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► **To cite this version:**

Daniele Angela, Georges Dloussky, Adriano Tomassini. On Bott-Chern cohomology of compact complex surfaces. 2014. hal-01067265v1

HAL Id: hal-01067265

<https://hal.science/hal-01067265v1>

Preprint submitted on 23 Sep 2014 (v1), last revised 1 Apr 2016 (v2)

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ON BOTT-CHERN COHOMOLOGY OF COMPACT COMPLEX SURFACES

DANIELE ANGELLA, GEORGES DLOUSSKY, AND ADRIANO TOMASSINI

ABSTRACT. We study Bott-Chern cohomology on compact complex non-Kähler surfaces. In particular, we compute such a cohomology for compact complex surfaces in class VII and for compact complex surfaces diffeomorphic to solvmanifolds.

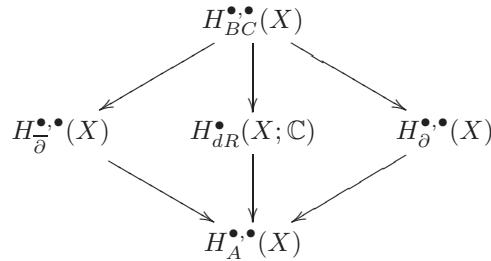
INTRODUCTION

For a given complex manifold X , many cohomological invariants can be defined, and many are known for compact complex surfaces.

Among these, one can consider *Bott-Chern and Aepli cohomologies*. They are defined as follows:

$$H_{BC}^{\bullet,\bullet}(X) := \frac{\ker \partial \cap \ker \bar{\partial}}{\text{im } \partial \bar{\partial}} \quad \text{and} \quad H_A^{\bullet,\bullet}(X) := \frac{\ker \partial \bar{\partial}}{\text{im } \partial + \text{im } \bar{\partial}}.$$

Note that the identity induces natural maps



where $H_{\bar{\partial}}^{\bullet,\bullet}(X)$ denotes the Dolbeault cohomology and $H_{\partial}^{\bullet,\bullet}(X)$ its conjugate, and the maps are morphisms of (graded or bi-graded) vector spaces. For compact Kähler manifolds, the natural map $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ is an isomorphism.

Assume that X is compact. The Bott-Chern and Aepli cohomologies are isomorphic to the kernel of suitable 4th-order differential elliptic operators, see [19, §2.b, §2.c]. In particular, they are finite-dimensional vector spaces. In fact, fixed a Hermitian metric g , its associated \mathbb{C} -linear Hodge- $*$ -operator induces the isomorphism

$$H_{BC}^{p,q}(X) \xrightarrow{\cong} H_A^{n-q,n-p}(X),$$

for any $p, q \in \{0, \dots, n\}$, where n denotes the complex dimension of X . In particular, for any $p, q \in \{0, \dots, n\}$, one has

$$\dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_{BC}^{q,p}(X) = \dim_{\mathbb{C}} H_A^{n-p,n-q}(X) = \dim_{\mathbb{C}} H_A^{n-q,n-p}(X).$$

For the Dolbeault cohomology, the Frölicher inequality relates the Hodge numbers and the Betti numbers: for any $k \in \{0, \dots, 2n\}$,

$$\sum_{p+q=k} \dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}(X) \geq \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

Similarly, for Bott-Chern cohomology, the following inequality *à la* Frölicher has been proven in [3, Theorem A]: for any $k \in \{0, \dots, n\}$,

$$\sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) \geq 2 \dim_{\mathbb{C}} H_{dR}^k(X; \mathbb{C}).$$

2010 *Mathematics Subject Classification.* 32C35, 57T15, 32J15.

Key words and phrases. compact complex surfaces, Bott-Chern cohomology, class VII, solvmanifold.

This work was supported by the Project PRIN “Varietà reali e complesse: geometria, topologia e analisi armonica”, by the Project FIRB “Geometria Differenziale e Teoria Geometrica delle Funzioni”, by GNSAGA of INdAM, and by ANR “Méthodes nouvelles en géométrie non kählerienne”.

The equality in the Frölicher inequality characterizes the degeneration of the Frölicher spectral sequence at the first level. This always happens for compact complex surfaces. On the other side, in [3, Theorem B], it is proven that the equality in the inequality *à la* Frölicher for the Bott-Chern cohomology characterizes the validity of the $\partial\bar{\partial}$ -Lemma, namely, the property that every ∂ -closed $\bar{\partial}$ -closed d-exact form is $\partial\bar{\partial}$ -exact too, [8]. The validity of the $\partial\bar{\partial}$ -Lemma implies that the first Betti number is even, which is equivalent to Kählerness for compact complex surfaces. Therefore the positive integer numbers

$$\Delta^k := \sum_{p+q=k} (\dim_{\mathbb{C}} H_{BC}^{p,q}(X) + \dim_{\mathbb{C}} H_A^{p,q}(X)) - 2b_k \in \mathbb{N},$$

varying $k \in \{1, 2\}$, measure the non-Kählerness of compact complex surfaces X .

Compact complex surfaces are divided in seven classes, according to the Kodaira and Enriques classification, see, e.g., [4]. In this note, we compute Bott-Chern cohomology for some classes of compact complex (non-Kähler) surfaces. In particular, we are interested in studying the relations between Bott-Chern cohomology and de Rham cohomology, looking at the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$. This can be intended as a weak version of the $\partial\bar{\partial}$ -Lemma, compare also [10].

More precisely, we start by proving that the non-Kählerness for compact complex surfaces is encoded only in Δ^2 , namely, Δ^1 is always zero. This gives a partial answer to a question by T. C. Dinh to the third author.

Theorem 1.1. *Let X be a compact complex surface. Then:*

- (i) *the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\partial}^{2,1}(X)$ induced by the identity is injective;*
- (ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

For compact complex surfaces in class VII, we show the following result, where we denote $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X)$ for $p, q \in \{0, 1, 2\}$.

Theorem 2.2. *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & & h_{BC}^{0,1} = 0 & \\ h_{BC}^{2,0} = 0 & h_{BC}^{1,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & h_{BC}^{0,2} = 0 & \\ & h_{BC}^{2,1} = 1 & & & & h_{BC}^{1,2} = 1 & \\ & & & h_{BC}^{2,2} = 1 & & & \end{array}$$

Finally, we compute the Bott-Chern cohomology for compact complex surfaces diffeomorphic to solv-manifolds, according to the list given by K. Hasegawa in [11], see Theorem 4.1. More precisely, we prove that the cohomologies can be computed by using just left-invariant forms. Furthermore, for such complex surfaces, we note that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ is injective, see Theorem 4.2.

We note that the above classes do not exhaust the set of compact complex non-Kähler surfaces, the cohomologies of elliptic surfaces being still unknown.

Acknowledgments. The first and third authors would like to thank the Aix-Marseille University for its warm hospitality.

1. NON-KÄHLERNES OF COMPACT COMPLEX SURFACES AND BOTT-CHERN COHOMOLOGY

We recall that, for a compact complex manifold of complex dimension n , for $k \in \{0, \dots, 2n\}$, we define the “non-Kählerness” degrees, [3, Theorem A],

$$\Delta^k := \sum_{p+q=k} (h_{BC}^{p,q} + h_{BC}^{n-q, n-p}) - 2b_k \in \mathbb{N},$$

where we use the duality in [19, §2.c] giving $h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X) = \dim_{\mathbb{C}} H_A^{n-q, n-p}(X)$. According to [3, Theorem B], $\Delta^k = 0$ for any $k \in \{0, \dots, 2n\}$ if and only if X satisfies the $\partial\bar{\partial}$ -Lemma, namely, every ∂ -closed $\bar{\partial}$ -closed d-exact form is $\partial\bar{\partial}$ -exact too. In particular, for a compact complex surface X , the condition $\Delta^1 = \Delta^2 = 0$ is equivalent to X being Kähler, the first Betti number being even, [14, 17, 20], see also [15, Corollaire 5.7], and [5, Theorem 11].

We prove that Δ^1 is always zero for any compact complex surface. In particular, a sufficient and necessary condition for compact complex surfaces to be Kähler is $\Delta^2 = 0$.

Theorem 1.1. *Let X be a compact complex surface. Then:*

- (i) the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is injective;
(ii) $\Delta^1 = 0$.

In particular, the non-Kählerness of X is measured by just $\Delta^2 \in \mathbb{N}$.

Proof. (i) Let $\alpha \in \wedge^{2,1}X$ be such that $[\alpha] = 0 \in H_{\bar{\partial}}^{2,1}(X)$. Let $\beta \in \wedge^{2,0}X$ be such that $\alpha = \bar{\partial}\beta$. Fix a Hermitian metric g on X , and consider the Hodge decomposition of β with respect to the Dolbeault Laplacian $\bar{\square}$: let $\beta = \beta_h + \bar{\partial}^*\lambda$ where $\beta_h \in \wedge^{2,0}X \cap \ker \bar{\square}$, and $\lambda \in \wedge^{2,1}X$. Therefore we have

$$\alpha = \bar{\partial}\beta = \bar{\partial}\bar{\partial}^*\lambda = -\bar{\partial}*\underbrace{(\partial*\lambda)}_{\in \wedge^{2,0}X} = -\bar{\partial}(\partial*\lambda) = \partial\bar{\partial}(*\lambda),$$

where we have used that any $(2,0)$ -form is primitive and hence, by the Weil identity, is self-dual. In particular, α is $\partial\bar{\partial}$ -exact, so it induces a zero class in $H_{BC}^{2,1}(X)$.

(ii) On the one hand, note that

$$\begin{aligned} H_{BC}^{1,0}(X) &= \frac{\ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0}X}{\text{im } \partial\bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \wedge^{1,0}X \\ &\subseteq \ker \bar{\partial} \cap \wedge^{1,0}X = \frac{\ker \bar{\partial} \cap \wedge^{1,0}X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X). \end{aligned}$$

It follows that

$$\dim_{\mathbb{C}} H_{BC}^{0,1}(X) = \dim_{\mathbb{C}} H_{BC}^{1,0}(X) \leq \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) = b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use that the Frölicher spectral sequence degenerates, hence in particular $b_1 = \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X)$.

On the other hand, by the assumption, we have

$$\dim_{\mathbb{C}} H_{BC}^{1,2}(X) = \dim_{\mathbb{C}} H_{BC}^{2,1}(X) \leq H_{\bar{\partial}}^{2,1}(X) = \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X),$$

where we use the Kodaira and Serre duality $H_{\bar{\partial}}^{2,1}(X) \simeq H^1(X; \Omega_X^2) \simeq H^1(X; \mathcal{O}_X) \simeq H_{\bar{\partial}}^{0,1}(X)$.

By summing up, we get

$$\begin{aligned} \Delta^1 &= \dim_{\mathbb{C}} H_{BC}^{0,1}(X) + \dim_{\mathbb{C}} H_{BC}^{1,0}(X) + \dim_{\mathbb{C}} H_{BC}^{1,2}(X) + \dim_{\mathbb{C}} H_{BC}^{2,1}(X) - 2b_1 \\ &= 2 \left(b_1 - \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) - b_1 \right) = 0, \end{aligned}$$

concluding the proof. \square

2. CLASS VII SURFACES

In this section, we compute Bott-Chern cohomology for compact complex surfaces in class VII.

Let X be a compact complex surface. By Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is always injective. Consider now the case when X is in class VII. If X is minimal, we prove that the same holds for cohomology with values in a line bundle. We will also prove that the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ is not injective.

Proposition 2.1. *Let X be a compact complex surface in class VII₀. Let $L \in H^1(X; \mathbb{C}^*) = \text{Pic}^0(X)$. The natural map $H_{BC}^{2,1}(X; L) \rightarrow H_{\bar{\partial}}^{2,1}(X; L)$ induced by the identity is injective.*

Proof. Let $\alpha \in \wedge^{2,1}X \otimes L$ be a $\bar{\partial}_L$ -exact $(2,1)$ -form. We need to prove that α is $\partial_L \bar{\partial}_L$ -exact too. Consider $\alpha = \bar{\partial}_L \vartheta$, where $\vartheta \in \wedge^{2,0}X \otimes L$. In particular, $\partial_L \vartheta = 0$, hence $\bar{\vartheta}$ defines a class in $H_{\bar{\partial}}^{0,2}(X; L)$. Note that $H_{\bar{\partial}}^{0,2}(X; L) \simeq H^2(X; \mathcal{O}_X(L)) \simeq H^0(X; K_X \otimes L^{-1}) = \{0\}$ for surfaces of class VII₀, [9, Remark 2.21]. It follows that $\bar{\vartheta} = -\bar{\partial}_L \bar{\eta}$ for some $\eta \in \wedge^{1,0}X \otimes L$. Hence $\alpha = \partial_L \bar{\partial}_L \eta$, that is, α is $\partial_L \bar{\partial}_L$ -exact. \square

We now compute the Bott-Chern cohomology of class VII surfaces.

Theorem 2.2. *The Bott-Chern numbers of compact complex surfaces in class VII are:*

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & & h_{BC}^{0,1} = 0 & \\ & h_{BC}^{1,0} = 0 & & & & & \\ h_{BC}^{2,0} = 0 & & h_{BC}^{1,1} = b_2 + 1 & & & h_{BC}^{0,2} = 0 & \\ & h_{BC}^{2,1} = 1 & & h_{BC}^{1,2} = 1 & & & \\ & & h_{BC}^{2,2} = 1 & & & & \end{array}$$

Proof. It holds $H_{BC}^{1,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \Lambda^{1,0} X}{\text{im } \partial \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \Lambda^{1,0} X \subseteq \ker \bar{\partial} \cap \Lambda^{1,0} X = \frac{\ker \bar{\partial} \cap \Lambda^{1,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{1,0}(X) = \{0\}$ hence $h_{BC}^{1,0} = h_{BC}^{0,1} = 0$.

On the other side, by Theorem 1.1, $0 = \Delta^1 = 2 \left(h_{BC}^{1,0} + h_{BC}^{2,1} - b_1 \right) = 2 \left(h_{BC}^{2,1} - 1 \right)$ hence $h_{BC}^{2,1} = h_{BC}^{1,2} = 1$.

Similarly, it holds $H_{BC}^{2,0}(X) = \frac{\ker \partial \cap \ker \bar{\partial} \cap \Lambda^{2,0} X}{\text{im } \partial \bar{\partial}} = \ker \partial \cap \ker \bar{\partial} \cap \Lambda^{2,0} X \subseteq \ker \bar{\partial} \cap \Lambda^{2,0} X = \frac{\ker \bar{\partial} \cap \Lambda^{2,0} X}{\text{im } \bar{\partial}} = H_{\bar{\partial}}^{2,0}(X) = \{0\}$ hence $h_{BC}^{2,0} = h_{BC}^{0,2} = 0$.

Note that, from [3, Theorem A], we have $0 \leq \Delta^2 = 2 \left(h_{BC}^{2,0} + h_{BC}^{1,1} + h_{BC}^{0,2} - b_2 \right) = 2 \left(h_{BC}^{1,1} - b_2 \right)$ hence $h_{BC}^{1,1} \geq b_2$. More precisely, from [3, Theorem B] and Theorem 1.1, we have that $h_{BC}^{1,1} = b_2$ if and only if $\Delta^2 = 0$ if and only if X satisfies the $\partial \bar{\partial}$ -Lemma, in fact X is Kähler, which is not the case.

Finally, we prove that $h_{BC}^{1,1} = b_2 + 1$. Consider the following exact sequences from [21, Lemma 2.3]. More precisely, the sequence

$$0 \rightarrow \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \rightarrow H_{BC}^{1,1}(X) \rightarrow \text{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) \rightarrow 0$$

is clearly exact. Furthermore, fix a Gauduchon metric g . Denote by $\omega := g(J, \cdot)$ the $(1,1)$ -form associated to g , where J denotes the integrable almost-complex structure. By definition of g being Gauduchon, we have $\partial \bar{\partial} \omega = 0$. The sequence

$$0 \rightarrow \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \xrightarrow{\langle \cdot | \omega \rangle} \mathbb{C}$$

is exact. Indeed, firstly note that for $\eta = \partial \bar{\partial} f \in \text{im } \partial \bar{\partial} \cap \Lambda^{1,1} X$, we have

$$\langle \eta | \omega \rangle = \int_X \partial \bar{\partial} f \wedge \bar{*} \omega = \int_X \partial \bar{\partial} f \wedge \omega = \int_X f \partial \bar{\partial} \omega = 0$$

by applying twice the Stokes theorem. Then, we recall the argument in [21, Lemma 2.3(ii)] for proving that the map

$$\langle \cdot | \omega \rangle : \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \rightarrow \mathbb{C}$$

is injective. Take $\alpha = d\beta \in \text{im } d \cap \Lambda^{1,1} X \cap \ker \langle \cdot | \omega \rangle$. Then

$$\langle \Lambda \alpha | 1 \rangle = \langle \alpha | \omega \rangle = 0,$$

where Λ is the adjoint operator of $\omega \wedge \cdot$ with respect to $\langle \cdot | \cdot \rangle$. Then $\Lambda \alpha \in \ker \langle \cdot | 1 \rangle = \text{im } \Lambda \partial \bar{\partial}$, by extending [16, Corollary 7.2.9] by \mathbb{C} -linearity. Take $u \in \mathcal{C}^\infty(X; \mathbb{C})$ such that $\Lambda \alpha = \Lambda \partial \bar{\partial} u$. Then, by defining $\alpha' := \alpha - \partial \bar{\partial} u$, we have $[\alpha'] = [\alpha] \in \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}}$, and $\Lambda \alpha' = 0$, and $\alpha' = d\beta'$ where $\beta' := \beta - \bar{\partial} u$. In particular, α' is primitive. Since α' is primitive and of type $(1,1)$, then it is anti-self-dual by the Weil identity. Then

$$\|\alpha'\|^2 = \langle \alpha' | \alpha' \rangle = \int_X \alpha' \wedge \bar{*} \alpha' = - \int_X \alpha' \wedge \bar{\alpha}' = - \int_X d\beta' \wedge d\bar{\beta}' = - \int_X d(\beta' \wedge d\bar{\beta}') = 0$$

and hence $\alpha' = 0$, and therefore $[\alpha] = 0$.

Since the space $\frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}}$ is finite-dimensional, being a sub-space of $H_{BC}^{1,1}(X)$, and since the space $\text{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right)$ is finite-dimensional, being a sub-space of $H_{dR}^2(X; \mathbb{C})$, we get that

$$\dim_{\mathbb{C}} \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \leq \dim_{\mathbb{C}} \mathbb{C} = 1,$$

and hence

$$b_2 < \dim_{\mathbb{C}} H_{BC}^{1,1}(X) = \dim_{\mathbb{C}} \text{im} \left(H_{BC}^{1,1}(X) \rightarrow H_{dR}^2(X; \mathbb{C}) \right) + \dim_{\mathbb{C}} \frac{\text{im } d \cap \Lambda^{1,1} X}{\text{im } \partial \bar{\partial}} \leq b_2 + 1.$$

We get that $\dim_{\mathbb{C}} H_{BC}^{1,1}(X) = b_2 + 1$. \square

Finally, we prove that the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ is not injective.

Proposition 2.3. *Let X be a compact complex surface in class VII. Then the natural map $H_{BC}^{1,2}(X) \rightarrow H_{\bar{\partial}}^{1,2}(X)$ induced by the identity is the zero map and not an isomorphism.*

where we have listed the harmonic representatives with respect to the Bott-Chern Laplacian of g .

In particular, the Bott-Chern numbers $\{h_{BC}^{p,q} := \dim_{\mathbb{C}} H_{BC}^{p,q}(X)\}_{p,q \in \{0,1,2\}}$ are

$$\begin{array}{ccccccc} & & & h_{BC}^{0,0} = 1 & & & \\ & & & & h_{BC}^{0,1} = 0 & & \\ h_{BC}^{2,0} = 0 & h_{BC}^{1,0} = 0 & & h_{BC}^{1,1} = 1 & & h_{BC}^{0,2} = 0 & . \\ & h_{BC}^{2,1} = 1 & & & h_{BC}^{1,2} = 1 & & \\ & & & h_{BC}^{2,2} = 1 & & & \end{array}$$

By [19, §2.c], we have

$$H_A^{\bullet,\bullet}(X) = \mathbb{C}\langle 1 \rangle \oplus \mathbb{C}\langle [\varphi^2] \rangle \oplus \mathbb{C}\langle [\varphi^{\bar{2}}] \rangle \oplus \mathbb{C}\langle [\varphi^{2\bar{2}}] \rangle \oplus \mathbb{C}\langle [\varphi^{12\bar{1}\bar{2}}] \rangle ,$$

where we have listed the harmonic representatives with respect to the Aeppli Laplacian of g .

In particular, the Aeppli numbers $\{h_A^{p,q} := \dim_{\mathbb{C}} H_A^{p,q}(X)\}_{p,q \in \{0,1,2\}}$ are

$$\begin{array}{ccccccc} & & & h_A^{0,0} = 1 & & & \\ & & & & h_A^{0,1} = 1 & & \\ h_A^{2,0} = 0 & h_A^{1,0} = 1 & & h_A^{1,1} = 1 & & h_A^{0,2} = 0 & . \\ & h_A^{2,1} = 0 & & & h_A^{1,2} = 0 & & \\ & & & h_A^{2,2} = 1 & & & \end{array}$$

Summarizing, we have the following.

Proposition 2.4. *Let $X := \mathbb{S}^1 \times \mathbb{S}^3$ be endowed with the complex structure of Calabi-Eckmann. The non-zero dimensions of the Dolbeault and Bott-Chern cohomologies are the following:*

$$h_{\bar{\partial}}^{0,0}(X) = h_{\bar{\partial}}^{0,1}(X) = h_{\bar{\partial}}^{2,1}(X) = h_{\bar{\partial}}^{2,2}(X) = 1$$

and

$$h_{BC}^{0,0}(X) = h_{BC}^{1,1}(X) = h_{BC}^{2,1}(X) = h_{BC}^{1,2}(X) = h_{BC}^{2,2}(X) = 1 .$$

Note in particular that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is an isomorphism, and that the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity is injective.

3. COMPLEX SURFACES DIFFEOMORPHIC TO SOLVMANIFOLDS

Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$. By [11, Theorem 1], X is (A) either a complex torus, (B) or a hyperelliptic surface, (C) or a Inoue surface of type \mathcal{S}_M , (D) or a primary Kodaira surface, (E) or a secondary Kodaira surface, (F) or a Inoue surface of type \mathcal{S}^{\pm} , and, as such, it is endowed with a left-invariant complex structure.

In each case, we recall the structure equations of the group G , see [11]. More precisely, take a basis $\{e_1, e_2, e_3, e_4\}$ of the Lie algebra \mathfrak{g} naturally associated to G . We have the following commutation relations, according to [11]:

(A) differentiable structure underlying a *complex torus*:

$$[e_j, e_k] = 0 \quad \text{for any } j, k \in \{1, 2, 3, 4\} ;$$

(hereafter, we write only the non-trivial commutators);

(B) differentiable structure underlying a *hyperelliptic surface*:

$$[e_1, e_4] = e_2 , \quad [e_2, e_4] = -e_1 ;$$

(C) differentiable structure underlying a *Inoue surface of type \mathcal{S}_M* :

$$[e_1, e_4] = -\alpha e_1 + \beta e_2 , \quad [e_2, e_4] = -\beta e_1 - \alpha e_2 , \quad [e_3, e_4] = 2\alpha e_3 ,$$

where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$;

(D) differentiable structure underlying a *primary Kodaira surface*:

$$[e_1, e_2] = -e_3 ;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$[e_1, e_2] = -e_3 , \quad [e_1, e_4] = e_2 , \quad [e_2, e_4] = -e_1 ;$$

(F) differentiable structure underlying a *Inoue surface of type \mathcal{S}^{\pm}* :

$$[e_2, e_3] = -e_1 , \quad [e_2, e_4] = -e_2 , \quad [e_3, e_4] = e_3 .$$

Denote by $\{e^1, e^2, e^3, e^4\}$ the dual basis of $\{e_1, e_2, e_3, e_4\}$. We recall that, for any $\alpha \in \mathfrak{g}^*$, for any $x, y \in \mathfrak{g}$, it holds $d\alpha(x, y) = -\alpha([x, y])$. Hence we get the following structure equations:

(A) differentiable structure underlying a *complex torus*:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = 0 \\ de^4 = 0 \end{cases} ;$$

(B) differentiable structure underlying a *hyperelliptic surface*:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = 0 \\ de^4 = 0 \end{cases} ;$$

(C) differentiable structure underlying a *Inoue surface of type \mathcal{S}_M* :

$$\begin{cases} de^1 = \alpha e^1 \wedge e^4 + \beta e^2 \wedge e^4 \\ de^2 = -\beta e^1 \wedge e^4 + \alpha e^2 \wedge e^4 \\ de^3 = -2\alpha e^3 \wedge e^4 \\ de^4 = 0 \end{cases} ;$$

(D) differentiable structure underlying a *primary Kodaira surface*:

$$\begin{cases} de^1 = 0 \\ de^2 = 0 \\ de^3 = e^1 \wedge e^2 \\ de^4 = 0 \end{cases} ;$$

(E) differentiable structure underlying a *secondary Kodaira surface*:

$$\begin{cases} de^1 = e^2 \wedge e^4 \\ de^2 = -e^1 \wedge e^4 \\ de^3 = e^1 \wedge e^2 \\ de^4 = 0 \end{cases} ;$$

(F) differentiable structure underlying a *Inoue surface of type \mathcal{S}^\pm* :

$$\begin{cases} de^1 = e^2 \wedge e^3 \\ de^2 = e^2 \wedge e^4 \\ de^3 = -e^3 \wedge e^4 \\ de^4 = 0 \end{cases} .$$

In cases (A), (B), (C), (D), (E), consider the G -left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and} \quad Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 \quad \text{and} \quad Je_4 := -e_3 .$$

Consider the G -left-invariant $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + ie^2 \\ \varphi^2 := e^3 + ie^4 \end{cases} .$$

In case (F), consider the G -left-invariant almost-complex structure J on X defined by

$$Je_1 := e_2 \quad \text{and} \quad Je_2 := -e_1 \quad \text{and} \quad Je_3 := e_4 - qe_2 \quad \text{and} \quad Je_4 := -e_3 - qe_1 ,$$

where $q \in \mathbb{R}$. Consider the G -left-invariant $(1, 0)$ -forms

$$\begin{cases} \varphi^1 := e^1 + i e^2 + i q e^4 \\ \varphi^2 := e^3 + i e^4 \end{cases}.$$

With respect to the G -left-invariant coframe $\{\varphi^1, \varphi^2\}$ for the holomorphic tangent bundle $T^{1,0} \Gamma \backslash G$, we have the following structure equations. (As for notation, we shorten, e.g., $\varphi^{1\bar{2}} := \varphi^1 \wedge \bar{\varphi}^2$.)

(A) *torus*:

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = 0 \end{cases}$$

(B) *hyperelliptic surface*:

$$\begin{cases} d\varphi^1 = -\frac{1}{2}\varphi^{12} + \frac{1}{2}\varphi^{1\bar{2}} \\ d\varphi^2 = 0 \end{cases}$$

(C) *Inoue surface \mathcal{S}_M* :

$$\begin{cases} d\varphi^1 = \frac{\alpha-i\beta}{2i}\varphi^{12} - \frac{\alpha-i\beta}{2i}\varphi^{1\bar{2}} \\ d\varphi^2 = -i\alpha\varphi^{2\bar{2}} \end{cases}$$

(where $\alpha \in \mathbb{R} \setminus \{0\}$ and $\beta \in \mathbb{R}$);

(D) *primary Kodaira surface*:

$$\begin{cases} d\varphi^1 = 0 \\ d\varphi^2 = \frac{i}{2}\varphi^{1\bar{1}} \end{cases}$$

(E) *secondary Kodaira surface*:

$$\begin{cases} d\varphi^1 = -\frac{1}{2}\varphi^{12} + \frac{1}{2}\varphi^{1\bar{2}} \\ d\varphi^2 = \frac{i}{2}\varphi^{1\bar{1}} \end{cases}$$

(F) *Inoue surface \mathcal{S}^\pm* :

$$\begin{cases} d\varphi^1 = \frac{1}{2i}\varphi^{12} + \frac{1}{2i}\varphi^{2\bar{1}} + \frac{q}{2}i\varphi^{2\bar{2}} \\ d\varphi^2 = \frac{1}{2i}\varphi^{2\bar{2}} \end{cases}.$$

4. COHOMOLOGIES OF COMPLEX SURFACES DIFFEOMORPHIC TO SOLVMANIFOLDS

In this section, we compute the Dolbeault and Bott-Chern cohomologies of the compact complex surfaces diffeomorphic to a solvmanifold.

We prove the following theorem.

Theorem 4.1. *Let X be a compact complex surface diffeomorphic to a solvmanifold $\Gamma \backslash G$; denote the Lie algebra of G by \mathfrak{g} . Then the inclusion $(\wedge^{\bullet, \bullet} \mathfrak{g}^*, \partial, \bar{\partial}) \hookrightarrow (\wedge^{\bullet, \bullet} X, \partial, \bar{\partial})$ induces an isomorphism both in Dolbeault and in Bott-Chern cohomologies. In particular, the dimensions of the de Rham, Dolbeault, and Bott-Chern cohomologies and the degrees of non-Kählerness are summarized in Table 5.*

Proof. Firstly, we compute the cohomologies of the sub-complex of G -left-invariant forms. The computations are straightforward from the structure equations.

(p, q)	$H_{\mathbb{C}}^{E,q}$	(A) torus			(B) hyperelliptic				(C) Inoue \mathcal{S}_M			
		$\dim_{\mathbb{C}} H_{\mathbb{C}}^{E,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\mathbb{C}}^{E,q}$	$\dim_{\mathbb{C}} H_{\mathbb{C}}^{E,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\mathbb{C}}^{E,q}$	$\dim_{\mathbb{C}} H_{\mathbb{C}}^{E,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0, 0)	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
(1, 0)	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^1, \varphi^2 \rangle$	2	$\langle \varphi^2 \rangle$	1	$\langle \varphi^2 \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(0, 1)	$\langle \bar{\varphi}^1, \bar{\varphi}^2 \rangle$	2	$\langle \bar{\varphi}^1, \bar{\varphi}^2 \rangle$	2	$\langle \bar{\varphi}^2 \rangle$	1	$\langle \bar{\varphi}^2 \rangle$	1	$\langle \bar{\varphi}^2 \rangle$	1	$\langle 0 \rangle$	0
(2, 0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(1, 1)	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}} \rangle$	4	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle 0 \rangle$	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0, 2)	$\langle \bar{\varphi}^{1\bar{2}} \rangle$	1	$\langle \bar{\varphi}^{1\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
(2, 1)	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
(1, 2)	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle 0 \rangle$	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
(2, 2)	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 1. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

(p, q)	(D) primary Kodaira				(E) secondary Kodaira				(F) Inoue S_{\pm}			
	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$	$H_{\bar{\partial}}^{p,q}$	$\dim_{\mathbb{C}} H_{\bar{\partial}}^{p,q}$	$H_{BC}^{p,q}$	$\dim_{\mathbb{C}} H_{BC}^{p,q}$
(0, 0)	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1	(1)	1
(1, 0)	$\langle \varphi^1 \rangle$	1	$\langle \varphi^1 \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(0, 1)	$\langle \varphi^1, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^1 \rangle$	1	$\langle \varphi^{\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{\bar{2}} \rangle$	1	(0)	0
(2, 0)	$\langle \varphi^{12} \rangle$	1	$\langle \varphi^{12} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(1, 1)	$\langle \varphi^{12}, \varphi^{2\bar{1}} \rangle$	2	$\langle \varphi^{1\bar{1}}, \varphi^{12}, \varphi^{2\bar{1}} \rangle$	3	(0)	0	$\langle \varphi^{1\bar{1}} \rangle$	1	(0)	0	$\langle \varphi^{2\bar{2}} \rangle$	1
(0, 2)	$\langle \varphi^{1\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{2}} \rangle$	1	(0)	0	(0)	0	(0)	0	(0)	0
(2, 1)	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1	$\langle \varphi^{12\bar{1}} \rangle$	1
(1, 2)	$\langle \varphi^{2\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	2	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1	(0)	0	$\langle \varphi^{1\bar{1}\bar{2}} \rangle$	1
(2, 2)	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 2. Dolbeault and Bott-Chern cohomologies of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

k	(A) torus		(B) hyperelliptic		(C) Inoue \mathcal{S}_M	
	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^2, \varphi^{\bar{1}}, \varphi^{\bar{2}} \rangle$	4	$\langle \varphi^2, \varphi^{\bar{2}} \rangle$	2	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{1}}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{2\bar{2}}, \varphi^{1\bar{2}} \rangle$	6	$\langle \varphi^{1\bar{1}}, \varphi^{2\bar{2}} \rangle$	2	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{1}}, \varphi^{12\bar{2}}, \varphi^{1\bar{1}\bar{2}}, \varphi^{2\bar{1}\bar{2}} \rangle$	4	$\langle \varphi^{12\bar{1}}, \varphi^{1\bar{1}\bar{2}} \rangle$	2	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 3. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 1.

k	(D) primary Kodaira		(E) secondary Kodaira		(F) Inoue S^{\pm}	
	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$	H_{dR}^k	$\dim_{\mathbb{C}} H_{dR}^k$
0	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1	$\langle 1 \rangle$	1
1	$\langle \varphi^1, \varphi^{\bar{1}}, \varphi^2 - \varphi^{\bar{2}} \rangle$	3	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1	$\langle \varphi^2 - \varphi^{\bar{2}} \rangle$	1
2	$\langle \varphi^{12}, \varphi^{1\bar{2}}, \varphi^{2\bar{1}}, \varphi^{1\bar{2}} \rangle$	4	$\langle 0 \rangle$	0	$\langle 0 \rangle$	0
3	$\langle \varphi^{12\bar{2}}, \varphi^{2\bar{1}\bar{2}}, \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	3	$\langle \varphi^{12\bar{1}} - \varphi^{1\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}} - q\varphi^{12\bar{2}} - \varphi^{1\bar{1}\bar{2}} + q\varphi^{2\bar{1}\bar{2}} \rangle$	1
4	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1	$\langle \varphi^{12\bar{1}\bar{2}} \rangle$	1

TABLE 4. de Rham cohomology of compact complex surfaces diffeomorphic to solvmanifolds, part 2.

In Tables 1 and 2 and in Tables 3 and 4, we summarize the results of the computations. The sub-complexes of left-invariant forms are depicted in Figure 1 (each dot represents a generator, vertical arrows depict the $\bar{\partial}$ -operator, horizontal arrows depict the ∂ -operator, and trivial arrows are not shown.) The dimensions are listed in Table 5.

On the one side, recall that the inclusion of left-invariant forms into the space of forms induces an injective map in Dolbeault and Bott-Chern cohomologies, see, e.g., [7, Lemma 9], [1, Lemma 3.6]. On

the other side, recall that the Frölicher spectral sequence of a compact complex surface X degenerates at the first level, equivalently, the equalities

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{1,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,1}(X) = \dim_{\mathbb{C}} H_{dR}^1(X; \mathbb{C})$$

and

$$\dim_{\mathbb{C}} H_{\bar{\partial}}^{2,0}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{1,1}(X) + \dim_{\mathbb{C}} H_{\bar{\partial}}^{0,2}(X) = \dim_{\mathbb{C}} H_{dR}^2(X; \mathbb{C})$$

hold. By comparing the dimensions in Table 5 with the Betti numbers case by case, we find that the left-invariant forms suffice in computing the Dolbeault cohomology for each case. Then, by [1, Theorem 3.7], see also [2, Theorem 1.3, Theorem 1.6], it follows that also the Bott-Chern cohomology is computed using just left-invariant forms. \square

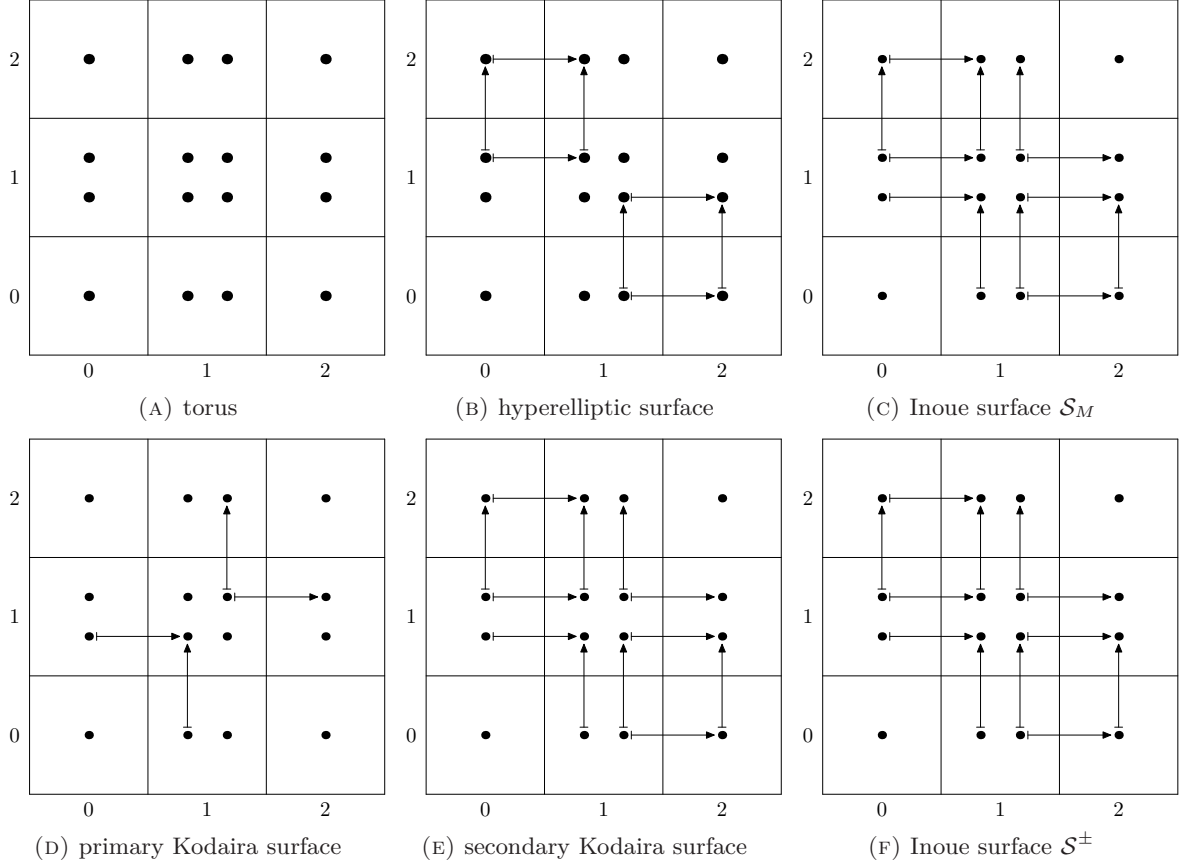


FIGURE 1. The double-complexes of left-invariant forms over 4-dimensional solvmanifolds.

(p, q)	(A) torus				(B) hyperell				(C) Inoue S_M				(D) prim Kod				(E) sec Kod				(F) Inoue S^\pm							
	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k	$h_{\bar{\partial}}^{p,q}$	$h_{BC}^{p,q}$	b_k	Δ^k				
$(0, 0)$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0
$(1, 0)$	2	2	4	0	1	1	2	0	0	0	1	0	1	1	3	0	0	0	1	0	0	0	1	0	0	0	1	0
$(0, 1)$	2	2			1	1			1	0			2	1			1	0			1	0			1	0		
$(2, 0)$	1	1	6	0	0	0	2	0	0	0	0	2	1	1	4	2	0	0	0	2	0	0	0	2	0	0	0	2
$(1, 1)$	4	4			2	2			0	1			2	3			0	1			0	1			0	1		
$(0, 2)$	1	1			0	0			0	0			1	1			0	0			0	0			0	0		
$(2, 1)$	2	2	4	0	1	1	2	0	1	1	1	0	2	2	3	0	1	1	1	0	1	1	1	0	1	1	1	0
$(1, 2)$	2	2			1	1			0	1			1	2			0	1			0	1			0	1		
$(2, 2)$	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0	1	1	1	0

TABLE 5. Summary of the dimensions of de Rham, Dolbeault, and Bott-Chern cohomologies and of the degree of non-Kählerness for compact complex surfaces diffeomorphic to solvmanifolds.

According to Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is injective for any compact complex surface. We are now interested in studying the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity, at least for compact complex surfaces diffeomorphic to solvmanifolds. In fact, by definition, the property of satisfying the $\partial\bar{\partial}$ -Lemma, [8], is equivalent to the natural map $\bigoplus_{p+q=\bullet} H_{BC}^{p,q}(X) \rightarrow H_{dR}^{\bullet}(X; \mathbb{C})$ being injective. Note that, for a compact complex manifold of complex dimension n , the injectivity of the map $H_{BC}^{n,n-1}(X) \rightarrow H_{dR}^{2n-1}(X; \mathbb{C})$ implies the $(n-1, n)$ -th weak $\partial\bar{\partial}$ -Lemma in the sense of J. Fu and S.-T. Yau, [10, Definition 5].

We prove the following result.

Theorem 4.2. *Let X be a compact complex surface diffeomorphic to a solvmanifold. Then the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ induced by the identity is an isomorphism, and the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ induced by the identity is injective.*

Proof. By the general result in Theorem 1.1, the natural map $H_{BC}^{2,1}(X) \rightarrow H_{\bar{\partial}}^{2,1}(X)$ is injective. In fact, it is an isomorphism as follows from the computations summarized in Tables 1 and 2. As for the injectivity of the natural map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$, it is a straightforward computation from Tables 1 and 2 and Tables 3 and 4.

As an example, we offer an explicit calculation of the injectivity of the map $H_{BC}^{2,1}(X) \rightarrow H_{dR}^3(X; \mathbb{C})$ for the Inoue surfaces of type 0, see [13], see also [22]. We will change a little bit the notation. Recall the construction of Inoue surfaces: let $M \in \mathrm{SL}(3; \mathbb{Z})$ be a unimodular matrix having a real eigenvalue $\lambda > 1$ and two complex eigenvalues $\mu \neq \bar{\mu}$. Take a real eigenvector $(\alpha_1, \alpha_2, \alpha_3)$ and an eigenvector $(\beta_1, \beta_2, \beta_3)$ of M . Let $\mathbb{H} = \{z \in \mathbb{C} \mid \Im z > 0\}$; on the product $\mathbb{H} \times \mathbb{C}$ consider the following transformations defined as

$$\begin{aligned} f_0(z, w) &:= (\lambda z, \mu w) \\ f_j(z, w) &:= (z + \alpha_j, w + \beta_j) \quad \text{for } j \in \{1, 2, 3\}. \end{aligned}$$

Denote by Γ_M the group generated by f_0, \dots, f_3 ; then Γ_M acts in a properly discontinuous way and without fixed points on $\mathbb{H} \times \mathbb{C}$, and $\mathcal{S}_M := \mathbb{H} \times \mathbb{C} / \Gamma_M$ is an Inoue surface of type 0, as in case (C) in [11]. Denoting by $z = x + iy$ and $w = u + iv$, consider the following differential forms on $\mathbb{H} \times \mathbb{C}$:

$$e^1 := \frac{1}{y} dx, \quad e^2 := \frac{1}{y} dy, \quad e^3 := \sqrt{y} du, \quad e^4 := \sqrt{y} dv.$$

(Note that e^1 and e^2 , and $e^3 \wedge e^4$ are Γ_M -invariant, and consequently they induce global differential forms on \mathcal{S}_M .) We obtain

$$de^1 = e^1 \wedge e^2, \quad de^2 = 0, \quad de^3 = \frac{1}{2} e^2 \wedge e^3, \quad de^4 = \frac{1}{2} e^2 \wedge e^4.$$

Consider the natural complex structure on \mathcal{S}_M induced by $\mathbb{H} \times \mathbb{C}$. Locally, we have

$$Je^1 = -e^2 \quad \text{and} \quad Je^2 = e^1 \quad \text{and} \quad Je^3 = -e^4 \quad \text{and} \quad Je^4 = e^3.$$

Considering the Γ_M -invariant $(2, 1)$ -Bott-Chern cohomology of \mathcal{S}_M , we obtain that

$$H_{BC}^{2,1}(\mathcal{S}_M) = \mathbb{C} \langle [e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4] \rangle.$$

Clearly $\bar{\partial}(e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4) = 0$ and $e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4 = -2i e^1 \wedge e^3 \wedge e^4 + 2d(e^3 \wedge e^4)$, therefore the de Rham cohomology class $[e^1 \wedge e^3 \wedge e^4 + ie^2 \wedge e^3 \wedge e^4] = -2i [e^1 \wedge e^3 \wedge e^4] \in H_{dR}^3(\mathcal{S}_M)$ is non-zero. \square

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