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ON BOTT-CHERN COHOMOLOGY OF COMPACT COMPLEX SURFACES

DANIELE ANGELLA, GEORGES DLOUSSKY, AND ADRIANO TOMASSINI

ABSTRACT. We study Bott-Chern cohomology on compact complex non-Kähler surfaces. In particular, we compute such a cohomology for compact complex surfaces in class VII and for compact complex surfaces diffeomorphic to solvmanifolds.

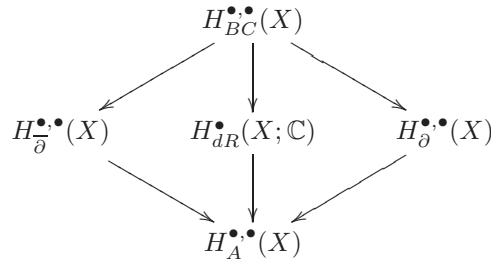
INTRODUCTION

For a given complex manifold X , many cohomological invariants can be defined, and many are known for compact complex surfaces.

Among these, one can consider *Bott-Chern and Aepli cohomologies*. They are defined as follows:

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial \bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}}.$$

Note that the identity induces natural maps



where $H_{\bar{\partial}}^{\bullet,\bullet}(X)$ denotes the Dolbeault cohomology and $H_{\partial}^{\bullet,\bullet}(X)$ its conjugate, and the maps are morphisms of (graded or bi-graded) vector spaces. For compact Kähler manifolds, the natural map $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ is an isomorphism.

Assume that X is compact. The Bott-Chern and Aepli cohomologies are isomorphic to the kernel of suitable 4th-order differential elliptic operators, see [19, §2.b, §2.c]. In particular, they are finite-dimensional vector spaces. In fact, fixed a Hermitian metric g , its associated \mathbb{C} -linear Hodge- $*$ -operator induces the isomorphism

$$H_{BC}^{p,q}(X) \xrightarrow{\cong} H_A^{n-q,n-p}(X),$$

for any $p, q \in \{0, \dots, n\}$, where n denotes the complex dimension of X . In particular, for any $p, q \in \{0, \dots, n\}$, one has

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_{BC}^{q,p}(X) = \dim_{\mathbb{C}} H_A^{n-p,n-q}(X) = \dim_{\mathbb{C}} H_A^{n-q,n-p}(X).$$

For the Dolbeault cohomology, the Frölicher inequality relates the Hodge numbers and the Betti numbers: for any $k \in \{0, \dots, 2n\}$,

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \geq \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

Similarly, for Bott-Chern cohomology, the following inequality *à la* Frölicher has been proven in [3, Theorem A]: for any $k \in \{0, \dots, n\}$,

$$\sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) \geq 2 \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

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The equality in the Frölicher inequality characterizes the degeneration of the Frölicher spectral sequence at the first level. This always happens for compact complex surfaces. On the other side, in [3, Theorem B], it is proven that the equality in the inequality *à la* Frölicher for the Bott-Chern cohomology characterizes the validity of the $\partial\bar{\partial}$ -Lemma, namely, the property that every ∂ -closed $\bar{\partial}$ -closed d-exact form is $\partial\bar{\partial}$ -exact too, [8]. The validity of the $\partial\bar{\partial}$ -Lemma implies that the first Betti number is even, which is equivalent to Kählerness for compact complex surfaces. Therefore the positive integer numbers

$$\Delta^k := \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) - 2b_k \in \mathbb{N},$$

varying $k \in \{1, 2\}$, measure the non-Kählerness of compact complex surfaces X .

Compact complex surfaces are divided in seven classes, according to the Kodaira and Enriques classification, see, e.g., [4]. In this note, we compute Bott-Chern cohomology for some classes of compact complex (non-Kähler) surfaces. In particular, we are interested in studying the relations between Bott-Chern cohomology and de Rham cohomology, looking at the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$. This can be intended as a weak version of the $\partial\bar{\partial}$ -Lemma, compare also [10].

More precisely, we start by proving that the non-Kählerness for compact complex surfaces is encoded only in Δ^2 , namely, Δ^1 is always zero. This gives a partial answer to a question by T. C. Dinh to the third author.

Theorem 1.1. *Let X be a compact complex surface. Then:*

- (i) *the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\partial}^{2,1}(X)$ induced by the identity is injective;*
- (ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

For compact complex surfaces in class VII, we show the following result, where we denote $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X)$ for $p, q \in \{0, 1, 2\}$.

Theorem 2.2. *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & & h_{BC}^{0,1} = 0 & \\ h_{BC}^{2,0} = 0 & h_{BC}^{1,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & h_{BC}^{0,2} = 0 & \\ & h_{BC}^{2,1} = 1 & & & & h_{BC}^{1,2} = 1 & \\ & & & h_{BC}^{2,2} = 1 . & & & \end{array}$$

Finally, we compute the Bott-Chern cohomology for compact complex surfaces diffeomorphic to solv-manifolds, according to the list given by K. Hasegawa in [11], see Theorem 4.1. More precisely, we prove that the cohomologies can be computed by using just left-invariant forms. Furthermore, for such complex surfaces, we note that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ is injective, see Theorem 4.2.

We note that the above classes do not exhaust the set of compact complex non-Kähler surfaces, the cohomologies of elliptic surfaces being still unknown.

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1. NON-KÄHLERNESS OF COMPACT COMPLEX SURFACES AND BOTT-CHERN COHOMOLOGY

We recall that, for a compact complex manifold of complex dimension n , for $k \in \{0, \dots, 2n\}$, we define the “non-Kählerness” degrees, [3, Theorem A],

$$\Delta^k := \sum_{p+q=k} (h_{BC}^{p,q} + h_{BC}^{n-q, n-p}) - 2b_k \in \mathbb{N},$$

where we use the duality in [19, §2.c] giving $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_A^{n-q, n-p}(X)$. According to [3, Theorem B], $\Delta^k = 0$ for any $k \in \{0, \dots, 2n\}$ if and only if X satisfies the $\partial\bar{\partial}$ -Lemma, namely, every ∂ -closed $\bar{\partial}$ -closed d-exact form is $\partial\bar{\partial}$ -exact too. In particular, for a compact complex surface X , the condition $\Delta^1 = \Delta^2 = 0$ is equivalent to X being Kähler, the first Betti number being even, [14, 17, 20], see also [15, Corollaire 5.7], and [5, Theorem 11].

We prove that Δ^1 is always zero for any compact complex surface. In particular, a sufficient and necessary condition for compact complex surfaces to be Kähler is $\Delta^2 = 0$.

Theorem 1.1. *Let X be a compact complex surface. Then:*

- (i) the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is injective;
(ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

Proof. (i) Let $\alpha \in \wedge^{2,1}X$ be such that $[\alpha] = 0 \in H_{\bar{\partial}}^{2,1}(X)$. Let $\beta \in \wedge^{2,0}X$ be such that $\alpha = \bar{\partial}\beta$. Fix a Hermitian metric g on X , and consider the Hodge decomposition of β with respect to the Dolbeault Laplacian $\bar{\square}$: let $\beta = \beta_h + \bar{\partial}^*\lambda$ where $\beta_h \in \wedge^{2,0}X \cap \ker \bar{\square}$, and $\lambda \in \wedge^{2,1}X$. Therefore we have

$$\alpha = \bar{\partial}\beta = \bar{\partial}\bar{\partial}^*\lambda = -\bar{\partial}*\underbrace{(\partial*\lambda)}_{\in \wedge^{2,0}X} = -\bar{\partial}(\partial*\lambda) = \partial\bar{\partial}(*\lambda),$$

where we have used that any $(2,0)$ -form is primitive and hence, by the Weil identity, is self-dual. In particular, α is $\partial\bar{\partial}$ -exact, so it induces a zero class in $H_{BC}^{2,1}(X)$.

(ii) On the one hand, note that

$$\begin{aligned} H_{BC}^{1,0}(X) &= \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0}X}{\text{im } \partial\bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0}X \\ &\subseteq \ker \bar{\partial} \cap \wedge^{1,0}X = \frac{\ker \bar{\partial} \cap \wedge^{1,0}X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X). \end{aligned}$$

It follows that

$$\dim_{\mathbb{C}} H_{BC}^{0,1}(X) = \dim_{\mathbb{C}} H_{BC}^{1,0}(X) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) = b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use that the Frölicher spectral sequence degenerates, hence in particular $b_1 = \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X)$.

On the other hand, by the assumption, we have

$$\dim_{\mathbb{C}} H_{BC}^{1,2}(X) = \dim_{\mathbb{C}} H_{BC}^{2,1}(X) \leq H_{\bar{\partial}}^{2,1}(X) = \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use the Kodaira and Serre duality $H_{\bar{\partial}}^{2,1}(X) \simeq H^1(X; \Omega_X^2) \simeq H^1(X; \mathcal{O}_X) \simeq H_{\bar{\partial}}^{0,1}(X)$.

By summing up, we get

$$\begin{aligned} \Delta^1 &= \dim_{\mathbb{C}} H_{BC}^{0,1}(X) + \dim_{\mathbb{C}} H_{BC}^{1,0}(X) + \dim_{\mathbb{C}} H_{BC}^{1,2}(X) + \dim_{\mathbb{C}} H_{BC}^{2,1}(X) - 2b_1 \\ &= 2 \left(b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) - b_1 \right) = 0, \end{aligned}$$

concluding the proof. \square

2. CLASS VII SURFACES

In this section, we compute Bott-Chern cohomology for compact complex surfaces in class VII.

Let X be a compact complex surface. By Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is always injective. Consider now the case when X is in class VII. If X is minimal, we prove that the same holds for cohomology with values in a line bundle. We will also prove that the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ is not injective.

Proposition 2.1. *Let X be a compact complex surface in class VII₀. Let $L \in H^1(X; \mathbb{C}^*) = \text{Pic}^0(X)$. The natural map $H_{BC}^{2,1}(X; L) \rightarrow H_{\bar{\partial}}^{2,1}(X; L)$ induced by the identity is injective.*

Proof. Let $\alpha \in \wedge^{2,1}X \otimes L$ be a $\bar{\partial}_L$ -exact $(2,1)$ -form. We need to prove that α is $\partial_L \bar{\partial}_L$ -exact too. Consider $\alpha = \bar{\partial}_L \vartheta$, where $\vartheta \in \wedge^{2,0}X \otimes L$. In particular, $\partial_L \vartheta = 0$, hence $\bar{\vartheta}$ defines a class in $H_{\bar{\partial}}^{0,2}(X; L)$. Note that $H_{\bar{\partial}}^{0,2}(X; L) \simeq H^2(X; \mathcal{O}_X(L)) \simeq H^0(X; K_X \otimes L^{-1}) = \{0\}$ for surfaces of class VII₀, [9, Remark 2.21]. It follows that $\bar{\vartheta} = -\bar{\partial}_L \bar{\eta}$ for some $\eta \in \wedge^{1,0}X \otimes L$. Hence $\alpha = \partial_L \bar{\partial}_L \eta$, that is, α is $\partial_L \bar{\partial}_L$ -exact. \square

We now compute the Bott-Chern cohomology of class VII surfaces.

Theorem 2.2. *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & & h_{BC}^{0,1} = 0 & \\ & h_{BC}^{1,0} = 0 & & & & & \\ h_{BC}^{2,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & & h_{BC}^{0,2} = 0 & \\ & h_{BC}^{2,1} = 1 & & h_{BC}^{1,2} = 1 & & & \\ & & h_{BC}^{2,2} = 1 & & & & \end{array}$$

Proof. It holds $H_{BC}^{1,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \Lambda^{1,0} X}{\text{im } \partial \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \Lambda^{1,0} X \subseteq \ker \bar{\partial} \cap \Lambda^{1,0} X = \frac{\ker \bar{\partial} \cap \Lambda^{1,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X) = \{0\}$ hence $h_{BC}^{1,0} = h_{BC}^{0,1} = 0$.

On the other side, by Theorem 1.1, $0 = \Delta^1 = 2 \left(h_{BC}^{1,0} + h_{BC}^{2,1} - b_1 \right) = 2 \left(h_{BC}^{2,1} - 1 \right)$ hence $h_{BC}^{2,1} = h_{BC}^{1,2} = 1$.

Similarly, it holds $H_{BC}^{2,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \Lambda^{2,0} X}{\text{im } \partial \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \Lambda^{2,0} X \subseteq \ker \bar{\partial} \cap \Lambda^{2,0} X = \frac{\ker \bar{\partial} \cap \Lambda^{2,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{2,0}(X) = \{0\}$ hence $h_{BC}^{2,0} = h_{BC}^{0,2} = 0$.

Note that, from [3, Theorem A], we have $0 \leq \Delta^2 = 2 \left(h_{BC}^{2,0} + h_{BC}^{1,1} + h_{BC}^{0,2} - b_2 \right) = 2 \left(h_{BC}^{1,1} - b_2 \right)$ hence $h_{BC}^{1,1} \geq b_2$. More precisely, from [3, Theorem B] and Theorem 1.1, we have that $h_{BC}^{1,1} = b_2$ if and only if $\Delta^2 = 0$ if and only if X satisfies the $\partial \bar{\partial}$ -Lemma, in fact X is Kähler, which is not the case.

Finally, we prove that $h_{BC}^{1,1} = b_2 + 1$. Consider the following exact sequences from [21, Lemma 2.3]. More precisely, the sequence

$$0 \rightarrow \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \rightarrow H_{BC}^{1,1}(X) \rightarrow \text{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) \rightarrow 0$$

is clearly exact. Furthermore, fix a Gauduchon metric g . Denote by $\omega := g(J, \cdot)$ the $(1,1)$ -form associated to g , where J denotes the integrable almost-complex structure. By definition of g being Gauduchon, we have $\partial \bar{\partial} \omega = 0$. The sequence

$$0 \rightarrow \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \xrightarrow{\langle \cdot | \omega \rangle} \mathbb{C}$$

is exact. Indeed, firstly note that for $\eta = \partial \bar{\partial} f \in \text{im } \partial \bar{\partial} \cap \Lambda^{1,1} X$, we have

$$\langle \eta | \omega \rangle = \int_X \partial \bar{\partial} f \wedge \bar{*} \omega = \int_X \partial \bar{\partial} f \wedge \omega = \int_X f \partial \bar{\partial} \omega = 0$$

by applying twice the Stokes theorem. Then, we recall the argument in [21, Lemma 2.3(ii)] for proving that the map

$$\langle \cdot | \omega \rangle : \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \rightarrow \mathbb{C}$$

is injective. Take $\alpha = d\beta \in \text{im } d \cap \Lambda^{1,1} X \cap \ker \langle \cdot | \omega \rangle$. Then

$$\langle \Lambda \alpha | 1 \rangle = \langle \alpha | \omega \rangle = 0,$$

where Λ is the adjoint operator of $\omega \wedge \cdot$ with respect to $\langle \cdot | \cdot \rangle$. Then $\Lambda \alpha \in \ker \langle \cdot | 1 \rangle = \text{im } \Lambda \partial \bar{\partial}$, by extending [16, Corollary 7.2.9] by \mathbb{C} -linearity. Take $u \in \mathcal{C}^\infty(X; \mathbb{C})$ such that $\Lambda \alpha = \Lambda \partial \bar{\partial} u$. Then, by defining $\alpha' := \alpha - \partial \bar{\partial} u$, we have $[\alpha'] = [\alpha] \in \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}}$, and $\Lambda \alpha' = 0$, and $\alpha' = d\beta'$ where $\beta' := \beta - \bar{\partial} u$. In particular, α' is primitive. Since α' is primitive and of type $(1,1)$, then it is anti-self-dual by the Weil identity. Then

$$\|\alpha'\|^2 = \langle \alpha' | \alpha' \rangle = \int_X \alpha' \wedge \bar{*} \alpha' = - \int_X \alpha' \wedge \bar{\alpha}' = - \int_X d\beta' \wedge d\bar{\beta}' = - \int_X d(\beta' \wedge d\bar{\beta}') = 0$$

and hence $\alpha' = 0$, and therefore $[\alpha] = 0$.

Since the space $\frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}}$ is finite-dimensional, being a sub-space of $H_{BC}^{1,1}(X)$, and since the space $\text{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right)$ is finite-dimensional, being a sub-space of $H_{dR}^2(X; \mathbb{C})$, we get that

$$\dim_{\mathbb{C}} \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \leq \dim_{\mathbb{C}} \mathbb{C} = 1,$$

and hence

$$b_2 < \dim_{\mathbb{C}} H_{BC}^{1,1}(X) = \dim_{\mathbb{C}} \text{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) + \dim_{\mathbb{C}} \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \leq b_2 + 1.$$

We get that $\dim_{\mathbb{C}} H_{BC}^{1,1}(X) = b_2 + 1$. \square

Finally, we prove that the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ is not injective.

Proposition 2.3. *Let X be a compact complex surface in class VII. Then the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ induced by the identity is the zero map and not an isomorphism.*

Proof. Note that, for class VII surfaces, the pluri-genera are zero. In particular, $H_{\bar{\partial}}^{1,2}(X) \simeq H_{\bar{\partial}}^{1,0}(X) = \{0\}$, by Kodaira and Serre duality. By Theorem 2.2, one has $H_{BC}^{1,2}(X) \neq \{0\}$. \square

2.1. Cohomologies of Calabi-Eckmann surface. In this section, as an explicit example, we compute the cohomologies of a compact complex surface in class VII: namely, we consider the Calabi-Eckmann structure on the differentiable manifolds underlying the Hopf surfaces.

Consider the differentiable manifold $X := \mathbb{S}^1 \times \mathbb{S}^3$. As a Lie group, $\mathbb{S}^3 = SU(2)$ has a global left-invariant co-frame $\{e^1, e^2, e^3\}$ such that $de^1 = -2e^2 \wedge e^3$ and $de^2 = 2e^1 \wedge e^3$ and $de^3 = -2e^1 \wedge e^2$. Hence, we consider a global left-invariant co-frame $\{f, e^1, e^2, e^3\}$ on X with structure equations

$$\begin{cases} df & = & 0 \\ de^1 & = & -2e^2 \wedge e^3 \\ de^2 & = & 2e^1 \wedge e^3 \\ de^3 & = & -2e^1 \wedge e^2 \end{cases} .$$

Consider the left-invariant almost-complex structure defined by the $(1,0)$ -forms

$$\begin{cases} \varphi^1 & := & e^1 + i e^2 \\ \varphi^2 & := & e^3 + i f \end{cases} .$$

By computing the complex structure equations, we get

$$\begin{cases} \partial\varphi^1 & = & i \varphi^1 \wedge \varphi^2 \\ \partial\varphi^2 & = & 0 \end{cases} \quad \text{and} \quad \begin{cases} \bar{\partial}\varphi^1 & = & i \varphi^1 \wedge \bar{\varphi}^2 \\ \bar{\partial}\varphi^2 & = & -i \varphi^1 \wedge \bar{\varphi}^1 \end{cases} .$$

We note that the almost-complex structure is in fact integrable.

The manifold X is a compact complex manifold not admitting Kähler metrics. It is bi-holomorphic to the complex manifold $M_{0,1}$ considered by Calabi and Eckmann, [6], see [18, Theorem 4.1].

Consider the Hermitian metric g whose associated $(1,1)$ -form is

$$\omega := \frac{i}{2} \sum_{j=1}^2 \varphi^j \wedge \bar{\varphi}^j .$$

As for the de Rham cohomology, from the Künneth formula we get

$$H_{dR}^{\bullet}(X; \mathbb{C}) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle \varphi^2 - \bar{\varphi}^2 \rangle \oplus \mathbb{C} \langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}2} \rangle \oplus \mathbb{C} \langle \varphi^{12\bar{1}\bar{2}} \rangle ,$$

(where, here and hereafter, we shorten, e.g., $\varphi^{12\bar{1}} := \varphi^1 \wedge \varphi^2 \wedge \bar{\varphi}^1$).

By [12, Appendix II, Theorem 9.5], one has that a model for the Dolbeault cohomology is given by

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) \simeq \bigwedge \langle x_{2,1}, x_{0,1} \rangle ,$$

where $x_{i,j}$ is an element of bi-degree (i,j) . In particular, we recover that the Hodge numbers $\left\{ h_{\bar{\partial}}^{p,q} := \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \right\}_{p,q \in \{0,1,2\}}$ are

$$\begin{array}{cccc} & & h_{\bar{\partial}}^{0,0} = 1 & \\ & & & h_{\bar{\partial}}^{0,1} = 1 \\ h_{\bar{\partial}}^{2,0} = 0 & h_{\bar{\partial}}^{1,0} = 0 & h_{\bar{\partial}}^{1,1} = 0 & h_{\bar{\partial}}^{0,2} = 0 \\ & h_{\bar{\partial}}^{2,1} = 1 & h_{\bar{\partial}}^{1,2} = 0 & \\ & & h_{\bar{\partial}}^{2,2} = 1 & \end{array} .$$

We note that the sub-complex

$$\iota: \bigwedge \langle \varphi^1, \varphi^2, \bar{\varphi}^1, \bar{\varphi}^2 \rangle \hookrightarrow \wedge^{\bullet,\bullet} X$$

is such that $H_{\bar{\partial}}(\iota)$ is an isomorphism. More precisely, we get

$$H_{\bar{\partial}}^{\bullet,\bullet}(X) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle [\varphi^2] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Dolbeault Laplacian of g .

By [2, Theorem 1.3, Proposition 2.2], we have also $H_{BC}(\iota)$ isomorphism. In particular, we get

$$H_{BC}^{\bullet,\bullet}(X) = \mathbb{C} \langle 1 \rangle \oplus \mathbb{C} \langle [\varphi^{1\bar{1}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}}] \rangle \oplus \mathbb{C} \langle [\varphi^{1\bar{1}\bar{2}}] \rangle \oplus \mathbb{C} \langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Bott-Chern Laplacian of g .

In particular, the Bott-Chern numbers $\{h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X)\}_{p,q \in \{0,1,2\}}$ are

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & h_{BC}^{0,1} = 0 & & \\ h_{BC}^{2,0} = 0 & h_{BC}^{1,0} = 0 & & h_{BC}^{1,1} = 1 & & h_{BC}^{0,2} = 0 & . \\ & h_{BC}^{2,1} = 1 & & & h_{BC}^{1,2} = 1 & & \\ & & & h_{BC}^{2,2} = 1 & & & \end{array}$$

By [19, §2.c], we have

$$H_A^{\bullet,\bullet}(X) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle [\varphi^2] \rangle \oplus \mathbb{C}\langle [\varphi^{\bar{2}}] \rangle \oplus \mathbb{C}\langle [\varphi^{2\bar{2}}] \rangle \oplus \mathbb{C}\langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Aeppli Laplacian of g .

In particular, the Aeppli numbers $\{h_A^{p,q} := \dim_{\mathbb{C}} H_A^{p,q}(X)\}_{p,q \in \{0,1,2\}}$ are

$$\begin{array}{ccccccc} & & & h_A^{0,0} = 1 & & & \\ & & & & h_A^{0,1} = 1 & & \\ h_A^{2,0} = 0 & h_A^{1,0} = 1 & & h_A^{1,1} = 1 & & h_A^{0,2} = 0 & . \\ & h_A^{2,1} = 0 & & & h_A^{1,2} = 0 & & \\ & & & h_A^{2,2} = 1 & & & \end{array}$$

Summarizing, we have the following.

Proposition 2.4. *Let $X := \mathbb{S}^1 \times \mathbb{S}^3$ be endowed with the complex structure of Calabi-Eckmann. The non-zero dimensions of the Dolbeault and Bott-Chern cohomologies are the following:*

$$h_{\bar{\partial}}^{0,0}(X) = h_{\bar{\partial}}^{0,1}(X) = h_{\bar{\partial}}^{2,1}(X) = h_{\bar{\partial}}^{2,2}(X) = 1$$

and

$$h_{BC}^{0,0}(X) = h_{BC}^{1,1}(X) = h_{BC}^{2,1}(X) = h_{BC}^{1,2}(X) = h_{BC}^{2,2}(X) = 1 .$$

Note in particular that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is an isomorphism, and that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity is injective.

3. COMPLEX SURFACES DIFFEOMORPHIC TO SOLVMANIFOLDS

Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$. By [11, Theorem 1], X is (A) either a complex torus, (B) or a hyperelliptic surface, (C) or a Inoue surface of type \mathcal{S}_M , (D) or a primary Kodaira surface, (E) or a secondary Kodaira surface, (F) or a Inoue surface of type \mathcal{S}^{\pm} , and, as such, it is endowed with a left-invariant complex structure.

In each case, we recall the structure equations of the group G , see [11]. More precisely, take a basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra \mathfrak{g} naturally associated to G . We have the following commutation relations, according to [11]:

(A) differentiable structure underlying a *complex torus*:

$$[e_j, e_k] = 0 \quad \text{for any } j, k \in \{1, 2, 3, 4\} ;$$

(hereafter, we write only the non-trivial commutators);

(B) differentiable structure underlying a *hyperelliptic surface*:

$$[e_1, e_4] = e_2 , \quad [e_2, e_4] = -e_1 ;$$

(C) differentiable structure underlying a *Inoue surface of type \mathcal{S}_M* :

$$[e_1, e_4] = -\alpha e_1 + \beta e_2 , \quad [e_2, e_4] = -\beta e_1 - \alpha e_2 , \quad [e_3, e_4] = 2\alpha e_3 ,$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$;

(D) differentiable structure underlying a *primary Kodaira surface*:

$$[e_1, e_2] = -e_3 ;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$[e_1, e_2] = -e_3 , \quad [e_1, e_4] = e_2 , \quad [e_2, e_4] = -e_1 ;$$

(F) differentiable structure underlying a *Inoue surface of type \mathcal{S}^{\pm}* :

$$[e_2, e_3] = -e_1 , \quad [e_2, e_4] = -e_2 , \quad [e_3, e_4] = e_3 .$$

Denote by $\{e^1, e^2, e^3, e^4\}$ the dual basis of $\{e_1, e_2, e_3, e_4\}$. We recall that, for any $\alpha \in \mathfrak{g}^*$, for any $x, y \in \mathfrak{g}$, it holds $d\alpha(x, y) = -\alpha([x, y])$. Hence we get the following structure equations:

(A) differentiable structure underlying a *complex torus*:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = 0 \\ de^4 = 0 \end{cases} ;$$

(B) differentiable structure underlying a *hyperelliptic surface*:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = 0 \\ de^4 = 0 \end{cases} ;$$

(C) differentiable structure underlying a *Inoue surface of type \mathcal{S}_M* :

$$\begin{cases} de^1 = \alpha e^1 \wedge e^4 + \beta e^2 \wedge e^4 \\ de^2 = -\beta e^1 \wedge e^4 + \alpha e^2 \wedge e^4 \\ de^3 = -2\alpha e^3 \wedge e^4 \\ de^4 = 0 \end{cases} ;$$

(D) differentiable structure underlying a *primary Kodaira surface*:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = e^1 \wedge e^2 \\ de^4 = 0 \end{cases} ;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = e^1 \wedge e^2 \\ de^4 = 0 \end{cases} ;$$

(F) differentiable structure underlying a *Inoue surface of type \mathcal{S}^\pm* :

$$\begin{cases} de^1 = e^2 \wedge e^3 \\ de^2 = e^2 \wedge e^4 \\ de^3 = -e^3 \wedge e^4 \\ de^4 = 0 \end{cases} .$$

In cases (A), (B), (C), (D), (E), consider the G -left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and} \quad Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 \quad \text{and} \quad Je_4 := -e_3 .$$

Consider the G -left-invariant $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + ie^2 \\ \varphi^2 := e^3 + ie^4 \end{cases} .$$

In case (F), consider the G -left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and} \quad Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 - qe_2 \quad \text{and} \quad Je_4 := -e_3 - qe_1 ,$$

where $q \in \mathbb{R}$. Consider the G -left-invariant $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + i e^2 + i q e^4 \\ \varphi^2 := e^3 + i e^4 \end{cases}.$$

With respect to the G -left-invariant coframe $\{\varphi^1, \varphi^2\}$ for the holomorphic tangent bundle $T^{1,0} \Gamma \backslash G$, we have the following structure equations. (As for notation, we shorten, e.g., $\varphi^{1\bar{2}} := \varphi^1 \wedge \bar{\varphi}^2$.)

(A) *torus*:

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = 0 \end{cases}$$

(B) *hyperelliptic surface*:

$$\begin{cases} d\varphi^1 = -\frac{1}{2}\varphi^{12} + \frac{1}{2}\varphi^{1\bar{2}} \\ d\varphi^2 = 0 \end{cases}$$

(C) *Inoue surface \mathcal{S}_M* :

$$\begin{cases} d\varphi^1 = \frac{\alpha-i\beta}{2i}\varphi^{12} - \frac{\alpha-i\beta}{2i}\varphi^{1\bar{2}} \\ d\varphi^2 = -i\alpha\varphi^{2\bar{2}} \end{cases}$$

(where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$);

(D) *primary Kodaira surface*:

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = \frac{i}{2}\varphi^{1\bar{1}} \end{cases}$$

(E) *secondary Kodaira surface*:

$$\begin{cases} d\varphi^1 = -\frac{1}{2}\varphi^{12} + \frac{1}{2}\varphi^{1\bar{2}} \\ d\varphi^2 = \frac{i}{2}\varphi^{1\bar{1}} \end{cases}$$

(F) *Inoue surface \mathcal{S}^\pm* :

$$\begin{cases} d\varphi^1 = \frac{1}{2i}\varphi^{12} + \frac{1}{2i}\varphi^{2\bar{1}} + \frac{q}{2}i\varphi^{2\bar{2}} \\ d\varphi^2 = \frac{1}{2i}\varphi^{2\bar{2}} \end{cases}.$$

4. COHOMOLOGIES OF COMPLEX SURFACES DIFFEOMORPHIC TO SOLVMANIFOLDS

In this section, we compute the Dolbeault and Bott-Chern cohomologies of the compact complex surfaces diffeomorphic to a solvmanifold.

We prove the following theorem.

Theorem 4.1. *Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$; denote the Lie algebra of G by \mathfrak{g} . Then the inclusion $(\wedge^{\bullet, \bullet} \mathfrak{g}^*, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet, \bullet} X, \partial, \bar{\partial})$ induces an isomorphism both in Dolbeault and in Bott-Chern cohomologies. In particular, the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies and the degrees of non-Kählerness are summarized in Table 5.*

Proof. Firstly, we compute the cohomologies of the sub-complex of G -left-invariant forms. The computations are straightforward from the structure equations.

(p, q)	$H_{\mathbb{C}}^{E,q}$	(A) torus			(B) hyperelliptic				(C) Inoue \mathcal{S}_M			
		$\dim_{\mathbb{C}} H_{\mathbb{C}}^{E,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\mathbb{C}}^{E,q}$	$\dim_{\mathbb{C}} H_{\mathbb{C}}^{E,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\mathbb{C}}^{E,q}$	$\dim_{\mathbb{C}} H_{\mathbb{C}}^{E,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0, 0)	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
(1, 0)	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^2 \rangle$	1	$\langle \varphi^2 \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(0, 1)	$\langle \bar{\varphi}^1, \bar{\varphi}^2 \rangle$	2	$\langle \bar{\varphi}^1, \bar{\varphi}^2 \rangle$	2	$\langle \bar{\varphi}^2 \rangle$	1	$\langle \bar{\varphi}^2 \rangle$	1	$\langle \bar{\varphi}^2 \rangle$	1	$\langle 0 \rangle$	0
(2, 0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(1, 1)	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle 0 \rangle$	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0, 2)	$\langle \bar{\varphi}^{1\bar{2}} \rangle$	1	$\langle \bar{\varphi}^{1\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(2, 1)	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
(1, 2)	$\langle \varphi^{11\bar{2}}, \varphi^{21\bar{2}} \rangle$	2	$\langle \varphi^{11\bar{2}}, \varphi^{21\bar{2}} \rangle$	2	$\langle \varphi^{11\bar{2}} \rangle$	1	$\langle \varphi^{11\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle \varphi^{11\bar{2}} \rangle$	1
(2, 2)	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 1. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

(p, q)	(D) primary Kodaira				(E) secondary Kodaira				(F) Inoue S_{\pm}			
	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0, 0)	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1
(1, 0)	$\langle \varphi^1 \rangle$	1	$\langle \varphi^1 \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(0, 1)	$\langle \varphi^1, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^1 \rangle$	1	$\langle \varphi^{\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{\bar{2}} \rangle$	1	(0)	0
(2, 0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(1, 1)	$\langle \varphi^{12}, \varphi^{2\bar{1}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{12}, \varphi^{2\bar{1}} \rangle$	3	(0)	0	$\langle \varphi^{1\bar{1}} \rangle$	1	(0)	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0, 2)	$\langle \varphi^{\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{\bar{1}\bar{2}} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(2, 1)	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
(1, 2)	$\langle \varphi^{2\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
(2, 2)	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 2. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

k	(A) torus		(B) hyperelliptic		(C) Inoue \mathcal{S}_M	
	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^2, \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	4	$\langle \varphi^2, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}}, \varphi^{\bar{1}\bar{2}} \rangle$	6	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}}, \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	4	$\langle \varphi^{12\bar{1}}, \varphi^{1\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 3. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

k	(D) primary Kodaira		(E) secondary Kodaira		(F) Inoue S^{\pm}	
	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^{\bar{1}}, \varphi^2 - \varphi^{\bar{2}} \rangle$	3	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{\bar{1}\bar{2}} \rangle$	4	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{2}}, \varphi^{2\bar{1}\bar{2}}, \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	3	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}} - q\varphi^{12\bar{2}} - \varphi^{1\bar{1}\bar{2}} + q\varphi^{2\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 4. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

In Tables 1 and 2 and in Tables 3 and 4, we summarize the results of the computations. The sub-complexes of left-invariant forms are depicted in Figure 1 (each dot represents a generator, vertical arrows depict the $\bar{\partial}$ -operator, horizontal arrows depict the ∂ -operator, and trivial arrows are not shown.) The dimensions are listed in Table 5.

On the one side, recall that the inclusion of left-invariant forms into the space of forms induces an injective map in Dolbeault and Bott-Chern cohomologies, see, e.g., [7, Lemma 9], [1, Lemma 3.6]. On

the other side, recall that the Frölicher spectral sequence of a compact complex surface X degenerates at the first level, equivalently, the equalities

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) = \dim_{\mathbb{C}} H_{dR}^1(X; \mathbb{C})$$

and

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{2,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,2}(X) = \dim_{\mathbb{C}} H_{dR}^2(X; \mathbb{C})$$

hold. By comparing the dimensions in Table 5 with the Betti numbers case by case, we find that the left-invariant forms suffice in computing the Dolbeault cohomology for each case. Then, by [1, Theorem 3.7], see also [2, Theorem 1.3, Theorem 1.6], it follows that also the Bott-Chern cohomology is computed using just left-invariant forms. \square

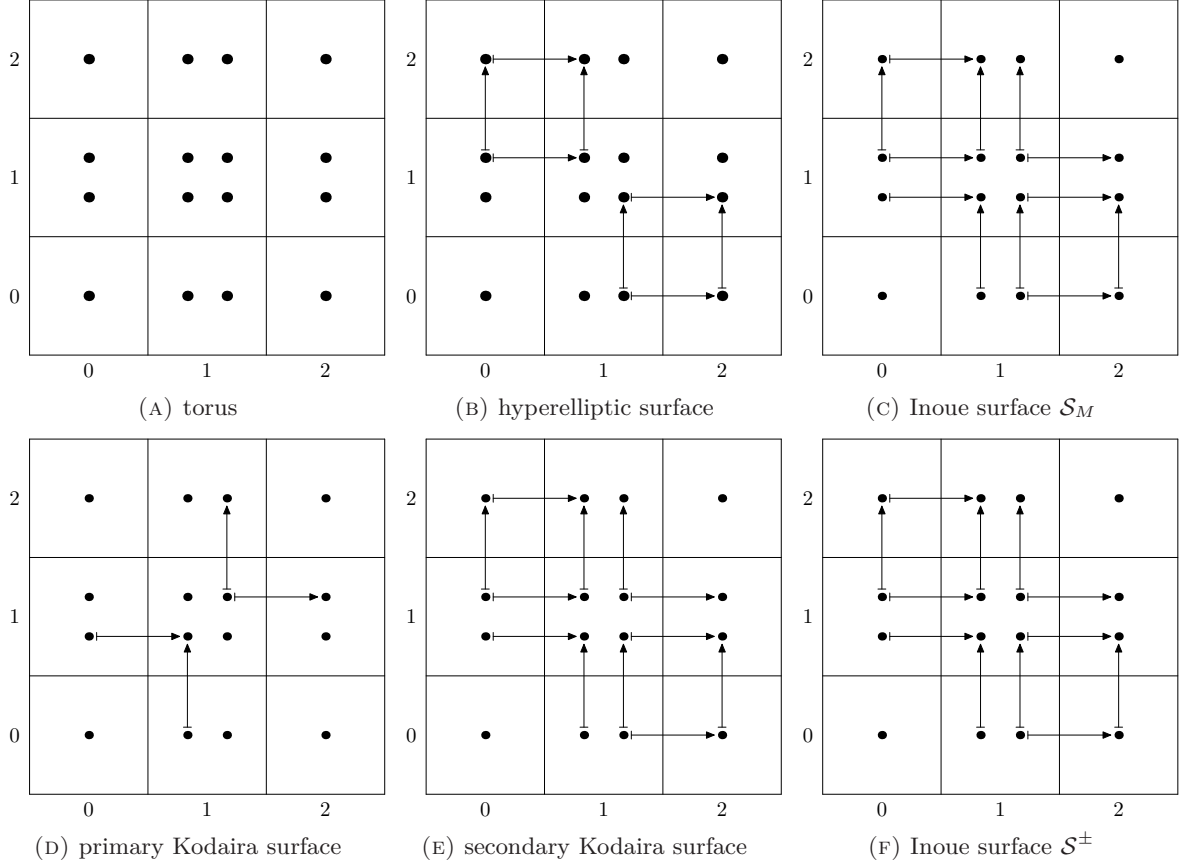


FIGURE 1. The double-complexes of left-invariant forms over 4-dimensional solvmanifolds.

(p, q)	(A) torus				(B) hyperell				(C) Inoue S_M				(D) prim Kod				(E) sec Kod				(F) Inoue S^\pm							
	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k				
$(0, 0)$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0
$(1, 0)$	2	2	4	0	1	1	2	0	0	0	1	0	1	1	3	0	0	0	1	0	0	0	1	0	0	0	1	0
$(0, 1)$	2	2			1	1			1	0			2	1			1	0			1	0			1	0		
$(2, 0)$	1	1	6	0	0	0	2	0	0	0	0	2	1	1	4	2	0	0	0	2	0	0	0	2	0	0	0	2
$(1, 1)$	4	4			2	2			0	1			2	3			0	1			0	1			0	1		
$(0, 2)$	1	1			0	0			0	0			1	1			0	0			0	0			0	0		
$(2, 1)$	2	2	4	0	1	1	2	0	1	1	1	0	2	2	3	0	1	1	1	0	1	1	1	0	1	1	1	0
$(1, 2)$	2	2			1	1			0	1			1	2			0	1			0	1			0	1		
$(2, 2)$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0

TABLE 5. Summary of the dimensions of de Rham, Dolbeault, and Bott-Chern cohomologies and of the degree of non-Kählerness for compact complex surfaces diffeomorphic to solvmanifolds.

According to Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is injective for any compact complex surface. We are now interested in studying the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity, at least for compact complex surfaces diffeomorphic to solvmanifolds. In fact, by definition, the property of satisfying the $\partial\bar{\partial}$ -Lemma, [8], is equivalent to the natural map $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ being injective. Note that, for a compact complex manifold of complex dimension n , the injectivity of the map $H_{BC}^{n,n-1}(X) \rightarrow H_{dR}^{2n-1}(X; \mathbb{C})$ implies the $(n-1, n)$ -th weak $\partial\bar{\partial}$ -Lemma in the sense of J. Fu and S.-T. Yau, [10, Definition 5].

We prove the following result.

Theorem 4.2. *Let X be a compact complex surface diffeomorphic to a solvmanifold. Then the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is an isomorphism, and the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity is injective.*

Proof. By the general result in Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is injective. In fact, it is an isomorphism as follows from the computations summarized in Tables 1 and 2. As for the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$, it is a straightforward computation from Tables 1 and 2 and Tables 3 and 4.

As an example, we offer an explicit calculation of the injectivity of the map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ for the Inoue surfaces of type 0, see [13], see also [22]. We will change a little bit the notation. Recall the construction of Inoue surfaces: let $M \in \mathrm{SL}(3; \mathbb{Z})$ be a unimodular matrix having a real eigenvalue $\lambda > 1$ and two complex eigenvalues $\mu \neq \bar{\mu}$. Take a real eigenvector $(\alpha_1, \alpha_2, \alpha_3)$ and an eigenvector $(\beta_1, \beta_2, \beta_3)$ of M . Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$; on the product $\mathbb{H} \times \mathbb{C}$ consider the following transformations defined as

$$\begin{aligned} f_0(z, w) &:= (\lambda z, \mu w) \\ f_j(z, w) &:= (z + \alpha_j, w + \beta_j) \quad \text{for } j \in \{1, 2, 3\}. \end{aligned}$$

Denote by Γ_M the group generated by f_0, \dots, f_3 ; then Γ_M acts in a properly discontinuous way and without fixed points on $\mathbb{H} \times \mathbb{C}$, and $\mathcal{S}_M := \mathbb{H} \times \mathbb{C} / \Gamma_M$ is an Inoue surface of type 0, as in case (C) in [11]. Denoting by $z = x + iy$ and $w = u + iv$, consider the following differential forms on $\mathbb{H} \times \mathbb{C}$:

$$e^1 := \frac{1}{y} dx, \quad e^2 := \frac{1}{y} dy, \quad e^3 := \sqrt{y} du, \quad e^4 := \sqrt{y} dv.$$

(Note that e^1 and e^2 , and $e^3 \wedge e^4$ are Γ_M -invariant, and consequently they induce global differential forms on \mathcal{S}_M .) We obtain

$$de^1 = e^1 \wedge e^2, \quad de^2 = 0, \quad de^3 = \frac{1}{2} e^2 \wedge e^3, \quad de^4 = \frac{1}{2} e^2 \wedge e^4.$$

Consider the natural complex structure on \mathcal{S}_M induced by $\mathbb{H} \times \mathbb{C}$. Locally, we have

$$Je^1 = -e^2 \quad \text{and} \quad Je^2 = e^1 \quad \text{and} \quad Je^3 = -e^4 \quad \text{and} \quad Je^4 = e^3.$$

Considering the Γ_M -invariant $(2, 1)$ -Bott-Chern cohomology of \mathcal{S}_M , we obtain that

$$H_{BC}^{2,1}(\mathcal{S}_M) = \mathbb{C} \langle [e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4] \rangle.$$

Clearly $\bar{\partial}(e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4) = 0$ and $e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4 = -2i e^1 \wedge e^3 \wedge e^4 + 2d(e^3 \wedge e^4)$, therefore the de Rham cohomology class $[e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4] = -2i [e^1 \wedge e^3 \wedge e^4] \in H_{dR}^3(\mathcal{S}_M)$ is non-zero. \square

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