The Goh necessary optimality conditions for the Mayer problem with control constraints
Hélène Frankowska, Daniela Tonon

To cite this version:

Hélène Frankowska, Daniela Tonon. The Goh necessary optimality conditions for the Mayer problem with control constraints. Decision and Control, 2013, pp.538-543. hal-01067270

HAL Id: hal-01067270
https://hal.science/hal-01067270
Submitted on 23 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
The Goh necessary optimality conditions for the Mayer problem with control constraints

Hélène Frankowska and Daniela Tonon†

Abstract

The well known Goh second order necessary optimality conditions in optimal control theory concern singular optimal controls taking values in the interior of a set of controls U. In this paper we investigate these conditions for the Mayer problem when U is a convex polytope or a closed subset of class C² for an integrable optimal control \( \bar{u}(\cdot) \) that may take values in the boundary of U. This is indeed a frequent situation in optimal control and for this reason the understanding of this issue is crucial for the theory of second order optimality conditions. Applying the Goh transformation we derive necessary conditions on tangent subspace to U at \( \bar{u}(t) \) for almost all t’s.

In the presence of an endpoint constraint, if the Mayer problem is calm, then similar second order necessary optimality conditions are satisfied whenever the maximum principle is abnormal. If it is normal, then analogous results hold true on some smaller subspaces.

1. Introduction

Let \( U \subset \mathbb{R}^m \), \( K \subset \mathbb{R}^n \) be closed, \( x_0 \in \mathbb{R}^n \), \( f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \) and \( \varphi : \mathbb{R}^n \to \mathbb{R} \) be sufficiently regular. Consider the Mayer optimal control problem

\[
\min \{ \varphi(x(1)) : x(\cdot) \in S_{[0,1]}(x_0), \ x(1) \in K \},
\]

where \( S_{[0,1]}(x_0) \) is the set of all absolutely continuous solutions of the control system

\[
\dot{x}(t) = f(t,x(t),u(t)), \ u(t) \in U, \ x(0) = x_0.
\]

*This work was co-funded by the European Union under the 7th Framework Programme "FP7-PEOPLE-2010-ITN", grant agreement number 264735-SADCO.
†Hélène Frankowska is with Institut de Mathématiques de Jussieu, Université Pierre et Marie Curie, 4 place Jussieu, 75252 Paris, France frankowska@math.jussieu.fr
‡Daniela Tonon is with Department of Electrical and Electronic Engineering, Imperial College London, Exhibition Road, London, SW7 2BT, UK. dtonon@imperial.ac.uk

Define the Hamiltonian \( H : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} :\)

\[
H(t,x,p,u) = \langle p, f(t,x,u) \rangle.
\]

We first consider the Mayer problem without endpoint constraints:

\[
\min \{ \varphi(x(1)) : x(\cdot) \in S_{[0,1]}(x_0) \}.
\]

Let \( (\bar{x}(\cdot),\bar{u}(\cdot)) \) be a strong local minimizer of (3). By the maximum principle, for an absolutely continuous \( p : [0, 1] \to \mathbb{R}^n \) satisfying

\[
- p'(t) = H_{\bar{u}}(t,\bar{x}(t),p(t),\bar{u}(t)) \quad \text{for a.e.} \ t \in [0,1]
\]

and

\[
- p(1) = \nabla \varphi(\bar{x}(1))
\]

the following maximum principle holds true for a.e. \( t :\)

\[
\langle p(t), f(t,\bar{x}(t),\bar{u}(t)) \rangle = \max_{u \in U} \langle p(t), f(t,\bar{x}(t),u) \rangle.
\]

Thus for a.e. \( t \in [0,1] \) such that the optimal control \( \bar{u}(t) \) lies in the interior of U, \( H_{\bar{u}}[t] = 0 \) and the Legendre-Clebsch condition \( H_{uu}[t] \leq 0 \) is satisfied, where \( [t] := (t,\bar{x}(t),p(t),\bar{u}(t)) \). If \( \bar{u}(\cdot) \) is singular, in the sense that \( H_{uu}[\cdot] = 0 \), the Legendre-Clebsch condition doesn’t bring any additional information about optimal controls. Different, but related, definitions of singular controls and discussions on their importance can be found for instance in [1, 2, 3], and the references therein.

Assume that \( H_{uu}[t] = 0 \) a.e. in \( [t_1,t_2] \). According to [4], if \( U \) is an open convex polytope, then the matrix \( H_{uu}[t]f_u[t] \) is symmetric and

\[
R(t) := \frac{1}{2} f_u[t]^* H_{uu}[t] f_u[t] \frac{d}{dt} H_{uu}[t] f_u[t]
\]

is negative definite for a.e. \( t \in [t_1,t_2] \). The first property, known in the literature as the Goh condition, in the case of affine control systems reduces to a condition on the Lie brackets of the flux functions, as we recall...
in Remark 3.2. The second property can be seen as a
generalized Legendre-Clebsch condition.

To deal with closed control constraints Goh also
suggested to use variations of the optimal control which
are equal to zero when  \( \bar{u}(t) \in \partial U \). However, this can’t
be applied to recover second-order optimality conditions
when  \( \bar{u}(t) \in \partial U \) almost everywhere.

We present here a different approach, based on
non-zero variations of the optimal control even when
\( \bar{u}(\cdot) \) is a boundary control.

For every  \( u \in U \) let us consider the maximal sub-
space  \( P_u \) contained in the tangent cone to  \( U \) at  \( u \).

Below, Theorems 4.1 and 5.2 provide pointwise
second order necessary optimality conditions respectively in the case when  \( U \) is a convex polytope or a
subset of class  \( C^2 \) for a singular optimal control  \( \bar{u}(\cdot) \) being merely integrable. Considering  \( \mathcal{H}_u[t]f_u[t] \) and
\( R(t) \) as bilinear operators on the subspace  \( P_{\bar{u}(t)} \) in both cases, we still have symmetry of  \( \mathcal{H}_u[t]f_u[t] \). In the former
case we also get the negative definiteness of  \( R(t) \) on
\( P_{\bar{u}(t)} \).

In Corollary 4.2 we prove that if  \( U \) is a closed con-
convex polytope in  \( \mathbb{R}^m \) and  \( f \) is affine with respect to  \( u \):

\[
f(x,u) = f_0(x) + g(x)u,
\]

with  \( g(.) \in C^2(\mathbb{R}^n;M(n \times m)) \) and if  \( g_k(x) \) denotes the
\( k \)-th column of the matrix  \( g(x) \), then the matrix valued
mapping  \( \Theta : [0,1] \to M(m \times m) \), given by

\[
\theta_{k,l}(t) := \langle p(t), [g_k, g_l](\tilde{x}(t)) \rangle \quad \forall k, l \in \{1, \ldots, m\},
\]

enjoys the following property: for a.e.  \( t \in [0,1] \), the
subspace  \( \Theta(t)(P_{\bar{u}(t)}) \) (the image of  \( P_{\bar{u}(t)} \) by the linear
operator  \( \Theta(t) \)) is orthogonal to the subspace  \( P_{\bar{u}(t)} \). In
particular we recover the well known Goh condition in the
geometric optimal control:  \( \Theta(t) = 0 \) for a.e.  \( t \) such that
\( \bar{u}(t) \in \text{Int} \, U \).

For the Mayer problem with the endpoint constraint
\( K \), we assume calmness at  \( \tilde{x}(\cdot) \). This allows to reduce
the problem to the one without endpoint constraints but
involving a nondifferentiable cost function and so the
classical first and second order necessary conditions are
not satisfied. Still we are able to derive second order
conditions through a variational approach similar to the
one without end point constraints.

Since the very first work by Goh, [4], the Goh condi-
tion was investigated by many researchers in the
geometric optimal control theory following fundamental
works by Krener, cf. [5] and [6], on the Goh condi-
tion (and its higher order extensions) when the optimal
control is piecewise  \( C^0 \). His approach was taken
over by several authors and the Goh condition became
mostly known in the form involving Lie brackets, as a
second-order necessary optimality condition for the ab-
normal maximum principle. See for instance [7, 8] and
[3, pp.314-319].

Several authors made an extensive use of the Goh
transformation to recover necessary and sufficient sec-
second order optimality conditions in an integral and a
pointwise form, see for instance [9], [10] and the bib-
liographies contained therein. However, in these lat-
ter works many additional structural restrictions on the
control system, control sets and also on the optimal con-
trol are imposed, as for instance in Assumptions 1, 2
from [9] concerning the structure optimal controls .

The outline of this paper is as follows. In Section 2
we introduce some notations and provide some prelimi-
ary results. Section 3 is devoted to the second order
necessary optimality conditions for an arbitrary set  \( U \)
and any integrable optimal control at times when it be-
longs to the interior of  \( U \). Section 4 deals with the case
when  \( U \) is a convex polytope and Section 5 with the
case when  \( U \) is of class  \( C^2 \). The last section adresses
the second order necessary optimality conditions for the
Mayer problem involving endpoint constraints.

Most of results presented here are given without
proofs, because they are quite long and technical and
will be published elsewhere.

2. Preliminaries and notations

The norm in  \( \mathbb{R}^n \) is denoted by  \( \| \cdot \| \) and the inner
product by  \( \langle \cdot, \cdot \rangle \). Let  \( \mathcal{B} := \{ x \in \mathbb{R}^n : \|x\| < 1 \} \) be the
open unit ball in  \( \mathbb{R}^n \). For a nonempty subset  \( K \) of  \( \mathbb{R}^n \),
\( \partial K \) denotes its boundary, \( \text{Int} \, K \) its interior and  \( K^c \) its
complement. The distance function  \( d_K : \mathbb{R}^n \to \mathbb{R}_+ \) is
defined by  \( d_K(x) := \inf_{y \in K} \|x-y\| \) for all  \( x \in \mathbb{R}^n \) and
the oriented distance  \( b_K : \mathbb{R}^n \to \mathbb{R} \) from  \( K \neq \mathbb{R}^n \) by
\( b_K(x) := d_K(x) - d_K^r(x) \) for all  \( x \in \mathbb{R}^n \). We set  \( b_K(\cdot) = 0 \)
if  \( K = \mathbb{R}^n \). Finally, by  \( M(k \times r) \) we denote the space of
matrices having  \( k \) lines and  \( r \) columns.

Definition 2.1 Let  \( C : \mathbb{R}^m \to \mathbb{R}^m \) be a linear mapping
and  \( F \) be a subspace of  \( \mathbb{R}^m \). We say that  \( C \) is symmetric
on  \( F \) if  \( \langle Cu, v \rangle = \langle u, Cv \rangle \) for all  \( u, v \in F \) and that  \( C \) is
negative definite on  \( F \) if  \( \langle Cu, u \rangle \leq 0 \) for all  \( u \in F \).

Let  \( U \) be a nonempty closed subset of  \( \mathbb{R}^m \). For  \( u_0 \in U \)
the adjacent tangent cone to  \( U \) at  \( u_0 \) is defined by

\[
T_U^1(u_0) := \left\{ u \in \mathbb{R}^n : \lim_{h \to 0^+} \frac{d_U(u_0 + hu)}{h} = 0 \right\}.
\]

The second-order adjacent subset to  \( U \) at  \( (u_0, u) \in \mathbb{R}^m \times \mathbb{R}^m \) is the set defined by

\[
T_U^{(2)}(u_0, u) := \left\{ v \in \mathbb{R}^n : \lim_{h \to 0^+} \frac{d_U(u_0 + hu + h^2v)}{h^2} = 0 \right\}.
\]
For any \( u \in U \) we denote by
\[
P_u \quad \text{the largest subspace contained in } T_U^0(u) \tag{u}.
\]

Let \( f : [0, 1] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n \), \( f'(t,x,u) \) be the derivative of \( f(t,\cdot,\cdot) \) at \((x,u)\), \( f_+(t,x,u) \) and \( f_+(t,x,u) \) be the partial derivatives of \( f \) with respect to \( x \) and \( u \). \( f''(t,x,u) \) denotes the Hessian of \( f(t,\cdot,\cdot) \) at \((x,u)\), while \( f_{xx}, f_{ux} \) and \( f_{uu} \) denote the second-order partial derivatives of \( f \) with respect to \( x \) and \( u \).

Fix a trajectory-control pair \((\bar{x}(\cdot), \bar{u}(\cdot))\) of (2) and set \( f(t) = f(t, \bar{x}(t), \bar{u}(t)) \). Let \( f_u[t], f_x[t] \) and \( f''[t] \) be defined in a similar way. Throughout the whole paper we impose the following assumptions:

(a) \( f(t,\cdot,\cdot) \in C^2 \) on \((\bar{x}(t)+\delta B) \times (\bar{u}(t)+\delta B) \), \( \forall t \); 
(b) \( ||f_u[t]||, ||f'_x[t]||, ||f''[t]|| \in L^1([0,1];\mathbb{R}^+) \) and \( \exists a(\cdot) \in L^1([0,1];\mathbb{R}^+) \) such that the mappings \( f'_x(t,\cdot,\cdot) \) and \( f''_x(t,\cdot,\cdot) \) are \( a(t) \)-Lipschitz on \((\bar{x}(t)+\delta B) \times (\bar{u}(t)+\delta B) \) for \( a.e. \, t \in [0,1] \); 
(c) \( f(t,x,U) \) is closed for all \( t \in [0,1], x \in \bar{x}(t)+\delta B \); 
(d) \( \varphi(\cdot) \in C^2 \) on \( \bar{x}(t)+\delta B \).

Recall that \((\bar{x}(\cdot), \bar{u}(\cdot))\) is a strong local minimizer for (3) (resp. (1)) if there exists \( \eta > 0 \) such that \( \bar{x}(\cdot) \) minimizes \( \varphi \) over all \( x(\cdot) \in S_{[0,1]}(x_0) \) (resp. over all \( x(\cdot) \in \mathbb{S}_{[0,1]}(x_0) \) with \( x(1) = K \)) satisfying \( ||\bar{x}(\cdot) - x(\cdot)||_\infty \leq \eta \).

Remark 2.2 Results contained in Sections 3-5 below are stated only for strong local minimizers. However they are also valid for the so called weak local minimizers. In contrast, Section 6 concerns only strong local minimizers. Not to switch between these different notions, in this paper we investigate only strong local minimizers.

Theorem 2.3 (Maximum Principle, [11]) Let \((\bar{x}(\cdot), \bar{u}(\cdot))\) be a strong local minimizer of the Mayer problem (3). Then there exists an absolutely continuous \( p : [0,1] \to \mathbb{R}^n \) satisfying (4), (5) and (6). Furthermore, for \( a.e. \, t \in [0,1] \) and for any \( u \in P_{\bar{u}(t)} \) we have \( H_u[t]u = 0 \).

Define \( A := \{ t \in [0,1] : \bar{u}(t) \in \partial U, \mathcal{H}_u[t] = 0 \} \) and \( \mathcal{U}_A = \{ u(\cdot) \in L^\infty([0,1];\mathbb{R}^m) : \exists c > 0, h_0 > 0 \text{ s.t. } \forall h \in [0,h_0], \forall t \in A, d_U(\bar{u}(t)+hw(t)) \leq ch^2 \} \).

Note that for any \( u(\cdot) \in \mathcal{U}_A, u(t) \in T_U^0(\bar{u}(t)) \) a.e. in \( A \).

Theorem 2.4 (Integral second order conditions) Consider a strong local minimizer \((\bar{x}(\cdot), \bar{u}(\cdot))\) of the Mayer problem (3). Then for all \( u(\cdot), v(\cdot) \in L^\infty([0,1];\mathbb{R}^m) \) such that \( u(\cdot) \in \mathcal{U}_A \) and \( v(\cdot) \in T_U^0(\bar{u}(t),u(\cdot)) \) for \( a.e. \, t \in A \), and for \( p(\cdot) \) as in the maximum principle either \( \int_0^1 H_u[t]u(t)dt < 0 \) or \( \mathcal{H}_u[t]u(t) = 0 \) a.e. in \([0,1] \)
\[
0 \geq \int_0^1 (\mathcal{H}_u[t]v(t) + \frac{1}{2}y(t)^*H_u[t]y(t) + u(t)^*\mathcal{H}_u[t]u(t) + \frac{1}{2}y(t)^*H_u[t]y(t))dt \tag{8}
\]
\[
-\frac{1}{2}y(1)^*\varphi^0(\bar{x}(1))y(1),
\]
where \( y(\cdot) \) solves the linear system
\[
\begin{aligned}
y'(t) &= f_u[t]y(t) + f_x[t]u(t) \\
y(0) &= 0.
\end{aligned}
\tag{9}
\]
The proof follows the same lines as the one in [12].

3. Second order conditions in the interior of control constraints

Consider a strong local minimizer \((\bar{x}(\cdot), \bar{u}(\cdot))\) of the Mayer problem (3) and \( p(\cdot) \) as in the maximum principle of Theorem 2.3. Let the matrix valued mapping \( R(t) \) be defined as in the introduction.

Theorem 3.1 Assume that for some \( 0 \leq t_1 < t_2 \leq 1 \), \( \bar{u}(\cdot) \) is integrable on \([t_1,t_2] \) and \( \mathcal{H}_u[t] = 0 \) a.e. in \([t_1,t_2] \); \( \mathcal{H}_u[t], f_{x}^{\cdot}[\cdot] \) are Lipschitz on \([t_1,t_2] \); \( \mathcal{H}_x[t], f_{u}^{\cdot}[\cdot] \) are essentially bounded on \([t_1,t_2] \).

Then, for \( a.e. \, t \in [t_1,t_2] \) such that \( \bar{u}(t) \in Int U \), the matrix \( \mathcal{H}_u[t]f_{u}^{\cdot}[\cdot] \) is symmetric and \( R(t) \leq 0 \).

The statement of the above theorem is proved for the Lebesgue points \( t \) of \((u(\cdot),R(\cdot))\) such that \( \bar{u}(t) \in Int U \). The proof is long and technical and will be published elsewhere. It involves the Goh transformation and some advanced properties of Lebesgue points.

Remark 3.2 If \( f \) is affine with respect to controls, the symmetry of the matrix \( \mathcal{H}_u[t]f_{u}^{\cdot}[\cdot] \) implies that the adjoint state is orthogonal to the Lie brackets of the flux. Indeed let \( f \) be as in (7), where \( g(\cdot) \in C^2(\mathbb{R}^n;\mathbb{M}(n \times m)) \) and \( U \) is a closed nonempty subset of \( \mathbb{R}^m \) and let \( g_k(x) \) denote the \( k \)-th column of \( g(x) \). Then for all \( t \) such that \( \mathcal{H}_u[t]f_{u}^{\cdot}[\cdot] \) is symmetric and for any \( k,l \in \{1,\ldots,m\} \),
\[
0 = (\mathcal{H}_u[t]f_{u}^{\cdot}[\cdot] - f_{u}^{\cdot}[\cdot]^{\ast}[\mathcal{H}_u[t]]^{\ast})_{k,l}
\]
\[
= \sum_{j=1}^{n} p_j(t) \sum_{j=1}^{n} \left( \frac{\partial g_{k,j}}{\partial x_l}[t]g_{l,j}[t] - \frac{\partial g_{l,j}}{\partial x_k}[t]g_{k,j}[t] \right),
\]
where \( g_{j,k} \) is the \( j \)-th element of \( g_k \). Hence for any \( k,l \in \{1,\ldots,m\} \),
\[
\begin{aligned}
\left( p(t), \frac{\partial g_{k}}{\partial x_l}[t] - \frac{\partial g_{l}}{\partial x_k}[t]g_{k}[t] \right) &= \left( p(t), [g_k,g_l][t] \right) = 0.
\end{aligned}
\]
4. Second order conditions on convex polytopes

A nonempty subset $U \subset \mathbb{R}^m$ is called a convex polytope in $\mathbb{R}^m$ if $U$ is an intersection of a finite family of affine half-spaces of $\mathbb{R}^m$. Then the adjacent tangent cone $T^+_U(u_0)$ is equal to the tangent cone of convex analysis to $U$ for any $u_0 \in U$.

**Theorem 4.1** Let $U \subset \mathbb{R}^m$ be a convex polytope in $\mathbb{R}^m$. Consider a strong local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the Mayer problem (3) and $p(\cdot)$ as in the maximum principle of Theorem 2.3. Assume that $\bar{u}(\cdot)$ is integrable on $[t_1, t_2]$ for some $0 \leq t_1 < t_2 \leq 1$ and (10) holds true. Then $\mathcal{H}_u[t]f_u[t]$ is symmetric on the subspace $P_{\bar{u}[t]}$ and $R(t)$ is negative definite on $P_{\bar{u}[t]}$ for a.e. $t \in [t_1, t_2]$.

The proof follows the same lines as the proof of Theorem 3.1, but a change of base is involved and the control variations take values in the tangent to $U$ at $\bar{u}(t)$.

The following corollary generalizes the classical Goh condition to boundary controls.

**Corollary 4.2** Under all the assumptions of Theorem 4.1 suppose that $f$ is as in (7), where $g(\cdot) \in C^2(\mathbb{R}^n; M(n \times m))$ and $U$ is a closed convex polytope in $\mathbb{R}^m$ and let $g_k(x)$ denote the $k$-th column of $g(x)$. Consider the matrix valued mapping $\Theta : [0, 1) \to M(m \times m)$, whose elements are

$$\theta_{k,l}(t) := \langle p(t), [g_k, g_l](\bar{x}(t)) \rangle \quad \forall k, l \in \{1, \ldots, m\}.$$  

Then for a.e. $t \in [t_1, t_2]$, the subspace $\Theta(t) \cdot (P_{\bar{u}[t]})$ is orthogonal to $P_{\bar{u}[t]}$. In particular, $\Theta(t)$ is equal to zero for a.e. $t \in [t_1, t_2]$ such that $\bar{u}(t) \in \text{Int} U$.

**Proof.** By Theorem 4.1 for a.e. $t \in [t_1, t_2]$, $\mathcal{H}_u[t]f_u[t]$ is symmetric on $P_{\bar{u}[t]}$. Fix such $t$. Since for all $a, b \in P_{\bar{u}[t]}$, $a^*(\mathcal{H}_u[t]f_u[t] - (\mathcal{H}_u[t]f_u[t])^*)b = 0$, we get

$$0 = \sum_{k=1}^m \sum_{l=1}^m a_k(\mathcal{H}_u[t]f_u[t] - (\mathcal{H}_u[t]f_u[t])^*)_{k,l}b_l.$$  

Consequently $\langle p(t), \sum_{k=1}^m \sum_{l=1}^m a_k[g_k, g_l](\bar{x}(t))b_l \rangle = 0$ and therefore $0 = \sum_{k=1}^m \sum_{l=1}^m a_k(p(t), [g_k, g_l](\bar{x}(t)))b_l = \langle a, \Theta(t)b \rangle$, as claimed.

5. Second order conditions for smooth control constraints

In this section we consider a proper closed set $U \subset \mathbb{R}^m$ of class $C^2$, see for instance [13]. For a strong local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the Mayer problem (3) define the following matrix valued mapping $C(\cdot)$ on $[0, 1]$

$$C(t) = \left\{ \begin{array}{ll} b^*_u(\bar{u}(t)) & \text{if } \bar{u}(t) \in \partial U \\ 0 & \text{otherwise} \end{array} \right.$$  

**Remark 5.1** If $U$ is of class $C^2$, then

a) $\forall u_0 \in \partial U$ there exists a neighborhood $W(u_0)$ of $u_0$ such that the oriented distance $b_U(\cdot) \in C^2(W(u_0))$.

b) $\forall u_0 \in \partial U$, $T^+_U(u_0) = \{ v \in \mathbb{R}^m : \langle \nabla b_U(u_0), v \rangle \leq 0 \}$ and the subspace

$$P_{u_0} := \{ v \in \mathbb{R}^m : \langle \nabla b_U(u_0), v \rangle = 0 \} = \partial T^+_U(u_0)$$

has the dimension equal to $m-1$.

c) $\forall u_0 \in \partial U$, $\nabla b_U(u_0)$ is the unit outward normal to $U$ at $u_0$ and, applying the inverse mapping theorem to $b_U$, it is not difficult to show that for any $u \in \partial T^+_U(u_0)$ we have $-\frac{1}{2} \langle b^*_U(u_0), u \rangle \nabla b_U(u_0) \in T^{\perp}_U(u_0, u)$.

For a trajectory-control pair $(\bar{x}(\cdot), \bar{u}(\cdot))$ and for every $t \in [0, 1]$ such that $\nabla b_U(\bar{u}(t))$ does exist, define the $m-1$ dimensional subspace

$$S(t) := \{ u \in \mathbb{R}^m : \langle \nabla b_U(\bar{u}(t)), u \rangle = 0 \}.$$  

**Theorem 5.2** Let $U$ be of class $C^2$. Consider a strong local minimizer $(\bar{x}(\cdot), \bar{u}(\cdot))$ of the Mayer problem (3) and $p(\cdot)$ as in the maximum principle of Theorem 2.3. Assume that for some $0 \leq t_1 < t_2 \leq 1$, $\bar{u}(\cdot)$ is integrable on $[t_1, t_2]$.

Then for a.e. $t \in [t_1, t_2]$ such that $\bar{u}(t) \in \partial U$ and for some $\delta > 0$

$$u^* \left( \mathcal{H}_u[s] - \| \mathcal{H}_u[t] \| b^*_U(\bar{u}(s)) \right) u = 0$$

for all $u \in S(s)$ and for a.e. $s \in [t - \delta, t + \delta]$, satisfying $\| \bar{u}(t) - \bar{u}(s) \| \leq \delta$;

$\mathcal{H}_u[\cdot], f_u[\cdot]$ are Lipschitz on $[t - \delta, t + \delta]$;

$\mathcal{H}_u[\cdot], f_u[\cdot]$ are essentially bounded on $[t - \delta, t + \delta]$;

we have $\mathcal{H}_u[t]f_u[t]$ is symmetric on the subspace $P_{\bar{u}[t]}$.

6. Mayer’s problem with the endpoint constraints

Let $K$ be a proper closed nonempty subset of $\mathbb{R}^n$ of class $C^2$. Then $b_K(\cdot) \in C^2$ on a neighborhood of $\partial K$, cf. [13]. We study here the Mayer optimal control problem (1) involving the end point constraints.

Let $(\bar{x}(\cdot), \bar{u}(\cdot))$ be a strong local minimizer of the above problem and $P_{\bar{u}[t]}$, be defined as in Section 2. If $\bar{x}(1)$ belongs to $\text{Int} K$, then the previously developed second order variational analysis does apply as though there were no endpoint constraints. This is due to the fact that small perturbations of $\bar{u}(\cdot)$ lead to feasible trajectories. For this reason we have to investigate only the case $\bar{x}(1) \in \partial K$. We deduce, for instance from [14], the following maximum principle.
Theorem 6.1 Consider a strong local minimizer \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) of (I) with \(\tilde{x}(1) \in \partial K\). Then there exists an absolutely continuous function \(p : [0,1] \rightarrow \mathbb{R}^n\) and \(\lambda \in \{0,1\}, \mu \geq 0\), satisfying \(\lambda + \mu > 0\), (4), (6) and the transversality condition

\[-p(1) = \lambda \nabla \varphi(\tilde{x}(1)) + \mu \nabla b_K(\tilde{x}(1)).\]

The above maximum principle may be abnormal, that is \(\lambda = 0\). In this case \(p(1)\) may be taken equal to \(-\nabla b_K(\tilde{x}(1))\).

Lemma 6.2 Consider a strong local minimizer \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) of (I) with \(\tilde{x}(1) \in \partial K\) for which an abnormal maximum principle of Theorem 6.1 holds true for some \(\mu > 0\). Then for every measurable bounded selection \(u(t) \in P_{\tilde{u}(t)}\) a.e. in \([0,1]\), the solution \(y(\cdot)\) of (9) satisfies \(|\nabla b_K(\tilde{x}(1)), y(1)| = 0\) and \(\langle p(t), f_{\tilde{u}}[t]u(t)\rangle = 0\) a.e. in \([0,1]\).

If the maximum principle is abnormal, then the cost is not involved in the first order optimality conditions. To apply the variational approach developed in the previous sections, we use the exact penalization which allows to deal with a normal maximum principle.

Definition 6.3 The Mayer problem (1) is called calm at \(\tilde{x}(\cdot)\) if there exist \(\varepsilon > 0\) and \(M > 0\) such that \(\forall v \in (0, \varepsilon)\) and \(x(\cdot) \in S_{[0,1]}(x_0)\) satisfying \(d_k(x(1)) \leq v \) and \(||x(\cdot) - \tilde{x}(\cdot)||_\infty \leq \varepsilon\) we have \(\varphi(x(\cdot)) \geq \varphi(\tilde{x}(\cdot)) - Mv\).

If the Mayer problem (1) is calm at \(\tilde{x}(\cdot)\), then by Proposition 6.4.3 from [15], there exists \(k > 0\) such that \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) is a strong local minimizer of the following minization problem:

\[
\min \left\{ \varphi(x(\cdot)) + kd_k(x(1)) : x(\cdot) \in S_{[0,1]}(x_0) \right\}.
\] (11)

Since \(\varphi(\cdot) + kd_k(\cdot)\) is no longer \(C^2\), we need the following (non smooth) maximum principle that can be deduced, for instance, from [14].

Theorem 6.4 Consider a strong local minimizer \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) of (11) with \(\tilde{x}(1) \in \partial K\). Then there exists an absolutely continuous function \(p : [0,1] \rightarrow \mathbb{R}^n\) satisfying (4), (6) and the transversality condition

\[-p(1) \in \nabla \varphi(\tilde{x}(1)) + k[0, \nabla b_K(\tilde{x}(1))],\] (12)

where \([0, \nabla b_K(\tilde{x}(1))] = \{\mu \nabla b_K(\tilde{x}(1)) : \mu \in [0,1]\}\).

Denote by \(q^1(\cdot)\) the solution of (4) for \(p(1) = -\nabla \varphi(\tilde{x}(1))\) and by \(q^2(\cdot)\) the solution of (4) for \(p(1) = -\nabla \varphi(\tilde{x}(1)) - k\nabla b_K(\tilde{x}(1))\). Note that \(q^1(1)\) are extremal points in the transversality condition (12). The mappings \(q^1(\cdot)\) do not satisfy, in general, the maximum principle (6) but are used below to express the second order necessary optimality conditions. Define for \(i = 1, 2\)

\[
\varphi^1(\cdot) := \varphi(\cdot), \quad \varphi^2(\cdot) := \varphi(\cdot) + k\nabla b_K(\tilde{x}(1)),
\]

\[
\mathcal{P}_{\xi}^0[\cdot] := \langle \mathcal{P}_{\xi}(\cdot), \tilde{x}(\cdot), q^i(\cdot), \tilde{u}(\cdot) \rangle.
\]

The mappings \(\mathcal{P}_{\xi}^0[\cdot]\) and \(\mathcal{P}_{\xi}^1[\cdot]\) are defined similarly.

Theorem 6.5 Consider a strong local minimizer \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) of (II) with \(\tilde{x}(1) \in \partial K\) and \(p(1)\) as in Theorem 6.4. Then for every \(u(\cdot) \in \mathcal{U}_A\) such that \(\mathcal{P}_{\xi}^0[\cdot]u(\cdot) = 0\) and the corresponding solution \(y(\cdot)\) of (9) verifies \(|\nabla b_K(\tilde{x}(1)), y(1)| = 0\) and for any \(v(\cdot) \in L^\infty([0,1]; \mathbb{R}^m)\) satisfying \(v(t) \in \mathcal{P}_{\xi}^0(\tilde{u}(t), u(t))\) for a.e. \(t \in A\), there exists \(i = 1, 2\) such that

\[
0 \geq \int_0^1 \left( \mathcal{P}_{\xi}^1[v(t)]v(t) + \frac{1}{2} v(t)^* \mathcal{P}_{\xi}^0[v(t)]v(t) \right) + u(t)^* \mathcal{P}_{\xi}^1[t]v(t) + \frac{1}{2} u(t)^* \mathcal{P}_{\xi}^0[t]u(t) dt \quad (13)
\]

Consider the solution \(q(\cdot)\) of the linear system (4) such that \(q(1) = -\nabla b_K(\tilde{x}(1))\) and define the subspaces \(\mathcal{F}_\xi \subset P_{\tilde{u}(t)}\)

\[
\mathcal{F}_\xi := \{ u \in P_{\tilde{u}(t)} : \langle q(t), f_{\tilde{u}}[t]u(t) \rangle = 0 \}.
\]

By Lemma 6.2, if the abnormal maximum principle of Theorem 6.1 holds true, then \(\mathcal{F}_\xi = P_{\tilde{u}(t)}\) a.e. in \([0,1]\) and \(q(\cdot)\) satisfies the maximum principle. Observe next that for every measurable essentially bounded selection \(u(t) \in \mathcal{F}_\xi\) and the corresponding solution \(y(\cdot)\) of (9) we have \(|\nabla b_K(\tilde{x}(1)), y(1)| = 0\). Therefore, by Theorem 6.5, (13) holds true for every \(u(\cdot) \in \mathcal{U}_A\) such that \(u(t) \in \mathcal{F}_\xi\) a.e. in \([0,1]\). This and similar, but lengthy proofs as those of results from Sections 3,4,5 allow to deduce from (13) the following three theorems.

Theorem 6.6 Consider a strong local minimizer \((\tilde{x}(\cdot), \tilde{u}(\cdot))\) of (II) such that \(\tilde{x}(1) \in \partial K\). Assume that (1) is calm at \(\tilde{x}(\cdot)\), that for some \(0 \leq t_1 < t_2 \leq 1\), \(\tilde{u}(\cdot)\) is integrable on \([t_1, t_2]\) and for \(j = 1, 2\)

\[
\mathcal{P}_{\xi}^j[\cdot] = 0 \quad \text{a.e. in } [t_1, t_2];
\]

\[
\mathcal{P}_{\xi}^j[\cdot], f_{\tilde{u}}[\cdot] \quad \text{are Lipschitz on } [t_1, t_2];
\]

\[
\mathcal{P}_{\xi}^j[\cdot], f_{\tilde{u}}[\cdot] \quad \text{are essentially bounded on } [t_1, t_2].
\] (14)

Then for a.e. \(t \in [t_1, t_2]\) with \(\tilde{u}(t) \in \text{Int} U\) there exists \(i \in \{1, 2\}\) such that \(\mathcal{P}_{\xi}^i[t]f_{\tilde{u}}[t]\) is symmetric on \(\mathcal{F}_\xi\) and

\[
R(t) := \frac{1}{2} f_{\tilde{u}}[t]^* \mathcal{P}_{\xi}^j[t] f_{\tilde{u}}[t] - \frac{1}{2} \mathcal{P}_{\xi}^i[t] f_{\tilde{u}}[t]^* \mathcal{P}_{\xi}^i[t] f_{\tilde{u}}[t]
\]

is negative definite on \(\mathcal{F}_\xi\).
Theorem 6.7 Let $U$ be a convex polytope and $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ be a strong local minimizer of (1) such that $\tilde{x}(1) \in \partial K$. Assume that (1) is calm at $\tilde{x}(\cdot)$ and for some $0 \leq t_1 < t_2 \leq 1$, $\tilde{u}(\cdot)$ is integrable on $[t_1, t_2]$ and (14) holds true for $j=1, 2$. Then for a.e. $t \in [t_1, t_2]$ there exists $i \in \{1, 2\}$ such that $\mathcal{H}_{ui}[t] f_u[t]$ is symmetric on $\mathcal{T}_t$ and $\mathcal{R}(t)$ is negative definite on $\mathcal{T}_t$.

Corollary 6.8 Under all the assumptions of Theorem 6.7 suppose that $f$ is as in (7), where $g(\cdot) \in C^2(\mathbb{R}^n; M(n \times m))$ and $U$ is a closed convex polytope in $\mathbb{R}^m$ and let $g_k(x)$ denote the $k$-th column of $g(x)$. For $i=1, 2$ define the matrix valued mappings $\Theta^i : [0, 1] \to M(m \times m)$, where

$$\theta^i_{j,k}(t) := \langle q^i(t), [g_k, g_l](\tilde{x}(t)) \rangle \quad \forall k,l \in \{1, \ldots, m\}.$$ Then for a.e. $t \in [t_1, t_2]$ there exists $i \in \{1, 2\}$ such that $\Theta^i(t)(\mathcal{T}_t)$ is orthogonal to $\mathcal{T}_t$.

Consider subspaces $S(\cdot)$ defined as in Section 5.

Theorem 6.9 Let $U$ be of class $C^2$ and $(\tilde{x}(\cdot), \tilde{u}(\cdot))$ be a strong local minimizer of the Mayer problem (1) such that $\tilde{x}(1) \in \partial K$. Assume that (1) is calm at $\tilde{x}(\cdot)$ and for some $0 \leq t_1 < t_2 \leq 1$, $\tilde{u}(\cdot)$ is integrable on $[t_1, t_2]$.

Then for a.e. $t \in [t_1, t_2]$ such that $\tilde{u}(t) \in \partial U$ and for some $\delta > 0$ and $j=1, 2$ the following relations are satisfied:

$$u^* \left( \mathcal{H}_{ua}[s] - \mathcal{H}_{ua}[a] b^u_0(\tilde{u}(s)) \right) = 0 \quad \forall u \in \mathcal{T}_s$$
and a.e. $s \in (t-\delta, t+\delta)$ with $|\tilde{u}(t) - \tilde{u}(s)| \leq \delta$;

$$\mathcal{H}_{ua}[\cdot], f_u[\cdot]$$ are Lipschitz on $(t-\delta, t+\delta)$;

$$\mathcal{H}_{ua}[\cdot], f_u[\cdot]$$ are essentially bounded on $(t-\delta, t+\delta)$

we can find $i \in \{1, 2\}$ for which $\mathcal{H}_{ui}[t] f_u[t]$ is symmetric on $\mathcal{T}_t$.

Remark 6.10 Similar results can be obtained when $K$ is an intersection of a finite family of closed sets $K_j$ of class $C^2$. Then the assumptions and statements will involve Hamiltonians $\mathcal{H}^j$ for each active index $j$.

7. Conclusions and future works

For a nonlinear Bolza optimal control problem we proposed pointwise second order necessary optimality conditions extending those of Goh to the case of a closed control constraint given by a polytope or by a $C^2$ subset of $\mathbb{R}^m$ and for integrable optimal controls that may take values on the boundary of control constraints. Furthermore, we have shown that on the interior of the control constraints Goh conditions hold true for integrable optimal controls without imposing any structural properties on the control set.

This was done by investigating first the free endpoint problem and then, the calmness assumption allowed us to reduce the Mayer problem with endpoint constraints to the one without them, but involving a non-differentiable cost function. For this equivalent problem we obtained similar pointwise conditions using the developed variational approach for the free endpoint problems. Our future work will concern a larger class of end-point constraints and more general control sets.

References