{p,1})}\|\delta\Theta^\varepsilon\|_{L^2(\dot B^{(d+1)/p-1/2}_{p,1}+\dot B^{d/p-1}_{p,1})},\\
\|\mathrm{div}\,(\mathcal{Q}u^\varepsilon\Theta^\varepsilon)\|_{L^1(\dot B^{(d+1)/p-3/2}_{p,1}+\dot B^{d/p-2}_{p,1})} &\lesssim \|\mathcal{Q}u^\varepsilon\|_{L^2(\dot B^{(d+1)/p-1/2}_{p,1})}\big(\|\Theta^\varepsilon\|^{\ell,\widetilde\varepsilon}_{L^2(\dot B^{d/2}_{2,1})}+\|a^\varepsilon\|^{h,\widetilde\varepsilon}_{L^2(\dot B^{d/p}_{p,1})}\big)\\
&\quad+\|\mathcal{Q}u^\varepsilon\|_{L^2(\dot B^{d/p}_{p,1})}\|\vartheta^\varepsilon\|^{h,\widetilde\varepsilon}_{L^2(\dot B^{d/p-1}_{p,1})},\end{aligned}$$

$$\begin{aligned}\|J(\varepsilon a^\varepsilon)\Delta(\vartheta^\varepsilon)^{\ell,\widetilde\varepsilon}\|_{L^1(\dot B^{(d+1)/p-3/2}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|_{L^\infty(\dot B^{(d+1)/p-1/2}_{p,1})}\|\Delta\vartheta^\varepsilon\|^{\ell,\widetilde\varepsilon}_{L^1(\dot B^{d/2-1}_{2,1})},\\
\|J(\varepsilon a^\varepsilon)\Delta(\vartheta^\varepsilon)^{h,\widetilde\varepsilon}\|_{L^1(\dot B^{d/p-2}_{p,1})} &\lesssim \|\varepsilon a^\varepsilon\|_{L^\infty(\dot B^{d/p}_{p,1})}\|\Delta\vartheta^\varepsilon\|^{h,\widetilde\varepsilon}_{L^1(\dot B^{d/p-2}_{p,1})},\\
\big\|\tfrac{\varepsilon}{1+\varepsilon a^\varepsilon}\big(2\mu|Du^\varepsilon|^2+\lambda(\mathrm{div}\,u^\varepsilon)^2\big)\big\|_{L^1(\dot B^{d/p-2}_{p,1})} &\lesssim \varepsilon(1+\|\varepsilon a\|_{L^\infty(\dot B^{d/p}_{p,1})})\|Du^\varepsilon\|^2_{L^2(\dot B^{d/p-1}_{p,1})}.\end{aligned}$$

Putting all the above inequalities together, remembering of (5.51) and (5.53), and setting

$$\delta X^\varepsilon := \|\delta\Theta^\varepsilon\|_{L^\infty(\dot B^{(d+1)/p-3/2}_{p,1}+\dot B^{d/p-2}_{p,1})}+\|\delta\Theta^\varepsilon\|_{L^2(\dot B^{(d+1)/p-1/2}_{p,1})+L^1(\dot B^{d/p}_{p,1})},$$

we eventually get

$$\delta X^\varepsilon \lesssim \|\Theta_0^\varepsilon - \Theta_0\|_{\dot B^{(d+1)/p-3/2}_{2,1}+\dot B^{d/p-2}_{p,1}} + \widetilde{\varepsilon} C_0^{\varepsilon,\nu} + \nu^{-1}(1+\nu^{-1}C_0^{\varepsilon,\nu})\widetilde{\varepsilon}^{1/2-1/p}(C_0^{\varepsilon,\nu})^2 \\ + \|\delta u^\varepsilon\|_{L^\infty(\dot B^{(d+1)/p-3/2}_{p,1}+\dot B^{d/p-2}_{p,1})}\|\nabla\Theta\|_{L^1(\dot B^{d/p}_{p,1})} + \nu^{-1}C_0^{\varepsilon,\nu}\delta X^\varepsilon,$$

which allows to conclude to (5.55). □

6. Appendix

In this short appendix, we recall the definition of paraproduct and remainder operators, and give some technical estimates that have been used throughout in the paper.

To start with, let us recall that, in the homogeneous setting, the paraproduct and remainder operators T and R are formally defined as follows:

$$T_u v := \sum_{j\in\mathbb{Z}} \dot S_{j-1}u\,\dot\Delta_j v \quad\text{and}\quad R(u,v) := \sum_{j\in\mathbb{Z}} \dot\Delta_j u\,\big(\dot\Delta_{j-1}+\dot\Delta_j+\dot\Delta_{j+1}\big)v$$

where $\dot S_k$ stands for the low-frequency cut-off operator defined by $\dot S_k := \chi(2^{-k}D)$.

The fundamental observation is that the general term of $T_u v$ is spectrally localized in the annulus $\big\{\xi\in\mathbb{R}^d\,,\ 1/12\le 2^{-j}|\xi|\le 10/3\big\}$, and that the general term of $R(u,v)$ is localized in the ball $B(0,2^j.20/3)$ (of course the values 1/12, 10/3 and 20/3 do not matter).

The main interest of the above definition lies in the following Bony's decomposition (first introduced in [3]):

$$uv = T_u v + R(u,v) + T_v u,$$

that has been used repeatedly in the present paper.

The following lemma has been used to get appropriate estimates of the solution both in the barotropic and in the polytropic cases:

Lemma 6.1. *Let $A(D)$ be a 0-order Fourier multiplier, and $j_0\in\mathbb{Z}$. Let $s<1$, $\sigma\in\mathbb{R}$ and $1\le p,p_1,p_2\le\infty$ with $1/p=1/p_1+1/p_2$. Then there exists a constant C depending only on j_0 and on the regularity parameters such that*

$$\|[\dot S_{j_0}A(D),T_a]b\|_{\dot B^{\sigma+s}_{p,1}} \le C\|\nabla a\|_{\dot B^{s-1}_{p_1,1}}\|b\|_{\dot B^{\sigma}_{p_2,\infty}}.$$

In the limit case $s=1$, we have

$$\|[\dot S_{j_0}A(D),T_a]b\|_{\dot B^{\sigma+1}_{p,1}} \le C\|\nabla a\|_{L^{p_1}}\|b\|_{\dot B^{\sigma}_{p_2,1}}.$$

Proof. We just treat the case $s<1$. By the definition of paraproduct, we have

$$[\dot S_{j_0}A(D),T_a]b = \sum_{j\in\mathbb{Z}}[\dot S_{j_0}A(D),\dot S_{j-1}a]\dot\Delta_j b.$$

Using that $A(D)$ is homogeneous of degree 0 and the properties of localization of operators $\dot S_k$ and $\dot\Delta_k$, we get for some smooth function $\widetilde\phi$ and for $j\le j_0-4$,

$$[\dot S_{j_0}A(D),\dot S_{j-1}a]\dot\Delta_j b = \sum_{k\le j-2}[\widetilde\phi(2^{-j}D),\dot\Delta_k a]\dot\Delta_j b.$$

Applying Lemma 2.97 of [2] yields

$$\|[\widetilde\phi(2^{-j}D),\dot\Delta_k a]\dot\Delta_j b\|_{L^p} \lesssim 2^{-j}\|\dot\Delta_k a\|_{L^{p_1}}\|\dot\Delta_j b\|_{L^{p_2}}. \tag{6.76}$$

In the case where j is close to j_0 (say $|j-j_0|\leq 4$), one may still find some smooth function ψ supported in an annulus, and such that

$$[\dot S_{j_0}A(D),\dot S_{j-1}a]\dot\Delta_j b=[\psi(2^{-j_0}D),\dot S_{j-1}a]\dot\Delta_j b,$$

which allows to get again (6.76). Summing up over j and k, and using convolution inequalities for series, it is easy to conclude to the desired inequality. □

Finally, we recall the following composition result.

Proposition 6.1. *Let G be a smooth function defined on some open interval I of $\mathbb{R}$ containing 0. Assume that $G(0)=0$. Then for all $s>0$, bounded interval $J\subset I$, $1\leq m\leq\infty$, and function a valued in J, the following estimates hold true:*

$$\|G(a)\|_{\dot B^s_{p,1}}\leq C\|a\|_{\dot B^s_{p,1}}\quad\text{and}\quad\|G(a)\|_{\widetilde L^m(\dot B^s_{p,1})}\leq C\|a\|_{\widetilde L^m(\dot B^s_{p,1})}.$$

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