From locally conformally Kähler to bi-hermitian structures on non-Kähler complex surfaces

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FROM LOCALLY CONFORMALLY KÄHLER TO BI-HERMITIAN STRUCTURES ON NON-KÄHLER COMPLEX SURFACES

VESTISLAV APOSTOLOV, MICHAEL BAILEY, AND GEORGES DLOUSSKY

Abstract. We prove that locally conformally Kähler metrics on certain compact complex surfaces with odd first Betti number can be deformed to new examples of bi-Hermitian metrics.

1. Introduction

A *bi-Hermitian* structure on a complex manifold $S = (M, J)$ consists of a pair $(J_+ = J, J_-)$ of integrable complex structures, inducing the same orientation, each of which is orthogonal with respect to a common riemannian metric g. We are generally only interested in the conformal class $c = [g]$. Furthermore, the case when $J_+ \equiv J_-$ or $J_+ \equiv -J_-$ is considered trivial, so we shall also assume that $J_+(x) \neq \pm J_-(x)$ for at least one point $x \in M$.

Bi-Hermitian geometry has attracted a great deal of interest recently through its link with *generalized Kähler geometry*, a natural extension of Kähler geometry first studied by Gualtieri [22] in the context of generalized complex structures introduced by N. J. Hitchin [27]. It is shown in [22] that a generalized Kähler structure is equivalent to the data of a bi-Hermitian structure (g, J_+, J_-), satisfying the relations

$$d^c_+ F_+ = -d^c_- F_- = dB, \tag{1.1}$$

for some 2-form B, where $F_\pm(\cdot,\cdot) = g(J_\pm\cdot,\cdot)$ are the corresponding fundamental 2-forms of the Hermitian structures $(g, J_\pm)$, and $d^c_\pm = i(\bar\partial_\pm - \partial_\pm)$ are the associated complex operators. We may (trivially) represent a Kähler structure (J, ω) by taking $J_\pm = \pm J, F_\pm = \pm\omega$ while recent work of Goto [20] provides a way to deform Kähler structures to non-Kähler, generalized Kähler structures.

This work is a part of the larger problem of the existence of (conformal classes of) bi-Hermitian structures on compact complex surfaces. In this case, to each bi-Hermitian structure (c, J_+, J_-) on $S = (M, J = J_+)$, one can associate (using the commutator $[J_+, J_-] = J_+J_- - J_+J_-$ and a reference metric $g \in c$) a non-trvial holomorphic section $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}^*_S \otimes \mathcal{L})$ of the anti-canonical bundle $\mathcal{K}^*_S$ of S, twisted with a topologically trivial flat holomorphic line bundle $\mathcal{L}$, see [3, Lemma 3]. Furthermore, bi-Hermitian structures (c, J_+, J_-) on compact real 4-dimensional (connected) manifolds M can be divided into three different classes as follows:

(i) Everywhere on M, $J_+ \neq J_-$ and $J_- \neq -J_-$;

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(ii) Everywhere on M, $J_+ \neq J_-$ (resp. $J_+ \neq -J_-$), but for at least one $x \in M$, $J_+(x) = -J_-(x)$ (resp. $J_+(x) = J_-(x)$, though—by replacing J_- with $-J_-$ if necessary—we can assume without loss of generality that in this class J_+ and J_- never agree but J_+ and $-J_-$ sometimes do);

(iii) There are points on M where $J_+ = J_-$ and also points where $J_+ = -J_-$.

Recall [35, 36, 12, 29] that on a compact complex surface $S = (M, J)$ a Kähler metric exists if and only if the first Betti number is even. Similarly, by [4, Cor. 1 and Prop. 4], a bi-Hermitian conformal structure (c, J_+, J_-) corresponds to a generalized Kähler structure for some $g \in c$ if and only if $b_1(M)$ is even. Furthermore, in this case the flat holomorphic line bundle $\mathcal{L}$ mentioned above is trivial ([3, Lemma 4]) and the bi-Hermitian structures are either of type (i) or (ii) ([3, Prop. 4]). The first case corresponds to Kähler surfaces with trivial canonical bundle (see [3]), i.e. tori and $K3$ surfaces. The classification in the second case follows by [3, 6] and a recent result in [19]: S must be then a Kähler surface of negative Kodaira dimension whose anti-canonical bundle $\mathcal{K}_S^*$ has a non-trivial section and any Kähler metric on $S = (M, J_+)$ can be deformed to a non-trivial bi-Hermitian structure (c, J_+, J_-) of the class (ii).

In the case when S doesn't admit Kähler metrics (i.e. the first Betti number of S is odd), the complex surfaces supporting bi-Hermitian structures in the class (i) are classified in [2].

Finally, another case for which the existence theory is fairly complete by [17, 13] consists of the bi-Hermitian complex surfaces arising from *twisted generalized Kähler* structures, i.e., those for which relation (1.1) is weakened to $d^c_+ F_+ = -d^c_- F_- = H$ for some closed 3-form H: when the deRham class $[H] \in H^3_{dR}(M)$ is trivial, we recover the generalized Kähler case discussed above, while when $[H] \neq 0$ one gets bi-Hermitian structures with $\mathcal{L} \cong \mathcal{O}$ of the class (iii) ([3, Prop. 4]) on compact complex surfaces in the Kodaira class VII ([1, Thm. 1]).

Thus motivated, in this note we narrow our focus to the existence of compatible bi-Hermitian structures of the class (ii) on compact complex surfaces $S = (M, J)$ with odd first Betti number. It is shown in [1] that S then must be a complex surface in the Kodaira class VII (i.e. S has Kodaira dimension $-\infty$ and $b_1(S) = 1$) while [14] provides a complete list of possibilities for the minimal model of S. A more exhaustive taxonomy of bi-Hermitian complex surfaces with odd first Betti number is provided in the appendix A.

One may regard a general bi-Hermitian structure (c, J_+, J_-) on a compact 4-manifold M as relaxing the generalized Kähler compatibility relation (1.1). Even when $b_1(M)$ is odd, a choice of metric in c satisfying (1.1) exists *locally* (see [3, Lemma 1] and [13, Prop. 6]); thus, compatible bi-Hermitian conformal classes on S are always locally conformal to generalized Kähler structures. It turns out that under the assumption (ii), one can further relate the bi-Hermitian structures to locally conformally Kähler metrics, in a similar way that non-Kähler generalized Kähler structures arise as deformations of genuine Kähler ones [19, 20]. This is the context for our main result.

Recall that a *locally conformally Kähler* (or *lcK*) metric on a complex manifold $S = (M, J)$ may be defined by a positive-definite $(1,1)$-form F satisfying $dF = \theta \wedge F$ for a closed 1-form θ. The 1-form θ is uniquely determined and is referred to as *the Lee form* of F. The corresponding Hermitian metric $g(\cdot, \cdot) = F(\cdot, J\cdot)$ defines a conformal class c on M. Changing the Hermitian metric $\tilde{g} = e^f g$ within c amounts to transform the Lee form by $\tilde{\theta} = \theta + df$, showing that the deRham class $[\theta]$ is an invariant of the conformal class c. The study of lcK metrics, which goes back to foundational works by F. Tricerri and I. Vaisman, is a natural extension of the theory of Kähler metrics to certain classes of non-Kählerian complex manifolds, see e.g. [16, 32] for an overview of the theory. Of particular interest is the case of compact complex surfaces, where recent works [8, 10, 11, 21] showed that lcK metric exists for all known (and conjecturally for all) compact complex surfaces with odd first Betti number, with the one exception of certain Inoue surfaces with zero second Betti number described in [8].

Let $S = (M, J)$ be a compact complex surfaces in the class VII. By the well-known isomorphism (see e.g. [7])

$$H^1_{dR}(S, \mathbb{C}) \simeq H^1(S, \mathcal{O}) \stackrel{\exp 2\pi i \bullet}{\longrightarrow} Pic^0(S) \simeq H^1(S, \mathbb{C}^\star) \simeq \mathbb{C}^\star \tag{1.2}$$

for any deRham class $a \in H^1_{dR}(S, \mathbb{C})$ there exists a unique flat holomorphic line bundle $\mathcal{L}_a$ over S. In the case where a is real, i.e., where it belongs to $H^1_{dR}(S, \mathbb{R})$, $\mathcal{L}_a$ is the complexification of a real flat bundle L_a over S, and in the sequel we will tacitly identify $\mathcal{L}_a$ with L_a, referring to such flat holomorphic bundles as being of real type. Then we can make the following conjecture:

Conjecture. Let $S = (M, J)$ be a compact complex surface in the class VII such that $H^0(S, \mathcal{K}^*_S \otimes \mathcal{L}) \neq 0$ for a flat holomorphic bundle of real type $\mathcal{L}$ with $H^0(S, \mathcal{L}^\ell) = 0$ for all $\ell \geq 1$. Then the following two conditions are equivalent:

- There exists a bi-Hermitian structure (g, J_+, J_-) of the class (ii) on (M, J), such that $J = J_+$ and $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}^*_S \otimes \mathcal{L})$.
- There exists a lcK metric with Lee form $-\theta$ whose deRham class in $H^1_{dR}(S, \mathbb{C})$ corresponds to the flat bundle $\mathcal{L}^*$.

The assumptions are justified by the fact that, by [3, Proposition 4], [1, Theorem 1], and the degree computation of [1, p. 561], the two cohomological conditions in the above Conjecture are necessary for the existence of a bi-Hermitian metric satisfying (ii), while $H^0(S, \mathcal{L}^\ell) = 0$ is necessary for the existence of a lcK metric with Lee form corresponding to L^*.

We will establish one direction of the conjectured correspondence by extending, from the Kähler case to the strictly lcK case, certain deformation arguments due to R. Goto [19], N. J. Hitchin [26] and M. Gualtieri [23].

Theorem 1.1. *Let $S = (M, J)$ be a compact complex surface in the class VII such that $H^0(S, \mathcal{K}^*_S \otimes \mathcal{L}) \neq 0$ for a flat holomorphic line bundle of real type $\mathcal{L}$. Let $[\theta] \in H^1_{dR}(S, \mathbb{R})$ be the real deRham class corresponding to $\mathcal{L}$ and suppose that S admits a lcK metric g with Lee form $-\theta$. Then S also admits a bi-Hermitian conformal structure (c, J_+, J_-) with $J_+ = J$ and $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}^*_S \otimes \mathcal{L})$.*

We use Theorem 1.1 to give new examples of bi-Hermitian metrics in the class (ii) on certain Hopf surfaces.

2. Preliminaries

For a closed 1-form θ on M, we denote by $L = L_\theta$ the flat real line bundle determined by the class $[\theta] \in H^1_{dR}(M)$. The differential operator $d_\theta = d - \theta\wedge$ then defines the Novikov complex

$$\cdots \stackrel{d_\theta}{\to} \bigwedge^{k-1} M \stackrel{d_\theta}{\to} \bigwedge^{k} M \stackrel{d_\theta}{\to} \cdots$$

and the corresponding cohomology groups $H^k_\theta(M)$. Let $\mathfrak{U} = (U_i)$ be an open covering such that $\theta_{|U_i} = df_i$. Then, $\mathfrak{U}$ defines a trivialization for L with (constant) transition functions $e^{f_i - f_j}$ on $U_{ij} = U_i \cap U_j$; furthermore, (U_i, e^{-f_i}) defines an isomorphism, denoted by e^{-f}, between the Novikov complex and L-valued de Rham complex

$$\cdots \stackrel{d_L}{\to} \bigwedge^{k-1} M \otimes L \stackrel{d_L}{\to} \bigwedge^{k} M \otimes L \stackrel{d_L}{\to} \cdots$$

which acts at degree k by $e^{-f}(\alpha) = (e^{-f_i}\alpha_{|U_i})$ for any $\alpha \in \Omega^k(M)$, and thus

$$d_\theta = e^{f_i} d_{L|U_i} e^{-f_i}.$$

In particular, we have an isomorphism between the cohomology groups

$$H^k_\theta(M) \simeq H^k(M, L).$$

Considering the Dolbeault cohomology groups of S with values in the flat holomorphic line bundle $\mathcal{L} = L \otimes \mathbb{C}$, we have

$$d_L = \partial_{\mathcal{L}} + \bar\partial_{\mathcal{L}}, \quad \text{and} \quad d_\theta = \partial_\theta + \bar\partial_\theta$$

with

$$\partial_\theta = \partial - \theta^{1,0} \wedge \quad \text{and} \quad \bar\partial_\theta = \bar\partial - \theta^{0,1} \wedge,$$

giving rise to the isomorphisms

$$H^{p,q}_{\bar\partial_\theta}(S) \simeq H^{p,q}(S, \mathcal{L}).$$

Similarly, the space of holomorphic sections $H^0(S, \mathcal{K}^*_S \otimes \mathcal{L})$ can be naturally identified with the space of smooth sections of $\bigwedge^2(T^{1,0}M)$ in the kernel of the twisted Cauchy–Riemann operator

$$\bar\partial_\theta \sigma = \bar\partial\sigma + \theta^{0,1} \otimes \sigma.$$

We shall use the following vanishing result.

Proposition 2.1. *Let S be a compact complex surface in the class VII and $\mathcal{L}$ a flat holomorphic line bundle over S, such that $H^0(S, \mathcal{K}^*_S \otimes \mathcal{L}) \neq 0$ and $H^0(S, \mathcal{L}^{\otimes 2}) = 0$. Then $H^{0,2}(S, \mathcal{L}^*) = 0$. In particular, for any $(0,2)$-form with values in $\mathcal{L}^*$, α, there exists a $(0,1)$-form with values in $\mathcal{L}^*$, β such that $\alpha = \bar\partial_{\mathcal{L}^*}\beta$.*

Proof. As $H^0(S, \mathcal{L}^{\otimes 2}) = 0$ and $H^0(S, \mathcal{K}^*_S \otimes \mathcal{L}) \neq 0$, it follows that $H^0(S, \mathcal{K}_S \otimes \mathcal{L}) = 0$. By Serre duality, $H^2(S, \mathcal{L}^*) \cong H^{0,2}(S, \mathcal{L}^*) = 0$. As $\Omega^{0,3}(M, \mathcal{L}^*) = 0$, $\alpha = \bar\partial_{\mathcal{L}^*}\beta$. □

Remark 2.2. If S is a minimal complex surface in the class VII with $b_2(S) > 0$ and $H^0(S, \mathcal{K}^*_S \otimes \mathcal{L}) \neq 0$, then $H^{0,2}(S, \mathcal{F}) = 0$ for any flat bundle $\mathcal{F}$ by [14], [15, Lemma 2.1] and Serre duality.

Our proof of Theorem 1.1 will rely on the following proposition, which should be regarded as a straightforward generalization of [23, Theorem 6.2], which deals with generalized Kähler structures, to the case of *locally conformal generalized Kähler structures* [37].

Proposition 2.3. [23] *Suppose $S = (M, J)$ is a compact complex surface as in Theorem 1.1, endowed with a holomorphic section $\sigma \in H^0(S, \mathcal{K}^*_S \otimes \mathcal{L})$, where $\mathcal{L}$ is a flat holomorphic line bundle of real type corresponding to a deRham class $[\theta] \in H^1_{dR}(M, \mathbb{R})$. Let $Q = \mathrm{Re}(\sigma)$, and $\omega \in \Omega^2(M, L^*)$ be a d_{L^*}-closed 2-form with values in L^* such that*

- *the J-invariant part of ω is positive definite;*

- *$\omega J - J^*\omega + \omega Q \omega = 0$, where $\omega : TM \to T^*M \otimes L^*$, $Q : T^*M \to TM \otimes L$ and J^* acts on T^*M by $J^*\alpha(\cdot) = -\alpha(J\cdot)$.*

Then,

(i) *$J_- := -J - Q\omega$ is an integrable complex structure on M;*
(ii) *$g = -\frac{1}{2}\omega(J - J_-)$ is a symmetric tensor field with values in L^* which defines a conformal class $c = [g]$ of riemannian metrics on M;*
(iii) *$J_+ := J$ and J_- are orthogonal with respect to c and $J_+(x) \neq J_-(x)$ on M.*

Proof. Let $\mathfrak{U} = (U_i)$ be an open covering such that $\theta_{|U_i} = df_i$. This defines a trivialization for L with positive constant transition functions $(U_{ij}, e^{f_i - f_j})$. We can then write $\omega = (\omega_i)$ and $\sigma = (\sigma_i)$ with respect to $\mathfrak{U}$, with ω_i (resp σ_i) being closed 2-forms (resp. holomorphic Poisson structures) on each U_i such that $\omega_j = e^{f_j - f_i}\omega_i$ (resp. $\sigma_j = e^{f_i - f_j}\sigma_i$). Putting $Q_i = \mathrm{Re}(\sigma_i)$, by [23, Theorem 6.2] the J-invariant part of ω_i gives rise to a bi-Hermitian metric g_i on each U_i with $g_j = e^{f_j - f_i}g_i$. Thus, the conformal structures $([g_i], U_i)$ extend to a global conformal class of riemannian metrics on M; similarly, $(U_i, Q_i\omega_i)$ is a well-defined tensor field on M, showing that J_- is an integrable almost complex structure on M. Finally, in order to verify (iii), suppose that $J_+(x) = J_-(x)$ for some $x \in M$. It follows from (i) that the endomorphism $Q\omega$ of T_xM commutes with J. As the J-invariant part of ω is positive-definite (and therefore non-degenerate) while Q anti-commutes with J, one concludes that Q must vanishes at x. But then, according to (i), $J_-(x) = -J(x) = -J_+(x)$, a contradiction. □

Conversely, the general theory of bi-Hermitian complex surfaces [3] implies

Proposition 2.4. *Any bi-Hermitian structure (c, J_+, J_-) on a compact 4-manifold M, such that $J_+(x) \neq J_-(x)$ for each $x \in M$ and $J_+(x) \neq -J_-(x)$ for at least one point x, arises from Proposition 2.3.*

Proof. With respect to a reference metric $g \in c$, let $F_\pm(\cdot,\cdot) = g(J_\pm\cdot,\cdot)$ denote the fundamental 2-forms of the Hermitian structures (g, J_+) and (g, J_-), respectively, and $\theta_\pm$ the corresponding Lee forms defined by $dF_\pm = \theta_\pm \wedge F_\pm$. By [3, Lemma 1], $\theta_+ + \theta_-$ is closed (as M is compact). Let J denote one of the complex structures, J_+ say, and $S = (M, J)$ the corresponding complex surface. By [3, Lemma 3], the $(1,1)$ tensor

$$\Phi := \frac{1}{2}(J_+J_- - J_-J_+)$$

can be transformed, via the metric g, to a $(2,0)$ tensor Q. The later defines a smooth section σ of $\bigwedge^2(T^{1,0}S)$ with $\mathrm{Re}(\sigma) = Q$, which belongs to the kernel of $\bar\partial_\theta = \bar\partial + \theta^{0,1}\otimes$ with $\theta = \frac{1}{2}(\theta_+ + \theta_-)$. Thus, by the discussion at beginning of this section, σ can be equally seen as an element of $H^0(S, \mathcal{K}_S^* \otimes \mathcal{L})$ where $\mathcal{L} = L \otimes \mathbb{C}$ is the flat holomorphic line bundle of real type, corresponding to the deRham class $[\theta]$.

Letting $p = -\frac{1}{4}\mathrm{trace}(J_+J_-)$, one has (see e.g. [3, Eq. (2)]) that p is a smooth function on M with values in $[-1, 1]$. Furthermore, $p(x) = \pm 1$ if and only if $J_+(x) = \pm J_-(x)$. Thus, our assumption is that $p < 1$ on M, so that the 2-form

$$\omega(\cdot,\cdot) := F_+(\cdot,\cdot) - \frac{1}{1-p}g(\Phi J\cdot,\cdot), \tag{2.1}$$

is well-defined on M and is manifestly self-dual with respect to g. The co-differential of ω has been computed in the proof of [3, Proposition 4] to be $\delta^g\omega(\cdot) = -\frac{1}{2}\omega((\theta_+ + \theta_-)^\sharp, \cdot)$, where $\sharp$ stands for the vector field corresponding to a 1-form via the metric g. As ω is selfdual, the last equality equivalently reads as

$$d\omega = \frac{1}{2}(\theta_+ + \theta_-) \wedge \omega,$$

showing that ω is $d_{-\theta}$ closed and thus can be identified with a d_{L^*}-closed 2-form with values in L^*. Note that the J-invariant part of ω is, by construction, the positive-definite fundamental 2-form F_+. The relations (i) and (ii) between J_+, J_-, g and ω are checked easily. □

3. Proof of Theorem 1.1

We start with a compact complex surface $S = (M, J)$ in the class VII, endowed with a section $\sigma \in H^0(S, \mathcal{K}_S^* \otimes \mathcal{L})$, for a flat holomorphic line bundle $\mathcal{L} = L \otimes \mathbb{C}$, where L is a flat real line bundle corresponding to a class $[\theta] \in H^1_{dR}(S)$. Let F be the fundamental form of a lcK metric on S with Lee form $-\theta$. Thus, F is $d_{-\theta}$ closed, positive definite J-invariant 2-form on S, which we will also identify with a d_{L^*}-closed positive definite J-invariant 2-form with values in L^* (still denoted by F). Note that the degree of $\mathcal{L}$ with respect to a Gauduchon metric g in the conformal class of the lcK structure F is $\deg_g(\mathcal{L}) = -\frac{1}{2\pi}\int_M |\theta|^2 dv_g < 0$ (see e.g. [2, Eq. (5)]), so that $H^0(S, \mathcal{L}^{\otimes \ell}) = 0$, $\forall \ell \geq 1$. It follows that for any $(0,2)$-form with values in $\mathcal{L}^*$, $\alpha = \bar\partial_{\mathcal{L}^*}\beta$, see Proposition 2.1.

We wish to find a family ω_t of closed 2-forms with values in L^*, such that $\omega_0 = 0$, $\dot\omega_0 = F$ and, for sufficiently small t, $(\omega_t, Q = \mathrm{Re}(\sigma))$ satisfy the two conditions of Proposition 2.3. Note that the boundary condition at $t = 0$ for ω_t implies that the J-invariant part of ω_t will be positive definite for t sufficiently small, so we have to deal with the second condition relating ω_t and Q. To this end, we suppose ω_t is expressed as a power series in t,

$$\omega_t = t\omega_1 + t^2\omega_2 + \dots,$$

where each ω_n is a closed real 2-form with values in L^* and $\omega_1 = F$.

The equation

$$\omega_t J - J^*\omega_t + \omega_t Q \omega_t = 0 \tag{3.1}$$

relates to the $(2,0)+(0,2)$ part of ω_t. In other words, it may be expressed as

$$2(J^*\omega_t^{2,0+0,2}) - \omega_t Q \omega_t = 0. \tag{3.2}$$

If we decompose this term-by-term, we have (factoring out t^n)

$$J^*\omega_n^{2,0+0,2} = \frac{1}{2}\sum_{i+j=n} \omega_i Q \omega_j. \tag{3.3}$$

Since $\omega_1 = F$ is $(1,1)$, this is satisfied for $n = 1$. Given ω_i for all $i < n$, (3.3) fixes $\omega_n^{2,0+0,2}$.

Since we need that $d_{L^*}\omega_t = 0$, in particular we must have $\bar\partial_{\mathcal{L}^*}\omega_n^{0,2} = 0$ for the $\omega_n^{0,2}$ thus determined: in complex dimension 2 this is automatic. By Proposition 2.1, there exists a $(0,1)$-form with values in $\mathcal{L}^*$, β_n, such that

$$\partial_{\mathcal{L}^*}\omega_n^{0,2} = \partial_{\mathcal{L}^*}\bar\partial_{\mathcal{L}^*}\beta_n. \tag{3.4}$$

If we let

$$\omega_n^{1,1} = \partial_{\mathcal{L}^*}\beta_n + \bar\partial_{\mathcal{L}^*}\bar\beta_n \tag{3.5}$$

then it follows that $d_{L^*}\omega_n = 0$.

Thus, in order to show that our choices of ω_i satisfy equation (3.3), it remains to be shown that the $(1,1)$-part of (3.3) vanishes, i.e.,

Lemma 3.1. *Given the ω_n as defined above,*

$$\sum_{i+j=n} (\omega_i Q \omega_j)^{1,1} = 0 \tag{3.6}$$

Proof. In other words, we want to show that

$$\sum_{i+j=n} (J^* \omega_i Q \omega_j + \omega_i Q \omega_j J) = 0$$

Since Q anti-commutes with J, i.e. $JQ = -QJ^*$, this is

$$\begin{aligned} &\sum_{i+j=n} (J^* \omega_i Q \omega_j - \omega_i J Q \omega_j - \omega_i Q J^* \omega_j + \omega_i Q \omega_j J) \\ = &\sum_{i+j=n} (J^* \omega_i - \omega_i J) Q \omega_j - \sum_{i+j=n} \omega_i Q (J^* \omega_j - \omega_j J) \end{aligned}$$

Now let us assume, inductively, that (3.6)—or, equivalently, (3.3)—holds for all $n' < n$. Then we make the substitution

$$J^* \omega_i - \omega_i J = \sum_{k+l=i} \omega_k Q \omega_l$$

(and likewise for ω_j), so that we finally get

$$\sum_{i+j=n} (J^* \omega_i Q \omega_j + \omega_i Q \omega_j J) = \sum_{j+k+l=n} \omega_k Q \omega_l Q \omega_j \; - \sum_{i+k+l=n} \omega_i Q \omega_k Q \omega_l = 0.$$

□

In this way we may build a formal power series for a real d_{L^*}-closed form ω_t with values in L^*, which satisfies (3.1). It remains to be shown that this series has a positive radius of convergence. This is rather standard, by using Hodge theory as in [30]. Thus, let g be a Hermitian metric on S (we can take for instance the lcK metric corresponding to F) and h a Hermitian metric on the holomorphic line bundle $\mathcal{L}^*$ (parallel with respect to the flat connection on L^*). Denote by $\bar{\Box}_{\mathcal{L}^*} = \bar{\partial}_{\mathcal{L}^*} \bar{\partial}^*_{\mathcal{L}^*} + \bar{\partial}^*_{\mathcal{L}^*} \bar{\partial}_{\mathcal{L}^*}$ the resulting Laplacian acting on smooth sections of $\wedge^{0,2} S \otimes \mathcal{L}^*$. As $H^{0,2}(S, \mathcal{L}^*) = 0$, $\bar{\Box}_{\mathcal{L}^*}$ is invertible on $C^\infty(\wedge^{0,2} S \otimes \mathcal{L}^*)$ with inverse denoted by $\mathbb{G}$. Then, letting

$$\beta_n = \bar{\partial}^*_{\mathcal{L}^*} \mathbb{G}(\omega_n^{0,2}),$$

(so that β_n manifestly solves (3.4)) we get, for $n > 1$,

$$\omega_n^{1,1} = \partial_{\mathcal{L}^*} \bar{\partial}^*_{\mathcal{L}^*} \mathbb{G}(\omega_n^{0,2}) + \text{complex conjugate},$$

and therefore

$$\omega_n = \omega_n^{0,2} + \partial_{\mathcal{L}^*} \bar{\partial}^*_{\mathcal{L}^*} \mathbb{G}(\omega_n^{0,2}) + \text{complex conjugate},$$

where $\omega_n^{0,2}$ is inductively defined by (3.3). Schauder estimates for the Laplacian imply that in $C^{k,\alpha}(M)$ (for given $k \geq 2, 0 < \alpha < 1$)

$$\|\omega_n\|_{k,\alpha} \leq C_{k,\alpha} \sum_{i+j=n} \|\omega_i\|_{k,\alpha} \|\omega_j\|_{k,\alpha},$$

for some positive constant $C_{k,\alpha}$. We can conclude that the power series $\omega_t = \sum_{n=1}^\infty \omega_n t^n$ converges for small t, by showing as in [30, Chapter 4, Thm. 2.1] that $||\omega_n||_{k,\alpha} \leq$

$\frac{1}{16C_{k,\alpha}}S_n$, where $S_n = \frac{b_{k,\alpha}^n}{n^2}$ with $b_{k,\alpha} = 16C_{k,\alpha}||F||_{k,\alpha}$ (therefore the series converges in $C^{k,\alpha}(M)$ for $t \in [0, 1/b_{k,\alpha})$).

In order to establish smoothness of ω_t, we use elliptic regularity as in [30]. The real 2-form ω_t satisfies $\omega_0 \equiv 0, \omega_1 \equiv F$ as well as the equation (3.2) and

$$\omega_t^{1,1} = tF + 2\mathrm{Re}\{\partial_{\mathcal{L}^*}\bar{\partial}^*_{\mathcal{L}^*}\mathbb{G}\omega_t^{0,2}\}. \tag{3.7}$$

Substituting (3.7) into (3.2) and taking only the $(0,2)$ part for simplicity, we get

$$2i\omega_t^{0,2} = \omega_t^{0,2}Q\omega_t^{0,2} + \Big(\big(tF + 2\mathrm{Re}\{\partial_{\mathcal{L}^*}\bar{\partial}^*_{\mathcal{L}^*}\mathbb{G}\omega_t^{0,2}\}\big)Q\big(tF + 2\mathrm{Re}\{\partial_{\mathcal{L}^*}\bar{\partial}^*_{\mathcal{L}^*}\mathbb{G}\omega_t^{0,2}\}\big)\Big)^{0,2}.$$

Letting $\Theta_t = \mathbb{G}\omega_t^{0,2}$, thus

$$2i\bar{\Box}_{\mathcal{L}^*}\Theta_t = E(\Theta_t), \tag{3.8}$$

where

$$E(\Theta_t) = \Big(\big((\bar{\Box}_{\mathcal{L}^*} + 2\mathrm{Re}\circ\partial_{\mathcal{L}^*}\bar{\partial}^*_{\mathcal{L}^*})\Theta_t + tF\big)Q\big((\bar{\Box}_{\mathcal{L}^*} + 2\mathrm{Re}\circ\partial_{\mathcal{L}^*}\bar{\partial}^*_{\mathcal{L}^*})\Theta_t + tF\big)\Big)^{0,2}$$

is a non-linear second-order differential operator with smooth coefficients acting on sections of $\wedge^{0,2}S \otimes \mathcal{L}^*$. As $\Theta_0 \equiv 0$, it follows that for small $t > 0$, the non-linear equation (3.8) is elliptic at Θ_t, so that Θ_t must be C^∞ (see e.g. [9, p. 467, Thm. 41]).

Remark 3.2. Our method of proof works without change in the case when $S = (M, J)$ is a compact complex surface of Kodaira dimension $-\infty$ with $b_1(M)$ even, endowed with a non-trivial section σ of the anti-canonical bundle $\mathcal{K}_S^*$ and a Kähler metric ω: this formally is the case when $\mathcal{L} = \mathcal{O}$. In this case, using the Hodge isomorphism, we have $H^{0,2}(S) \cong H^0(S, \mathcal{K}_S) = 0$ as the Kodaira dimension is negative. We thus recast [19, Theorem 6.1] entirely within the framework of bi-Hermitian geometry. A more general approach to the deformation theory of generalized Kähler structures of any dimension has been independently developed by M. Gualtieri and N. J. Hitchin [24].

Remark 3.3. We point out that as a by-product of Theorem 1.1, we obtain non-obstructness of the class $[QF] \in H^1(S, T^{1,0}S)$, for any lcK metric F with Lee form corresponding to $\mathcal{L}^*$, should it exist.

4. Towards a converse

In order to further motivate the conjecture in the introduction, recall that by Proposition 2.4, any bi-Hermitian structure (c, J_+, J_-) on $S = (M, J)$ with $J = J_+$ and $J_+(x) \neq J_-(x)$ for each $x \in M$, gives rise to a d_{L^*}-closed form ω whose J-invariant part is positive-definite: in other words, J is tamed by a locally conformally symplectic 2-form ω with Lee form corresponding to L^*. Note that the flat line bundle L^* is the dual of the flat bundle L for which $\sigma \in H^0(S, \mathcal{K}_S^* \otimes \mathcal{L})$. As computed in [1] (see also [2, Eq. (6)]), another necessary condition for L^* is that $\deg_g(\mathcal{L}^*) > 0$ with respect to the Gauduchon metric of (c, J), in particular $H^0(S, \mathcal{L}^\ell) = 0$ for $\ell \geq 1$. On a given minimal complex surface in the Class VII with a global spherical shell and second Betti number $b_2(M) > 0$, one can show that there is a finite number of such line bundles L. We therefore ask the following more general

Question 4.1. Let $S = (M, J)$ be a minimal compact complex surface in the class VII, with a global shperical shell, and $\mathcal{L}$ a flat holomorphic line bundle of real type such that $H^0(S, \mathcal{L}^\ell) = 0$ for any $\ell \geq 1$. Suppose there exists a d_{L^*}-closed 2-form with values in L^*, ω, whose J-invariant part is positive definite and denote $\Omega = [\omega] \in H^2(M, L^*)$ the corresponding Novikov cohomology class. Does Ω contain a positive-definite J-invariant 2-form F?

Note that the vanishing $H^0(S, \mathcal{L}^\ell) = 0$ for $\ell \geq 1$ is a necessary condition as the degree of $\mathcal{L}$ with respect to a Gauduchon metric g in the conformal class of the lcK structure is $\deg_g(\mathcal{L}) = -\frac{1}{2\pi}\int_M |\theta|^2 dv_g < 0$ (see e.g. [2, Eq. (5)]).

For a compact complex surface with $b_1(S)$ even and $H^0(S, \mathcal{K}_S) = 0$ (and $\mathcal{L} = \mathcal{O}$), the analogous statement is known to be true as $H^2_{dR}(S) \cong H^{1,1}(S, \mathbb{R})$ and therefore the deRham class $\Omega = [\omega]$ of a symplectic form taming J defines a Kähler class by a result of Buchdahl [12] and Lamari [29].

Similar question has been raised in [10, Remark 1] and in [33]. The general existence results in [10, 11] show that lcK metrics exist for L^* corresponding to an interval of *big* Lee forms in $H^1_{dR}(S)$ (i.e. for Lee forms with deRham classes ta, $0 \neq a \in H^1_{dR}(M)$, $t > \varepsilon(S) \gg 0$) while the stability results in [18, 10] imply that the existence of lcK metrics in $H^2(M, L^*)$ is stable under complex deformations of S.

5. Examples

In this section we give new examples of bi-Hermitian metrics on primary Hopf surfaces S, such that $J_+(x) \neq J_-(x)$ on S and $J_+ = -J_-$ on an elliptic curve $E \subset S$. To the best of our knowledge, these are new: Indeed, they are not strongly bi-Hermitian (as $J_+ = -J_-$ on E) nor are they ASD (see [34]) as $J_+ \neq J_-$ everywhere. According to [13], these examples generate bi-Hermitian structures of the same kind on the blow-ups of S at E.

Recall that a *diagonal* primary Hopf surface S is defined as the quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by a contraction

$$\gamma(z_1, z_2) = (a_1 z_1, a_2 z_2), \quad 0 < |a_1| \leq |a_2| < 1. \tag{5.1}$$

Letting $a = (a_1, a_2)$ we denote by S_a the resulting diagonal Hopf surface. As $S_a \cong S^1 \times S^3$, any holomorphic line bundle is topologically trivial. The diagonal Hopf surface S_a admits two elliptic curves, E_1 and E_2, which are respectively the projections of the axes $\{z_1 = 0\}$ and $\{z_2 = 0\}$ in $\mathbb{C}^2$ under the contraction (5.1). A holomorphic section of the anti-canonical bundle $\mathcal{K}^*_{S_a}$ is induced by the γ-invariant bi-vector

$$z_1 z_2 \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2},$$

showing that

$$\mathcal{K}^*_{S_a} = [E_1 + E_2].$$

Without loss of generality, we can choose the identification (1.2) so that $\mathcal{K}^*_{S_a}$ corresponds to $a_1 a_2 \in \mathbb{C}^*$ and thus $[E_i]$ corresponds to a_i. It follows that the flat bundles

$$\mathcal{L}_{p_1,p_2} \cong p_1[E_1] + p_2[E_2], \; p_i \geq 0,$$

which correspond to $a_1^{p_1} a_2^{p_2}$ under (1.2), all admit holomorphic sections while $\mathcal{K}^*_{S_a} \otimes \mathcal{L}_{p_1,p_2}$ has a non-trivial sections for $p_i \geq -1$. Furthermore, it is not difficult to show (see e.g. [2, Lemma 4]) that $\mathcal{L}_{p_1,p_2}$ is of real type if and only if $a_1^{p_1} a_2^{p_2}$ is a real number. Finally, the condition $H^0(S_a, \mathcal{L}^\ell_{p_1,p_2}) = 0$ implies $p_1 \leq -1$ or $p_2 \leq -1$. We thus get two families of flat bundles $\mathcal{L}_p = \mathcal{L}_{-1,p}$, $p \geq -1$ (resp. $\tilde{\mathcal{L}}_q = \mathcal{L}_{q,-1}, q \geq -1$) possibly satisfying the necessary conditions in Theorem 1.1, subject to the constraint $a_2^p / a_1 \in \mathbb{R}$

(resp. $a_1^q/a_2 \in \mathbb{R}$). It remains to investigate whether or not there is a lcK metric on S_a in $H^2(S_a, L_p^*)$ or $H^2(S_a, \tilde{L}_q^*)$.

In the case $p = -1$ (or equivalently $q = -1$), the existence of lcK structure in $H^2(S_a, \mathcal{L}_p^*) = H^2(S_a, \mathcal{K}_{S_a}^*)$ (with $a_1 a_2 \in \mathbb{R}$) is established in [21] while the existence of (strongly) bi-Hermitian deformations was observed in [2]. So we shall assume $p, q \geq 0$.

It is shown in [21] that any diagonal Hopf surface S_a admits a Vaisman lcK metric, i.e. a lcK Hermitian metric g_0 whose Lee form θ_0 is parallel. As observed in [2], the deRham class $[\theta_0]$ of the Gauduchon–Ornea lcK metrics corresponds to the real number $c_0 = |a_1||a_2|$. The Vaisman lcK metrics always come in families, called 0-type deformation in [8],

$$g_t = g_0 + \frac{(t-1)}{|\theta_0|^2}(\theta_0 \otimes \theta_0 + J\theta_0 \otimes J\theta_0),\ t > 0,$$

with Lee forms $\theta_t = t\theta_0$, so that the deRham class of $t\theta_0$ corresponds to c_0^t (see e.g. [8, Eq. [7)] or [5, Eq. (21) & (22)]). It follows that for each flat bundle of real type, $\mathcal{L}_\mu$, corresponding (via (1.2)) to a positive real number $\mu > 1$, there exists a lcK metric with Lee form corresponding to L_μ^*. Thus, if for some $p \geq 0$ we have $a_2^p/a_1 > 1$ (resp. for some $q \geq 0$ we have $a_1^q/a_2 > 1$), we can apply Theorem 1.1 with $\mathcal{L} = \mathcal{L}_p$ (resp. $\mathcal{L} = \tilde{\mathcal{L}}_q$) in order to construct bi-Hermitian metrics metrics on $S^1 \times S^3$. As a special case, we can take $a_1 = a_2 = \lambda \in \mathbb{R}$ and $\mathcal{L} = \mathcal{L}_0 \cong \tilde{\mathcal{L}}_0$ (i.e. $p = 0 = q$).

Another class of (primary) Hopf surfaces are the non-diagonal ones, when $S = S_{b,\lambda,m}$ is obtained as a quotient of $\mathbb{C}^2 \setminus \{(0,0)\}$ by the contraction

$$\gamma(z_1, z_2) = (b^m z_1 + \lambda z_2^m, bz_2),\ 0 < |b| < 1, \lambda \neq 0, m \geq 1.$$

The deformation argument in [21] shows that such S still admits lcK metrics in each $H^2(S, L_\mu^*)$ where L_μ is a flat bundle corresponding to $\mu > 1$. Furthermore, the axe $z_2 = 0$ of $\mathbb{C}^2$ defines an elliptic curve $E \subset S$ with $\mathcal{K}_S^* \cong (m+1)E$ and corresponding complex number b^{m+1}. If b is real, then Theorem 1.1 applies for the flat bundles $\mathcal{L} = \mathcal{L}_p \cong -p[E],\ 0 \leq p \leq m$, corresponding to b^{-p}.

Remark 5.1. It is well-known (see e.g. [32]) that the Vaisman lcK metrics we have used to produce our examples of bi-Hermitian metrics admit potentials, i.e. there exists a smooth (real) section f of L^* such that $F = 2i\partial_{\mathcal{L}^*}\bar{\partial}_{\mathcal{L}^*} f = d_{L^*} d^c_{L^*} f$. This allows to construct bi-Hermitian metrics via a hamiltonian flow, originally due to Hitchin [25], and re-casted in the case of Hopf surfaces in our previous work [2] (in order to obtain strongly bi-Hermitian metrics). Indeed, for any such potential f, $X_f = Q(df)$ is a smooth vector field on M whose flow φ_s defines a family of 2-forms

$$\omega_t := \int_0^t \varphi_s^*(d_{L^*} d^c_{L^*} f) ds,$$

satisfying the second relation in Proposition 2.3 and $\left(\frac{d\omega_t}{dt}\right)_{t=0} = d_{L^*} d^c_{L^*} f = F$ (see [23]). Thus, we obtain a familly of bi-Hermitian structures with $J_+ = J$ and $J_-^t = -\varphi_t(J)$.

Appendix A. A rough classification of bi-Hermitian complex surfaces in the class VII_0

In this section, we recast the list obtained in [14] of the minimal compact complex surfaces $S = (M, J)$ in the Kodaira class VII, possibly admitting compatible bi-Hermitian structures, in terms of the classification of bi-Hermitian structures in the three classes (i)–(iii) from the introduction (defined as a function of the number of connected components of the divisor determined by $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}_S^* \otimes \mathcal{L})$).

As in [14], we will assume that the algebraic dimension of S is zero, i.e. that there are no non-constant meromorphic functions on S, and that the fundamental group $\pi_1(S) \cong \mathbb{Z}$. These assumptions exclude only the cases of elliptic primary Hopf surfaces (i.e. diagonal Hopf surfaces with $a_1^{p_1} = a_2^{p_2}$ for some $p_1, p_2 \in \mathbb{Z}$, see [28]), and the secondary Hopf surfaces (which are finitely covered by a primary Hopf surface described in Section 5). We then have the following

Proposition A.1. *Let $S = (M, J)$ be a minimal compact complex surface in the Kodaira class VII of algebraic dimension* 0*, endowed with a compatible bi-Hermitian structure $(c, J_+ = J, J_-)$. By replacing S with a finite covering if necessary, assume also the fundamental group of S is $\mathbb{Z}$. Then one of the following must hold.*

(i) *Everywhere on M, $J_+ \neq J_-$ and $J_- \neq -J_-$. Then $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}_S^* \otimes \mathcal{K}_S)$ never vanishes, and S is a primary Hopf surface described in* [2, Thm. 1].

(ii) *Everywhere on M, $J_+ \neq J_-$ but for at least one $x \in M$, $J_+(x) = -J_-(x)$. Then $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}_S^* \otimes \mathcal{L})$ with $\mathcal{L} \neq \mathcal{O}$, and S must be either a primary Hopf surface, a parabolic Inoue surface, or a surface with GSS of intermediate type.*

(iii) *There are points on M where $J_+ = J_-$ and also points where $J_+ = -J_-$. Then $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}_S^*)$ and S must be a primary Hopf surface, a parabolic Inoue surface or an even Inoue-Hirzebruch surface.*

Proof. The case (i) is treated in [2].

We will next establish (iii). As the zero set of the holomorphic section $\sigma = [J_+, J_-]^\sharp \in H^0(S, \mathcal{K}_S^* \otimes \mathcal{L})$ consists of the points where either $J_+(x) = J_-(x)$ or $J_+(x) = -J_-(x)$ (see e.g. [3]), we conclude that $\mathcal{K}_S^* \otimes \mathcal{L}$ is represented by an effective divisor D with at least two connected components. By [14, Thm. 0.4], S must be then either a primary Hopf surface, a (parabolic) Inoue surface, or an Inoue–Hirzebruch surface. The Inoue–Hirzebruch surfaces come in two families, called *even* or *odd* in [14], and it has been already observed in the proof of [14, Cor. 3.45] that the odd Inoue–Hirzebruch surfaces cannot appear as the effective divisor representing $\mathcal{K}_S^* \otimes \mathcal{L}$ for some flat bundle $\mathcal{L}$ is connected (it is given by one cycle of rational curves). It remains therefore to show that in the three cases for S, we must have $\mathcal{L} = \mathcal{O}$. Suppose for contradiction that $\mathcal{L} \neq \mathcal{O}$: by the degree computation in [1], we must then have $H^0(S, \mathcal{L}^\ell) = 0$ for $\ell \geq 1$.

If S is a diagonal Hopf surface as in Section 5 with $[D] \cong \mathcal{K}_S^* \otimes \mathcal{L}$ and $H^0(S, \mathcal{L}^\ell) = 0$ for $\ell \geq 1$, we have already noticed that $[D] \cong (p+1)[E_2], p \geq 0$ or $[D] \cong (q+1)[E_1], q \geq 0$. As we assume that S doesn't have non-trivial meromorphic functions, it follows $D = (p+1)E_2$ or $D = (q+1)E_1$, a contradiction as D has at least two connected components. Similarly, if S is a non-diagonal primary Hopf surface, then $\mathcal{K}_S^* \cong (m+1)[E]$ and therefore $D = (p+1)E, p \geq 0$, a contradiction.

If S is a parabolic Inoue surface or an even Inoue-Hirzebruch surface, then $\mathcal{K}_S^* \cong [A + B]$ for a cycle A of rational curves and a smooth elliptic curve B, or for two cycles of rational curves A, B, respectively (see [14, Prop. 2.7] and [31]). In the first case, the cycle A represents a flat bundle, and therefore, by [14, Lemma 2.26], we must have $D = B$, a contradiction as B is connected; in the second case, neither A nor B represents a flat bundle, so we obtain a contradiction as we assumed $\mathcal{L} \neq \mathcal{O}$.

Finally, we consider the case (ii): the only additional point with respect to [14, Thm. 0.4] is that Inoue–Hirzebruch surfaces cannot support bi-Hermitian structures of the class (ii). Indeed, as we have already explained in the introduction, a necessary condition for the existence of such bi-Hermitian structures is that $H^0(S, \mathcal{K}_S^* \otimes \mathcal{L}) \neq 0$ for a non-trivial flat bundle with $H^0(S, \mathcal{L}^\ell) = 0$ for $\ell \geq 1$. For the odd Inoue–Hirzebruch surfaces, $H^0(S, \mathcal{K}_S^* \otimes \mathcal{L}) \neq 0$ for a (unique) a flat bundle $\mathcal{L}$ which satisfy $\mathcal{L}^2 \cong \mathcal{O}$ while for the even Inoue–Hirzebruch surfaces the only flat bundle $\mathcal{L}$ with $H^0(S, \mathcal{K}_S^* \otimes \mathcal{L}) \neq 0$ is the trivial one, [31, 14]. □

Remark A.2. The general existence problem for bi-Hermitian structures can be reduced to the minimal case by [14, Lemma 3.43] and the construction in [13]. From the list above, the existence is now fully established in the case (i) by [2] and in the case (iii) by [17]. The construction in this paper provides the first existence results in the case (ii), but a complete resolution is still to come. We also note (see [4, Prop. 3]) that the bi-Hermitian minimal complex surfaces in the class (iii) are precisely the ones arising from twisted generalized Kähler structures.

References

[1] V. Apostolov, *Bihermitian surfaces with odd first Betti number*, Math. Z. **238** (2001), 555–568.

[2] V. Apostolov and G. Dloussky, *Bihermitian metrics on Hopf surfaces*, Math. Res. Lett. **15** (2008), 827–839.

[3] V. Apostolov, P. Gauduchon, and G. Grantcharov, *Bihermitian structures on complex surfaces*, Proc. London Math. Soc. (3) **79** (1999), 414–428. *Corrigendum*, **92** (2006), 200–202.

[4] V. Apostolov, M. Gualtieri, *Generalized Kähler manifolds, commuting complex structures and split tangent bundles*, Commun. Math. Phys. **271** (2007), 561–575.

[5] V. Apostolov and O. Muškarov, *Weakly Einstein Hermitian surfaces*, Ann. Inst. Fourier (Grenoble) **49** (1999), 1673–1692.

[6] C. Bartocci and E. Macrì, *Classification of Poisson surfaces*, Comm. Contemp. Math. **7** (2005), 89–95.

[7] W. Barth, K. Hulek, C. Peters, and A. Van de Ven, Compact Complex Surfaces, *Springer, Heidelberg, Second Edition, 2004.*

[8] F. Belgun, *On the metric structure of non-Kähler complex surfaces*, Math. Ann. **317** (2000), 1–40.

[9] A. L. Besse, *Einstein manifolds*, Ergeb. Math. Grenzgeb. **3**, Springer-Verlag, Berlin, Heidelberg, New York, 1987.

[10] M. Brunella, *Locally conformally Kähler metrics on certain non-Kählerian surfaces*, Math. Ann. (2010) **346** 629–639.

[11] M. Brunella, *Locally conformally Kähler metrics on Kato surfaces*, Nagoya Math. J. **202**, 77–81.

[12] N. Buchdahl, *On compact Kähler surfaces*, Ann. Inst. Fourier **49** (1999), 287–302.

[13] G. Cavalcanti and M. Gualtieri, *Blowing up generalized Kähler* 4*-manifolds*, Bull. Braz. Math. soc. New Sreies **42**, 537–557.

[14] G. Dloussky, *On surfaces of class VII_0^+ with numerically anticanonical devisor*, Amer. J. Math. **128** (2006), 639–670.

[15] G. Dloussky, K. Oeljeklaus, and M.Toma, *Class VII_0 surfaces with b_2 curves*, Tohoku Math. J. **55** (2003), 283–309.

[16] S. Dragomir and L. Ornea, Locally coformal Kähler geometry, *Birkhauser 1998.*

[17] A. Fujiki, M. Pontecorvo *Anti-self-dual bihermitian structures on Inoue surfaces*, J. Differential Geom. **85** (2010), 15–71.

[18] R. Goto *On the stability of locally conformal Khler structures*, available at arxiv 1012.2285v1 (2010).

[19] R. Goto, *Unobstructed K-deformations of generalized complex structures and bi-Hermitian structures*, Adv. Math. **231** (2012), 1041–1067.

[20] R. Goto, *Deformations of generalized complex and generalized Kähler structures*, J. Differential Geom. **84** (2010), 525–560.

[21] P. Gauduchon and L. Ornea *Locally conformally Kähler metrics on Hopf surfaces*, Ann. Inst. Fourier (Grenoble), **48** (1998), 11071127.

[22] M. Gualtieri, *Generalized Kähler geometry*, available at arXiv:1007.3485.

[23] M. Gualtieri, *Branes on Poisson varieties*, in "The Many facets of Geometry: A Tribute to Nigel Hitchin" (eds. J.-P. Bourguignon, O. Garcia-Prada and S. Salamon), Oxford University Press, 2010.

[24] M. Gualtieri, N. J. Hitchin, private communication.

[25] N. J. Hitchin, *Bihermitian metrics on del Pezzo surfaces*, J. Symplectic Geom. **5** (2007), 1–8.

[26] N. J. Hitchin, *Deformations of holomorphic Poisson manifolds*, Mosc. Math. J. **12** (2012), 567–591, available at arXiv:1105.4775.

[27] N. J. Hitchin, *Generalized Calabi-Yau manifolds*, Q. J. Math. **54** (2003), 281–308.

[28] K. Kodaira, *On the structure of complex analytic surfaces II, III*, Am. J. Math **86** (1966), 751–798; Am. J. Math. **88** (1966), 682–721.

[29] A. Lamari, *Courants kählériens et surfaces compactes*, Ann. Inst. Fourier **49** (1999), 263–285.

[30] J. Morrow, K. Kodaira, Complex manifolds. *Holt, Rinehart and Winston, Inc., New York-Montreal, Que.-London*, 1971.

[31] I. Nakamura, *On surfaces of class VII_0 with curves*, Invent. math. **78** (1984), 393–343.

[32] L. Ornea and M. Verbitsky *A report on locally conformally Kähler manifolds. Harmonic maps and differential geometry*, 135–149, Contemp. Math. **542**, Amer. Math. Soc., Providence, RI, 2011.

[33] L. Ornea and M. Verbitsky, *Morse-Novikov cohomology of locally conformally K̈hler manifolds*, J. Geom. Phys. **59** (2009), 295–305.

[34] M. Pontecorvo, *Complex structures on Riemannian* 4*-manifolds*, Math. Ann. **309** (1997), 159–177.

[35] Y.-T. Siu, *Every $K3$ surface is Kähler*, Invent. Math. **73** (1983), 139–150.

[36] A. Todorov, *Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of $K3$ surfaces*, Invent. Math. **61** (1980), 251–265.

[37] I. Vaisman, *Conformal changes of generalized complex structures*, An. Sti. Univ. "Al. I. Cuza" Iasi, Sect. I, Mat. (N.S.), **54** (2008), 1-14.

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