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POSITIVE CONES IN BANACH SPACES

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ABSTRACT. We discuss the Haar negligibility of the positive cone associated with a basic sequence in a separable Banach space. In particular, we show that up to equivalence, the canonical basis of c_0 is the only normalized subsymmetric unconditional basic sequence whose positive cone is not Haar null, and the only normalized unconditional basic sequence whose positive cone contains a translate of every compact set. We also show that an unconditional basic sequence with a non–Haar null positive cone has to be c_0-saturated in a very strong sense, and that every quotient of the space generated by such a sequence is c_0-saturated.

1. INTRODUCTION

The main topic of this paper is *Haar negligibility*, a quite well-known but still rather mysterious notion of smallness introduced by J. P. R. Christensen [Chr1] in the 1970's (and rediscovered much later in [HSY]). A Borel subset A of a Polish abelian group G is said to be *Haar null* if there exists a (Borel) probability measure μ on G such that $\mu(A + x) = 0$ for every $x \in G$. Any such measure μ is called a *test measure* for A.

If the group G is locally compact, Haar negligibility turns out to be equivalent to negligibility with respect to the Haar measure of G. Haar null sets always have empty interior, and they form a σ-ideal, i.e. any countable union of Haar null sets is again Haar null. Also, a "Pettis' Lemma" is available: if a Borel set A is not Haar null, then $A - A$ is a neighbourhood of 0; in particular, every compact subset of G is Haar null if G is not locally compact. Finally, it follows from the inner regularity of measures that if a Borel set $A \subset G$ contains a translate of every compact subset of G, then A is not Haar null. (This is essentially the only known way of showing that a given set is not Haar null). We refer to [BeLi, Chapter 6] for more information.

In the present paper, we study Haar negligibility in the framework of separable Banach spaces.

Our work is motivated by beautiful results obtained by Matouskova and Stegall, which provide the following surprising link between the geometry of a Banach space and the Haar negligibility of its closed convex subsets with empty interior: *A separable Banach space X reflexive if and only if every closed convex subset of X with empty interior is Haar null.* More precisely: *If X is reflexive then every closed convex subset of X with*

empty interior is Haar null ([M3]); *and if X is not reflexive, there exists a closed convex subset of X with empty interior which contains a translate of every compact subset of X* ([MS]).

Thus, we see that as far as Haar negligibility is concerned, closed convex sets in reflexive Banach spaces are "uninteresting". However, this does not rule out the possibility that closed convex sets with empty interior taken from an interesting specific class turn out to be Haar null in some nonreflexive Banach spaces, and non–Haar null in some others.

With this in mind, a quite natural class of examples to consider is that of *positive cones* associated with basic sequences. If $\mathbf{e} = (e_i)_{i \geq 1}$ is a basic sequence in a Banach space X, the positive cone associated with $\mathbf{e}$ is the set of all $x \in X$ which can be written as $x = \sum_1^\infty x_i e_i$ with *nonnegative* coefficients x_i. This positive cone will be denoted by $Q^+(\mathbf{e})$. (That $Q^+(\mathbf{e})$ has indeed empty interior is easy to check).

In some sense, the real starting point of our study is the following well-known fact: the positive cone of c_0 is not Haar-null, whereas the positive cone of ℓ_1 *is* Haar null. More generally, positive cones show-up naturally in the study of Haar negligibility, and play a crucial role in the proofs of the Matouskova and Stegall results. They are the main objects of study in the present paper.

Throughout the paper, we shall say that a positive cone $Q^+(\mathbf{e})$ in a Banach space X *is Haar null* if it is Haar null in $[\mathbf{e}]$, the closed subspace of X generated by $\mathbf{e}$.

Ideally, one would like to characterize the basic sequences $\mathbf{e}$ whose positive cone is Haar null. (Note that since Haar negligibility is preserved under linear isomorphisms, this can be done only up to equivalence). However, this seems rather too ambitious, and we mostly concentrate on *unconditional* basic sequences.

Note that if $\mathbf{e}$ is unconditional then $Q^+(\mathbf{e}) - Q^+(\mathbf{e}) = [\mathbf{e}]$; so one would rather expect positive cones to be non–Haar null. Nevertheless, all our results suggest that the following "conjecture" might be true: *up to equivalence, the only unconditional normalized basic sequence whose positive cone is not Haar null is the canonical basis of c_0.*

We have been unable to prove this in full generality, but we believe that some of our results do support this conjecture rather convincingly. The most "important" ones seem to be the following.

- The conjecture holds true for *subsymmetric* unconditional basic sequences, and also for all unconditional basic sequences if "non–Haar negligibility" is replaced by the stronger property "to contain a translate of every compact set".
- A normalized unconditional basic sequence $\mathbf{e}$ whose positive cone is not Haar null has to be "c_0-saturated" in quite a strong sense: every normalized block-sequence of $\mathbf{e}$ has a subsequence equivalent to the canonical basis of c_0.
- If a Banach space X has an unconditional basis whose positive cone is not Haar null, then every quotient of X is c_0-saturated.

The paper is organized as follows.

In Section 2, we introduce the notions and tools which will be used throughout the paper, and we recall some known results concerning the Haar negligibility of positive cones.

In Section 3 we give a characterization of those basic sequences whose positive cone contains a translate of every compact set, and we use it to show that the canonical basis of c_0 is (up to equivalence) the only normalized unconditional basic sequence with this property. We also observe that unconditionality cannot be dispensed with, one counterexample being the positive cone of the classical James space.

In Section 4 we obtain several results concerning unconditional basic sequences whose positive cone is not Haar null. We show in 4.1 that any block-sequence of such a sequence $\mathbf{e}$ also has a non–Haar null mositive cone, from which it follows that all block-sequences of $\mathbf{e}$ are both shrinking and non–boundedly complete. In 4.2, we isolate a simple necessary condition for the non–Haar negligibility of a positive cone, which turns out to be crucial in our study. We use this condition to prove in a very elementary way that the positive cone of the Schreier space is Haar null, and that the canonical basis of c_0 is the only *symmetric* sequence with a non–Haar null positive cone. The c_0-saturation property stated above is established in 4.3, and the result concerning subsymmetric sequences follows immediately. In 4.4, we prove a stronger c_0-saturation result. However, in 4.5 we give an example showing that this kind of c_0-saturation is not even sufficient to ensure the non–Haar negligibility of the positive cone.

In Section 5 we show that non–Haar negligibility of the positive cone associated with an unconditional basis entails c_0-saturation of the quotients.

Section 6 is independent of the rest of the paper. We give simple examples of Haar null positive cones admitting a *Gaussian* test measure. On the other hand, we observe that the positive cone associated to an arbitrary Schauder basis is never *Gauss null*.

Finally, we list a few natural questions in Section 7.

This work is based on Chapter 3 of the third author's PhD thesis [Mo].

2. Background

2.1. **Notations and terminology.** We start by fixing some notations and recalling a few quite well-known notions from Banach space theory. Our references are [LiTz] and [AK].

Throughout the paper, the symbol X will denote a Banach space with norm $\| \cdot \|$. As a rule, all Banach spaces are real, separable and infinite-dimensional.

For any family of vectors $\mathbf{e} = (e_i)_{i \in I}$ in X, we denote by $[e_i; \ i \in I]$, or simply by $[\mathbf{e}]$, the closed linear span of $\{e_i; \ i \in I\}$.

A sequence $\mathbf{e} = (e_i)_{i \geq 1} \subset X$ is a *basic sequence* if it is a Schauder basis of its closed linear span $[\mathbf{e}]$, i.e. every $x \in [\mathbf{e}]$ can be written in a unique way as $x = \sum_{i \geq 1} x_i e_i$, where $x_i \in \mathbb{R}$ and the series is norm-convergent. If, further, every (convergent) series $\sum x_i e_i$ is unconditionnally convergent, the basic sequence $\mathbf{e}$ is said to be *unconditional*. A basic

sequence (e_i) is said to be *normalized* if $\|e_i\| = 1$ for all i, and *semi-normalized* if it is bounded and $\inf_i \|e_i\| > 0$.

If $\mathbf{e} = (e_i)$ is a basic sequence, we denote by (e_i^*) the associated sequence of coordinate functionals on $[\mathbf{e}]$. Note that if $\inf_i \|e_i\| > 0$ then (e_i^*) is w^*-null, i.e. $\langle e_i^*, x \rangle \to 0$ for every $x \in [\mathbf{e}]$.

The *basis constant* of $\mathbf{e} = (e_i)_{i \geq 1}$ is the number $K(\mathbf{e}) := \sup \{\|P_n\|, n \geq 1\}$, where P_n is the canonical projection of $[\mathbf{e}]$ onto $[e_i; \ 1 \leq i \leq n]$. This depends heavily on the given norm $\|\cdot\|$, but the formula $\|\|x\|\| := \sup \{\|P_n(x)\|, n \geq 1\}$ defines an equivalent norm on $[\mathbf{e}]$ with respect to which the basis constant of $\mathbf{e}$ is equal to 1. Since Haar negligibility is preserved under linear isomorphisms, it follows that for our purpose, we can safely restrict ourselves to basic sequences with basis constant equal to 1.

Likewise, if $\mathbf{e}$ is unconditional, the *unconditional basis constant* of $\mathbf{e}$ is the number $K_u(\mathbf{e}) := \sup \{\|P_I\|; \ I \subset \mathbb{N}\}$, where P_I is the canonical projection of $[\mathbf{e}]$ onto $[e_i; \ i \in I]$; and this can be made equal to 1 by replacing the original norm with the equivalent norm $\|\|x\|\| := \sup \{\|P_I(x)\|; \ I \subset \mathbb{N}\}$. When $K_u(\mathbf{e}) = 1$, the basic sequence $\mathbf{e}$ is said to be *1-unconditional*.

If $\mathbf{e}$ is a basic sequence in X, we denote $\|\cdot\|^{\mathbf{e}}$ the restriction of $\|\cdot\|$ to the closed subspace $[\mathbf{e}] \subset X$. If $\mathbf{e}$ is normalized, we denote by $\|\cdot\|_\infty^{\mathbf{e}}$ the "ℓ_∞ norm built on $\mathbf{e}$", i.e. the norm on $[\mathbf{e}]$ defined by

$$\left\| \sum_{i \geq 1} x_i e_i \right\|_\infty^{\mathbf{e}} := \sup_{i \geq 1} |x_i|.$$

When $\mathbf{e}$ is a basis of X, we write $\|\cdot\|_\infty$ instead of $\|\cdot\|_\infty^{\mathbf{e}}$.

It is important to keep in mind that the original norm always dominates the ℓ_∞ norm built on a normalized basic sequence $\mathbf{e}$; more precisely, we have

$$\|\cdot\|_\infty^{\mathbf{e}} \leq 2K(\mathbf{e})\|\cdot\|^{\mathbf{e}}.$$

Two basic sequences $(e_i)_{i \geq 1}$ and $(f_i)_{i \geq 1}$ (living on possibly different Banach spaces) are said to be *equivalent* if, for every sequence of scalars $(x_i)_{i \geq 1}$, the convergence of the series $\sum x_i e_i$ is equivalent to the convergence of the series $\sum x_i f_i$; in other words, if there is an isomorphism from $[e_i; \ i \geq 1]$ onto $[f_i; \ i \geq 1]$ sending e_i to f_i. Note that the property "to have a Haar null positive cone" is invariant under equivalence.

If $\mathbf{e} = (e_i)_{i \geq 1}$ is a basic sequence, a *block-sequence* built on $\mathbf{e}$ is a sequence $\mathbf{f} = (f_j)_{j \geq 1}$ of the form

$$f_j = \sum_{i=p_{j-1}}^{p_j - 1} a_i e_i \,,$$

where $(p_j)_{j \geq 0}$ is an increasing sequence of integers and (a_i) is a sequence of scalars. Any such sequence $\mathbf{f}$ is basic, with $K(\mathbf{f}) \leq K(\mathbf{e})$; and if $\mathbf{e}$ is unconditional then so is $\mathbf{f}$, with $K_u(\mathbf{f}) \leq K_u(\mathbf{e})$.

Finally, recall that a basic sequence $\mathbf{e} = (e_i)$ is said to be *shrinking* if the linear span of (e_i^*) is dense in the dual space of $[\mathbf{e}]$, and *boundedly complete* if the convergence of a series $\sum x_i e_i$ is equivalent to the boundedness of its partial sums.

By a famous result due to R. C. James, a basic sequence generates a reflexive Banach space if and only if it is both shrinking and boundedly complete. Moreover, an *unconditional* basic sequence is non–shrinking if and only if it has a subsequence equivalent to the canonical basis of ℓ_1, and non–boundedly complete if and only if it has a subsequence equivalent to the canonical basis of c_0.

2.2. Three useful facts.

For the sake of future reference, we now recall three extremely useful results concerning basic sequences.

The first one is merely an observation:

Fact 2.1. *If* $\mathbf{e} = (e_i)_{i \geq 1}$ *is an unconditional basic sequence, then every bounded sequence of scalars* $\mathbf{a} = (a_i)_{i \geq 1}$ *defines a "bounded multiplier" on* $[\mathbf{e}]$*; more precisely, for any convergent series* $\sum x_i e_i$*, the series* $\sum a_i x_i e_i$ *is convergent with*

$$\left\| \sum_{i \geq 1} a_i x_i e_i \right\| \leq K_u(\mathbf{e}) \, \|\mathbf{a}\|_\infty \left\| \sum_{i \geq 1} x_i e_i \right\|.$$

In particular, any fixed change of signs on the e_i's defines an isomorphism of $[\mathbf{e}]$. Since Haar negligibility is preserved under linear isomorphisms, it follows that the property "to have a Haar null positive cone" is invariant under changes of signs: if an unconditional basic sequence (e_i) has a Haar null positive cone, then so does any sequence of the form $(\pm e_i)$.

The second result we want to state explicitly is the "principle of small perturbations", which is a standard tool for establishing the equivalence of two basic sequences:

Lemma 2.2. *Let* $\mathbf{e} = (e_i)_{i \geq 1}$ *be a normalized basic sequence in* X*. If* $\mathbf{f} = (f_i)_{i \geq 1}$ *is a sequence in* X *such that* $\eta := \sum_1^\infty \|e_i - f_i\| < \frac{1}{2K(\mathbf{e})}$*, then* $\mathbf{f}$ *is basic and equivalent to* $\mathbf{e}$*. Moreover, for every convergent series* $\sum x_i f_i$*, one has the following estimate:*

$$(1 - 2K(\mathbf{e})\eta) \left\| \sum_{i=1}^\infty x_i e_i \right\| \leq \left\| \sum_{i=1}^\infty x_i f_i \right\| \leq 2K(\mathbf{e}) \left\| \sum_{i=1}^\infty x_i e_i \right\|.$$

The above estimate is not explicitly stated in [AK] or [LiTz], but it follows easily from the proofs of the principle of small perturbations given therein.

Finally, let us recall the so-called "Bessaga-Pelczynski selection principle":

Lemma 2.3. *If* $\mathbf{e}$ *is a basic sequence, then any normalized sequence* $(y_n) \subset [\mathbf{e}]$ *such that* $\langle e_i^*, y_n \rangle \xrightarrow[n \to \infty]{} 0$ *for all* $i \in \mathbb{N}$ *admits a basic subsequence equivalent to a (normalized) block-sequence of* $\mathbf{e}$*.*

This is often stated with a *weakly null* sequence (y_n), but the greater generality is useful; see [AK].

2.3. **Some Haar null and non–Haar null positive cones.** To conclude this background section, we now state some known results concerning the Haar negligibility of positive cones.

Proposition 2.4. *Let* $\mathbf{e} = (e_i)_{i \geq 1}$ *be a normalized basic sequence.*

(1) *If* $\mathbf{e}$ *is equivalent to the canonical basis of* c_0*, then* $Q^+(\mathbf{e})$ *contains a translate of every compact subset of* $[\mathbf{e}]$*, and so* $Q^+(\mathbf{e})$ *is not Haar null.*

(2) *If* $\mathbf{e}$ *is equivalent to the canonical basis of* ℓ_p*,* $1 \leq p < \infty$*, then* $Q^+(\mathbf{e})$ *is Haar null. More generally,* $Q^+(\mathbf{e})$ *is Haar null as soon as* $\mathbf{e}$ *dominates the canonical basis of* ℓ_p*, i.e. there exists a finite constant* M *such that*

$$\sum_i |x_i|^p \leq M \left\| \sum_i x_i e_i \right\|^p$$

for every finite sequence of scalars (x_i)*.*

(3) *If* $[\mathbf{e}]$ *is reflexive, then* $Q^+(\mathbf{e})$ *is Haar null.*

(4) *If* $\mathbf{e}$ *is unconditional and satisfies*

$$\lim_{k \to \infty} \inf_{n_1 < \cdots < n_k} \left\| \sum_{i=1}^k e_{n_i} \right\| = \infty,$$

then $Q^+(\mathbf{e})$ *is Haar null.*

Part (1) is almost obvious; see [BeLi, p. 132]. Part (2) can be found in [BorN] and [M1] (see also [BeLi]). Part (3) follows from the main result of [M3], and (4) can be found in [M2].

Concerning (1), will show in Section 3 that the canonical basis of c_0 is in fact the only unconditional basic sequence for which the positive cone contains a translate of every compact set. We will also see that (2), (3) and (4) can be obtained as applications of Proposition 4.6 and Theorem 4.9 below. (However, the direct proof of (3) given in [BeLi, Prop. 6.8] provides an explicit test measure for the positive cone).

3. Positive cones containing a translate of every compact set

In this section, we study the positive cones $Q^+(\mathbf{e})$ *containing a translate of every compact set.* By this, we mean of course that $Q^+(\mathbf{e})$ contains a translate of every compact subset of $[\mathbf{e}]$. As already mentioned, this is a strong way of being non–Haar null.

The main result of this section is the following.

Theorem 3.1. *Let* $\mathbf{e} = (e_i)_{i \geq 1}$ *be a normalized basic sequence. The positive cone* $Q^+(\mathbf{e})$ *contains a translate of every compact set if and only if there exists a sequence of real numbers* $(\lambda_i)_{i \geq 1}$*, with* $\lambda_i \geq 1$ *for all* i*, such that*

$$\sup_{n \geq 1} \left\| \sum_{i=1}^n \lambda_i e_i \right\| < \infty.$$

From this, it is easy to deduce

Corollary 3.2. *Up to equivalence, the only normalized* unconditional *basic sequence whose positive cone contains a translate of every compact set is the canonical basis of c_0.*

Proof. Let $\mathbf{e}$ be a normalized unconditional basic sequence such that $Q^+(\mathbf{e})$ contains a translate of every compact set, and set $X := [\mathbf{e}]$. Without loss of generality, we may assume that $\mathbf{e}$ is 1-unconditional.

We already know that the original norm $\|\cdot\|$ of X dominates the norm $\|\cdot\|_\infty$, so we just have to show that $\|\cdot\|_\infty$ also dominates $\|\cdot\|$. Let $(\lambda_i)_{i\geq 1}$ be the sequence given by Theorem 3.1, and set

$$M := \sup_{n\geq 1}\left\|\sum_{i=1}^n \lambda_i e_i\right\| < \infty.$$

For every $x = \sum_{i\geq 1} x_i e_i \in X \setminus \{0\}$ and all $n \geq 1$, we have

$$P_n(x) := \sum_{i=1}^n x_i e_i = \sum_{i=1}^n \frac{x_i}{\lambda_i}\lambda_i e_i\,.$$

If we set $a_i := x_i/\lambda_i$, then $|a_i| \leq \|x\|_\infty$ and so, by Fact 2.1:

$$\|P_n(x)\| \leq \|x\|_\infty\left\|\sum_{i=1}^n \lambda_i e_i\right\| \leq M\,\|x\|_\infty\,.$$

Hence, we obtain $\|x\| = \lim_{n\to\infty}\|P_n(x)\| \leq M\,\|x\|_\infty.$ □

The proof of Theorem 3.1 relies on the following lemma. Here and afterwards, for any positive number r, we set

$$Q_r^+(\mathbf{e}) := Q^+(\mathbf{e}) \cap \overline{B}(0,r)\,.$$

Lemma 3.3. *Let $\mathbf{e}$ be a Schauder basis of a Banach space X, and let D be any dense subset of $\overline{B}(0,1)$. The following are equivalent.*
  (i) *The cone $Q^+(\mathbf{e})$ contains a translate of every compact set.*
  (ii) *There exists $R > 0$ such that $Q_R^+(\mathbf{e})$ contains a translate of every compact subset of $\overline{B}(0,1)$.*
  (iii) *There exists $M > 0$ such that $Q_M^+(\mathbf{e})$ contains a translate of every finite subset of $\overline{B}(0,1) \cap D$.*

Moreover if condition (iii) *is satisfied, then* (ii) *holds with $R = 2M$.*

The proof of this lemma is quite nontrivial; but it is essentially the same as that of [MS, Theorem 3] and is therefore omitted.

Proof of Theorem 3.1. Without loss of generality, we may assume that the basis constant of $\mathbf{e}$ is equal to 1.

Assume first that $Q^+(\mathbf{e})$ contains a translate of every compact subset of $X := [\mathbf{e}]$. By Lemma 3.3, there exists $M > 0$ such that Q_M^+ contains a translate of every finite subset of $D := \overline{B}(0, 1) \cap \mathrm{span}\,(e_i;\ i \geq 1)$.

For every $k \geq 1$, set $F_k := \{-e_i; 1 \leq i \leq k\}$. This is a finite subset of D, so there exists $z_k = \sum_{i=1}^{\infty} z_{i,k} e_i \in X$ such that $z_k + F_k \subset Q_M^+(\mathbf{e})$. Then $\|z_k\| \leq M + 1$, and so $|z_{i,k}| \leq 2(M + 1)$ since $K(\mathbf{e}) = 1$. Moreover, since $z_k + F_k \subset Q^+(\mathbf{e})$ we have $z_{i,k} \geq 1$ whenever $k \geq i$; hence $1 \leq z_{i,k} \leq 2(M + 1)$ for $i \geq 1$ and $k \geq i$.

By a diagonal process, one can find an increasing sequence of integers $(p_k)_{k \geq 1}$ such that for every $i \geq 1$, the sequence $(z_{i,p_k})_{k \geq 1}$ converges to some real number $\lambda_i \geq 1$ as $k \to \infty$. Let us show that the sequence $(\lambda_i)_{i \geq 1}$ has the required property, i.e. that the partial sums of the series $\sum \lambda_i e_i$ are uniformly bounded.

For any $n, k \geq 1$, we have

$$
\begin{aligned}
\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| &\leq \left\| \sum_{i=1}^{n} (\lambda_i - z_{i,p_n}) e_i \right\| + \| P_n(z_{p_k}) \| \\
&\leq n \sup \{ |(\lambda_i - z_{i,p_k})| ;\ 1 \leq i \leq n \} + \| z_{p_k} \| \\
&\leq n \sup \{ |(\lambda_i - z_{i,p_k})| ;\ 1 \leq i \leq n \} + M + 1.
\end{aligned}
$$

Since this inequality holds for every $k \geq 1$, we obtain

$$
\left\| \sum_{i=1}^{n} \lambda_i e_i \right\| \leq M + 1 \quad \text{for all } n \geq 1.
$$

We now prove the converse. Suppose that there exists $M > 0$ and a sequence of real numbers $(\lambda_i)_{i \geq 1}$ with $\lambda_i \geq 1$, such that $\|\sum_{i=1}^{n} \lambda_i e_i\| \leq M$ for every $n \geq 1$.

Set again $D := \overline{B}(0, 1) \cap \mathrm{span}\,(e_i;\ i \geq 1)$. Then $R := 1 + 2M$ satisfies property (iii) of Lemma 3.3. Indeed let F be any finite subset of D. One can find $m \geq 1$ such that $\langle e_n^*, x \rangle = 0$ for every $x \in F$ if $n \geq m$. Since $|\langle e_n^*, x \rangle| \leq 2$ for all $n \geq 1$, if $x \in F$ (because $\|x\| \leq 1$), we see that $F + 2 \sum_{i=1}^{m} \lambda_i e_i \subset Q_{1+2M}^+(\mathbf{e})$. $\qquad\square$

We conclude this section by a simple example showing that Corollary 3.2 breaks down if we drop the unconditionality assumption.

Example 3.4. *The positive cone associated to the canonical basis of the* James space $\mathcal{J}$ *contains a translate of every compact set.*

Proof. Recall that $\mathcal{J}$ is the space of all sequences of real numbers $x = (x_i)_{i \geq 1}$ which converge to 0 and satisfy the following condition:

$$
\|x\|_{\mathcal{J}} := \sup_{m \in \mathbb{N}} \sup \left\{ \left[(x_{p_2} - x_{p_1})^2 + \cdots + (x_{p_m} - x_{p_{m-1}})^2 \right]^{1/2} ;\ 1 \leq p_1 \cdots < p_m \right\} < \infty.
$$

We refer to e.g. [LiTz, p. 25] for more details on the James space.

The canonical basis $\mathbf{e} = (e_i)_{i \geq 0}$ is a normalized Schauder basis of $\mathcal{J}$, and for every $n \geq 1$ we have

$$\left\| \sum_{i=1}^{n} e_i \right\|_{\mathcal{J}} = 1 \,.$$

It follows then from Theorem 3.1 that $Q^+(\mathbf{e})$ contains a translate of every compact set. $\qquad \square$

4. Positive cones which are not Haar null

We now address the question of characterizing the basic sequences whose positive cone is not Haar null. We obtain a complete answer in the case of subsymmetric unconditional sequences: within this class of sequences, the canonical basis of c_0 is the only one whose positive cone is not Haar null. We also show that an unconditional basic sequence whose positive cone is not Haar null has to be "very close" to the canonical basis of c_0, in a sense that will be made precise.

4.1. **Haar negligibility and block-sequences.** The following result shows that for unconditional basic sequences, Haar negligibility of the positive cone can be detected by looking at any block-sequence.

Proposition 4.1. *Let $\mathbf{e} = (e_i)_{i \geq 1}$ be an unconditional basic sequence. If $\mathbf{e}$ admits a block-sequence $\mathbf{f}$ whose positive cone is Haar null, then $Q^+(\mathbf{e})$ is Haar null.*

Proof. Set $X := [\mathbf{e}]$, and let Y be the closed subspace of X generated by the block-sequence $\mathbf{f} = (f_j)_{j \geq 1}$. There exists an increasing sequence of integers $(p_j)_{j \geq 0}$, with $p_0 = 1$, such that $f_j = \sum_{i=p_{j-1}}^{p_j - 1} a_i e_i$; and since Haar negligibility is invariant under any change of signs on the e_i's, we may assume that $a_i \geq 0$ for all i.

By assumption, one can find a probability measure μ on Y such that $\mu(Q^+(\mathbf{f}) + y) = 0$ for every $y \in Y$. Denote by ν the measure μ viewed as a probability measure on X; that is, $\nu(A) = \mu(A \cap Y)$ for every Borel set $A \subset X$. We show that ν is a test measure for $Q^+(\mathbf{e})$.

Since $\mathbf{e}$ is unconditional, we can write any $x \in X$ as $x = x^+ + x^-$, where $x^+ \in Q^+(\mathbf{e})$ and $x^- \in Q^-(\mathbf{e})$. Then $x + Q^+(\mathbf{e}) \subset x^- + Q^+(\mathbf{e})$, and so

$$\nu(x + Q^+) \leq \nu(x^- + Q^+).$$

Hence, it is enough to check that $\nu(x + Q^+(\mathbf{e})) = 0$ for every $x \in Q^-(\mathbf{e})$. This follows immediately from the following

Fact. Given $x \in Q^-(\mathbf{e})$, one can find $w \in X$ such that $\big(x + Q^+(\mathbf{e})\big) \cap Y \subset w + Q^+(\mathbf{f})$.

Indeed, once this is known one just has to write

$$\nu(x + Q^+(\mathbf{e})) = \mu((x + Q^+(\mathbf{e})) \cap Y) \leq \mu(w + Q^+(\mathbf{f})) = 0.$$

Proof of Fact. Fix $x \in Q^-(\mathbf{e})$, and let $z \in (x + Q^+(\mathbf{e})) \cap Y$ be arbitrary. Write

$$z = \sum_{i=1}^{\infty}(x_i + \lambda_i)e_i = \sum_{j=1}^{\infty} z_j f_j = \sum_{j=1}^{\infty} z_j \left(\sum_{i=p_{j-1}}^{p_j - 1} a_i e_i \right),$$

where $\lambda_i \geq 0$ for all i. Then $z_j a_i - x_i = \lambda_i \geq 0$ for every $j \geq 1$ and all $i \in [p_{j-1}, p_j)$. So we get for $j \geq 1$,

$$(4.1) \qquad w_j := \sup \left\{ \frac{x_i}{a_i}; \ p_{j-1} \leq i < p_j, \ a_i \neq 0 \right\} \leq z_j.$$

Since $x_i \leq 0$ for all i, the series $\sum |x_i| e_i$ is convergent and $\sum_{i=1}^{\infty} |x_i| e_i = -x$. It follows from Fact 2.1 that the series $\sum_{j \geq 1} \left(\sum_{i=p_{j-1}}^{p_j - 1} |w_j| a_i e_i \right) = \sum |w_j| y_j$ is convergent, and so the series $\sum w_j y_j$ is also convergent in Y. Set $w := \sum_{j=1}^{\infty} w_j y_j$. Note that w depends only on x and $\mathbf{f}$ (not on the point z we are considering).

Using (4.1), we see that $z = w + \sum_{j=1}^{\infty}(z_j - w_j)y_j \in w + Q^+(\mathbf{f})$, which concludes the proof.   □

□

We know that the positive cone of ℓ_1 is Haar null, and that positive cones in reflexive spaces are also Haar null. By James' characterization of reflexivity, we deduce

Corollary 4.2. *If* $\mathbf{e}$ *is an unconditional basic sequence such that* $Q^+(\mathbf{e})$ *is not Haar null, then every block-sequence of* $\mathbf{e}$ *is both shrinking and non–boundedly complete.*

Since an unconditional basic sequence is non–boundedly complete if and only if it has a block-sequence equivalent to the canonical basis of c_0, it follows that if $\mathbf{e}$ is unconditional and $Q^+(\mathbf{e})$ is not Haar null, then every block-sequence of $\mathbf{e}$ admits a block-sequence equivalent to the canonical basis of c_0. We will see in Section 4.4 that $\mathbf{e}$ has in fact a much stronger "c_0-saturation" property.

4.2. **The** (δ, R)**-condition.** Matouskova observed in [M2] that a Borel subset A of a Polish abelian group G is Haar null if and only if the following holds: *for every* $\delta > 0$ *and every* $r > 0$, *there exists a probability measure* μ *supported by the open ball* $B(0, r)$ *such that* $\mu(A + g) \leq \delta$ *for every* $g \in G$. From this, it is easy to deduce

Lemma 4.3. *Let* $\mathbf{e}$ *be a normalized basic sequence. The cone* $Q^+(\mathbf{e})$ *is not Haar null if and only if there exist* $\delta > 0$ *and* $R > 0$ *satisfying the following property:*

($*$) For every probability measure μ *on* X *supported on* $\overline{B}(0, 1)$, *there exists* $x \in X$ *such that* $\mu(x + Q_R^+(\mathbf{e})) \geq \delta$.

In this situation we have $\|x\| \leq 1 + R$.

Proof. Since a countable union of Haar null sets is Haar null, $Q^+(\mathbf{e})$ is not Haar null if and only if $Q_n^+(\mathbf{e})$ is not Haar null for some integer $n \geq 1$.

If $Q_n^+(\mathbf{e})$ is not Haar null, it follows from Matouskova's result mentioned above that there exist $\delta > 0$ and $r > 0$ such that for every probability mesure μ on X of support contained in $\overline{B}(0, r)$, there exists $x \in X$ such that $\mu(x + Q_n^+) \geq \delta$. This gives the required condition with $R := n/r$. Conversely if $(*)$ holds then Matouskova's result implies that $Q_R^+(\mathbf{e})$ is not Haar null, and so $Q^+(\mathbf{e})$ is not Haar null either.

The inequality $\|x\| \leq 1 + R$ follows from the triangle inequality. $\square$

In what follows we will consider a weakened version of condition $(*)$ above, where we only allow a special type of measures.

Definition 4.4. Let $\delta > 0$ and $R > 0$. We say that a normalized basic sequence $\mathbf{e} = (e_i)_{i \geq 1}$ satisfies the (δ, R)-*condition* if for every probability measure μ having finite support contained in $\{-e_i; \; i \geq 1\}$, there exists $x \in X$ (necessarily satisfying $\|x\| \leq 1 + R$) such that $\mu(x + Q_R^+) \geq \delta$.

The following observation shows the relevance of this condition to our matters.

Observation 4.5. *Let* $\mathbf{e} = (e_i)$ *be a normalized basic sequence.*
  (i) *If* $Q^+(\mathbf{e})$ *is not Haar null, then* $\mathbf{e}$ *satisfies the* (δ, R)-*condition for some* $\delta, R > 0$.
  (ii) *If* $\mathbf{e}$ *is unconditional and if* $Q^+(\mathbf{e})$ *is not Haar null, there exist* $\delta, R > 0$ *such that all normalized block-sequences of* $\mathbf{e}$ *satisfy the* (δ, R)-*condition.*

Proof. Part (i) follows immediately from Lemma 4.3. To prove (ii), argue by contradiction and use Proposition 4.1. $\square$

When $\mathbf{e}$ is unconditional, this implies a condition on the norm of all finite sums of the form $\sum_{i \in J} e_i$.

Proposition 4.6. *Let* $\mathbf{e} = (e_i)$ *be a normalized, 1-unconditional basic sequence, and assume that* $\mathbf{e}$ *satisfies the* (δ, R)-*condition for some* $\delta, R > 0$. *Then the following facts hold.*
  (a) *For every finite set* $I \subset \mathbb{N}$, *one can find* $J \subset I$ *such that*

$$|J| \geq \delta \, |I| \quad \text{and} \quad \left\| \sum_{i \in J} e_i \right\| \leq 1 + R \, .$$

  (b) *If* $\delta = 1$, *then* $\mathbf{e}$ *is equivalent to the canonical basis of* c_0.
  (c) *If* $\delta < 1$ *then, for any finite set* $I \subset \mathbb{N}$, *one has the estimate*

$$\left\| \sum_{i \in I} e_i \right\| \leq \frac{1 + R}{-\log(1 - \delta)} \log(|I|) + 2(1 + R) \, .$$

Proof. (a) Set $\lambda_i = \frac{1}{|I|}$ for $i \in I$. Then $\mu = \sum_{i \in I} \lambda_i \delta_{-e_i}$ is a probability measure with finite support contained in $\{-e_i, i \geq 1\}$. So there exists $x \in X$ such that $\mu\left(x + Q_R^+\right) \geq \delta$. This implies that there exists $J \subset I$ such that $|J| \geq \delta\,|I|$ and such that $-e_j \in x + Q_R^+$ for all $j \in J$. So $\langle e_j^*, x \rangle \leq -1$ for all $j \in J$ (and hence $|\langle e_j^*, x \rangle| \geq 1$). Since $\mathbf{e}$ is 1-unconditional and since $\|x\| \leq 1 + R$, it follows from Fact 2.1 that $\left\|\sum_{i \in J} e_i\right\| \leq 1 + R$.

(b) If $\delta = 1$, then for every finite set I we have $\left\|\sum_{i \in I} e_i\right\| \leq 1 + R$. By 1-unconditionality, this implies that $\|\cdot\| \leq (1 + R)\,\|\cdot\|_\infty$, and hence $\mathbf{e}$ is equivalent to the canonical basis of c_0.

(c) If $\delta < 1$, we construct a family of disjoint subsets of I in the following way: set $I_0 := I$ and choose $J_0 \subset I_0$ according to (i). Then set $I_1 = I_0 - J_0$ and we choose $J_1 \subset I_1$ in the same way, and so on. At step k, we thus have $|I_k| \leq (1-\delta)\,|I_{k-1}|$, $|I_k| \leq (1-\delta)^k\,|I|$ and $\left\|\sum_{i \in J_k} e_i\right\| \leq 1 + R$

We stop when I_{k+1} is empty, so at the latest when $(1 - \delta)^k\,|I| \leq 1$, which gives $k \geq \frac{\log(|I|)}{-\log(1-\delta)}$. So we can choose $k \leq \frac{\log(|I|)}{-\log(1-\delta)} + 1$. This gives

$$\left\|\sum_{i \in I} e_i\right\| \leq \sum_{j=0}^{k} \left\|\sum_{i \in J_j} e_i\right\| \leq (k+1)(1+R) \leq \frac{1+R}{-\log(1-\delta)} \log(|I|) + 2(1+R)\,.$$

$\square$

Remark 1. Part (4) of Proposition 2.4 follows immediately from (a).

Remark 2. If $\mathbf{e} = (e_i)$ satisfies the assumption of Proposition 4.6, then (c) shows that

$$\lim_{|I| \to \infty} \left\|\frac{1}{|I|} \sum_{i \in I} e_i\right\| = 0\,.$$

Sequences $\mathbf{e}$ satisfying this property are called *Blum-Hanson sequences* in [LMP]. They are quite interesting objects of study.

Remark 3. For every normalized basic sequence $\mathbf{e} = (e_i)$ and any $M \in \mathbb{R}^+$, let us denote by $\mathcal{I}_M(\mathbf{e})$ the family of all subsets σ of $\mathbb{N}$ such that

$$\left\|\sum_{i \in I} e_i\right\| \leq M \quad \text{for every finite set } I \subset \sigma\,.$$

Note that if $\mathbf{e}$ is unconditional and if $\sigma \in \mathcal{I}_M(\mathbf{e})$ is infinite, then $(e_i)_{i \in \sigma}$ is equivalent to the canonical basis of c_0. So our "main conjecture" would be proved if one could show the following: *if $\mathbf{e}$ is a normalized basic sequence such that $Q^+(\mathbf{e})$ is not Haar null, then one can partition $\mathbb{N}$ into finitely many sets from $\bigcup_{M>0} \mathcal{I}_M(\mathbf{e})$.* Even though this is of course very unlikely to be relevant, it is hard not to notice the formal analogy with the

"paving" formulation of the famous and recently solved Kadison-Singer Problem. Note that, by standard arguments, it would be enough to show that any ultrafilter $\mathcal{U}$ on $\mathbb{N}$ contains a set from $\bigcup_{M>0} \mathcal{I}_M(\mathbf{e})$.

Assuming that $\mathbf{e}$ is unconditional and that $Q^+(\mathbf{e})$ is not Haar null (so that $\mathbf{e}$ satisfies some (δ, R)-condition), Proposition 4.6 (a) shows that there exists a constant M such that $\mathcal{I}_M(\mathbf{e})$ has an interesting "largeness" property, namely that inside any finite set $I \subset \mathbb{N}$, one can find a set $J \in \mathcal{I}_M(\mathbf{e})$ with comparable cardinality; but this is definitely not enough to get a partition result.

In the same spirit (and under the same assumptions on $\mathbf{e}$), the argument of [BouTz, Corollary 1.4] shows the following: there exists a constant M, a probability measure $\mathfrak{m}$ on the compact set $\mathcal{I}_M(\mathbf{e}) \subset 2^{\mathbb{N}}$ and a constant $\alpha > 0$ such that

$$\mathfrak{m}(\{\sigma \in \mathcal{I}_M(\mathbf{e}); \ i \in \sigma\}) \geq \alpha \quad \text{for every } i \in \mathbb{N}.$$

This is another "largeness" property of the family $\mathcal{I}_M(\mathbf{e})$, that makes perhaps plausible the existence of a suitable partition.

We now show that Proposition 4.6 can be used to characterize the *symmetric* basic sequences whose positive cone is Haar null. Recall that a basic sequence (e_i) is said to be symmetric if, for any permutation π of the integers, the sequence $(e_{\pi(i)})$ is equivalent to (e_i). In this case (see [LiTz, Chapter 3]), there exists a constant $C > 0$ such that for every convergent series $\sum_{i \geq 1} x_i e_i$ and every permutation $\pi : \mathbb{N} \to \mathbb{N}$ we have

$$\frac{1}{C} \left\| \sum_{i \geq 1} x_i e_{\pi(i)} \right\| \leq \left\| \sum_{i \geq 1} x_i e_i \right\| \leq C \left\| \sum_{i \geq 1} x_i e_{\pi(i)} \right\|.$$

Every symmetric sequence is unconditional, but not conversely (consider any enumeration of the "canonical" basis of $l_1 \oplus c_0$).

Corollary 4.7. *Up to equivalence, the only normalized and symmetric basic sequence whose positive cone is not Haar null is the canonical basis of c_0.*

Proof. Let $\mathbf{e} = (e_i)$ be a normalized, symmetric sequence such that $Q^+(\mathbf{e})$ is not Haar null. Assume without loss of generality that $\mathbf{e}$ is 1-unconditional. Choose $\delta, R > 0$ such that $\mathbf{e}$ satisfies the (δ, R)-condition. By proposition 4.6 and since $K_u(\mathbf{e}) = 1$, one can find for every integer $p \geq 1$ a finite set $I_p \subset \mathbb{N}$ of cardinality p such that $\left\| \sum_{i \in I_p} e_i \right\| \leq 1 + R$.

Since $\mathbf{e}$ symmetric, there exists a finite constant C such that, for *every* finite set I of integers of cardinality p, we have

$$\left\| \sum_{i \in I} e_i \right\| \leq C \left\| \sum_{i \in I_p} e_i \right\|.$$

So the finite sums $\sum_{i \in I} e_i$ are uniformly bounded, which implies (by unconditionality) that the given norm $\| \cdot \|^{\mathbf{e}}$ on $[\mathbf{e}]$ is dominated by the norm $\| \cdot \|^{\mathbf{e}}_{\infty}$. Hence these two norms are equivalent, and $\mathbf{e}$ is equivalent to the canonical basis of c_0. $\qquad \square$

As another of illustration of Proposition 4.6, we now show that the positive cone associated with the canonical basis of the *Schreier space* $\mathcal{S}$ is Haar null.

Let us first recall the definition of $\mathcal{S}$. Let c_{00} be the linear space of all real sequences with finite support. For $x = (x_i)_{i \geq 1} \in c_{00}$, set

$$\|x\|_{\mathcal{S}} := \sup \left\{ \sum_{i=1}^{p} |x_{k_i}| : p \geq 1, \ p \leq k_1 < \ldots < k_p \right\}.$$

Then $\| \cdot \|_{\mathcal{S}}$ is a norm on c_{00}, and the Schreier space $\mathcal{S}$ is the completion of c_{00} with respect to this norm. The canonical basis (e_i) of c_{00} is a normalized, 1-unconditional basis of $\mathcal{S}$, and it is also shrinking.

Corollary 4.8. *The positive cone of the Schreier space is Haar null.*

Proof. Let $n \geq 1$, and let $I \subset \mathbb{N}$ be a set of cardinality $2n$. Then the set $I \cap [n, \infty)$ contains at least n elements of I. It follows from the definition of $\| \cdot \|_{\mathcal{S}}$ that we have:

$$\left\| \sum_{i \in I} e_i \right\|_{\mathcal{S}} \geq n.$$

Since n is arbitrary, the result now follows from Proposition 4.6. $\qquad \square$

4.3. **Extracting c_0-subsequences.** In this section, we show that a normalized unconditional basic sequence whose positive cone is not Haar null has a wealth of subsequences equivalent to the canonical basis of c_0. This will follow easily from the following result.

Theorem 4.9. *Let $\mathbf{e}$ be a normalized basic sequence, and assume that $\mathbf{e}$ satisfies the (δ, R)-condition for some $\delta, R > 0$. Then there exists an increasing sequence of integers $(m_j)_{i \geq 1}$ and a sequence $(x_n)_{n \geq 1} \subset \overline{B}(0, 1 + R)$ such that*

$$\left\langle e^*_{m_j}, x_n \right\rangle \geq 1 \quad \text{for every } n \geq 1 \text{ and all } j \in \{1, \ldots, n\}.$$

Remark. It follows immediately from this result that a normalized Schauder basis of a reflexive Banach space cannot satisfy a (δ, R)-condition, which gives part (3) of Proposition 2.4. Indeed let X be a Banach with a normalized Schauder basis $\mathbf{e} = (e_i)_{i \geq 1}$, and assume that $\mathbf{e}$ satisfies the (δ, R)-condition for some $\delta, R > 0$. If the space X were reflexive, it would be possible to extract from the sequence $(x_n)_{n \geq 1}$ given by theorem 4.9 a subsequence converging weakly to some $x \in X$. Then we would have $\left\langle e^*_{m_j}, x \right\rangle \geq 1$ for every $j \geq 1$, a contradiction since $\left\langle e^*_{m_j}, x \right\rangle \to 0$ (recall that $\mathbf{e}$ is normalized).

Part (2) of Proposition 2.4 is also a consequence of Theorem 4.9. Indeed, let $\mathbf{e} = (e_i)$ be a normalized basic sequence which dominates the canonical basis of ℓ_p. If $\mathbf{e}$ were to satisfy some (δ, R)-condition, then the sequence $(x_n)_{n \geq 1}$ given by 4.9 would satisfy

$\sum_{i\geq 1}|\langle e_i^*, x_n\rangle|^p \geq n$ for every $n \geq 1$, which is impossible because (x_n) is bounded. Hence $\mathbf{e}$ does not satisfy any (δ, R)-condition, so $Q^+(\mathbf{e})$ is Haar null.

Theorem 4.9 is in fact a simple consequence of a classical combinatorial lemma due to V. Ptak, which was already an essential tool in [M3].

Lemma 4.10. *Let W be an infinite set, and let $\mathfrak{V}$ be a family of subsets of W. Denote by $\mathbf{P}(W)$ be the set of all finitely supported probability measures on W. Assume that*

$$\inf_{\lambda \in \mathbf{P}(W)} \sup_{V \in \mathfrak{V}} \lambda(V) > 0 \,.$$

Then, there exists a sequence $(V_j)_{j\geq 1}$ of elements of $\mathfrak{V}$ and a sequence $(w_j)_{j\geq 1}$ of pairwise distinct elements of W such that $\{w_1, ..., w_n\} \subset V_n$ for every $n \geq 1$.

For a proof of Ptak's lemma, see [Pt] or [T].

Proof of Theorem 4.9. Let $W := \{-e_i;\ i \in \mathbb{N}\}$ and denote by $\mathfrak{V}$ the family of all subsets V of W such that $x + V \subset Q_R^+$ for some $x \in \overline{B}(0, 1+R)$.

Let $\lambda \in \mathbf{P}(W)$, and write $\lambda(w)$ instead of $\lambda(\{w\})$, $w \in W$. Set

$$I_\lambda := \{i \in \mathbb{N};\ \lambda(-e_i) > 0\} \,.$$

Then I_λ is a finite set and $\sum_{i \in I_\lambda} \lambda(-e_i) = 1$.

Since $\mathbf{e}$ satisfies the (δ, R)-condition, there exists $J_\lambda \subset I_\lambda$ and $x \in \overline{B}(0, 1+R)$ such that $\sum_{i \in J_\lambda} \lambda(-e_i) \geq \delta > 0$ and $x + V \subset Q_R^+$, where $V := \{-e_i;\ i \in J_\lambda\}$. Then $V \in \mathfrak{V}$ and $\lambda(J) \geq \delta$.

By Ptak's lemma, there exists an increasing sequence of integers $(m_j)_{n\geq 1}$ and a sequence (V_j) of elements of $\mathfrak{V}$ such that $\{-e_{m_1}, ..., -e_{m_n}\} \subset V_n$ for every $n \geq 1$. So there exists a sequence $(x_n) \subset \overline{B}(0, 1+R)$ such that $x_n + \{-e_{m_1}, ..., -e_{m_n}\} \subset Q_R^+$ for all $n \geq 1$. Hence for every $n \geq 1$ and all $j \in \{1, \ldots, n\}$, we have $\langle e_{m_j}^*, x_n\rangle \geq 1$. $\qquad\square$

From Theorem 4.9, we easily deduce

Corollary 4.11. *If $\mathbf{e}$ is a normalized, 1-unconditional basic sequence satisfying some (δ, R)-condition, then $\mathbf{e}$ has a subsequence which is $(1 + R)$-equivalent to the canonical basis of c_0.*

Proof. Let (m_j) and (x_n) be given by Theorem 4.9, and set $\mathbf{f} := (e_{m_j})_{j\geq 1}$. By Fact 2.1, we have

$$\left\| \sum_{j=1}^n e_{m_j} \right\| \leq \|x_n\| \leq 1 + R$$

for all $n \geq 1$, which implies that $\|\cdot\|^{\mathbf{f}} \leq (1 + R)\|\cdot\|_\infty^{\mathbf{f}}$. On the other hand, since $\mathbf{f}$ is normalized we also have $\|\cdot\|_\infty^{\mathbf{f}} \leq \|\cdot\|^{\mathbf{f}}$, so the subsequence $\mathbf{f}$ is $(1 + R)$-equivalent to the canonical basis of c_0. $\qquad\square$

Remark. Of course, the same result holds if $\mathbf{e}$ is only assumed to be semi-normalized, replacing $1 + R$ by some constant M depending only on R and on $\inf_{i \geq 1} \|e_i\|$.

If $\mathbf{e}$ is an unconditional basic sequence with a non–Haar null positive cone, then, after the usual renorming to make it 1-unconditional and keeping in mind Proposition 4.1, one can apply Corollary 4.11 to any normalized block-sequence of $\mathbf{e}$. This yields

Corollary 4.12. *If $\mathbf{e}$ is an unconditional basic sequence whose positive cone is not Haar null, then every normalized block-sequence $\mathbf{f}$ of $\mathbf{e}$ is c_0-saturated; that is, any subsequence of $\mathbf{f}$ has a further subsequence equivalent to the canonical basis of c_0, with uniform bounds on the isomorphism constants.*

Another immediate consequence of Corollary 4.11 is the extension of Corollary 4.7 to the case of *subsymmetric* unconditional basic sequences. Recall that a basic sequence $\mathbf{e}$ is said to be subsymmetric if it is equivalent to all its subsequences. Every symmetric sequence is subsymmetric, but not conversely; see [LiTz].

Corollary 4.13. *Up to equivalence, the canonical basis of c_0 is the only normalized and subsymmetric unconditional basic sequence whose positive cone is not Haar null.*

Proof. Let $\mathbf{e}$ be such a sequence. Since $\mathbf{e}$ is unconditional, Corollary 4.11 implies that $\mathbf{e}$ has a subsequence equivalent to the canonical basis of c_0. Since $\mathbf{e}$ is equivalent to all its subsequences, the result follows. $\qquad\square$

Finally, we quote the following "decomposition" result for Banach spaces with a non–Haar null positive cone.

Corollary 4.14. *Let $\mathbf{e} = (e_i)_{i \geq 1}$ be a normalized unconditional basic sequence, and set $X := [\mathbf{e}]$. Assume that $Q^+(\mathbf{e})$ is not Haar null. Then one can find some constant M and a partition $(I_\lambda)_{\lambda \in \Lambda}$ of $\mathbb{N}$ such that each space $X_\lambda := [e_i;\ i \in I_\lambda]$ is M-isomorphic to c_0 and $X = \oplus_{\lambda \in \Lambda} X_\lambda$, where $\oplus$ denotes an unconditional Schauder decomposition.*

Proof. By Corollary 4.12, one can find a constant C such that every subsequence of $\mathbf{e}$ has a further subsequence which is C-equivalent to the canonical basis of c_0. By an obvious transfinite induction argument, it follows that one can partition $\mathbb{N}$ as $\mathbb{N} = I_0 \cup \bigcup_{\lambda \in \Lambda} I_\lambda$, where I_0 is finite (possibly empty) and each space X_λ is M-isomorphic to c_0. Replacing one set I_λ by $I_\lambda \cup I_0$ and using the unconditionality of $\mathbf{e}$, the result follows. $\qquad\square$

4.4. **More on c_0-saturation.** We saw in Corollary 4.12 that unconditional basic sequences with a non–Haar null positive cone have a strong "c_0-saturation" property. In this section, we elaborate a little bit more on this.

For the sake of brevity, we adopt the following ad hoc terminology: given $M \geq 1$, a normalized basic sequence $\mathbf{e}$ is said to be *of type Mc_0* if $\|\cdot\|^{\mathbf{e}} \leq M\|\cdot\|^{\mathbf{e}}_\infty$. This, of course, implies that $\mathbf{e}$ is equivalent to the canonical basis of c_0.

Definition 4.15. Let $\mathbf{e} = (e_i)_{i \geq 1}$ be a normalized basic sequence. We say that

- **e** is Mc_0-*saturated* (for some $M \geq 1$) if every subsequence of **e** has a further subsequence of type Mc_0.
- **e** is *strongly Mc_0-saturated* (for some $M \geq 1$ if the following holds: from every sequence $(I_j)_{j \geq 1}$ of pairwise disjoint subsets of $\mathbb{N}$ for which there exists $N \geq 1$ such that all sequence $(e_i)_{i \in I_j}$ are of type Nc_0, it is possible to extract a subsequence $(I'_j)_{j \geq 1}$ such that $(e_i)_{i \in \bigcup_{j \geq 1} I'_j}$ is of type NMc_0.
- **e** is *(strongly) c_0-saturated* if it is (strongly) Mc_0-saturated for some constant M.

Note that taking each I_i to be a singleton in the above definition, we see that strong Mc_0-saturation indeed implies Mc_0-saturation.

Recall also that a *space X* is said to be c_0-saturated if every (infinite-dimensional) closed subspace of X admits a further subspace isomorphic to c_0. Using the Bessaga-Pelczynski selection principle (Lemma 2.3), it is fairly easy to check that if a Banach space X admits a c_0-saturated (normalized) Schauder basis, then X is c_0-saturated.

Our aim is to prove the following strengthening of Corollary 4.12.

Theorem 4.16. *Let* **e** *be a normalized, 1-unconditional basic sequence. Assume that there exists $\delta, R > 0$ such that all normalized block-sequences satisfy the (δ, R)-condition. Then all these block-sequences are strongly $(1 + R)c_0$-saturated.*

From this, we immediately get

Corollary 4.17. *If* **e** *is an unconditional basic sequence whose positive cone is not Haar null, then all normalized block-sequences of* **e** *are strongly c_0-saturated.*

The proof of Theorem 4.16 relies on the following lemma.

Lemma 4.18. *Let* $\mathbf{e} = (e_i)_{i \geq 1}$ *be a normalized, 1-unconditional basic sequence, and let $\delta, R > 0$. Assume that all normalized block-sequences of* **e** *satisfy the (δ, R)-condition. Let also* $\mathbf{y} = (y_i)_{i \geq 1}$ *be a normalized block-sequence of* **e**. *Finally, let n be a positive integer, let $(\lambda_j)_{1 \leq j \leq n}$ be a family of n nonnegative real numbers satisfying $\sum_j \lambda_j = 1$, and let $(I_j)_{1 \leq j \leq n}$ be a family of n pairwise disjoint infinite subsets of $\mathbb{N}$ such that each sequence $(y_i)_{i \in I_j}$ is of type Nc_0 for some $N \geq 1$. Then, one can find a set $J \subset \{1, \ldots, n\}$ such that $\sum_{j \in J} \lambda_j \geq \delta$ and the sequence $(y_i)_{i \in \bigcup_{j \in J} I_j}$ is of type $N(1 + R)c_0$.*

Proof. For $m \geq 1$ and $1 \leq j \leq n$, let I_j^m be the set consisting of the first m elements of I_j, with the convention $I_j^m := I_j$ if $m \geq |I_j|$. Set $z_{m,j} := \sum_{i \in I_j^m} y_i$ (so that $\|z_{m,j}\| \leq N$) and $\widehat{z}_{m,j} = \frac{z_{m,j}}{\|z_{m,j}\|}$.

For any fixed m, the sets I_j^m are pairwise disjoint. Hence $(\widehat{z}_{m,j})_{1 \leq j \leq n}$ is a (finite) block-sequence of $(y_i)_{i \geq 1}$ for every $m \geq 1$. So it is also a block-sequence of **e**, and it is normalized; hence it is 1-unconditional and satisfies the (finite version of) the (δ, R)-condition. Forgetting the dependance on m, we denote $\widehat{Q}^+$ its positive cone in $Z :=$ span $(\widehat{z}_{m,j}; 1 \leq j \leq n)$.

For $j \in \{1, \ldots, n\}$, denote by Δ_j the Dirac measure at $-\widehat{z}_{m,j}$. Then $\mu = \sum_{j=1}^{n} \lambda_j \Delta_j$ is a probability measure with finite support contained in $\{-\widehat{z}_{m,j}; \ 1 \leq j \leq n\}$. So there exists $z \in Z$ such that $\mu\left(z + \widehat{Q}_R^+\right) \geq \delta$, and hence there exists $J^m \subset \{1, \ldots, n\}$ such that $\langle \widehat{z}_{m,j}^*, z \rangle \leq -1$ for all $j \in J^m$ and $\sum_{j \in J^m} \lambda_j \geq \delta$. Since $(\widehat{z}_{m,j})_{1 \leq j \leq n}$ is 1-unconditional and $\|z\| \leq 1 + R$, we deduce from Fact 2.1 that $\left\| \sum_{j \in J^m} \widehat{z}_{m,j} \right\| \leq 1 + R$; and it follows again from Fact 2.1 that $\left\| \sum_{j \in J^m} z_{m,j} \right\| \leq N(1 + R)$.

Since there are finitely many sets J^m, one can find $J \subset \{1, \ldots, n\}$ such that $J = J^m$ for infinitely many m. Moreover, since the basis constant of $\mathbf{e}$ is equal to 1, we have, for every $m \geq 1$ and every $m' \geq m$,

$$\left\| \sum_{j \in J} z_{m,j} \right\| \leq \left\| \sum_{j \in J} z_{m',j} \right\|.$$

It follows that $\left\| \sum_{j \in J} z_{m,j} \right\| \leq N(1 + R)$ for every $m \geq 1$. Hence, all finite sums built on $(y_j)_{j \in \cup_{i \in J} I_i}$ are bounded in norm by $N(1 + R)$, which implies that $(y_j)_{j \in \cup_{i \in J} I_i}$ is of type $N(1 + R)c_0$. □

Proof of Theorem 4.16. Since every normalized block-sequence of $\mathbf{e}$ satisfies the same assumptions as $\mathbf{e}$, it is enough to show that $\mathbf{e}$ is strongly $(1 + R)c_0$-saturated.

Let $(I_j)_{j \geq 1}$ be a sequence of pairwise disjoint subsets of $\mathbb{N}$ such that each sequence $(e_i)_{i \in I_j}$ is of type Nc_0, for some fixed $N \geq 1$.

Denote by $\mathfrak{V}$ the family of all subsets V of $\mathbb{N}$ such that the basic sequence $(e_i)_{i \in \cup_{j \in V} I_j}$ is of type $N(1 + R)c_0$. Lemma 4.18 ensures that

$$\inf_{\lambda \in \mathbf{P}(\mathbb{N})} \sup_{V \in \mathfrak{V}} \lambda(V) \geq \delta > 0.$$

It follows then from Ptak's lemma that there exists a sequence $(V_j)_{i \geq 1}$ of elements of $\mathfrak{V}$, and a sequence $(w_j)_{j \geq 1}$ of pairwise distinct positive integers such that $\{w_1, \ldots, w_n\} \subset V_n$ for every $n \geq 1$. Then $(e_i)_{i \in \cup_{j \geq 1} I_{w_j}}$ is of type $N(1 + R)c_0$. □

4.5. **One example.** So far, we have shown that for a normalized unconditional basic sequence $\mathbf{e}$, the following implications hold true:

$Q^+(\mathbf{e})$ not Haar null $\implies$ there exists $\delta, R > 0$ such that every normalized block-sequence of $(e_i)_{i \geq 1}$ satisfies the (δ, R)-condition $\implies$ every normalized block-sequence of $\mathbf{e}$ is strongly c_0-saturated $\implies$ the space $X := [\mathbf{e}]$ is c_0-saturated.

With our "main conjecture" in mind, it is rather tempting to believe that the strong saturation of all block-sequences is already enough to ensure that $\mathbf{e}$ is equivalent to the canonical basis of c_0. The following result shows that this is not so.

Example 4.19. *There exists a normalized, 1-unconditional basic sequence $\mathbf{e}$ which is not isomorphic to the canonical basis of c_0 but has the property that all its block-sequences are strongly Mc_0-saturated for any $M \geq 1$.*

Proof. Not surprisingly, the sequence $\mathbf{e}$ will be defined as the "canonical basis" of a c_0-direct sum of suitably chosen finite-dimensional ℓ_p spaces.

Let (p_n) be an increasing sequence of real numbers to be chosen later, with $p_n \geq 1$ and $\lim_{n \to \infty} p_n = \infty$. We set $X_n := l_{p_n}(n)$, and we denote by $(e_i^n)_{1 \leq i \leq n}$ the canonical basis of X_n. Finally, we denote by X the c_0-direct sum of the spaces X_n,

$$X = \oplus_{c_0} \{X_n, n \in \mathbb{N}\}.$$

Denote by $(e_i)_{i \geq 1}$ the "canonical" enumeration of the vectors e_i^n; that is, $e_1 = e_1^1$, $e_2 = e_1^2$, $e_3 = e_2^2$, $e_4 = e_1^3$ an so on. Then (e_i) is a Schauder basis of X. This basis is normalized and 1-unconditional.

For $n \geq 1$, denote by $P_n : X \to X_n$ the canonical projection of X onto X_n. Since $K(\mathbf{e}) = 1$, we have $\|x\| = \sup \{\|P_n(x)\|, n \in \mathbb{N}\}$ for every $x \in X$..

We want (e_i) *not* to be equivalent to the canonical basis of c_0. For that it is enough to ensure that

$$\limsup_{m \to \infty} \left\| \sum_{i=1}^m e_i \right\| = \infty \,;$$

and this will hold if the sequence (p_n) satisfies

$$\lim_{n \to \infty} n^{1/p_n} = \infty \,.$$

Having fixed (p_n) in this way, we now show that every normalized block-sequence of $(e_i)_{i \geq 1}$ is strongly Rc_0-saturated for every $R \geq 1$. So let us fix $R \geq 1$, and let $(y_i)_{i \geq 1}$ be a normalized block-sequence of (e_i), and let $(I_i)_{i \geq 1}$ be a sequence of pairwise disjoint subsets of $\mathbb{N}$ such that, for some $N \geq 1$, all sequences $(y_j)_{j \in I_i}$ are of type Nc_0.

We claim that for any $k \geq 1$, one can find a positive integer n_k such that, for every $n \geq n_k$ and every family $(x_i)_{1 \leq i \leq k}$ of disjointly supported elements of $X_n \cap \overline{B}(0, N)$, we have

$$(4.2) \qquad \left\| \sum_{i=1}^k x_i \right\| \leq NR \,.$$

Indeed, denoting by J_i the support of x_i and writing $x_i = \sum_{q \in J_i} x_{i,q} e_q^n$, we have

$$\left\| \sum_{i=1}^k x_i \right\| = \left[\sum_{i=1}^k \left(\sum_{q \in J_i} |x_{i,q}^{p_n}| \right) \right]^{1/p_n} \leq \left(\sum_{i=1}^k N^{p_n} \right)^{1/p_n} \leq N \, k^{1/p_n} \,.$$

So inequality (4.2) is satisfied as soon as $k^{1/p_n} \leq R$, which holds true when n is sufficiently large since $p_n \to \infty$ as $n \to \infty$.

Since (y_i) is a block-sequence of (e_i), there exists for every $k \geq 1$ an integer $j_k \geq 1$ such that, for every $i \geq j_k$,
$$P_n(y_i) = 0 \quad \text{if } n < n_k \,.$$
Hence, for every $k \geq 1$, one can find an integer $i_k \geq 1$ such that
$$P_n(y_j) = 0 \quad \text{for } j \in I_{i_k} \quad \text{if } n < n_k \,.$$

To conclude the proof, it is now enough to show that the sequence $(y_j)_{j \in \cup_{k \geq 1} I_{i_k}}$ is of type NRc_0; and since this sequence is normalized and 1-unconditional, we just have to show that the norm of any finite sum built on it does not exceed NR.

Every finite sum w built on $(y_j)_{j \in \cup_{k \geq 1} I_{i_k}}$ can be written as
$$w = \sum_{1 \leq k \leq m} w_k \,,$$
where $m \geq 1$ is an integer and w_k is a finite sum built on $(y_j)_{j \in I_{i_k}}$. It follows then from the definition of the norm on X that we have
$$\|w\| = \left\| \sum_{1 \leq k \leq m} w_k \right\| = \sup_{n \in \mathbb{N}} \left\| P_n\Big(\sum_{1 \leq k \leq m} w_k \Big) \right\| \,.$$

For $n \geq n_m$ it follows from the definition of n_m that $\left\| P_n(\sum_{1 \leq k \leq m} w_k) \right\| \leq NR$.

For $n_{i-1} \leq n < n_i$, with $1 \leq i \leq m$, we have $P_n(\sum_{1 \leq k \leq m} w_k) = P_n(\sum_{1 \leq k \leq i-1} w_k)$, and so it follows from the definition of n_{i-1} that $\left\| P_n(\sum_{1 \leq k \leq m} w_k) \right\| \leq NR$. Hence, we do get $\|w\| \leq NR$ for any finite sum w built on $(y_j)_{j \in \cup_{k \geq 1} I_{i_k}}$, which concludes the proof. $\qquad \square$

Remark. As it turns out, the above sequence $\mathbf{e}$ does not satisfy any (δ, R)-condition, and hence $Q^+(\mathbf{e})$ is Haar null. Indeed if $\delta > 0$ is given then, for any set $I \subset \{1, \dots, n\}$ of cardinality δn we have $\left\| \sum_{i \in I} e_i^n \right\| \geq (\delta n)^{1/p_n}$, which tends to ∞ by the choice of the sequence (p_n). By Proposition 4.6, it follows that $\mathbf{e}$ cannot satisfy any (δ, R)-condition. Therefore, this example leaves our "main conjecture" open.

5. QUOTIENTS

We saw in the previous section that if a Banach space X admits an unconditional basis X whose positive cone is not Haar null, then X is c_0-saturated. One of the main objectives of the present section is to prove the following more precise result.

Theorem 5.1. *Let X be a Banach space, and assume that X admits an unconditional basis whose positive cone is not Haar null. Then X has c_0-saturated quotients; that is, every quotient space of X is c_0-saturated.*

To put this result into perspective, it is worth mentioning that the class of Banach spaces with c_0-saturated quotients seems far from being completely understood.

Odell showed in [O] that the Schreier space $\mathcal{S}$ has this property, and asked whether this holds true for every c_0-saturated Banach space having an unconditional and shrinking Schauder basis. Leung [Leu] first gave a negative answer to this question, by constructing a c_0-saturated space with a shrinking unconditional basis admitting a quotient isomorphic to ℓ_2. This was generalized by Gasparis [Ga], who obtained the same result for any ℓ_p, $1 < p < \infty$.

Theorem 5.1 just gives one class of examples, possibly consisting of the single space c_0 (!) In any event, it shows that the c_0-saturation of quotients is a property strictly weaker than the non–Haar negligibility of the positive cone (by Odell's result and Corollary 4.8), and that the positive cones of the spaces constructed in [Leu] and [Ga] are Haar null.

As a matter of fact, we will deduce Theorem 5.1 from a more general result having little to do with Haar negligibility. Let us first introduce the following terminology.

Definition 5.2. A Banach space X will be said to have *property* (P) if every weakly null semi-normalized sequence in X admits a subsequence equivalent to the canonical basis of c_0.

It is clear that in this definition, one can replace "semi-normalized" by "normalized".

Note that if X admits a Schauder basis $\mathbf{e}$ then, by Bessaga-Pelczynki's selection principle, property (P) can be tested by looking only at those weakly null normalized sequences which are block-sequence of $\mathbf{e}$. By Theorem 4.16 (or Corollary 4.12) it follows that if the basis $\mathbf{e}$ is unconditional and has a non–Haar null positive cone, then X has property (P).

Note also that if X has separable dual, then every quotient space E of X with property (P) is c_0-saturated: indeed, any infinite-dimensional subspace of E contains a weakly null normalized sequence (because it has separable dual), and hence a subspace isomorphic to c_0 by property (P).

From these two remarks, it is clear that the following result implies Theorem 5.1.

Theorem 5.3. *Let X be a Banach space admitting an unconditional shrinking Schauder basis. If X has property* (P) *then all quotients of X have property* (P).

Indeed, assume that X is a Banach space admitting an unconditional basis $\mathbf{e}$ whose positive cone is not Haar null. Then X has property (P) by Corollary 4.12. Moreover, the basis $\mathbf{e}$ is shrinking by corollary 4.2. By Theorem 5.3, all quotients of X have property (P), and hence all these quotients are c_0-saturated since X has separable dual.

The proof of Theorem 5.3 relies on the following result from [O].

Proposition 5.4. *Let X be a Banach admitting a normalized, shrinking and 1-unconditional basis $(e_i)_{i \geq 1}$. Let T be a continuous surjective linear operator from X onto another Banach space Y, and let C be any finite constant such that $\overline{B}_Y(0, 1) \subseteq C\, T\left(\overline{B}_X(0, 1)\right)$. Let also $(\varepsilon_i)_{i \geq 1}$ be a decreasing sequence of positive real numbers tending to 0.*

Then, given any weakly null sequence $(y_i)_{i \geq 1}$ *in the unit ball of* Y, *one can find a subsequence* (y_i') *of* (y_i) *and an increasing sequence of integers* $(p_i)_{i \geq 0}$ *with* $p_0 = 0$ *satisfying the following property:*

For every sequence of real numbers $(a_i)_{i \geq 1}$ *such that* $\left\| \sum\limits_{i=1}^{\infty} a_i y_i' \right\| \leq 2$, *there exists an increasing sequence of integers* $(r_i)_{i \geq 0}$ *with* $r_0 = 0$ *and a sequence* $(x_i)_{i \geq 1} \subset X$ *such that*

(1) $p_i < r_i < p_{i+1}$ *for every* $i \geq 1$;
(2) $x_i \in \text{span}\left(e_{r_{i-1}}, \dots, e_{r_i}\right)$;
(3) $\|T(x_i) - a_i y_i'\| \leq \varepsilon_i$;
(4) *the series* $\sum x_i$ *is convergent and* $\left\| \sum\limits_{i=1}^{\infty} x_i \right\| \leq 2C$.

Proof of theorem 5.3. Fix a normalized shrinking unconditional basis $\mathbf{e} = (e_i)_{i \geq 1}$ for X. Since property (P) is invariant under renormings, we may assume that $\mathbf{e}$ is 1-unconditional.

Let Y be any quotient of X, and let $T : X \to Y$ be the canonical quotient map. Note that $\overline{B}_Y(0, 1) \subseteq T\left(\overline{B}_X(0, 1)\right)$.

Let $(y_i)_{i \geq 1}$ be a weakly null normalized sequence in Y. Replacing this sequence by a subsequence we may suppose that (y_i) is a basic sequence, with basis constant K.

We apply Proposition 5.4 to (y_i), with $\varepsilon_i = \frac{1}{2K} 2^{-i}$; this gives a subsequence (y_i') of (y_i) and an increasing sequence of integers $(p_i)_{i \geq 1}$.

For $i, n \geq 1$ denote by $\delta_{i,n}$ the usual "Kronecker symbol". Then $\| \sum\limits_{i=1}^{\infty} \delta_{i,n} y_i' \| = 1 \leq 2$ for every $n \geq 1$. Hence one can find for each $n \geq 1$ a sequence $(x_{i,n})_{i \geq 1} \subset X$ such that

(1) $x_{i,n} \in [e_r; \ p_{i-1} < r < p_{i+1}]$;
(2) $\| \sum\limits_{i=1}^{\infty} x_{i,n} \| \leq 2$, and in particular $\|x_{n,n}\| \leq 2$;
(3) $\|T(x_{n,n}) - y_n'\| \leq \varepsilon_n$.

Set $u_i := x_{2i,2i}$ and $v_i := y_{2i}'$. Note that by (3), we have $\|u_i\| \geq \frac{1 - \varepsilon_{2i}}{\|T\|} \geq \frac{3}{4\|T\|}$, so that (u_i) is a semi-normalized block-sequence of (e_i).

By the definition of (u_i) and (v_i), we have

$$\sum_{i=1}^{\infty} \|T(u_i) - v_i\| \leq \sum_{i=1}^{\infty} \varepsilon_{2i} < \frac{1}{2K}.$$

Since $(v_i)_{i \geq 1}$ is a normalized basic sequence, it follows (by the principle of small perturbations, Lemma 2.2) that $(T(u_i))$ is a basic sequence equivalent to (v_i) and that for every sequence of real numbers (a_i) such that the series $\sum a_i v_i$ is convergent we have:

$$\left(1 - 2K \sum_{i=1}^{\infty} \|v_i - T(u_i)\|\right) \left\| \sum_{i=1}^{\infty} a_i v_i \right\| \leq \left\| \sum_{i=1}^{\infty} a_i T(u_i) \right\| \leq 2K \left\| \sum_{i=1}^{\infty} a_i v_i \right\|.$$

In particular, there exists a constant $c > 0$ such that for any such sequence (a_i):

$$c \left\| \sum_{i=1}^{\infty} a_i v_i \right\| \leq \|T\| \times \left\| \sum_{i=1}^{\infty} a_i u_i \right\|.$$

Now, observe that the sequence (u_i) is weakly null, being a bounded blok-sequence of the shrinking Schauder basis $\mathbf{e}$. Since (u_i) is also semi-normalized and X has property (P), it follows that (u_i) has a subsequence equivalent to the canonical basis of c_0. The above inequality then implies that the corresponding subsequence of (v_i) is a subsequence of (y_i) equivalent to the canonical basis of c_0. $\square$

To conclude this section (and although this has nothing to do with Haar negligibility), we now show that property (P) is "almost" a three spaces property.

Proposition 5.5. *Let X be a Banach space admitting a shrinking unconditional basis, and let F be a closed subspace of X. If F et X/F have property* (P)*, then X has property* (P).

Proof. Let $\mathbf{e} = (e_i)_{i \geq 1}$ be a normalized, shrinking unconditional basis for X, and assume without loss of generality that $\mathbf{e}$ is 1-unconditional. Also, set $Y := X/F$ and denote by $T : X \to Y$ the canonical quotient map.

Let $(x_i)_{i \geq 1}$ be a weakly null normalized sequence in X. To show that (x_i) admits a subsequence equivalent to the canonical basis of c_0, we distinguish two cases.

Case 1. $\inf_i \|T(x_i)\| = 0$.

In this case, we may assume (upon extracting a subsequence) that $\lim_{i \to \infty} \|T(x_i)\| = 0$. Then there exists a sequence $(f_i) \subset F$ such that $\|x_i - f_i\| \to 0$. The sequence (f_i) is weakly null, and since (x_i) is normalized we may assume that it is also normalized. Since F has property (P), (f_i) has a subsequence (f_i') equivalent to the canonical basis of c_0. By the principle of small perturbations (x_i) has a subsequence equivalent to some subsequence of (f_i'), which gives the required result.

Case 2. $\inf_i \|T(x_i)\| > 0$.

In this case, $(T(x_i))$ is a a weakly null semi-normalized sequence in Y. Since Y has property (P), we may assume that this sequence is equivalent to the canonical basis of c_0. Choose a constant $M > 0$ such that for every finite set $I \subset \mathbb{N}$, we have

$$\left\| \sum_{i \in I} \frac{1}{M} T(x_i) \right\| \leq 2.$$

Since $\overline{B}_Y(0,1) \subset T\left(\overline{B}_X(0,1)\right)$, we may apply Proposition 5.4 with $C := 1$ and $y_i := \frac{1}{M} T(x_i)$, taking e.g. $\varepsilon_i := 2^{-i}$. This gives a subsequence $(y_i')_{i \geq 1}$ of $\left(\frac{1}{M} T(x_i)\right)_{i \geq 1}$ and an

increasing sequence of positive integers $(p_i)_{i\geq 0}$. We denote by $(x'_i)_{i\geq 1}$ the subsequence of (x_i) such that

$$y'_i = \frac{1}{M}\, T(x'_i)\,.$$

For each $n \geq 1$, we have $\|\sum_{i=1}^{n} y'_i\| \leq 2$. So there exists a sequence $(x_{i,n})_{i\geq 1} \subset X$ such that

- $x_{i,n} \in E_i := [e_r;\ p_{2i-1} < r < p_{2i+1}]$;
- $\left\|T(x_{i,n}) - y'_i\right\| \leq \varepsilon_i$ if $i \leq n$;
- $\left\|\sum_{i=1}^{\infty} x_{i,n}\right\| \leq 2$, and hence $\left\|\sum_{i\in I} x_{i,n}\right\| \leq 2$ for every finite set $I \subset \mathbb{N}$.

For each fixed $i \geq 1$, the sequence $(x_{i,n})_{n\geq 1}$ is a bounded sequence in the finite-dimensional space E_i; in fact $\|x_{i,n}\| \leq 2$. Hence (by a diagonal argument) we may assume that $(x_{i,n})_{n\geq 1}$ converges to some $\bar{x}_i \in E_i$ for every $i \geq 1$. Then $\|\bar{x}_i\| \leq 2$ and $\|T(\bar{x}_i) - y'_i\| \leq \varepsilon_i$. Since (y') is semi-normalized, the latter inequality implies that $\liminf_i \|\bar{x}_i\| > 0$; hence, we may assume that $(\bar{x}_i)$ is semi-normalized.

Now set $u_i := \bar{x}_{2i}$ and $v_i := \frac{1}{M} x'_{2i}$. Observe that (u_i) is a semi-normalized block-sequence of $\mathbf{e}$. For every finite set $I \subset \mathbb{N}$, we have

$$\left\|\sum_{i\in I} u_i\right\| = \lim_{n\to\infty} \left\|\sum_{i\in I} x_{2i,n}\right\| \leq 2\,.$$

Since (u_i) is unconditional (being a block-sequence of $\mathbf{e}$) and semi-nomalized, it follows that (u_i) is equivalent to the canonical basis of c_0. Moreover, since $\|T(\bar{x}_i) - y'_i\| \leq \varepsilon_i$ for all i, we have

$$\lim_{i\to\infty} T(u_i - v_i) = 0\,.$$

Set $z_i := u_i - v_i$. If there exists a subsequence of $(z_i)_{i\geq 1}$ which converges to 0, the principle of small perturbations tells us that some subsequence of (v_i) is equivalent to (u_i), i.e to the canonical basis of c_0, and the conclusion follows since $v_i = \frac{1}{M} x_i$. Otherwise (disgarding finitely many terms) the sequence (z_i) is semi-normalized, and hence we may apply Case 1 to it; so there exists a subsequence of (z_i) which is equivalent to the canonical basis of c_0. It is then easily checked that the same property holds true for (v_i), and hence for (x_i). $\qquad\square$

6. Positive cones and Gaussian measures

6.1. **Gaussian test measures.** In this section we intend to produce "explicit" test measures for Haar null positive cones. More precisely we will see that under reasonable assumptions, it is possible to construct test measures which are *Gaussian* measures.

Gaussian measures on Banach spaces may be defined as follows: a Borel probability measure μ on a separable Banach space X is Gaussian if and only if it is the distribution

of an X-valued random variable of the form

$$\xi(\omega) = x_0 + \sum_{i=1}^{\infty} g_i(\omega)\, x_i\,,$$

where $x_i \in X$, the g_i are independent real-valued random variable (defined on the same probability space $(\Omega, \mathfrak{A}, \mathbb{P})$) having standard Gaussian distribution $\gamma = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt$, and the series is almost surely convergent. This means that for any Borel set $A \subset X$, we have

$$\mu(A) = \mathbb{P}(\xi \in A)\,.$$

A Gaussian measure μ is said to be *centred* if $x_0 = 0$ in the above representation, which means that $\int_X x^* d\mu = 0$ for every continuous linear functional $x^* \in X^*$. The measure μ is *nondegenerate* if $\mu(V) > 0$ for every open set $V \neq \emptyset$; in the above representation, this means that the linear span of the $x_i, i \geq 1$ is dense in X. For more information on Gaussian measures, we refer to [Bog] or [CTV].

In the remaining of this section, we fix a normalized basic sequence $\mathbf{e}$ whose positive cone is Haar null, and we denote by X the Banach space generated by $\mathbf{e}$. For the sake of notational simplicity, we write Q^+ instead of $Q^+(\mathbf{e})$. For technical reasons, we will in fact mostly deal with the *negative* cone $Q^- := -Q^+$.

We will consider Gaussian measures of a very special type. Let us fix once and for all a sequence independent real-valued random variable (defined on the same probability space $(\Omega, \mathfrak{A}, \mathbb{P})$) having standard Gaussian distribution. We denote by $\mathfrak{R}(\mathbf{e})$ the family of all sequences of positive real numbers $\mathfrak{r} = (r_i)_{i \geq 1}$ such that the random series $\sum r_i g_i e_i$ is almost surely convergent. If $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$, the corresponding Gaussian measure is denoted by $\mu_{\mathfrak{r}}$:

$$\mu_{\mathfrak{r}} \sim \sum r_i g_i e_i\,.$$

Any such measure will be called a *diagonal* Gaussian measure. Note that $\mu_{\mathfrak{r}}$ is centred, and also nondegenerate since $r_i \neq 0$ for all i.

We restrict ourselves to diagonal measure because this allows to compute very easily the measure of a translate of the negative cone. Indeed, if $x = \sum_1^{\infty} x_i e_i \in X$ then, by independance, we have

$$(6.1) \qquad\qquad \mu(x + Q^-) = \prod_{i \geq 1} \mathbb{P}\left(g_i \leq \frac{x_i}{r_i} \right).$$

Having set these notations, we are going to prove the following result.

Theorem 6.1. *Assume that the basic sequence $\mathbf{e}$ is unconditional. Let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$, and assume that there exists a permutation π_0 of the integers and a constant $\alpha > 0$ such that $r_{\pi_0(i)} = O\left(i^{-\alpha}\right)$. Then $\mu_{\mathfrak{r}}$ is a test measure for $Q^- = Q^-(\mathbf{e})$ if and only if the series $\sum \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)}$ is divergent.*

When $\mathbf{e}$ is the canonical basis of ℓ_p, it is well-known that a Gaussian series $\sum r_i g_i e_i$ is almost surely convergent in ℓ_p if and only if $\sum_1^\infty |r_i|^p < \infty$; see [CTV, Chapter V]. Applying Theorem 6.1, we get

Corollary 6.2. *Assume that $\mathbf{e}$ is the canonical basis of ℓ_p, and let $\mathfrak{r} = (r_i)$ be a sequence of positive real numbers such that $\sum_1^\infty r_i^p < \infty$. Denote by $(\bar{r}_i)$ the decreasing rearrangement of (r_i). Then, the measure $\mu_\mathfrak{r}$ is a test measure for Q^- if and only if*

$$\sum_{i \geq 1} \log(i)^{p/2} \, \bar{r}_i^p = \infty \, .$$

Proof. This follows immediately from Theorem 6.1 because $\bar{r}_i = o\left(i^{-1/p}\right)$.

$\square$

To prove Theorem 6.1, we basically have to determine whether an expression like (6.1) above can be positive or not. The following remark shows that it is enough to consider vectors $x \in X$ having *positive* coefficients on the basis $\mathbf{e}$. The set of all such vectors x will be denoted by Q^{++}:

$$Q^{++} = \left\{ x = \sum_{i=1}^\infty x_i e_i \in X; \ x_i > 0 \text{ for all } i \geq 1 \right\} \, .$$

Remark 6.3. *Let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$. For any $a \in X$, one can find $x \in Q^{++}$ such that $\mu_\mathfrak{r}(x + Q^-) \geq \mu_\mathfrak{r}(a + Q^-)$. In particular, if $\mu_\mathfrak{r}$ is not a test measure for Q^- then there exists $x \in Q^{++}$ such that $\mu_\mathfrak{r}(x + Q^-) > 0$.*

Proof. Write $a = \sum_1^\infty a_i e_i$ and assume without loss of generality that $\mu_\mathfrak{r}(a + Q^-) > 0$. Set $I_a^- = \{i \in \mathbb{N}; \ , a_i \leq 0\}$ and $I_a^+ = \{i \in \mathbb{N}; \ a_i > 0\}$; then

$$\mu_\mathfrak{r}(a + Q^-) = \prod_{i \in I_a^-} \mathbb{P}\left(g_i \leq \frac{a_i}{r_i}\right) \times \prod_{i \in I_a^+} \mathbb{P}\left(g_i \leq \frac{a_i}{r_i}\right) \, ,$$

and hence

$$\mu_\mathfrak{r}(a + Q^-) \leq \prod_{i \in I_a^-} \mathbb{P}\left(g_i \leq \frac{a_i}{r_i}\right) \leq 2^{-|I_a^-|} \, .$$

Hence, the set I_a^- is finite. So if we set $x_i := 1$ if $i \in I_a^-$ and $x_i := a_i$ if $i \in I_a^+$, then $x := \sum_{i \geq 1} x_i e_i$ is a well-defined element of Q^{++} such that $\mu_\mathfrak{r}(x + Q^-) \geq \mu_\mathfrak{r}(a + Q^-)$. $\square$

In view of this remark, in what follows we will only consider translates of the negative cone Q^- by elements of Q^{++}.

Lemma 6.4. *Let $\mathfrak{r} = (r_i) \in \mathfrak{R}(\mathbf{e})$, and let $x = \sum_{i \geq 1} x_i e_i \in Q^{++}$. Then*

$$\mu_\mathfrak{r}(x + Q^-) > 0 \quad \text{if and only if} \quad \sum_{i \geq 1} \frac{e^{-\frac{1}{2}(x_i/r_i)^2}}{(x_i/r_i)} < \infty \, .$$

Proof. We have $\mu(x + Q^-) = \prod_{i \geq 1} \mathbb{P}\left(g_i \leq \frac{x_i}{r_i}\right) = \prod_{i \geq 1}\left(1 - \mathbb{P}\left(g_i \geq \frac{x_i}{r_i}\right)\right)$. Therefore, $\mu(x + Q^-) > 0$ if and only if

$$\sum_{i \geq 1} \mathbb{P}\left(g_i \geq \frac{x_i}{r_i}\right) < \infty.$$

Set $g(t) := \frac{e^{-t^2/2}}{\sqrt{2\pi}}$ and $G(a) = \int_a^\infty g(t)dt$, $a > 0$. Then $\mathbb{P}\left(g_i \geq \frac{x_i}{r_i}\right) = G\left(\frac{x_i}{r_i}\right)$. Now, it is well-known that

$$G(a) \sim \frac{e^{-a^2/2}}{\sqrt{2\pi}\,a} \quad \text{as } a \to \infty,$$

and the lemma follows. $\qquad\square$

This lemma allows now to state the following "comparison criterion".

Lemma 6.5. *Let $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$, and let $x \in Q^{++}$. Let also π be a permutation of the integers.*
- *If $\limsup\limits_{i \to \infty}\left(\frac{x_{\pi(i)}}{r_{\pi(i)}}\log(i)^{-1/2}\right) < \sqrt{2}$, then $\mu_\mathfrak{r}(x + Q^-) = 0$.*
- *If $\liminf\limits_{i \to \infty}\left(\frac{x_{\pi(i)}}{r_{\pi(i)}}\log(i)^{-1/2}\right) > \sqrt{2}$, then $\mu_\mathfrak{r}(x + Q^-) > 0$.*

Proof. Set $b_i := \frac{x_i}{r_i}$ and $a_i = \frac{x_{\pi(i)}}{r_{\pi(i)}}$. Then $\mu(x + Q^-) > 0$ if and only if $\sum_1^\infty \frac{e^{(b_i)^2/2}}{b_i} < \infty$, which is equivalent to the convergence of the series $\sum \frac{e^{-(a_i)^2/2}}{a_i}$. We need the following

Fact. For any $M > 0$, set $h_M(t) := g(M \log(t)^{1/2})$, where $g(t) = \frac{e^{-t^2/2}}{t}$. Then the series $\sum_{i \geq 1} h_M(i)$ is convergent if $M > \sqrt{2}$, and divergent if $M < \sqrt{2}$

Proof of Fact. The fonction h_M is positive and decreasing on $(0, \infty)$, and so the convergence of the series $\sum_{i \geq 1} h_M(i)$ is equivalent to the convergence of the integral $\int_1^\infty h_M(t)dt$. A simple computation gives $h'_M(t) = -h_M(t)\left(\frac{M^2}{2t} + \frac{1}{2t\log(t)}\right)$, and so $\frac{h'_M(t)}{h_M(t)} \sim -\frac{M^2}{2t}$ as $t \to \infty$. It follows that the integral $\int_1^\infty h_M(t)$ is convergent if $M > \sqrt{2}$ and divergent if $M < \sqrt{2}$. $\qquad\square$

If $\liminf\limits_{i \to \infty}\left(a_i \log(i)^{-1/2}\right) > \sqrt{2}$, take $M > \sqrt{2}$ such that $a_i \geq M \log(i)^{1/2}$ for i sufficiently large, say $i \geq i_M$. Since $g(t) = \frac{e^{-t^2/2}}{t}$ is decreasing on $(0, \infty)$, we obtain

$$\sum_{i \geq i_M} \frac{e^{-(a_i)^2/2}}{a_i} = \sum_{i \geq i_M} g(a_i) \leq \sum_{i \geq i_M} g\left(M \log(i)^{1/2}\right) = \sum_{i \geq i_M} h_M(i),$$

and hence $\mu_\mathfrak{r}(x + Q^-) > 0$ by the above Fact.

If $\limsup\limits_{i \to \infty}\left(a_i \log(i)^{-1/2}\right) > \sqrt{2}$, one gets in the same way that $\mu_\mathfrak{r}(x + Q^-) > 0$.

$\qquad\square$

From Lemma 6.5, we immediately deduce

Corollary 6.6. *Let* $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$. *If there exists a permutation* π *of the integers such that the series* $\sum_{i \geq 1} \log(i)^{1/2} r_{\pi(i)} e_{\pi(i)}$ *converges in* X, *then* $\mu_{\mathfrak{r}}$ *is not a test measure for* Q^-.

Proof. If we set $x_M := M \sum_{i \geq 1} \log(i)^{1/2} r_{\pi(i)} e_{\pi(i)}$, then $\mu_{\mathfrak{r}}(x_M + Q^-) > 0$ for any $M > \sqrt{2}$. $\qquad\square$

The next lemma gives a kind of converse to Corollary 6.6.

Lemma 6.7. *Assume that the basic sequence* $\mathbf{e}$ *is unconditional, and let* $\mathfrak{r} \in \mathfrak{R}(\mathbf{e})$. *If* $\mu_{\mathfrak{r}}$ *is not a test measure for* Q^- *then, for every permutation* π *of the integers and every* $\alpha > 0$, *there exists a set* $I = I_{\pi,\alpha} \subset \mathbb{N}$ *such that* $\sum_{i \in I} \frac{1}{i^\alpha \log(i)^{1/2}} < \infty$ *and the series* $\sum_{i \notin I} \log(i)^{1/2} r_{\pi(i)} e_{\pi(i)}$ *is convergent.*

Proof. Assume that μ is not a test measure Q^-, and choose a point $x \in Q^{++}$ such that $\mu(x + Q^-) > 0$. Write $x = \sum_{i \geq 1} a_i r_i e_i$, so that $a_i > 0$ for all i. By Lemma 6.4, we know that the series $\sum \frac{e^{-(a_i)^2/2}}{a_i}$ is convergent.

Now, fix $\alpha > 0$ and a permutation π of the integers. Set

$$I := \left\{ i \in \mathbb{N}; \ a_{\pi(i)} \leq \sqrt{2\alpha} \log(i)^{1/2} \right\}.$$

By unconditionality and the definition of I, the series $\sum_{i \notin I} \log(i)^{1/2} r_{\pi(i)} e_{\pi(i)}$ is convergent. Moreover, we have

$$\sum_{i \in I} \frac{1}{i^\alpha \log(i)^{1/2}} = \sum_{i \in I} \frac{e^{-\alpha \log(i)}}{\log(i)^{1/2}} \leq \sum_{i \in I} \frac{e^{-(a_{\pi(i)})^2/2}}{a_{\pi(i)}} < \infty.$$

$\qquad\square$

We can now give the

Proof of Theorem 6.1. Taking α smaller if necessary, we may assume that in fact

$$r_{\pi_0(i)} = O\left(\frac{1}{i^\alpha \log(i)} \right).$$

By Corollary 6.6, we know that if $\mu_{\mathfrak{r}}$ is a test measure for Q^-, then $\sum \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)}$ is divergent.

Conversely, assume that $\mu_{\mathfrak{r}}$ is not a test measure for Q^-. Then, by Lemma 6.7, there exists a set $I \subset \mathbb{N}$ such that $\sum_{i \in I} \frac{1}{i^\alpha \log(i)^{1/2}} < \infty$ and the series $\sum_{i \notin I} \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)}$ is convergent.

Since $r_{\pi_0(i)} = O\left(\frac{1}{i^\alpha \log(i)} \right)$, we have $\sum_{i \in I} \| \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)} \| = \sum_{i \in I} \log(i)^{1/2} r_{\pi_0(i)} < \infty$. So the series $\sum_{i \in I} \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)}$ is convergent, and altogether the whole series $\sum \log(i)^{1/2} r_{\pi_0(i)} e_{\pi_0(i)}$ is convergent. $\qquad\square$

6.2. **Positive cones are not Gauss null.** Besides Haar negligibility, there are many natural notions of smallness for subsets of Banach spaces. One of the most useful ones is *Gauss negligibility*, which was introduced by R. R. Phelps in [Ph]. A Borel set A in a separable Banach space X is said to be *Gauss null* if $\mu(A) = 0$ for every nondegenerate Gaussian measure μ on X.

Since the family of all Gaussian measures on X is translation-invariant, any Gauss null set is obviously Haar null; more precisely, a Borel set $A \subset X$ is Gauss null exactly when every nondegenerate Gaussian measure on X is a test measure for A.

Gauss negligibility is in fact a *much* stronger notion of smallness than Haar negligibility. For example, a compact set needs not be Gauss null; see [BeLi]. More in the spirit of the present paper, we have the following result.

Proposition 6.8. *Positive cones determined by basic sequences are never Gauss null.*

Proof. Let $\mathbf{f} = (f_i)_{i \geq 1}$ be any normalized basic sequence, and set $X := [\mathbf{f}]$.

Let also $(e_i)_{i \geq 1}$ be the canonical basis of ℓ_2, and fix any vector $x = \sum_{i \geq 1} x_i e_i \in \ell_2$ such that $x_i \neq 0$ for all $i \geq 1$. Let (α_i) be a sequence of positive numbers such that the formula

$$T\left(\sum_{i \geq 1} y_i e_i\right) := \sum_{i \geq 1} \alpha_i \operatorname{sgn}(x_i) \, y_i f_i$$

defines a bounded operator $T : \ell_2 \to X$. (For example we can take any square-summable sequence (α_i)). Note that the operator T has dense range since $\alpha_i x_i \neq 0$ for all i.

Now, choose a sequence $(a_i)_{i \geq 1} \subset (0, 1)$ such that $\prod_{i \geq 1}(1 - a_i) > 0$, and a sequence of real numbers (r_i) such that $0 < \frac{r_i}{x_i} \leq \sqrt{a_i}$ for all $i \geq 1$. The series $\sum_{i \geq 1} r_i e_i$ is convergent in ℓ_2 and so the Gaussian series $\sum_{i \geq 1} r_i g_i e_i$ is almost surely convergent. Let ν be the associated Gaussian measure on ℓ_2:

$$\nu \sim \sum_{i=1}^{\infty} r_i g_i e_i \,.$$

Since the operator $T : \ell_2 \to X$ has dense range, the measure

$$\mu := \nu \circ T^{-1} \sim \sum_{i=1}^{\infty} r_i g_i T e_i$$

is a nondegenerate Gaussian measure on X.

Set

$$H := \left\{ y = \sum_{i \geq 1} y_i e_i \in \ell_2; \ \frac{y_i}{r_i} \geq 0 \text{ for all } i \right\} \,.$$

By the independence of the random variables g_i, we have

$$\nu(-x + H) = \prod_{i \geq 1} \mathbb{P}\big(g_i \geq -x_i/r_i\big)$$

$$\geq \prod_{i \geq 1} \mathbb{P}\big(|g_i| \leq x_i/r_i\big)$$

$$\geq \prod_{i \geq 1} \Big(1 - \mathbb{P}\big(|g_i| > x_i/r_i\big)\Big).$$

Since $\mathbb{P}(|g_i| > x_i/r_i) \leq r_i^2/x_i^2$ by Chebishev's inequality, it follows that

$$\nu(-x + H) \geq \prod_{i \geq 1} \left(1 - \frac{r_i^2}{x_i^2}\right) \geq \prod_{i \geq 1}(1 - a_i) > 0.$$

On the other hand, since $T(H) \subset Q^+(\mathbf{f})$ by the definition of T and since T is one-to-one, we have

$$\mu(T(-x) + Q^+(\mathbf{f})) \geq \mu\big(T(-x + H)\big) = \nu(-x + H).$$

So $\mu(T(-x) + Q^+(\mathbf{f})) > 0$, and hence $Q^+(\mathbf{f})$ is not Gauss null. $\qquad\square$

Remark. Another way of proving Proposition 6.8 is by using a deep result of M. Csörnyei, according to which Gauss negligibility is equivalent to *cube negligibility* (see [Cs] or [BeLi]). Indeed, it is not difficult to check directly that the positive cone of a Schauder basis is never cube-null; see [Mo, Prop 3.5.2]

7. Some questions

(1) Let us say that a Borel set A in a Polish abelian group G *compactivorous* if for every compact set $K \subset G$, one can find $x \in G$ and an open set V such that $V \cap K \neq \emptyset$ and $x + (V \cap K) \subset A$. Does there exist a non–Haar null positive cone (in some Banach space) which is not compactivorous?

(1') Does there exist *any* closed convex non–Haar null set which is not compactivorous? Note that for a quite large class of Polish abelian groups G, one can find a G_δ set in G which is non–Haar null and non–compactivorous (see [M1, Proposition 1.2.2]), but it does not seem clear that there always exists a *closed* set with these properties.

(2) What can be said of a Banach lattice whose positive cone is not Haar null?

(3) Let $\mathbf{e}$ a normalized unconditional basic sequence. Assume that there exists $\delta, R > 0$ such that all block-sequences of $\mathbf{e}$ satisfy the (δ, R)-condition. Does this imply that $\mathbf{e}$ has to be equivalent to the canonical basis of c_0?

(3') In the situation of (3), is it at least true that $Q^+(\mathbf{e})$ is not Haar null?

(4) Regarding our "main conjecture", one may ask if the following weaker result holds true: if $\mathbf{e}$ is an unconditional basic sequence whose positive cone is not Haar null, then the Banach space $[\mathbf{e}]$ is isomorphic to a *subspace* of c_0. The methods of [GKL] might be relevant here.

(5) Is property (P) a three spaces property?

(6) Let $\mathbf{e}$ be basic sequence with a Haar null positive cone. Assume that $Q^+(\mathbf{e})$ admits a Gaussian test measure. Does it admit a *diagonal* such measure?

(7) Does there exist a Haar null positive cone which does not admit any Gaussian test measure?

(8) It is known that if G is a non–locally compact Polish abelian group, then the family of all closed Haar null sets of G is extremely complicated from a descriptive point of view; see [So] and [SR]. What is the descriptive complexity of the family of all basic sequences having a Haar null positive cone?

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