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# Robust feedback switching control: dynamic programming and viscosity solutions\*

Erhan BAYRAKTAR<sup>†</sup>   Andrea COSSO<sup>‡</sup>   Huyên PHAM<sup>§</sup>

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## Abstract

We consider a robust switching control problem. The controller only observes the evolution of the state process, and thus uses feedback (closed-loop) switching strategies, a non standard class of switching controls introduced in this paper. The adverse player (nature) chooses open-loop controls that represent the so-called Knightian uncertainty, i.e., misspecifications of the model. The (half) game switcher versus nature is then formulated as a two-step (robust) optimization problem. We develop the stochastic Perron method in this framework, and prove that it produces a viscosity sub and supersolution to a system of Hamilton-Jacobi-Bellman (HJB) variational inequalities, which envelope the value function. Together with a comparison principle, this characterizes the value function of the game as the unique viscosity solution to the HJB equation, and shows as a byproduct the dynamic programming principle for robust feedback switching control problem.

**MSC Classification:** 60G40, 91A05, 49L20, 49L25.

**Keywords:** model uncertainty, optimal switching, feedback strategies, stochastic games, stochastic Perron's method, viscosity solutions.

## 1 Introduction

Optimal switching is a class of stochastic control problems that has attracted a lot of interest and generated important developments in applied and financial mathematics. Switching control consists in sequence of interventions that occur at random discrete times due to switching costs, and naturally arises in investment problems with fixed transaction costs or in real options. The literature on this topic is quite large and we refer e.g. to [25], [18], [9],

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[19], [2] for a treatment by dynamic programming and PDE methods, to [14], [15], [10] for the connection with reflected BSDE methods, and to [8], [7], [12] for various applications to finance and real options in energy markets.

The standard approach to the study of a switching control problem is to give an evolution for the controlled state process, with assigned drift and diffusion coefficients. These, however, are in practice obtained through estimation procedures and are unlikely to coincide with the *real* coefficients. For this reason, in the present work we study a switching control problem *robust* to a misspecification of the model for the controlled state process. This is formalized as follows: Given  $s \geq 0$ ,  $x \in \mathbb{R}^d$ , and a regime  $i \in \mathbb{I}_m := \{1, \dots, m\}$ , let us consider the controlled system of stochastic differential equations, for  $t \geq s$ :

$$\begin{cases} X_t = x + \int_s^t b(X_r, I_r, u_r) dr + \int_s^t \sigma(X_r, I_r, u_r) dW_r, \\ I_t = i 1_{\{s \leq t < \tau_0\}} + \sum_{n \in \mathbb{N}} \iota_n 1_{\{\tau_n \leq t < \tau_{n+1}\}}. \end{cases} \quad (1.1)$$

The piecewise constant process  $I$  denotes the regime value at any time  $t$ , whose evolution is determined by the controller through the switching control  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$ , while the process  $u$ , decided by nature, brings the uncertainty within the model. In the switching control problem with model uncertainty, the objective of the controller is the maximization of the following functional, over a finite time horizon  $T < \infty$ :

$$\begin{aligned} J(s, x, i; \alpha, u) &:= \mathbb{E} \left[ \int_s^T f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r) dr + g(X_T^{s,x,i;\alpha,u}, I_T^{s,x,i;\alpha,u}) \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau_n}^{s,x,i;\alpha,u}, I_{\tau_n^-}^{s,x,i;\alpha,u}, I_{\tau_n}^{s,x,i;\alpha,u}) 1_{\{s \leq \tau_n < T\}} \right], \end{aligned}$$

playing against nature, described by  $u$ . This leads to the “robust” optimization problem

$$\sup_{\alpha} \left( \inf_u J(s, x, i; \alpha, u) \right). \quad (1.2)$$

What definition for the switching control  $\alpha$  and for  $u$  should we adopt? As a first attempt, if we interpret (1.2) as a game between the controller and nature, it would be reasonable to formulate it in terms of *strategies against controls*, as in the seminal paper by Elliott & Kalton [11]. In this case,  $\alpha$  is a non-anticipating switching strategy, while  $u$  is an open-loop control. Then, the switcher knows the current and past choices made by the opponent (see Section 4.2 below for more details on this formulation). In the context of robust optimization, the controller does not know in general the choice made by nature. However, he knows the evolution of  $X$  and also of  $I$  (by keeping track of his previous actions). For this reason, inspired by [22], we take  $\alpha$  as a *feedback switching control* rather than a non-anticipating strategy (namely, we present a *feedback formulation* of a switching control problem, which is quite uncommon in the literature). On the other hand,  $u$  can be an *open-loop control* (nature is aware of the all information at disposal). This leads to the formulation of *robust feedback* switching control problem that we compare in the last section of this paper with the Elliott-Kalton one, in a specific example. Notice that, in both formulations, when the controller picks a strategy or a feedback control, then nature acts in a way to minimize the payoff knowing the controller’s action (as a consequence of

the  $\inf_u$  in (1.2)), which looks unrealistic, since nature is disinterested and has no-payoff. Nevertheless, the fact that nature takes a decision to contrast the controller's choice means that we look at the *worst-case scenario* for the switcher, consistently with the robustness characteristic of the optimization problem.

We develop the stochastic Perron method in this framework of robust feedback switching control. This method was initially introduced to analyze linear problems in [3], Dynkin games in [5] and *regular* control problems in [4]. Later on, it was adapted to analyze exit time problems in [21], control problems with state constraints in [20], singular control problems in [6], stochastic differential games in [23] and stochastic control with model uncertainty in [22]. Stochastic Perron's method is similar to a *verification theorem* and avoids having to go through the dynamic programming principle (DPP) first (which is not known a priori in this context) to show that the value function is a solution to the HJB equation. Actually, the DPP is obtained as a byproduct of the stochastic Perron method and comparison principle. Unlike the classical verification theorem, the stochastic Perron does not require the a priori smoothness of the value function. The method is to construct viscosity (semi-) solutions to the HJB equation, which envelope the value function, and relies on the comparison principle of the HJB equation to conclude that the value function is the unique viscosity solution. In order to carry out the construction, one needs to define a suitable family of stochastic supersolutions/subsolutions, the crucial property of which is closedness under minimization/maximization and whose members stay above/below the value function. The technical part of the proof is in showing that the infimum/supremum of the above families give a viscosity subsolution/supersolution to the HJB equation. One of the advantages of the stochastic Perron method is that it allows us to demonstrate that the information available to nature (whether it uses open-loop or feedback controls) does not affect the value of the game. We do this by constructing the class of stochastic supersolutions for an auxiliary problem, which by definition lies above our original value function. Our results here can be thought of as a generalization of the recent work [22], in which the controller uses elementary feedback strategies. In our setting changing the value of control has a switching cost. This changes the nature of the problem as the past action of the controller needs to be stored as a state variable. The presence of this additional state variable brings about several subtle technical issues, which we resolve in this paper. For example, concatenating the feedback switching strategies need to be done with care (not to incur an additional cost at the time of concatenation), which forces us to make appropriate changes in defining the stochastic subsolutions.

The rest of this paper is organized as follows. In Section 2, we provide a rigorous formulation of the robust feedback switching control problem. We develop in Section 3 the stochastic Perron method, and characterize the infimum (supremum) of the stochastic supersolutions (subsolutions) as the viscosity subsolution (supersolution) of the HJB equation. In Section 4, by using a comparison principle under a no free loop condition on the switching costs, we conclude that the value function is the unique viscosity solution to the HJB equation, and obtain as a byproduct the dynamic programming principle. We finally compare the two formulations: robust feedback/Elliott-Kalton, in a specific example, which then gives a counterexample to uniqueness for the HJB equation.

## 2 Modeling a robust switching control problem

### 2.1 Feedback switching system under model uncertainty

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a fixed probability space on which a  $d$ -dimensional Brownian motion  $W = (W_t)_{t \geq 0}$  is defined. For any  $s \geq 0$ , we consider the filtration  $\mathbb{F}^{W,s} = (\mathcal{F}_t^{W,s})_{t \geq s}$ , which is the augmented natural filtration generated by the Brownian increments starting at  $s$ , i.e.,

$$\mathcal{F}_t^{W,s} := \sigma(W_u - W_s, s \leq u \leq t) \vee \mathcal{N}(\mathbb{P}, \mathcal{F}), \quad t \geq s,$$

where  $\mathcal{N}(\mathbb{P}, \mathcal{F}) := \{N \in \mathcal{F} : \mathbb{P}(N) = 0\}$ . For each  $s \geq 0$ , we denote by  $\mathbb{F}^s = (\mathcal{F}_t^s)_{t \geq s}$  another filtration satisfying the usual conditions, which is larger than  $\mathbb{F}^{W,s}$  and keeps  $(W_t - W_s)_{t \geq s}$  a Brownian motion starting at  $s$ .

We fix a finite time horizon  $0 < T < \infty$ . For any  $s \in [0, T]$ , we denote by  $y(\cdot)$  or  $y$  a generic element of the space  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ , where  $\mathcal{L}([s, T]; \mathbb{I}_m)$  denotes the set of càglàd paths valued in  $\mathbb{I}_m$  (notice that the elements of  $\mathcal{L}([s, T]; \mathbb{I}_m)$  are indeed piecewise constant paths, since  $\mathbb{I}_m$  is a discrete set). Let  $\mathbb{B}^s = (\mathcal{B}_t^s)_{s \leq t \leq T}$ , with  $\mathcal{B}_t^s := \sigma(y(u), s \leq u \leq t)$  for any  $t \in [s, T]$ , be the natural filtration of  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . A map  $\tau : C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow [s, T]$  satisfying  $\{\tau \leq t\} \in \mathcal{B}_t^s, \forall t \in [s, T]$ , is called a *stopping rule*.  $\mathcal{T}^s$  denotes the family of all stopping rules starting at  $s$ . For any  $s \in [0, T]$  and  $\tau \in \mathcal{T}^s$ , we define, as usual,

$$\begin{aligned} \mathcal{B}_{\tau^+}^s &:= \{B \in \mathcal{B}_T^s : \forall t \in [s, T], B \cap \{y : \tau(y) \leq t\} \in \mathcal{B}_{t^+}^s\}, \\ \mathcal{B}_\tau^s &:= \{B \in \mathcal{B}_T^s : \forall t \in [s, T], B \cap \{y : \tau(y) \leq t\} \in \mathcal{B}_t^s\}, \end{aligned}$$

where  $\mathcal{B}_{t^+}^s := \cap_{r > t} \mathcal{B}_r^s, t \in [s, T)$ , and  $\mathcal{B}_{T^+}^s := \mathcal{B}_T^s$ . We also denote  $y(T^+) := y(T)$ , for any  $y \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ .

**Definition 2.1 (Feedback switching controls)** Fix  $s \in [0, T]$ . We say that the double sequence  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$  is a feedback switching control starting at  $s$  if:

- $\tau_n \in \mathcal{T}^s$ , for any  $n \in \mathbb{N}$ , and

$$s \leq \tau_0 \leq \dots \leq \tau_n \leq \dots \leq T.$$

Moreover,  $(\tau_n)_{n \in \mathbb{N}}$  satisfies the following property:  $\forall (y_n)_{n \in \mathbb{N}} \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ , with  $y_n(t) = y_{n+1}(t), t \in [s, \tau_n(y_n)]$ , then

$$\tau_n(y_n) = T, \quad \text{for } n \text{ large enough.}$$

- $\iota_n : C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{I}_m$  is  $\mathcal{B}_{\tau_n}^s$ -measurable, for any  $n \in \mathbb{N}$ .

$\mathcal{A}_{s,s}$  denotes the family of all feedback switching controls starting at  $s$ .

**Remark 2.1** Consider a sequence of paths  $(y_n)_{n \in \mathbb{N}}$  as in Definition 2.1. Then, the sequence  $(\tau_n(y_n))_{n \in \mathbb{N}}$  is nondecreasing. Indeed, from Lemma 2.1 below we have  $\tau_n(y_n) = \tau_n(y_{n+1})$ . Since  $\tau_n(y_{n+1}) \leq \tau_{n+1}(y_{n+1})$  from the nondecreasing property of the sequence  $(\tau_n)_{n \in \mathbb{N}}$ , the thesis follows. See also Remark 2.3 below, where the property “ $\tau_n(y_n) = T$ , for  $n$  large enough” is analyzed in detail.  $\square$

**Definition 2.2 (Open-loop controls)** Fix  $s \in [0, T]$ . An open-loop control  $u$  starting at  $s$ , for the nature, is an  $\mathbb{F}^s$ -adapted process  $u: [s, T] \times \Omega \rightarrow U$ . We denote by  $\mathcal{U}_{s,s}$  the collection of all possible open-loop controls, given the initial deterministic time  $s$ .

For any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ ,  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$ ,  $u \in \mathcal{U}_{s,s}$ , we can now write equation (1.1) on  $[0, T]$  as follows:

$$\begin{cases} X_t = x + \int_s^t b(X_r, I_r, u_r) dr + \int_s^t \sigma(X_r, I_r, u_r) dW_r, & s \leq t \leq T, \\ I_t = i 1_{\{s \leq t < \tau_0(X, I, -)\}} + \sum_{n \in \mathbb{N}} \iota_n(X, I, -) 1_{\{\tau_n(X, I, -) \leq t < \tau_{n+1}(X, I, -)\}}, & s \leq t < T, \\ I_T = I_{T-}, \end{cases} \quad (2.1)$$

with  $I_{s-} := I_s$ . Notice that the presence of  $I_{-}$  in place of  $I$  in the arguments of  $\tau_n, \iota_n$  is due to the fact that the choice of  $(\tau_n, \iota_n)$  by the controller is based only on the information coming from the previous switching actions  $(\tau_i, \iota_i)_{0 \leq i \leq n-1}$ . Moreover, the last equation  $I_T = I_{T-}$  in (2.1) means that there is no regime switching at the final time  $T$ . We impose the following assumptions on the coefficients  $b: \mathbb{R}^d \times \mathbb{I}_m \times U \rightarrow \mathbb{R}^d$  and  $\sigma: \mathbb{R}^d \times \mathbb{I}_m \times U \rightarrow \mathbb{R}^{d \times d}$  (in the sequel, we use the notation  $\|A\|^2 = \text{tr}(AA^\top)$  for the Hilbert-Schmidt norm of any matrix  $A$ ).

**(H1)**

- (i)  $b, \sigma$  are jointly continuous on  $\mathbb{R}^d \times \mathbb{I}_m \times U$ .
- (ii)  $b, \sigma$  are uniformly Lipschitz continuous in  $x$ , i.e.,

$$|b(x, i, u) - b(x', i, u)| + \|\sigma(x, i, u) - \sigma(x', i, u)\| \leq L_1 |x - x'|,$$

$\forall x, x' \in \mathbb{R}^d, i \in \mathbb{I}_m, u \in U$ , for some positive constant  $L_1$ .

**Remark 2.2** From Assumption **(H1)** it follows that  $b$  and  $\sigma$  satisfy a linear growth condition in  $x$ , i.e.,

$$|b(x, i, u)| + \|\sigma(x, i, u)\| \leq M_1(1 + |x|),$$

$\forall x \in \mathbb{R}^d, i \in \mathbb{I}_m, u \in U$ , for some positive constant  $M_1$ . □

**Remark 2.3** Fix  $s \in [0, T]$  and  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$ . Let us consider the following properties of the nondecreasing sequence  $(\tau_n)_{n \in \mathbb{N}}$ :

- (i) *Uniformly finite.* There exists  $N \in \mathbb{N}$  such that,  $\forall y \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ ,

$$\tau_n(y) = T, \quad \text{for } n \geq N.$$

- (ii) *Finite along every adaptive sequence.*  $\forall (y_n)_{n \in \mathbb{N}} \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ , with  $y_n(t) = y_{n+1}(t)$ ,  $t \in [s, \tau_n(y_n)]$ ,

$$\tau_n(y_n) = T, \quad \text{for } n \text{ large enough.}$$

(iii) *Finite along every path.*  $\forall y \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ ,

$$\tau_n(y) = T, \quad \text{for } n \text{ large enough.}$$

Condition (i) is the strongest, while (iii) is the weakest. In Definition 2.1 we imposed the intermediate property (ii), since it allows to have a well-posedness result for equation (2.1), which is no longer guaranteed if we require only (iii). To see this latter point, we construct a counter-example. Take  $s = 0$ ,  $T = 1$ , and  $m = 2$  so that  $\mathbb{I}_2 = \{1, 2\}$ . Consider the sequence  $(b_n)_{n \in \mathbb{N}} \subset [0, 1]$  given by

$$b_n = \sum_{j=0}^n \frac{1}{2^{j+2}}, \quad \forall n \in \mathbb{N}.$$

In particular, we have  $b_0 = \frac{1}{4}$ ,  $b_1 = \frac{1}{4} + \frac{1}{8}$ ,  $b_2 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16}$ ,  $\dots$ , and in general

$$b_n = \frac{2^{n+1} - 1}{2^{n+2}}, \quad \forall n \geq 0.$$

Notice that  $(b_n)_{n \in \mathbb{N}}$  is a strictly increasing sequence satisfying  $b_n \nearrow \frac{1}{2}$ , as  $n \rightarrow \infty$ . Now, for every  $y \in C([0, 1]; \mathbb{R}^d) \times \mathcal{L}([0, 1]; \mathbb{I}_2)$  we write  $y = (y^X, y^I)$  with  $y^X \in C([0, 1]; \mathbb{R}^d)$  and  $y^I \in \mathcal{L}([0, 1]; \mathbb{I}_2)$ . Then, we define the sequence  $(\tau_n)_{n \in \mathbb{N}}$  as follows:

$$\tau_n(y) = b_n \mathbf{1}_{\{y \in B_n\}} + \mathbf{1}_{\{y \in B_n^c\}}, \quad \forall y \in C([0, 1]; \mathbb{R}^d) \times \mathcal{L}([0, 1]; \mathbb{I}_2), \quad n \in \mathbb{N},$$

where

$$\begin{aligned} B_0 &= \{y \in C([0, 1]; \mathbb{R}^d) \times \mathcal{L}([0, 1]; \mathbb{I}_2) : y^I(t) = y^I(0), 0 < t \leq b_0\}, \\ B_n &= \{y \in B_{n-1} : y^I(t) = 3 - y^I(b_{n-1}), b_{n-1} < t \leq b_n\}, \quad \forall n \geq 1. \end{aligned}$$

Observe that, since  $y^I(t) \in \mathbb{I}_2$  then  $3 - y^I(t) \in \mathbb{I}_2$ ; moreover, when  $y^I(t) = 1$  then  $3 - y^I(t) = 2$ , while if  $y^I(t) = 2$  then  $3 - y^I(t) = 1$ . We also notice that  $B_n \in \mathcal{B}_{b_n}^0$ , therefore  $\tau_n \in \mathcal{T}^0$ . Furthermore,  $(\tau_n)_{n \in \mathbb{N}}$  is a nondecreasing sequence which verifies property (iii) above: this is due to the fact that every path  $y \in C([0, 1]; \mathbb{R}^d) \times \mathcal{L}([0, 1]; \mathbb{I}_2)$  has only a finite number of jumps, since  $\mathbb{I}_2$  is a discrete set; in other words, any  $y$  belongs to  $B_n^c$  when  $n$  is large enough (e.g., when  $n$  is strictly greater than the number of jumps of  $y$ ). However,  $(\tau_n)_{n \in \mathbb{N}}$  does not satisfy property (ii), as we shall prove below. We also define

$$\iota_n(y) = 3 - y^I(b_n), \quad \forall y \in C([0, 1]; \mathbb{R}^d) \times \mathcal{L}([0, 1]; \mathbb{I}_2), \quad n \in \mathbb{N}.$$

In other words, when  $y^I(b_n) = 1$  then  $\iota_n(y) = 2$ , while when  $y^I(b_n) = 2$  then  $\iota_n(y) = 1$ . Let  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$ , then  $\alpha$  satisfies Definition 2.1, but for property (ii) (see below), even if property (iii) is satisfied. Now, we solve equation (2.1) with  $x \in \mathbb{R}^d$ ,  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$ ,  $u \in \mathcal{U}_{0,0}$ , and  $i = 1 \in \mathbb{I}_2$ . Define the (deterministic) process  $I: [0, 1] \rightarrow \mathbb{I}_2$  as follows, for any  $t \in [0, \frac{1}{2})$ ,

$$I_t = \begin{cases} 1, & 0 \leq t \leq b_0, \\ 2, & b_0 < t \leq b_1, \\ 1, & b_1 < t \leq b_2, \\ 2, & b_2 < t \leq b_3, \\ \vdots & \end{cases}$$



On the other hand, we do not specify  $I$  on  $[\frac{1}{2}, 1]$ , we only require that the limit  $I_{1-} := \lim_{t \uparrow 1} I_t$  exists and we suppose that  $I_1 = I_{1-}$ . Notice that  $I_{\frac{1}{2}-}$  does not exist, therefore  $I \notin \mathcal{L}([0, 1]; \mathbb{I}_2)$ . However, the process  $I$  solves equation (2.1) (viceversa, every process satisfying (2.1) coincides with  $I$  on the interval  $[0, \frac{1}{2}]$ ; in particular, there does not exist a solution process with paths in  $\mathcal{L}([0, 1]; \mathbb{I}_2)$ ). Moreover, under Assumption **(H1)** we can also solve equation (2.1) for  $X$ . Since we did not specify the behavior of  $I$  on the entire interval  $[0, 1]$ , we can not have uniqueness of the solution for (2.1). Nevertheless, we notice that the sequence  $(\tau_n)_{n \in \mathbb{N}}$  does not satisfy property (ii) above. Indeed, let  $y_n(\cdot) := I_{\wedge b_n}$ ,  $n \in \mathbb{N}$ . Then,  $y_n \in \mathcal{L}([0, 1]; \mathbb{I}_2)$ , but  $\tau_n(y_n) < \frac{1}{2}$ , for any  $n$ . This shows that if we only require property (iii), then the well-posedness of equation (2.1) is no longer guaranteed.  $\square$

We now study the well-posedness of equation (2.1), for which we need the following two lemmata.

**Lemma 2.1** *Let  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ , and  $y_1, y_2 \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . If  $y_1(t) = y_2(t)$ ,  $s \leq t \leq \tau(y^1)$ , then:*

(i)  $\tau(y_1) = \tau(y_2)$ .

(ii)  $\iota(y_1) = \iota(y_2)$ , for any  $\mathcal{B}_\tau^s$ -measurable map  $\iota: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{I}_m$ .

**Proof** Let  $t^* := \tau(y_1)$ . We begin noting that if  $B \in \mathcal{B}_{t^*}^s$  and  $y_1 \in B$ , then  $y_2 \in B$ , as well. Since  $\tau$  is a stopping rule, the event  $B := \{y: \tau(y) = t^*\}$  belongs to  $\mathcal{B}_{t^*}^s$ . As  $y_1 \in B$ , we then see that  $y_2 \in B$ , i.e.,  $\tau(y_2) = \tau(y_1)$ .

Let now  $\iota: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{I}_m$  be  $\mathcal{B}_\tau^s$ -measurable. By definition of  $\iota$ , the event  $\tilde{B} := \{y: \iota(y) = \iota(y_1)\}$  belongs to  $\mathcal{B}_\tau^s$ . Therefore,  $B := \tilde{B} \cap \{\tau(y) \leq t^*\} \in \mathcal{B}_{t^*}^s$ . Since  $y_1 \in B$ , from the observation at the beginning of the proof it follows that  $y_2 \in B$ , which implies  $y_2 \in \tilde{B}$ , i.e.,  $\iota(y_2) = \iota(y_1)$ .  $\square$

**Lemma 2.2** *Let  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ , and  $Y = (Y_t)_{s \leq t \leq T}$  be an  $\mathbb{F}^s$ -adapted process valued in  $\mathbb{R}^d \times \mathbb{I}_m$ . Suppose that every path of  $Y$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . Then,  $\tau_Y: \Omega \rightarrow [s, T]$  defined as  $\tau_Y(\omega) := \tau(Y(\omega))$ ,  $\omega \in \Omega$ , is an  $\mathbb{F}^s$ -stopping time. Moreover, if  $\iota: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{I}_m$  is  $\mathcal{B}_\tau^s$ -measurable then  $\iota_Y(\omega) := \iota(Y(\omega))$ ,  $\omega \in \Omega$ , is  $\mathcal{F}_{\tau_Y}^s$ -measurable.*

**Proof.** For any  $t \in [s, T]$ , we notice that the map  $Y$  is measurable from  $(\Omega, \mathcal{F}_t^s)$  into  $(C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m), \mathcal{B}_t^s)$ . Then,  $\{\omega: \tau_Y(\omega) \leq t\} = \{\omega: \tau(Y(\omega)) \leq t\} = \{\omega: Y(\omega) \in \tau^{-1}([s, t])\}$ . Since  $\tau^{-1}([s, t]) \in \mathcal{B}_t^s$ , we have  $\{\omega: Y(\omega) \in \tau^{-1}([s, t])\} \in \mathcal{F}_t^s$ , which implies that  $\tau_Y$  is an  $\mathbb{F}^s$ -stopping time.

Let now  $\iota: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{I}_m$  be  $\mathcal{B}_\tau^s$ -measurable. We have to prove that  $\{\omega: \iota_Y(\omega) = \underline{i}\} \in \mathcal{F}_{\tau_Y}^s$ , for any  $\underline{i} \in \mathbb{I}_m$ , i.e.,  $\{\omega: \iota_Y(\omega) = \underline{i}\} \cap \{\omega: \tau_Y(\omega) \leq t\} \in \mathcal{F}_t^s$ , for any  $\underline{i} \in \mathbb{I}_m$  and  $t \in [s, T]$ . Then, fix  $\underline{i} \in \mathbb{I}_m$  and  $t \in [s, T]$ . We have

$$\begin{aligned} \{\omega: \iota_Y(\omega) = \underline{i}\} \cap \{\omega: \tau_Y(\omega) \leq t\} &= \{\omega: Y(\omega) \in \iota^{-1}(\underline{i})\} \cap \{\omega: Y(\omega) \in \tau^{-1}([s, t])\} \\ &= \{\omega: Y(\omega) \in \{y: \iota(y) = \underline{i}\} \cap \{y: \tau(y) \leq t\}\}. \end{aligned}$$

Since  $\iota$  is  $\mathcal{B}_\tau^s$ -measurable, then  $\{y: \iota(y) = \underline{i}\} \cap \{y: \tau(y) \leq t\} \in \mathcal{B}_t^s$ . Therefore, from the observation at the beginning of the proof, we get the thesis.  $\square$



**Proposition 2.1** *Let Assumption (H1) hold. For any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ ,  $\alpha \in \mathcal{A}_{s,s}$ ,  $u \in \mathcal{U}_{s,s}$ , there exists a unique (up to indistinguishability)  $\mathbb{F}^s$ -adapted process  $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u}) = (X_t^{s,x,i;\alpha,u}, I_t^{s,x,i;\alpha,u})_{s \leq t \leq T}$  to equation (2.1), such that every path of  $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u})$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . Moreover, for any  $p \geq 1$  there exists a positive constant  $C_{p,T}$ , depending only on  $p, T, M_1$  (independent of  $s, x, i, \alpha, u$ ), such that*

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} |X_t^{s,x,i;\alpha,u}|^p \right] \leq C_{p,T} (1 + |x|^p). \quad (2.2)$$

**Remark 2.4** In Proposition 2.1 we require that every path of  $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u})$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$  in order to guarantee that the maps  $\tau_n(X^{s,x,i;\alpha,u}(\omega), I^{s,x,i;\alpha,u}(\omega))$  and  $\iota_n(X^{s,x,i;\alpha,u}(\omega), I^{s,x,i;\alpha,u}(\omega))$  are well-defined for every  $\omega \in \Omega$ ,  $n \in \mathbb{N}$ .  $\square$

**Proof.** Fix  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ ,  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$ ,  $u \in \mathcal{U}_{s,s}$ .

**Step I. Existence.** We begin noting that, since the control  $\alpha$  is of feedback type, we have to construct the solution  $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u})$  and  $\alpha$  simultaneously. To do it we proceed as follows: for any  $N \in \mathbb{N}$ , we solve equation (2.1) controlled by  $u$  and the first  $N$  switching actions  $(\tau_n, \iota_n)_{0 \leq n \leq N-1}$ . This is done by induction on  $N$ . Then, noting that  $(X^N, I^N) = (X^{N-1}, I^{N-1})$  on the stochastic interval  $[s, \tau_{N-1})$ , by pasting together the various solutions we are able to construct a solution  $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u})$  to the original equation (2.1) with the entire switching control  $\alpha$ . We now report the rigorous arguments.

For any  $N \in \mathbb{N}$ , let  $\alpha^N = (\tau_n^N, \iota_n^N)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$  be given by

$$(\tau_n^N, \iota_n^N) := \begin{cases} (\tau_n, \iota_n), & 0 \leq n \leq N-1, \\ (T, \iota_n), & n \geq N. \end{cases}$$

Let  $N = 0$  and consider equation (2.1) controlled by  $\alpha^0$  and  $u$ . Notice that  $I$  is uncontrolled, in particular  $I_t = i$ ,  $s \leq t \leq T$ . Then, it is well-known that under Assumption (H1) there exists a unique (up to indistinguishability)  $\mathbb{F}^s$ -adapted solution  $(X_t^0, I_t^0)_{s \leq t \leq T}$  to this equation, with  $I_t^0 = i$  for any  $t \in [s, T]$ , such that every (not only  $\mathbb{P}$ -a.e., simply choosing an opportune indistinguishable version) path of  $(X^0, I^0)$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ .

Now, let us prove the inductive step. Let  $N \in \mathbb{N} \setminus \{0\}$  and suppose that there exists an  $\mathbb{F}^s$ -adapted solution  $(X^{N-1}, I^{N-1})$  to equation (2.1) controlled by  $\alpha^{N-1}$  and  $u$ , such that every path of  $(X^{N-1}, I^{N-1})$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . Our aim is to solve equation (2.1) controlled by  $\alpha^N$  and  $u$ . To this end, we define the process  $I^N = (I_t^N)_{s \leq t \leq T}$  as follows:

$$\begin{cases} I_t^N = I_t^{N-1} 1_{\{s \leq t < \tau_{N-1}(X^{N-1}, I^{N-1})\}} + \iota_{N-1}(X^{N-1}, I^{N-1}) 1_{\{\tau_{N-1}(X^{N-1}, I^{N-1}) \leq t < T\}}, \\ I_T^N = I_T^{N-1}. \end{cases}$$

From Lemma 2.2 we see that  $I^N$  is an  $\mathbb{F}^s$ -adapted process, with every path in  $\mathcal{L}([s, T]; \mathbb{I}_m)$ . Then, under Assumption (H1) there exists a unique (up to indistinguishability)  $\mathbb{F}^s$ -adapted solution  $(X_t^N, I_t^N)_{s \leq t \leq T}$  to equation (2.1), such that every path of  $(X^N, I^N)$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . Since  $(X^N, I^N)$  and  $(X^{N-1}, I^{N-1})$  solve the same equation on  $[s, \tau_{N-1}(X^{N-1}, I^{N-1}))$ , then  $(X_t^N, I_t^N) = (X_t^{N-1}, I_t^{N-1})$ ,  $t \in [s, \tau_{N-1}(X^{N-1}, I^{N-1}))$ . In

particular,  $(X_t^N, I_t^N) = (X_t^{N-1}, I_t^{N-1})$ , for any  $t \in [s, \tau_{N-1}(X^{N-1}, I_-^{N-1})]$ . From Lemma 2.1, it follows that

$$(\tau_n(X^{N-1}, I_-^{N-1}), \iota_n(X^{N-1}, I_-^{N-1})) = (\tau_n(X^N, I_-^N), \iota_n(X^N, I_-^N)), \quad 0 \leq n \leq N-1.$$

As a consequence,  $(X^N, I^N)$  solves equation (2.1) controlled by  $\alpha^N$  and  $u$ .

Finally, let us define (with the convention  $\tau_{-1} := s$ )

$$X_t^{s,x,i;\alpha,u} := \sum_{n \in \mathbb{N}} X_t^N \mathbf{1}_{\{\tau_{N-1}(X^{N-1}, I_-^{N-1}) \leq t < \tau_N(X^N, I_-^N)\}}, \quad (2.3)$$

$$I_t^{s,x,i;\alpha,u} := \sum_{n \in \mathbb{N}} I_t^N \mathbf{1}_{\{\tau_{N-1}(X^{N-1}, I_-^{N-1}) \leq t < \tau_N(X^N, I_-^N)\}}, \quad (2.4)$$

for any  $s \leq t < T$  and  $(X_T^{s,x,i;\alpha,u}, I_T^{s,x,i;\alpha,u}) := (X_{T^-}^{s,x,i;\alpha,u}, I_{T^-}^{s,x,i;\alpha,u})$ . For simplicity of notation, we denote  $(X, I) := (X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u})$ . Recalling that  $\tau_{N-1}(X^{N-1}, I_-^{N-1}) = \tau_{N-1}(X^N, I_-^N) \leq \tau_N(X^N, I_-^N)$ , we see that the sequence  $(\tau_N(X^N, I_-^N))_{N \geq -1}$  is nondecreasing, so that, for any  $t \in [s, T]$ , there is at most one term different from zero in the series appearing in (2.3) and (2.4). Moreover, from Definition 2.1, and, more precisely, from property (ii) of Remark 2.3, we have that, for every  $\omega \in \Omega$ ,  $\tau_N(X^N(\omega), I_-^N(\omega)) = T$ , for  $N$  large enough. In particular,  $X$  and  $I$  are well-defined over the entire interval  $[s, T]$  and they are  $\mathbb{F}^s$ -adapted. Furthermore, we notice that  $(X_t, I_t) = (X_t^N, I_t^N)$ ,  $t \in [s, \tau_N(X^N, I_-^N)]$ . Then, using again property (ii) of Remark 2.3, it follows that every path of  $(X, I_-)$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . In addition, since  $(X_t, I_{t-}) = (X_t^N, I_{t-}^N)$ ,  $t \in [s, \tau_N(X^N, I_-^N)]$ , from Lemma 2.1 we have

$$(\tau_N(X^N, I_-^N), \iota_N(X^N, I_-^N)) = (\tau_N(X, I_-), \iota_N(X, I_-)), \quad \forall N \in \mathbb{N}.$$

In particular,  $(X_t, I_t) = (X_t^N, I_t^N)$ ,  $t \in [s, \tau_N(X, I_-)]$ . This implies that  $(X, I)$  solves equation (2.1) on  $[s, \tau_N(X, I_-)]$ , for any  $N \in \mathbb{N}$ . Recalling property (ii) of Remark 2.3, we see that  $(X, I)$  solves equation (2.1) on  $[s, T]$ . Since, by definition,  $(X_T, I_T) = (X_{T^-}, I_{T^-})$ , it follows that  $(X, I)$  solves equation (2.1) on  $[s, T]$ .

**Step II. Uniqueness.** Let  $(X^1, I^1)$  and  $(X^2, I^2)$  be two solutions of (2.1). Set  $\underline{\tau}_0 := \tau_0(X^1, I_-^1) \wedge \tau_0(X^2, I_-^2)$ . Notice that  $(X^1, I^1)$  and  $(X^2, I^2)$  solve the same equation on  $[0, \underline{\tau}_0]$ . Therefore  $(X^1, I^1)$  and  $(X^2, I^2)$  are equal (up to indistinguishability) on  $[0, \underline{\tau}_0]$ . Consider  $\omega \in \Omega$  such that  $\underline{\tau}_0(\omega) = \tau_0(X^1(\omega), I_-^1(\omega))$ . Since  $(X_t^1(\omega), I_{t-}^1(\omega)) = (X_t^2(\omega), I_{t-}^2(\omega))$ ,  $t \in [s, \underline{\tau}_0(\omega)] = [s, \tau_0(X^1(\omega), I_-^1(\omega))]$ , from Lemma 2.1 it follows that  $\tau_0(X^1(\omega), I_-^1(\omega)) = \tau_0(X^2(\omega), I_-^2(\omega))$ . When  $\underline{\tau}_0(\omega) = \tau_0(X^2(\omega), I_-^2(\omega))$ , a similar argument shows that we still have  $\tau_0(X^1(\omega), I_-^1(\omega)) = \tau_0(X^2(\omega), I_-^2(\omega))$ . From the arbitrariness of  $\omega$ , we conclude that  $\underline{\tau}_0 = \tau_0(X^1, I_-^1) = \tau_0(X^2, I_-^2)$ . Using again Lemma 2.1, we also deduce  $\iota_0(X^1, I_-^1) = \iota_0(X^2, I_-^2)$ . By induction on  $n$ , we can prove that

$$\begin{aligned} (\tau_n(X^1, I_-^1), \iota_n(X^1, I_-^1)) &= (\tau_n(X^2, I_-^2), \iota_n(X^2, I_-^2)), & \forall n \in \mathbb{N}, \\ (X_t^1, I_t^1) &= (X_t^2, I_t^2), & \forall t \in [s, \tau_n(X^1, I_-^1)], n \in \mathbb{N}. \end{aligned}$$

From Definition 2.1, and, more precisely, from property (ii) of Remark 2.3, we have that, for any  $\omega \in \Omega$ ,  $\tau_n(X^1(\omega), I_-^1(\omega)) = T$  for  $n$  large enough. As a consequence,  $(X^1, I^1)$  and

$(X^2, I^2)$  are equal (up to indistinguishability) on  $[s, T]$ . Since  $(X_T^1, I_T^1) = (X_{T-}^1, I_{T-}^1)$  and  $(X_T^2, I_T^2) = (X_{T-}^2, I_{T-}^2)$ , we conclude that  $(X^1, I^1)$  and  $(X^2, I^2)$  are equal (up to indistinguishability) on  $[s, T]$ .

**Step III.** *Estimate* (2.2). Under **(H1)**, estimate (2.2) is well-known, see, e.g., Theorem 1.3.15 in [19].  $\square$

## 2.2 The Value function

The value function associated to the robust switching control problem is defined as follows:

$$V(s, x, i) := \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} J(s, x, i; \alpha, u), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m, \quad (2.5)$$

with

$$J(s, x, i; \alpha, u) := \mathbb{E} \left[ \int_s^T f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r) dr + g(X_T^{s,x,i;\alpha,u}, I_T^{s,x,i;\alpha,u}) - \sum_{n \in \mathbb{N}} c(X_{\tau_n}^{s,x,i;\alpha,u}, I_{\tau_n}^{s,x,i;\alpha,u}, I_{\tau_n}^{s,x,i;\alpha,u}) 1_{\{s \leq \tau_n < T\}} \right], \quad (2.6)$$

where  $\tau^n$  stands for  $\tau^n(X^{s,x,i;\alpha,u}, I_{-}^{s,x,i;\alpha,u})$ . We impose the following conditions on the functions  $g: \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$ ,  $f: \mathbb{R}^d \times \mathbb{I}_m \times U \rightarrow \mathbb{R}$ , and  $c: \mathbb{R}^d \times \mathbb{I}_m \times \mathbb{I}_m \rightarrow \mathbb{R}$ .

### (H2)

- (i)  $g, f, c$  are jointly continuous on their domains.
- (ii)  $c$  is nonnegative.
- (iii)  $g, f, c$  satisfy a polynomial growth condition in  $x$ , i.e.,

$$|g(x, i)| + |f(x, i, u)| + |c(x, i, j)| \leq M_2(1 + |x|^p),$$

$\forall x \in \mathbb{R}^d, i, j \in \mathbb{I}_m, u \in U$ , for some positive constants  $M_2$  and  $p \geq 1$ .

- (iv)  $g$  satisfies

$$g(x, i) \geq \max_{j \neq i} [g(x, j) - c(x, i, j)],$$

for any  $x \in \mathbb{R}^d$  and  $i \in \mathbb{I}_m$ .

**Remark 2.5** Notice that, given  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ , then  $\inf_{u \in \mathcal{U}_{s,s}} J(s, x, i; \alpha, u)$  might assume the value  $-\infty$  for some  $\alpha \in \mathcal{A}_{s,s}$ . However,  $V(s, x, i)$  is a finite number for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and satisfies a polynomial growth condition in  $x$ :

$$|V(s, x, i)| \leq C(1 + |x|^p), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m, \quad (2.7)$$

for some positive constant  $C$ , depending only on  $T, M_1, M_2$ , and with the same  $p$  as in Assumption **(H2)**(iii). Indeed, since  $c$  is nonnegative, we find

$$V(s, x, i) \leq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^T f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r) dr + g(X_T^{s,x,i;\alpha,u}) \right]. \quad (2.8)$$

On the other hand, let  $\alpha^* = (\tau_n^*, \iota_n^*)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$  be given by  $(\tau_n^*, \iota_n^*) = (T, \underline{i})$ ,  $\forall n \in \mathbb{N}$ , for some fixed  $\underline{i} \in \mathbb{I}_m$ . Then

$$\begin{aligned} V(s, x, i) &\geq \inf_{u \in \mathcal{U}_{s,s}} J(s, x, i; \alpha^*, u) \\ &= \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^T f(X_r^{s,x,i;\alpha^*,u}, I_r^{s,x,i;\alpha^*,u}, u_r) dr + g(X_T^{s,x,i;\alpha^*,u}) \right]. \end{aligned} \quad (2.9)$$

From (2.8) and (2.9), we obtain

$$|V(s, x, i)| \leq \sup_{\alpha \in \mathcal{A}_{s,s}} \sup_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^T |f(X_r^{s,x,i;\alpha,u}, I_r^{s,x,i;\alpha,u}, u_r)| dr + |g(X_T^{s,x,i;\alpha,u})| \right].$$

Now, from estimate (2.2) and the polynomial growth condition of  $f$  and  $g$  in **(H2)**(iii), we see that estimate (2.7) holds. As a consequence, in (2.5) we could take the supremum only over  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$  satisfying ( $\tau^n$  stands for  $\tau^n(X_{\cdot}^{s,x,i;\alpha,u}, I_{\cdot}^{s,x,i;\alpha,u})$ )

$$\inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ - \sum_{n \in \mathbb{N}} c(X_{\tau_n}^{s,x,i;\alpha,u}, I_{\tau_n^-}^{s,x,i;\alpha,u}, I_{\tau_n}^{s,x,i;\alpha,u}) 1_{\{s \leq \tau_n < T\}} \right] > -\infty.$$

□

Our aim is to prove that  $V$  is the unique viscosity solution to the dynamic programming equation associated to the robust switching control problem, which turns out to be a system of variational inequalities of Hamilton-Jacobi-Bellman type of the following form:

$$\begin{cases} \min \left\{ - \frac{\partial V}{\partial t}(s, x, i) - \inf_{u \in U} [\mathcal{L}^{i,u} V(s, x, i) + f(x, i, u)], \right. \\ \left. V(s, x, i) - \max_{j \neq i} [V(s, x, j) - c(x, i, j)] \right\} = 0, & (s, x, i) \in [0, T) \times \mathbb{R}^d \times \mathbb{I}_m, \\ V(T, x, i) = g(x, i), & (x, i) \in \mathbb{R}^d \times \mathbb{I}_m, \end{cases} \quad (2.10)$$

where

$$\mathcal{L}^{i,u} V(s, x, i) = b(x, i, u) \cdot D_x V(s, x, i) + \frac{1}{2} \text{tr} [\sigma \sigma^\top(x, i, u) D_x^2 V(s, x, i)].$$

We need the definition of (discontinuous) viscosity solution to equation (2.10), that we now provide. To this end, given a locally bounded function  $v: [0, T) \times \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$ , we define its lower semicontinuous (lsc for short) envelope  $v_*: [0, T] \times \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$ , and upper semicontinuous (usc for short) envelope  $v^*: [0, T] \times \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$ , by

$$v_*(s, x, i) = \liminf_{\substack{(s', x') \rightarrow (s, x) \\ (s', x') \in [0, T) \times \mathbb{R}^d}} v(s', x', i) \quad \text{and} \quad v^*(s, x, i) = \limsup_{\substack{(s', x') \rightarrow (s, x) \\ (s', x') \in [0, T) \times \mathbb{R}^d}} v(s', x', i),$$

for all  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ .

**Definition 2.3 (Viscosity solution to (2.10))**

(i) A lsc (resp. usc) function  $v$  on  $[0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  is called a viscosity supersolution (resp. subsolution) to (2.10) if

$$v(T, x, i) \geq (\text{resp. } \leq) g(x, i)$$

for any  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ , and

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(s, x) - \inf_{u \in U} [\mathcal{L}^{i,u} \varphi(s, x) + f(x, i, u)], \right. \\ \left. v(s, x, i) - \max_{j \neq i} [v(s, x, j) - c(x, i, j)] \right\} \geq \text{(resp. } \leq) 0$$

for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and any  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$  such that

$$v(s, x, i) - \varphi(s, x) = \min_{(s', x') \in [0, T] \times \mathbb{R}^d} [v(s', x', i) - \varphi(s', x')] \\ \left\{ \text{resp. } v(s, x, i) - \varphi(s, x) = \max_{(s', x') \in [0, T] \times \mathbb{R}^d} [v(s', x', i) - \varphi(s', x')] \right\}.$$

(ii) A locally bounded function  $v$  on  $[0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  is called a viscosity solution to (2.10) if  $v_*$  is a viscosity supersolution and  $v^*$  is a viscosity subsolution to (2.10).

### 3 Stochastic Perron's method

Our aim is to prove that  $V$  is a viscosity solution to the dynamic programming equation (2.10) and satisfies the dynamic programming principle. To derive these results, we exploit *stochastic Perron's method*, which allows to obtain the viscosity properties of  $V$  without relying on the dynamic programming principle, but by means of the comparison theorem for viscosity solutions to (2.10) (the dynamic programming principle will be obtained as a by-product of this procedure).

#### 3.1 An Auxiliary robust switching problem

We begin with the formulation of an auxiliary robust switching control problem where nature adopts feedback controls in place of open-loop controls. Using the comparison principle for equation (2.10), we shall see that the corresponding value function, denoted by  $\bar{V}$ , coincides with  $V$ . In other words, the information available to nature does not affect the value of the game. This is not the only motivation for the introduction of this auxiliary robust control problem. Indeed, in the implementation of the stochastic Perron method, and in particular in the study of the viscosity properties of stochastic supersolutions, we encountered the following difficulty: given two different controls  $u_1$  and  $u_2$ , for nature, we have to concatenate them at some stopping rule  $\tau = \tau(X, I, -)$ . If  $u^1$  and  $u^2$  are open-loop controls, the control  $u^1 \otimes_\tau u^2$  resulting from the concatenation of  $u^1$  and  $u^2$  at the stopping rule  $\tau$ , given by

$$(u^1 \otimes_\tau u^2)(t, \omega, y) = u^1(t, \omega) 1_{\{s \leq t \leq \tau(y)\}} + u^2(t, \omega) 1_{\{\tau(y) < t \leq T\}},$$

is no more of open-loop type, since it also depends on  $y$ . On the other hand, if  $u^1$  and  $u^2$  are feedback controls, then  $u^1 \otimes_\tau u^2$  is still a feedback control. For this technical reason, to study the original control problem with corresponding value function  $V$ , we also need to consider another robust switching control problem, in which nature adopts feedback controls. In particular, inspired by [23] and [22], it turns out that it is more convenient,

and it is enough, to consider only piecewise constant feedback controls, i.e., the *elementary feedback strategies* that we now define.

**Definition 3.1 (Elementary feedback strategies)** Fix  $s \in [0, T]$ . We say that  $u$  is an elementary feedback strategy starting at  $s$  if:

- $\tau_k \in \mathcal{T}^s$ , for any  $k = 1, \dots, n$ , and

$$s =: \tau_0 \leq \dots \leq \tau_k \leq \dots \leq \tau_n = T.$$

- $\xi_k: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow U$  is  $\mathcal{B}_{\tau_{k-1}}^s$ -measurable, for any  $k = 1, \dots, n$ .

The strategy  $u: [s, T] \times C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow U$  is given by

$$u(t, y) := \xi_1(y)1_{\{t=s\}} + \sum_{k=1}^n \xi_k(y)1_{\{\tau_{k-1}(y) < t \leq \tau_k(y)\}}.$$

$\mathcal{U}_{s,s}^E$  denotes the family of all elementary feedback strategies starting at  $s$ .

**Remark 3.1** We notice that Definition 3.1 is inspired by Definition 2.2 in [23] (see also Definition 2.1 in [22]), the only difference being that  $\xi_k$  is  $\mathcal{B}_{\tau_{k-1}}^s$ -measurable instead of  $\mathcal{B}_{\tau_{k-1}}^s$ -measurable. This implies that the map  $\xi_k = \xi_k(y)$  depends on  $y$  through the values  $\{y(t), s \leq t \leq \tau_{k-1}(y)^+\}$ , so that  $\xi_k$  can also depend on  $y(\tau_{k-1}(y)^+)$ . Recalling that in our setting  $y$  denotes a generic path of  $(X_t, I_{t-})_{s \leq t \leq T}$ , this means that  $\xi_k$  depends on  $(X_t, I_t)_{s \leq t \leq \tau_{k-1}(X, I_-)}$  rather than on  $(X_t, I_{t-})_{s \leq t \leq \tau_k(X, I_-)}$ . Therefore, nature reacts to the switcher using all the information at disposal at time  $\tau_{k-1} = \tau_{k-1}(X, I_-)$ , including  $I_{\tau_{k-1}}$  (in particular, if  $\tau_{k-1}$  coincides with a switching action, nature is aware of the action that the switcher has just performed).  $\square$

We have the following well-posedness result for equation (2.1) when  $u$  is an elementary feedback strategy (so that  $u_r$  stands for  $u(r, X, I_-)$ ), where the only difference with Proposition 2.1 is that now the solution is adapted to the smaller filtration  $\mathbb{F}^{W,s}$ , since  $\mathbb{F}^s$  plays no role when  $u \in \mathcal{U}_{s,s}^E$ .

**Proposition 3.1** Let Assumption (H1) hold. For any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ ,  $\alpha \in \mathcal{A}_{s,s}$ ,  $u \in \mathcal{U}_{s,s}^E$ , there exists a unique (up to indistinguishability)  $\mathbb{F}^{W,s}$ -adapted process  $(X^{s,x,i;\alpha,u}, I^{s,x,i;\alpha,u}) = (X_t^{s,x,i;\alpha,u}, I_t^{s,x,i;\alpha,u})_{s \leq t \leq T}$  to equation (2.1), such that every path of  $(X^{s,x,i;\alpha,u}, I_-^{s,x,i;\alpha,u})$  belongs to  $C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ . Moreover, for any  $p \geq 1$  there exists a positive constant  $C_{p,T}$ , depending only on  $p, T, M_1$  (independent of  $s, x, i, \alpha, u$ ), such that

$$\mathbb{E} \left[ \sup_{s \leq t \leq T} |X_t^{s,x,i;\alpha,u}|^p \right] \leq C_{p,T}(1 + |x|^p). \quad (3.1)$$

**Proof.** The proof can be done along the lines of the proof of Proposition 2.1. We simply notice that in Proposition 2.1 we used the following result: if  $u \in \mathcal{U}_{s,s}$  and  $I = (I_t)_{s \leq t \leq \tau}$  is

known up to a certain  $\mathbb{F}^s$ -stopping time  $\tau$ , then there exists a unique (up to indistinguishability)  $\mathbb{F}^s$ -adapted solution  $X = (X_t)_{s \leq t \leq \tau}$  to the equation

$$X_t = x + \int_s^t b(X_r, I_r, u_r) dr + \int_s^t \sigma(X_r, I_r, u_r) dW_r, \quad s \leq t \leq \tau, \quad (3.2)$$

such that every path of  $X$  belongs to  $C([s, T]; \mathbb{R}^d)$ . The validity of this result is well-known under **(H1)**. On the other hand, it is not immediately clear when  $u \in \mathcal{U}_{s,s}^E$  is an elementary feedback strategy. However, the result is still valid and follows from Proposition 2.4 in [23], see also Theorem 2.2 in [22]. Moreover, when  $u \in \mathcal{U}_{s,s}^E$  it turns out that the process  $X$  is adapted to the smaller filtration  $\mathbb{F}^{W,s}$ . Finally, under Assumption **(H1)**, estimate (3.1) is well-known, see, e.g., Theorem 1.3.15 in [19].  $\square$

We can finally introduce the value function for the robust switching control problem where nature adopts the elementary feedback strategies

$$\bar{V}(s, x, i) := \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}^E} J(s, x, i; \alpha, u), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m.$$

Notice that  $V(s, x, i) \leq \bar{V}(s, x, i)$ , for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ . Moreover, proceeding as in Remark 2.5, we can show that  $\bar{V}$  satisfies a polynomial growth condition in  $x$ :  $|\bar{V}(s, x, i)| \leq C(1 + |x|^p) < \infty$ , for some positive constant  $C$ , depending only on  $T, M_1, M_2$ , and with the same  $p$  as in Assumption **(H2)**(iii).

### 3.2 Concatenation of feedback controls

In the present section, we need to introduce the concept of feedback control starting at a certain stopping rule  $\tau$  and to define the notion of concatenation at  $\tau$  of two feedback controls, which will be crucial in the development of the stochastic Perron method.

**Definition 3.2 (Feedback switching controls starting strictly later than  $\tau$ )** Fix  $s$  in  $[0, T]$  and  $\tau \in \mathcal{T}^s$ . We say that the double sequence  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$  is a feedback switching control starting strictly later than  $\tau$  if:

- $\tau_n \in \mathcal{T}^s$ , for any  $n \in \mathbb{N}$ , and

$$\tau \leq \tau_0 \leq \dots \leq \tau_n \leq \dots \leq T,$$

with  $\tau < \tau_0$  on the set  $\{\tau < T\}$ . Moreover,  $(\tau_n)_{n \in \mathbb{N}}$  satisfies the following property:  $\forall (y_n)_{n \in \mathbb{N}} \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ , with  $y_n(t) = y_{n+1}(t)$ ,  $t \in [s, \tau_n(y_n)]$ , then

$$\tau_n(y_n) = T, \quad \text{for } n \text{ large enough.}$$

- $\iota_n: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{I}_m$  is  $\mathcal{B}_{\tau_n}^s$ -measurable, for any  $n \in \mathbb{N}$ .

$\mathcal{A}_{s,\tau+}$  denotes the family of all feedback switching controls for the controller, given the initial deterministic time  $s$  and starting strictly later than  $\tau$



**Remark 3.2** Definition 3.2 is inspired by Definition 2.7 in [23], with in addition the condition “ $\tau < \tau_0$  on the set  $\{\tau < T\}$ ”, which justifies the presence of the adverb *strictly* in the name. Indeed, our aim is to define the set  $\mathcal{A}_{s,\tau+}$  in such a way that when we concatenate two feedback switching controls  $\alpha \in \mathcal{A}_{s,s}$  and  $\tilde{\alpha} \in \mathcal{A}_{s,\tau+}$  at a stopping rule  $\tau \in \mathcal{T}^s$  (see Proposition 3.2 below) then  $\alpha \otimes_{\tau} \tilde{\alpha}$  coincides with  $\alpha$  at time  $\tau$  (this property plays an important role in the sequel, e.g., in the proof of Theorem 3.1). On the other hand, when we concatenate two elementary feedback strategies  $u \in \mathcal{U}_{s,s}^E$  and  $\tilde{u} \in \mathcal{U}_{s,\tau}^E$  (see Definition 3.3 below), then  $u \otimes_{\tau} \tilde{u}$  coincides with  $u$  at time  $\tau$ , simply adopting the same definition for  $\mathcal{U}_{s,\tau}^E$  as in [23] combined with Remark 3.1.  $\square$

Following [23], Definition 2.7, and recalling Remark 3.1, we now define the elementary feedback strategies starting at some stopping rule  $\tau$ .

**Definition 3.3 (Elementary feedback strategies starting at  $\tau$ )** Fix  $s \in [0, T]$  and  $\tau \in \mathcal{T}^s$ . We say that  $u$  is an elementary feedback strategy starting at  $\tau$  if:

- $\tau_k \in \mathcal{T}^s$ , for any  $k = 1, \dots, n$ , and

$$\tau =: \tau_0 \leq \dots \leq \tau_k \leq \dots \leq \tau_n = T.$$

- $\xi_k: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow U$  is  $\mathcal{B}_{\tau_{k-1}^+}^s$ -measurable, for any  $k = 1, \dots, n$ .

The elementary feedback strategy

$$u: \{(t, y) \in [s, T] \times (C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)) : \tau(y) \leq t \leq T\} \longrightarrow U$$

is given by

$$u(t, y) := \xi_1(y)1_{\{t=\tau(y)\}} + \sum_{k=1}^n \xi_k(y)1_{\{\tau_{k-1}(y) < t \leq \tau_k(y)\}}.$$

$\mathcal{U}_{s,\tau}^E$  denotes the family of all elementary feedback strategies, given the initial deterministic time  $s$  and starting at  $\tau$ .

Notice that, when  $\tau = s$  in Definition 3.3, we find exactly  $\mathcal{U}_{s,s}^E$ . As in [23], Lemma 2.8 and Proposition 2.9, we have the two following results, whose simple proof is only sketched for Lemma 3.1 and omitted for Proposition 3.2.

**Lemma 3.1** Fix  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ ,  $\alpha^1 = (\tau_n^1, \iota_n^1)_{n \in \mathbb{N}}$ ,  $\alpha^2 = (\tau_n^2, \iota_n^2)_{n \in \mathbb{N}} \in \mathcal{A}_{s,\tau+}$ ,  $u^1, u^2 \in \mathcal{U}_{s,\tau}^E$ , and  $B \in \mathcal{B}_{\tau+}^s$ .

- The double sequence  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$  given by

$$(\tau_n(y), \iota_n(y)) = (\tau_n^1(y), \iota_n^1(y))1_{\{y \in B\}} + (\tau_n^2(y), \iota_n^2(y))1_{\{y \in B^c\}}$$

is in  $\mathcal{A}_{s,\tau+}$ .

- The map

$$u: \{(t, y) \in [s, T] \times (C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)) : \tau(y) \leq t \leq T\} \longrightarrow U$$

given by

$$u(t, y) = u^1(t, y)1_{\{y \in B\}} + u^2(t, y)1_{\{y \in B^c\}}$$

is in  $\mathcal{U}_{s, \tau}^E$ .

**Proof.** We only prove the first item, where we focus on the two main points. In particular, the proof that  $\tau_n \in \mathcal{T}^s$  and  $\iota_n \in \mathcal{B}_{\tau_n}^s$  is based on the observation that  $B \in \mathcal{B}_{\tau^+}^s \subset \mathcal{B}_{\tau_n^+}^s, \mathcal{B}_{\tau_n^+}^s$ , for any  $n \in \mathbb{N}$ , which is a consequence of the property:  $\tau < \tau_0^1, \tau_0^2$  on the set  $\{\tau < T\}$ . The other non-trivial part is the proof that  $\alpha$  satisfies property (ii) of Remark 2.3. To prove it, consider  $(y_n)_{n \in \mathbb{N}} \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ , with  $y_n(t) = y_{n+1}(t)$ ,  $t \in [s, \tau_n(y_n)]$ . Since  $\tau_0 \leq \tau_n$  for any  $n \in \mathbb{N}$ , we have

$$y_0(t) = y_n(t), \quad \forall t \in [s, \tau_0(y_0)], n \in \mathbb{N}.$$

As  $\tau < \tau_0$  on the set  $\{\tau < T\}$ , it follows that

$$y_0(t^+) = y_n(t^+), \quad \forall t \in [s, \tau(y_0)], n \in \mathbb{N}. \quad (3.3)$$

In particular  $y_0(\tau(y_0)^+) = y_n(\tau(y_0)^+)$ . Moreover, from Lemma 2.1 we get  $\tau(y_0) = \tau(y_n)$ , so that  $y_0(\tau(y_0)^+) = y_n(\tau(y_n)^+)$ . Therefore,  $y_0 \in B$  if and only if  $y_n \in B$ , for any  $n \in \mathbb{N}$ . In conclusion, property (ii) of Remark 2.3 for  $(\tau_n)_{n \in \mathbb{N}}$  follows from the definitions of  $(\tau_n^1)_{n \in \mathbb{N}}$  and  $(\tau_n^2)_{n \in \mathbb{N}}$ .  $\square$

**Proposition 3.2 (Concatenation)** Fix  $s \in [0, T]$ ,  $\tau, \rho \in \mathcal{T}^s$  with  $\tau \leq \rho \leq T$ ,  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \rho^+}$ ,  $\tilde{u} \in \mathcal{U}_{s, \rho}^E$ . Then

- for each  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$  (resp.  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau^+}$ ), the double sequence  $\alpha \otimes_\rho \tilde{\alpha} = (\tau_n^{\otimes \rho}, \iota_n^{\otimes \rho})_{n \in \mathbb{N}}$  given by

$$(\tau_n^{\otimes \rho}(y), \iota_n^{\otimes \rho}(y)) = (\tau_n(y), \iota_n(y))1_{\{\tau_n(y) \leq \rho(y)\}} + (\tilde{\tau}_n(y), \tilde{\iota}_n(y))1_{\{\tau_n(y) > \rho(y)\}}$$

is in  $\mathcal{A}_{s, s}$  (resp.  $\mathcal{A}_{s, \tau^+}$ );

- for each  $u \in \mathcal{U}_{s, \tau}^E$ , the map

$$u \otimes_\rho \tilde{u}: \{(t, y) \in [s, T] \times (C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)) : \tau(y) \leq t \leq T\} \longrightarrow U$$

given by

$$(u \otimes_\rho \tilde{u})(t, y) = u(t, y)1_{\{\tau(y) \leq t \leq \rho(y)\}} + \tilde{u}(t, y)1_{\{\rho(y) < t \leq T\}}$$

is in  $\mathcal{U}_{s, \tau}^E$ .

### 3.3 Stochastic semisolutions and viscosity properties

We can now provide the definitions of stochastic sub and supersolution, which are the cornerstones of the stochastic Perron method.

**Definition 3.4 (Stochastic subsolutions)** *The set of stochastic subsolutions for equation (2.10), denoted by  $\mathcal{V}^-$ , is the set of functions  $v: [0, T] \times \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$  which have the following properties:*

- $v$  is continuous and satisfies the terminal condition  $v(T, x, i) \leq g(x, i)$ ,  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ , together with the polynomial growth condition

$$\sup_{(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|v(s, x, i)|}{1 + |x|^q} < \infty,$$

for some  $q \geq 1$ .

- For any  $s \in [0, T]$  and  $\tau, \rho \in \mathcal{T}^s$  with  $\tau \leq \rho \leq T$ , there exists  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau^+}$  (possibly depending on  $s, \tau, \rho$ ) such that, for any  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ ,  $u \in \mathcal{U}_{s, s}$ , and  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ , we have

$$\begin{aligned} v(\tau', X_{\tau'}, I_{\tau'}) &\leq \mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n}) \mathbf{1}_{\{\tau' \leq \tilde{\tau}'_n < \rho'\}} \right]_{\mathcal{F}_{\tau'}^s}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

with the shorthands  $X = X^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}, u}$ ,  $I = I^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}, u}$ ,  $\tau' = \tau(X, I, -)$ ,  $\rho' = \rho(X, I, -)$ , and  $\tilde{\tau}'_n = \tilde{\tau}_n(X, I, -)$ .

**Definition 3.5 (Stochastic supersolutions)** *The set of stochastic supersolutions for equation (2.10), denoted by  $\mathcal{V}^+$ , is the set of functions  $v: [0, T] \times \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$  which have the following properties:*

- $v$  is continuous and satisfies the terminal condition  $v(T, x, i) \geq g(x, i)$ ,  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ , together with the polynomial growth condition

$$\sup_{(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|v(s, x, i)|}{1 + |x|^q} < \infty,$$

for some  $q \geq 1$ .

- For any  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ , and  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ , there exists  $\tilde{u} \in \mathcal{U}_{s, \tau}^E$  (possibly depending on  $s, \tau, \alpha$ ) such that, for any  $u \in \mathcal{U}_{s, s}^E$ ,  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ , and  $\rho \in \mathcal{T}^s$ , with  $\tau \leq \rho \leq T$ , we have

$$\begin{aligned} v(\tau', X_{\tau'}, I_{\tau'}) &\geq \mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, I_t, \tilde{u}_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) \mathbf{1}_{\{\tau' \leq \tau'_n < \rho'\}} \right]_{\mathcal{F}_{\tau'}^s}, \quad \mathbb{P}\text{-a.s.} \end{aligned}$$

with the shorthands  $X = X^{s, x, i; \alpha, u \otimes_{\tau} \tilde{u}}$ ,  $I = I^{s, x, i; \alpha, u \otimes_{\tau} \tilde{u}}$ ,  $\tau' = \tau(X, I, -)$ ,  $\rho' = \rho(X, I, -)$ ,  $\tau'_n = \tau_n(X, I, -)$ , and  $\tilde{u}_t = \tilde{u}(t, X, I, -)$ .

**Remark 3.3** The definitions of stochastic sub and supersolution are inspired by [23], Definitions 3.1-3.2-3.3, but for the fact that in Definition 3.4 above we fix  $\rho$  before  $\tilde{\alpha}$ , so that  $\tilde{\alpha}$  can depend on  $\rho$ . This greater freedom in the choice of  $\tilde{\alpha}$  turns out to be fundamental in the implementation of the stochastic Perron method, Theorem 3.1, and it is due to the condition “ $\tau < \tau_0$  on the set  $\{\tau < T\}$ ” in the definition of  $\mathcal{A}_{s,\tau+}$ , already discussed in Remark 3.2. Indeed, using the set  $\mathcal{A}_{s,\tau+}$ , the existence of an “optimal” feedback switching control  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{l}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,\tau+}$ , which works for every  $\rho \in \mathcal{T}^s$ , with  $\tau \leq \rho \leq T$ , is not guaranteed. For example, it could happen that every “optimal” feedback switching control which works for all  $\rho$  has to satisfy  $\tilde{\tau}_0 = \tau$ , therefore it can not belong to  $\mathcal{A}_{s,\tau+}$ . To avoid this problem, firstly we fix  $\rho$ , then we choose an “optimal”  $\tilde{\alpha} \in \mathcal{A}_{s,\tau+}$ . Another possibility would be to look for an “ $\varepsilon$ -optimal”  $\tilde{\alpha} \in \mathcal{A}_{s,\tau+}$  which works for every  $\rho$ .  $\square$

We first notice that, as stated below, the two sets  $\mathcal{V}^-$  and  $\mathcal{V}^+$  are not empty, moreover every stochastic sub/supersolution satisfies half of the dynamic programming principle (also known as sub/superoptimality principle, see [24]).

**Lemma 3.2** *Let Assumptions (H1) and (H2) hold.*

- (i)  $\mathcal{V}^- \neq \emptyset$  and  $\mathcal{V}^+ \neq \emptyset$ .
- (ii) *Every stochastic subsolution  $v \in \mathcal{V}^-$  satisfies the half dynamic programming principle: for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and  $\rho \in \mathcal{T}^s$ ,*

$$v(s, x, i) \leq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right], \quad (3.4)$$

with the shorthands  $X = X^{s,x,i;\alpha,u}$ ,  $I = I^{s,x,i;\alpha,u}$ ,  $\rho' = \rho(X, I, -)$ , and  $\tau'_n = \tau_n(X, I, -)$ .

- (iii) *Every stochastic supersolution  $v \in \mathcal{V}^+$  satisfies the half dynamic programming principle: for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and  $\rho \in \mathcal{T}^s$ ,*

$$v(s, x, i) \geq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}^E} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right],$$

with the shorthands  $X = X^{s,x,i;\alpha,u}$ ,  $I = I^{s,x,i;\alpha,u}$ ,  $\rho' = \rho(X, I, -)$ ,  $\tau'_n = \tau_n(X, I, -)$ , and  $u_t = u(t, X, I, -)$ .

**Proof.** We begin proving that  $\mathcal{V}^- \neq \emptyset$ . Let us consider the function  $v: [0, T] \times \mathbb{R}^d \times \mathbb{I}_m \rightarrow \mathbb{R}$  given by

$$v(s, x, i) := -C e^{\lambda(T-s)} (1 + |x|^q), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m, \quad (3.5)$$

where  $q = \max\{4, p\}$ , with  $p$  as in Assumption (H2)(iii), and  $C, \lambda$  are positive constants to be determined later. Set  $h(x) = |x|^q$ . Notice that  $h \in C^2(\mathbb{R}^d)$  and there exists a positive

constant  $M_h$  (depending only on  $q$ ) such that  $|D_x h(x)| \leq M_h |x|^{q-1}$  and  $D_x^2 h(x) \leq M_h |x|^{q-2}$ ,  $\forall x \in \mathbb{R}^d$ .

From the polynomial growth condition of  $g$  in Assumption **(H2)**(iii), we see that  $v(T, x, i) \leq g(x, i)$  if we choose  $C$  large enough.

Now, we choose  $\lambda$  so that the stochastic subsolution property holds. Fix  $s \in [0, T]$  and  $\tau, \rho \in \mathcal{T}^s$  with  $\tau \leq \rho \leq T$ . We choose  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau+}$  as follows: for any  $n \in \mathbb{N}$ ,  $\tilde{\tau}_n \equiv T$  and  $\tilde{\iota}_n \equiv \dot{i}$ , for some fixed  $\dot{i} \in \mathbb{I}_m$ . Let  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ ,  $u \in \mathcal{U}_{s, s}$ , and  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ . Set  $X = X^{s, x, i; \alpha \otimes \tau \tilde{\alpha}, u}$ ,  $I = I^{s, x, i; \alpha \otimes \tau \tilde{\alpha}, u}$ ,  $\tau' = \tau(X, I)$ , and  $\rho' = \rho(X, I)$ . Then, noting that  $v(s, X_s, I_s)$  is constant with respect to  $I_s$ , and applying Itô's formula to  $\int_{\tau'}^s f(X_t, I_t, u_t) dt + v(s, X_s, I_s)$  between  $s = \tau'$  and  $s = \rho'$ , we obtain

$$\begin{aligned} & \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \\ = & \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\tau', X_{\tau'}, I_{\tau'}) - C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} D_x h(X_t) \cdot b(X_t, I_t, u_t) dt \\ & - C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} (D_x h(X_t))^\top \sigma(X_t, I_t, u_t) dW_t + \lambda C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} (1 + h(X_t)) dt \\ & - \frac{1}{2} C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} \text{tr}[\sigma \sigma^\top(X_t, I_t, u_t) D_x^2 h(X_t)] dt. \end{aligned}$$

Taking the conditional expectation with respect to  $\mathcal{F}_{\tau'}^s$ , using the linear growth conditions of  $b, \sigma, f$ , and the estimates on  $D_x h(x)$  and  $D_x^2 h(x)$ , we find

$$\begin{aligned} & \mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \middle| \mathcal{F}_{\tau'}^s \right] \\ \geq & v(\tau', X_{\tau'}, I_{\tau'}) + \mathbb{E} \left[ -M_2 \int_{\tau'}^{\rho'} (1 + |X_t|^p) dt - CM_h M_1 \int_{\tau'}^{\rho'} e^{\lambda(T-t)} |X_t|^{q-1} (1 + |X_t|) dt \right. \\ & \left. + \lambda C \int_{\tau'}^{\rho'} e^{\lambda(T-t)} (1 + |X_t|^q) dt - \frac{1}{2} CM_h M_1^2 \int_{\tau'}^{\rho'} e^{\lambda(T-t)} |X_t|^{q-2} (1 + |X_t|)^2 dt \middle| \mathcal{F}_{\tau'}^s \right]. \end{aligned}$$

We see that there exists a positive constant  $\bar{C}$  (depending only on  $C, M_h, M_1, M_2$ ) such that

$$\begin{aligned} & \mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \middle| \mathcal{F}_{\tau'}^s \right] \geq v(\tau', X_{\tau'}, I_{\tau'}) \\ & + (\lambda C - \bar{C}) \mathbb{E} \left[ \int_{\tau'}^{\rho'} e^{\lambda(T-t)} (1 + |X_t|^q) dt \middle| \mathcal{F}_{\tau'}^s \right]. \end{aligned}$$

Now, we choose  $\lambda \geq 0$  such that  $\lambda C - \bar{C} \geq 0$ . Then, we have

$$\mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \middle| \mathcal{F}_{\tau'}^s \right] \geq v(\tau', X_{\tau'}, I_{\tau'}).$$

From the definition of  $\tilde{\alpha}$ , we see that  $\sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}_n}, I_{(\tilde{\tau}_n)^-}, I_{\tilde{\tau}_n}) 1_{\{\tau' \leq \tilde{\tau}_n < \rho'\}} = 0$ . Therefore, it follows that  $v \in \mathcal{V}^-$ . In a similar way we can prove that  $-v \in \mathcal{V}^+$ , so that  $\mathcal{V}^+ \neq \emptyset$ .

Concerning (ii), let  $v \in \mathcal{V}^-$  and fix  $s \in [0, T]$ ,  $\tau, \rho \in \mathcal{T}^s$ , with  $s \equiv \tau \leq \rho \leq T$ . From the second item of the definition of stochastic subsolution, there exists  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s+}$

such that, for any  $u \in \mathcal{U}_{s,s}$  and  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$  (we choose  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$  with  $\tau_n \equiv T$  and  $\iota_n \equiv i$ , for any  $n \in \mathbb{N}$ ; with this choice we have  $(X^{s,x,i;\alpha \otimes_s \tilde{\alpha}, u}, I^{s,x,i;\alpha \otimes_s \tilde{\alpha}, u}) = (X^{s,x,i;\tilde{\alpha}, u}, I^{s,x,i;\tilde{\alpha}, u})$ ; in particular,  $I_s^{s,x,i;\alpha \otimes_s \tilde{\alpha}, u} = i$ ), we find

$$\begin{aligned} v(s, x, i) &\leq \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n}) 1_{\{s \leq \tilde{\tau}'_n < \rho'\}} \middle| \mathcal{F}_s^s \right], \quad \mathbb{P}\text{-a.s.} \end{aligned} \quad (3.6)$$

with the shorthands  $X = X^{s,x,i;\tilde{\alpha}, u}$ ,  $I = I^{s,x,i;\tilde{\alpha}, u}$ ,  $\rho' = \rho(X, I, -)$ , and  $\tilde{\tau}'_n = \tilde{\tau}'_n(X, I, -)$ . Taking the expectation in (3.6) and the infimum with respect to  $u \in \mathcal{U}_{s,s}$ , we get

$$\begin{aligned} v(s, x, i) &\leq \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}'_n}, I_{(\tilde{\tau}'_n)^-}, I_{\tilde{\tau}'_n}) 1_{\{s \leq \tilde{\tau}'_n < \rho'\}} \right] \\ &\leq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \\ &\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right]. \end{aligned}$$

In a similar way we can prove statement (iii).  $\square$

As stated below, every stochastic subsolution is less than every stochastic supersolution, while the value functions  $V$  and  $\bar{V}$  are squeezed between them.

**Lemma 3.3** *Let Assumptions (H1) and (H2) hold.*

(i)  $\sup_{v \in \mathcal{V}^-} v =: v^- \leq V \leq \bar{V} \leq v^+ := \inf_{v \in \mathcal{V}^+} v.$

(ii)  $v^-$  is lsc and satisfies the polynomial growth condition

$$\sup_{(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|v^-(s, x, i)|}{1 + |x|^q} < \infty, \quad (3.7)$$

for some  $q \geq 1$ .

(iii)  $v^+$  is usc and satisfies the polynomial growth condition

$$\sup_{(s,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|v^+(s, x, i)|}{1 + |x|^q} < \infty,$$

for some  $q \geq 1$ .

**Proof.** Statement (i) follows easily from the half dynamic programming principle in Lemma 3.2 satisfied by any element of  $\mathcal{V}^-$  and  $\mathcal{V}^+$ . Concerning (ii), we notice that  $v^-$  is lsc since it is the supremum of a family of lsc (actually, continuous) functions. Moreover, let  $\underline{v} \in \mathcal{V}^-$  and  $\bar{v} \in \mathcal{V}^+$ . From (i) it follows that  $\underline{v} \leq v^- \leq \bar{v}$ , and from the polynomial growth condition

of  $\underline{v}, \bar{v}$  we see that  $v^-$  satisfies the polynomial growth condition (3.7). Statement (iii) can be proved in a similar way.  $\square$

We can now state our main result.

**Theorem 3.1 (Stochastic Perron's method)** *Let Assumptions (H1) and (H2) hold. Then,  $v^-$  is a viscosity supersolution to equation (2.10) and  $v^+$  is a viscosity subsolution to equation (2.10).*

In order to prove Theorem 3.1, we need the following two lemmata. In particular, Lemma 3.4 states that  $\mathcal{V}^-$  (resp.  $\mathcal{V}^+$ ) is stable by supremum (resp. infimum), which gives the existence of a monotone approximating sequence for  $v^-$  (resp.  $v^+$ ) in Lemma 3.5.

**Lemma 3.4** *Let Assumptions (H1) and (H2) hold.*

- (i) *If  $v^1, v^2 \in \mathcal{V}^-$  then  $v := v^1 \vee v^2 \in \mathcal{V}^-$ .*
- (ii) *If  $v^1, v^2 \in \mathcal{V}^+$  then  $v := v^1 \wedge v^2 \in \mathcal{V}^+$ .*

**Proof.** Let us prove (i). As the first item of the definition of stochastic subsolution is clear, we prove that  $v$  satisfies the second item. To this end, fix  $s \in [0, T]$  and  $\tau, \rho \in \mathcal{T}^s$  with  $\tau \leq \rho \leq T$ . Let  $\tilde{\alpha}^1 = (\tilde{\tau}_n^1, \tilde{t}_n^1)_{n \in \mathbb{N}}, \tilde{\alpha}^2 = (\tilde{\tau}_n^2, \tilde{t}_n^2)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau+}$  be the two feedback switching controls, starting strictly later than  $\tau$ , corresponding to the stochastic subsolutions  $v^1$  and  $v^2$ . Now, consider the set  $B := \{(v^1 - v^2)(\tau(y), y(\tau(y)^+)) \geq 0\} \in \mathcal{B}_{\tau+}^s$  and define the double sequence  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{t}_n)_{n \in \mathbb{N}}$  as follows

$$(\tilde{\tau}_n(y), \tilde{t}_n(y)) := (\tilde{\tau}_n^1(y), \tilde{t}_n^1(y))1_{\{y \in B\}} + (\tilde{\tau}_n^2(y), \tilde{t}_n^2(y))1_{\{y \in B^c\}},$$

for any  $y \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$ ,  $n \in \mathbb{N}$ . From Lemma 3.1 it follows that  $\tilde{\alpha} \in \mathcal{A}_{s, \tau+}$ . Now, we prove that  $\tilde{\alpha}$  satisfies the condition in the second item of the definition of stochastic subsolution for  $v$ . Take  $\alpha = (\tau_n, t_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ ,  $u \in \mathcal{U}_{s, s}$ , and  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ . We adopt the shorthands:

$$\begin{aligned} X &= X^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}, u}, & X^1 &= X^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}^1, u}, & X^2 &= X^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}^2, u}, \\ I &= I^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}, u}, & I^1 &= I^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}^1, u}, & I^2 &= I^{s, x, i; \alpha \otimes_{\tau} \tilde{\alpha}^2, u}. \end{aligned}$$

We also denote  $\tau' = \tau(X, I, -)$ ,  $\rho' = \rho(X, I, -)$ ,  $\rho^{1'} = \rho(X^1, I^1, -)$ ,  $\rho^{2'} = \rho(X^2, I^2, -)$ ,  $\tilde{\tau}'_n = \tilde{\tau}_n(X, I, -)$ ,  $\tilde{\tau}_n^{1'} = \tilde{\tau}_n^1(X^1, I^1, -)$ , and  $\tilde{\tau}_n^{2'} = \tilde{\tau}_n^2(X^2, I^2, -)$ . Notice that  $(X_t, I_{t-}) = (X_t^1, I_{t-}^1) = (X_t^2, I_{t-}^2)$ ,  $t \in [s, \tau']$ . Therefore, from Lemma 2.1 we see that  $\tau' = \tau(X^1, I^1, -) = \tau(X^2, I^2, -)$ . Moreover, for any  $t \in [\tau', T]$ ,

$$(X_t, I_t) = (X_t^1, I_t^1)1_{\{(v^1 - v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} + (X_t^2, I_t^2)1_{\{(v^1 - v^2)(\tau', X_{\tau'}, I_{\tau'}) < 0\}}.$$

As a consequence,

$$\begin{aligned} \rho' &= \rho^{1'}1_{\{(v^1 - v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} + \rho^{2'}1_{\{(v^1 - v^2)(\tau', X_{\tau'}, I_{\tau'}) < 0\}}, \\ \tilde{\tau}'_n &= \tilde{\tau}_n^{1'}1_{\{(v^1 - v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} + \tilde{\tau}_n^{2'}1_{\{(v^1 - v^2)(\tau', X_{\tau'}, I_{\tau'}) < 0\}}. \end{aligned}$$



Therefore, from the previous identities and the stochastic subsolution property of  $v^1$ , we obtain

$$\begin{aligned}
& v^1(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} = v^1(\tau', X_{\tau'}^1, I_{\tau'}^1)1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} \\
& \leq \mathbb{E} \left[ \left( \int_{\tau'}^{\rho^{1'}} f(X_t^1, I_t^1, u_t) dt + v^1(\rho^{1'}, X_{\rho^{1'}}, I_{\rho^{1'}}) \right. \right. \\
& \quad \left. \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}_n^{1'}}, I_{(\tilde{\tau}_n^{1'})^-}, I_{\tilde{\tau}_n^{1'}}) 1_{\{\tau' \leq \tilde{\tau}_n^{1'} < \rho^{1'}\}} \right) 1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} \middle| \mathcal{F}_{\tau'}^s \right] \\
& \leq \mathbb{E} \left[ \left( \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \right. \\
& \quad \left. \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}_n'}, I_{(\tilde{\tau}_n')^-}, I_{\tilde{\tau}_n'}) 1_{\{\tau' \leq \tilde{\tau}_n' < \rho'\}} \right) 1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} \middle| \mathcal{F}_{\tau'}^s \right].
\end{aligned}$$

Concerning  $v^2$ , proceeding similarly we get

$$\begin{aligned}
v^2(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) < 0\}} & \leq \mathbb{E} \left[ \left( \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) \right. \right. \\
& \quad \left. \left. - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}_n'}, I_{(\tilde{\tau}_n')^-}, I_{\tilde{\tau}_n'}) 1_{\{\tau' \leq \tilde{\tau}_n' < \rho'\}} \right) 1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) < 0\}} \middle| \mathcal{F}_{\tau'}^s \right].
\end{aligned}$$

In conclusion, we find

$$\begin{aligned}
& v(\tau', X_{\tau'}, I_{\tau'}) \\
& = v^1(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) \geq 0\}} + v^2(\tau', X_{\tau'}, I_{\tau'})1_{\{(v^1-v^2)(\tau', X_{\tau'}, I_{\tau'}) < 0\}} \\
& \leq \mathbb{E} \left[ \int_{\tau'}^{\rho'} f(X_t, I_t, u_t) dt + v(\rho', X_{\rho'}, I_{\rho'}) - \sum_{n \in \mathbb{N}} c(X_{\tilde{\tau}_n'}, I_{(\tilde{\tau}_n')^-}, I_{\tilde{\tau}_n'}) 1_{\{\tau' \leq \tilde{\tau}_n' < \rho'\}} \middle| \mathcal{F}_{\tau'}^s \right],
\end{aligned}$$

which is the subsolution property for  $v$ .

A similar argument allows to prove the stability with respect to infimum of  $\mathcal{V}^+$  in (ii). In particular, fix  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ , and  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$ . Let  $\tilde{u}^1, \tilde{u}^2 \in \mathcal{U}_{s,\tau}^E$  be the two elementary feedback strategies, for the nature, starting at  $\tau$  and corresponding to the supersolutions  $v^1$  and  $v^2$ . Let  $B := \{(v^1 - v^2)(\tau(y), y(\tau(y)^+)) \leq 0\} \in \mathcal{B}_{\tau^+}^s$ . Then, from Lemma 3.1 we see that the map

$$\tilde{u}(t, y) := \tilde{u}^1(t, y)1_{\{y \in B\}} + \tilde{u}^2(t, y)1_{\{y \in B^c\}}$$

is an elementary feedback strategy starting at  $\tau$ , which allows to prove that  $v$  is a stochastic supersolution to equation (2.10).  $\square$

**Lemma 3.5** *Let Assumptions (H1) and (H2) hold.*

- (i) *There exists a nondecreasing sequence  $(v_n)_{n \in \mathbb{N}} \subset \mathcal{V}^-$  such that  $v_n \nearrow v^-$ .*
- (ii) *There exists a nonincreasing sequence  $(v_n)_{n \in \mathbb{N}} \subset \mathcal{V}^+$  such that  $v_n \searrow v^+$ .*

**Proof.** From Proposition 4.1 in [3] we can find a sequence  $(\tilde{v}_n)_{n \in \mathbb{N}} \subset \mathcal{V}^-$  satisfying  $v^- = \sup_{n \in \mathbb{N}} \tilde{v}_n$ . Set  $v_n := \tilde{v}_0 \vee \cdots \vee \tilde{v}_n$ ,  $n \in \mathbb{N}$ . Then  $v_n \nearrow v^-$  as  $n \rightarrow \infty$ , and from Lemma 3.4 we see that  $(v_n)_{n \in \mathbb{N}} \subset \mathcal{V}^-$ . In a similar way we can prove statement (ii).  $\square$

We are now in a position to prove Theorem 3.1. Firstly, we just state here, in the spirit of Lemma 2.4 in [5], the following technical result, which will be used several times in the proof of Theorem 3.1.

**Lemma 3.6** *Let  $\mathcal{C} \subset [0, T] \times \mathbb{R}^d$  be a compact set and consider a continuous function  $F: \mathbb{R}^m \times \mathcal{C} \rightarrow \mathbb{R}$ , which is nondecreasing in each of its first  $m$  components. If there exists  $\delta > 0$  such that  $\inf_{(t,x) \in \mathcal{C}} F(v^-(t, x, \cdot), t, x) > \delta$  (resp.  $\sup_{(t,x) \in \mathcal{C}} F(v^+(t, x, \cdot), t, x) < -\delta$ ), then*

$$\inf_{(t,x) \in \mathcal{C}} F(v(t, x, \cdot), t, x) > \delta$$

$$\left( \text{resp. } \sup_{(t,x) \in \mathcal{C}} F(v(t, x, \cdot), t, x) < -\delta \right)$$

for some  $v \in \mathcal{V}^-$  (resp.  $v \in \mathcal{V}^+$ ).

**Proof.** Notice that, from the strict inequality  $\inf_{(t,x) \in \mathcal{C}} F(v^-(t, x, \cdot), t, x) > \delta$  we can find  $\varepsilon > 0$  such that  $F(v^-(t, x, \cdot), t, x) > \delta + \varepsilon$ , for any  $(t, x) \in \mathcal{C}$ . Recall from Lemma 3.5 that there exists a nondecreasing sequence of stochastic subsolutions  $v_n \nearrow v^-$ . Let

$$A_n := \{(t, x) \in \mathcal{C}: F(v_n(t, x, \cdot), t, x) \leq \delta + \varepsilon/2\}.$$

Notice that  $A_n$  is closed,  $A_{n+1} \subset A_n$ , and  $\bigcap_{n=0}^{\infty} A_n = \emptyset$ . Since  $A_n \subset \mathcal{C}$ , using the compactness we see that there exists an  $n_0$  such that  $A_{n_0} = \emptyset$ , namely  $F(v_{n_0}(t, x, \cdot), t, x) > \delta + \varepsilon$ , for any  $(t, x) \in \mathcal{C}$ . In particular,  $\inf_{(t,x) \in \mathcal{C}} F(v_{n_0}(t, x, \cdot), t, x) > \delta$ . We then take  $v := v_{n_0}$ . In a similar way we can prove the statement for  $v^+$ .  $\square$

### Proof of Theorem 3.1.

**Step I.**  $v^-$  is a viscosity supersolution to the HJB equation (2.10).

**Step I(i).** *Interior viscosity supersolution property.* Let  $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ ,  $i \in \mathbb{I}_m$ , and consider a test function  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$  such that  $v^-(\cdot, \cdot, i) - \varphi(\cdot, \cdot)$  attains a strict global minimum equal to zero at  $(t_0, x_0)$ . Reasoning by contradiction, we assume that

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \inf_{u \in U} [\mathcal{L}^{i,u} \varphi(t_0, x_0) + f(x_0, i, u)], \right. \\ \left. v^-(t_0, x_0, i) - \max_{j \neq i} [v^-(t_0, x_0, j) - c(x_0, i, j)] \right\} < 0.$$

We distinguish two cases.

**Case a.**  $-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \inf_{u \in U} [\mathcal{L}^{i,u} \varphi(t_0, x_0) + f(x_0, i, u)] < 0$ . Then, there exists  $\varepsilon \in (0, T - t_0)$  such that

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \inf_{u \in U} [\mathcal{L}^{i,u} \varphi(t_0, x_0) + f(x_0, i, u)] < -\varepsilon.$$

From the continuity of  $b, \sigma, f$ , together with the compactness of  $U$ , we see that we can choose a smaller  $\varepsilon \in (0, T - t_0)$  such that

$$-\frac{\partial \varphi}{\partial t}(t, x) - \inf_{u \in U} [\mathcal{L}^{i,u} \varphi(t, x) + f(x, i, u)] < -\varepsilon, \quad \forall (t, x) \in B(t_0, x_0, \varepsilon),$$

where

$$B(t_0, x_0, \varepsilon) = \{(t, x) \in [0, T] \times \mathbb{R}^d : \max\{|t - t_0|, |x - x_0|\} < \varepsilon\}. \quad (3.8)$$

Since  $v^-(\cdot, \cdot, i) - \varphi(\cdot, \cdot)$  is lsc and strictly positive on the compact set  $\mathcal{C} := \overline{B(t_0, x_0, \varepsilon)} \setminus B(t_0, x_0, \varepsilon/2)$ , there exists  $\delta > 0$  such that  $\inf_{(t,x) \in \mathcal{C}} (v^-(t, x, i) - \varphi(t, x)) > \delta$ . Denoting  $F(p, t, x) := p - \varphi(t, x)$ , it follows from Lemma 3.6 that there exists  $v \in \mathcal{V}^-$  such that  $\varphi(t, x) + \delta < v(t, x, i)$  on  $\mathcal{C}$ . Now, define

$$v^\delta(t, x, i) = \begin{cases} (\varphi(t, x) + \delta) \vee v(t, x, i), & \text{on } \overline{B(t_0, x_0, \varepsilon)}, \\ v(t, x, i), & \text{outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Moreover,  $v^\delta(t, x, j) = v(t, x, j)$  for any  $(t, x, j) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ , with  $j \neq i$ . Our aim is to prove that  $v^\delta \in \mathcal{V}^-$ , which would give a contradiction, since  $v^\delta(t_0, x_0, i) > v^-(t_0, x_0, i)$ . Clearly,  $v^\delta$  satisfies the first item in Definition 3.4, therefore it remains to prove the second item. To this end, fix  $s \in [0, T]$  and  $\tau, \rho \in \mathcal{T}^s$ , with  $\tau \leq \rho \leq T$ . Let  $\tilde{\alpha}^0 = (\tilde{\tau}_n^0, \tilde{\iota}_n^0)_{n \in \mathbb{N}}$  be given by

$$(\tilde{\tau}_n^0, \tilde{\iota}_n^0) = (T, i), \quad \forall n \in \mathbb{N}.$$

Notice that  $\tilde{\alpha}^0 \in \mathcal{A}_{s, \tau+}$ . Introduce now the stopping rule  $\rho_1 : C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow [s, T]$ ,  $\tau \leq \rho_1 \leq T$ ,

$$\rho_1(y) = \inf \{t \in [\tau(y), T] : (t, y^X(t)) \notin B(t_0, x_0, \varepsilon/2)\} \wedge T. \quad (3.9)$$

We denote by  $\tilde{\alpha}^1 = (\tilde{\tau}_n^1, \tilde{\iota}_n^1)_{n \in \mathbb{N}} \in \mathcal{A}_{s, (\rho_1 \wedge \rho)+}$  the feedback switching control in Definition 3.4, corresponding to  $s, \rho_1 \wedge \rho, \rho$ , for the stochastic subsolution  $v$ . Then, we define  $\tilde{\alpha}^2 = \tilde{\alpha}^0 \otimes_{\rho_1 \wedge \rho} \tilde{\alpha}^1$ , which belongs to  $\mathcal{A}_{s, \tau+}$  thanks to Proposition 3.2. Moreover, let  $\tilde{\alpha}^3 = (\tilde{\tau}_n^3, \tilde{\iota}_n^3)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau+}$  be the feedback switching control corresponding to  $s, \tau, \rho$  for the stochastic subsolution  $v$ . Then, we define  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{\iota}_n)_{n \in \mathbb{N}}$  by (for any  $y \in C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m)$  we write  $y = (y^X, y^I)$  with  $y^X \in C([s, T]; \mathbb{R}^d)$  and  $y^I \in \mathcal{L}([s, T]; \mathbb{I}_m)$ )

$$\begin{aligned} (\tilde{\tau}_n(y), \tilde{\iota}_n(y)) &= (\tilde{\tau}_n^2(y), \tilde{\iota}_n^2(y)) 1_{\{(v-\varphi)(\tau(y), y(\tau(y)^+)) < \delta, y^I(\tau(y)^+) = i\}} \\ &\quad + (\tilde{\tau}_n^3(y), \tilde{\iota}_n^3(y)) 1_{\{(v-\varphi)(\tau(y), y(\tau(y)^+)) < \delta, y^I(\tau(y)^+) = i\}^c}. \end{aligned}$$

From Lemma 3.1 it follows that  $\tilde{\alpha} \in \mathcal{A}_{s, \tau+}$ . Moreover, the feedback switching control  $\tilde{\alpha}$  satisfies the condition in the definition of stochastic subsolution for  $v^\delta$ . To see this, fix  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ ,  $u \in \mathcal{U}_{s, s}$ , and  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ . We adopt the shorthands:

$$\begin{aligned} (X, I) &= (X^{s, x, i; \alpha \otimes_\tau \tilde{\alpha}, u}, I^{s, x, i; \alpha \otimes_\tau \tilde{\alpha}, u}), \\ (X^1, I^1) &= (X^{s, x, i; \alpha \otimes_\tau \tilde{\alpha}^2, u}, I^{s, x, i; \alpha \otimes_\tau \tilde{\alpha}^2, u}), \\ (X^2, I^2) &= (X^{s, x, i; \alpha \otimes_\tau \tilde{\alpha}^3, u}, I^{s, x, i; \alpha \otimes_\tau \tilde{\alpha}^3, u}). \end{aligned}$$

We also denote  $\tau' = \tau(X, I_-)$ ,  $\rho'_1 = \rho_1(X, I_-)$ , and  $\rho' = \rho(X, I_-)$ . Notice that

$$(X, I) = (X^1, I^1) 1_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) < \delta, I_{\tau'} = i\}} + (X^2, I^2) 1_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) < \delta, I_{\tau'} = i\}^c}.$$

In particular, it is useful to decompose  $v^\delta(\tau', X_{\tau'}, I_{\tau'})$  as follows

$$v^\delta(\tau', X_{\tau'}, I_{\tau'}) = (\varphi(\tau', X_{\tau'}^1) + \delta) 1_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) < \delta, I_{\tau'} = i\}} \quad (3.10)$$

$$+ v(\tau', X_{\tau'}^2, I_{\tau'}^2) \mathbf{1}_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) < \delta, I_{\tau'} = i\}}^c.$$

Then, to see that the stochastic subsolution property of  $v^\delta$  holds, we consider the two terms on the right-hand side of (3.10) individually. Regarding the first term, we apply Itô's formula to  $\varphi$  between  $\tau'$  and  $\rho'_1 \wedge \rho'$ , observing that  $I_t^1 = i$  for any  $t \in [\tau', \rho'_1 \wedge \rho']$ ; afterwards, we use the subsolution property of  $v$  with corresponding feedback switching control  $\tilde{\alpha}^1$ . Finally, concerning the other term in (3.10), the result follows from the stochastic subsolution property of  $v$  and the definition of  $\tilde{\alpha}^3$ .

**Case b.**  $v^-(t_0, x_0, i) < \max_{j \neq i} [v^-(t_0, x_0, j) - c(x_0, i, j)]$  and  $-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \inf_{u \in U} [\mathcal{L}^{i,u} \varphi(t_0, x_0) + f(x_0, i, u)] \geq 0$ . Since  $v^-$  is lsc and  $c$  is continuous, there exists  $\varepsilon \in (0, T - t_0)$  such that

$$v^-(t_0, x_0, i) + \varepsilon < \inf_{(t,x) \in \overline{B(t_0, x_0, \varepsilon)}} \max_{j \neq i} [v^-(t, x, j) - c(x, i, j)].$$

Set  $F(p, t, x) = \max_{j \neq i} [p_j - c(x, i, j)]$ , for any  $(p, t, x) \in \mathbb{R}^m \times \overline{B(t_0, x_0, \varepsilon)}$ . Then, from Lemma 3.6 it follows that there exists  $v \in \mathcal{V}^-$  such that  $F(v(t, x, \cdot), t, x) > v^-(t_0, x_0, i) + \varepsilon \geq v(t_0, x_0, i) + \varepsilon$ , for any  $(t, x) \in \overline{B(t_0, x_0, \varepsilon)}$ . We also suppose that the function  $v$  given by Lemma 3.6 satisfies  $v^-(t_0, x_0, i) - v(t_0, x_0, i) < \varepsilon/2$ . Since  $v$  is continuous on  $\overline{B(t_0, x_0, \varepsilon)}$ , we can find  $\delta > 0$  such that

$$\sup_{(t', x') \in \overline{B(t_0, x_0, \delta)}} v(t', x', i) + \varepsilon < \inf_{(t,x) \in B(t_0, x_0, \varepsilon)} \max_{j \neq i} [v(t, x, j) - c(x, i, j)]. \quad (3.11)$$

Let  $M > 0$  be an upper bound for the continuous function  $|f(x, i, u)|$  on the compact set  $\overline{B(t_0, x_0, \varepsilon)} \times \mathbb{I}_m \times U$ . We suppose that  $\delta \leq \varepsilon/(4M)$ . Now, define (we adopt the notation  $\|(t, x)\| = \max\{|t|, |x|\}$ )

$$v^\delta(t, x, i) = \begin{cases} v(t, x, i) + \frac{\varepsilon}{2\delta}(\delta - \|(t - t_0, x - x_0)\|), & \text{on } \overline{B(t_0, x_0, \delta)}, \\ v(t, x, i), & \text{outside } \overline{B(t_0, x_0, \delta)}. \end{cases}$$

Moreover,  $v^\delta(t, x, j) = v(t, x, j)$  for any  $(t, x, j) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ , with  $j \neq i$ . As  $v^\delta(t_0, x_0, i) > v^-(t_0, x_0, i)$ , we get a contradiction if we prove that  $v^\delta \in \mathcal{V}^-$ . In order to do so, fix  $s \in [0, T]$  and  $\tau, \rho \in \mathcal{T}^s$ , with  $\tau \leq \rho \leq T$ . We have to determine  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{t}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau+}$  which works for  $v^\delta$ . To this end, define  $\rho_1 \in \mathcal{T}^s$  as follows

$$\rho_1(y) = \inf \{t \in [\tau(y), T] : (t, y^X(t)) \notin B(t_0, x_0, \delta)\} \wedge T.$$

Let  $\tilde{\alpha}^0 = (\tilde{\tau}_n^0, \tilde{t}_n^0)_{n \in \mathbb{N}}$  be given by:  $(\tilde{\tau}_n^0, \tilde{t}_n^0) = (T, i)$  for any  $n \geq 1$ , and

$$\begin{aligned} \tilde{\tau}_0^0(y) &= (\rho_1 \wedge \rho)(y) \mathbf{1}_{\{(\tau(y), y^X(\tau(y))) \in B(t_0, x_0, \delta)\}} + T \mathbf{1}_{\{(\tau(y), y^X(\tau(y))) \notin B(t_0, x_0, \delta)\}}, \\ \tilde{t}_0^0(y) &= \min_{j \neq i} \{v(\tilde{\tau}_0^0(y), y^X(\tilde{\tau}_0^0(y)), j) - c(y^X(\tilde{\tau}_0^0(y)), i, j) = m(y)\}, \end{aligned}$$

where  $m: C([s, T]; \mathbb{R}^d) \times \mathcal{L}([s, T]; \mathbb{I}_m) \rightarrow \mathbb{R}$  is defined as

$$m(y) = \max_{j \neq i} [v(\tilde{\tau}_0^0(y), y^X(\tilde{\tau}_0^0(y)), j) - c(y^X(\tilde{\tau}_0^0(y)), i, j)].$$

Notice that  $m$  is  $\mathcal{B}_{\tilde{\tau}_0^0}^s$ -measurable, so that  $\tilde{t}_0^0$  is  $\mathcal{B}_{\tilde{\tau}_0^0}^s$ -measurable. Moreover,  $\tau < \tilde{\tau}_0^0$  on the set  $\{\tau < T\}$ . In particular,  $\tilde{\alpha}^0 \in \mathcal{A}_{\tau+}^s$ . Now, consider the feedback switching control  $\tilde{\alpha}^1 =$

$(\tilde{\tau}_n^1, \tilde{t}_n^1)_{n \in \mathbb{N}} \in \mathcal{A}_{s, (\tilde{\tau}_0^0 \wedge \rho)^+}$  in Definition 3.4, corresponding to  $s, \tilde{\tau}_0^0 \wedge \rho, \rho$ , for the stochastic subsolution  $v$ . We define  $\tilde{\alpha}^2 = \tilde{\alpha}^0 \otimes_{\tilde{\tau}_0^0 \wedge \rho} \tilde{\alpha}^1$ , which belongs to  $\mathcal{A}_{s, \tau^+}$  thanks to Proposition 3.2. Consider also the feedback switching control  $\tilde{\alpha}^3 = (\tilde{\tau}_n^3, \tilde{t}_n^3)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau^+}$ , corresponding to  $s, \tau, \rho$ , for the stochastic subsolution  $v$ . Then, let  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{t}_n)_{n \in \mathbb{N}}$  be given by

$$\begin{aligned} (\tilde{\tau}_n(y), \tilde{t}_n(y)) &= (\tilde{\tau}_n^2(y), \tilde{t}_n^2(y)) 1_{\{(\tau(y), y^X(\tau(y))) \in B(t_0, x_0, \delta), y^I(\tau(y)^+) = i\}} \\ &\quad + (\tilde{\tau}_n^3(y), \tilde{t}_n^3(y)) 1_{\{(\tau(y), y^X(\tau(y))) \in B(t_0, x_0, \delta), y^I(\tau(y)^+) = i\}^c}. \end{aligned}$$

From Lemma 3.1 it follows that  $\tilde{\alpha} \in \mathcal{A}_{s, \tau^+}$ . Moreover,  $\tilde{\alpha}$  is the feedback switching control which satisfies the condition in the definition of stochastic subsolution for  $v^\delta$ . To see this, fix  $\alpha = (\tau_n, t_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ ,  $u \in \mathcal{U}_{s, s}$ , and  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ . We adopt the shorthands introduced in Case *a*. Consider the event  $A := \{(\tau', X_{\tau'}) \in B(t_0, x_0, \delta), I_{\tau'} = i\}$ . On  $A^c$  the result follows from the stochastic subsolution property of  $v$  and the definition of  $\tilde{\alpha}^3$ . On the other hand, on  $A$  we have

$$\begin{aligned} v^\delta(\tau', X_{\tau'}, I_{\tau'}) 1_A &= v^\delta(\tau', X_{\tau'}^1, i) 1_A = \left[ v(\tau', X_{\tau'}^1, i) + \frac{\varepsilon}{2\delta} (\delta - \|(\tau' - t_0, X_{\tau'}^1 - x_0)\|) \right] 1_A \\ &\leq \left[ v(\tau', X_{\tau'}^1, i) + \frac{\varepsilon}{2} \right] 1_A. \end{aligned}$$

Using (3.11) and taking the conditional expectation with respect to  $\mathcal{F}_{\tau'}^s$ , we obtain (denoting  $\tilde{\tau}_0^{0, \prime} = \tilde{\tau}_0^0(X_\cdot, I_\cdot)$ )

$$v^\delta(\tau', X_{\tau'}, I_{\tau'}) 1_A \leq \mathbb{E} \left[ v(\tilde{\tau}_0^{0, \prime} \wedge \rho', X_{\tilde{\tau}_0^{0, \prime} \wedge \rho'}^1, I_{\tilde{\tau}_0^{0, \prime} \wedge \rho'}^1) - c(X_{\tilde{\tau}_0^{0, \prime} \wedge \rho'}^1, i, I_{\tilde{\tau}_0^{0, \prime} \wedge \rho'}^1) - \frac{\varepsilon}{2} \middle| \mathcal{F}_{\tau'}^s \right] 1_A.$$

Observe that  $\tilde{\tau}_0^{0, \prime} \leq \rho'$  on  $A$ . Therefore, the above inequality can be written as

$$v^\delta(\tau', X_{\tau'}, I_{\tau'}) 1_A \leq \mathbb{E} \left[ v(\tilde{\tau}_0^{0, \prime}, X_{\tilde{\tau}_0^{0, \prime}}^1, I_{\tilde{\tau}_0^{0, \prime}}^1) - c(X_{\tilde{\tau}_0^{0, \prime}}^1, i, I_{\tilde{\tau}_0^{0, \prime}}^1) - \frac{\varepsilon}{2} \middle| \mathcal{F}_{\tau'}^s \right] 1_A.$$

Adding and subtracting  $\int_{\tau'}^{\tilde{\tau}_0^{0, \prime}} f(X_t^1, I_t^1, u_t) dt$ , noting that  $(\tilde{\tau}_0^{0, \prime} - \tau') 1_A \leq 2\delta$  and  $2\delta M - \varepsilon/2 \leq 0$ , we find

$$v^\delta(\tau', X_{\tau'}, I_{\tau'}) 1_A \leq \mathbb{E} \left[ \int_{\tau'}^{\tilde{\tau}_0^{0, \prime}} f(X_t^1, I_t^1, u_t) dt + v(\tilde{\tau}_0^{0, \prime}, X_{\tilde{\tau}_0^{0, \prime}}^1, I_{\tilde{\tau}_0^{0, \prime}}^1) - c(X_{\tilde{\tau}_0^{0, \prime}}^1, i, I_{\tilde{\tau}_0^{0, \prime}}^1) \middle| \mathcal{F}_{\tau'}^s \right] 1_A.$$

Finally, using the stochastic subsolution property of  $v$ , with corresponding feedback switching control  $\tilde{\alpha}^1$ , and the inequality  $v \leq v^\delta$ , we derive the stochastic subsolution property of  $v^\delta$ .

**Step I(ii).** *Terminal condition.* Reasoning by contradiction, we assume that there exist  $x_0 \in \mathbb{R}^d$  and  $i \in \mathbb{I}_m$  such that

$$v^-(T, x_0, i) < g(x_0, i).$$

Since  $g$  is continuous, there exists  $\varepsilon > 0$  such that  $v^-(T, x_0, i) \leq g(x, i) - \varepsilon$  whenever  $|x - x_0| \leq \varepsilon$ . Consider the compact set

$$\mathcal{C} := \left( \overline{B(T, x_0, \varepsilon)} \setminus B(T, x_0, \varepsilon/2) \right) \cap ([0, T] \times \mathbb{R}^d),$$

where  $B(T, x_0, \varepsilon) = \{(t, x) \in [0, T] \times \mathbb{R}^d : \max\{|t - t_0|, |x - x_0|\} < \varepsilon\}$ . Since  $v^-$  is lsc, it is bounded from below on  $\mathcal{C}$ . Therefore, we can find  $\eta > 0$  small enough (possibly depending on  $\varepsilon$ ) such that

$$v^-(T, x_0, i) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x) \in \mathcal{C}} v^-(t, x, i).$$

From Lemma 3.6 with  $F(p, t, x) = p$  for any  $(p, t, x) \in \mathbb{R} \times \mathcal{C}$ , we can find a stochastic subsolution  $v \in \mathcal{V}^-$  such that

$$v^-(T, x_0, i) - \frac{\varepsilon^2}{4\eta} < -\varepsilon + \inf_{(t,x) \in \mathcal{C}} v(t, x, i). \quad (3.12)$$

For  $k > 0$  define

$$\varphi^{\eta, \varepsilon, k}(t, x) = v^-(T, x_0, i) - \frac{|x - x_0|^2}{\eta} - k(T - t).$$

Since  $b, \sigma, f$  are continuous, we can choose  $k$  large enough such that

$$-\frac{\partial \varphi^{\eta, \varepsilon, k}}{\partial t}(t, x) - \inf_{u \in U} [\mathcal{L}^{i, u} \varphi^{\eta, \varepsilon, k}(t, x) + f(x, i, u)] < 0, \quad \forall (t, x) \in \overline{B(T, x_0, \varepsilon)}.$$

From (3.12) it follows that  $\varphi^{\eta, \varepsilon, k}(t, x) < -\varepsilon + v(t, x, i)$  on  $\mathcal{C}$ . Moreover

$$\varphi^{\eta, \varepsilon, k}(T, x) \leq v^-(T, x_0, i) \leq g(x, i) - \varepsilon, \quad \text{whenever } |x - x_0| \leq \varepsilon.$$

Now, for  $\delta \in (0, \varepsilon)$  define

$$v^\delta(t, x, i) = \begin{cases} (\varphi^{\eta, \varepsilon, k}(t, x) + \delta) \vee v(t, x, i), & \text{on } \overline{B(t_0, x_0, \varepsilon)}, \\ v(t, x, i), & \text{outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Moreover,  $v^\delta(t, x, j) = v(t, x, j)$  for any  $(t, x, j) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ , with  $j \neq i$ . As  $v^\delta(T, x_0, i) > v^-(T, x_0, i)$ , we get a contradiction if we are able to prove that  $v^\delta \in \mathcal{V}^-$ . In particular, for any  $s \in [0, T]$  and  $\tau, \rho \in \mathcal{T}^s$  with  $\tau \leq \rho \leq T$ , we have to find  $\tilde{\alpha} = (\tilde{\tau}_n, \tilde{t}_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, \tau^+}$  which works for  $v^\delta$ . Consider the feedback switching control  $\tilde{\alpha}$  defined in Step I(i), Case *a*, with  $\rho_1$  the exit time from  $B(T, x_0, \varepsilon/2)$ . Then, proceeding as in Case *a* of Step I(i), we can prove that  $\tilde{\alpha}$  satisfies the condition in the definition of stochastic subsolution for  $v^\delta$ .

**Step II.**  $v^+$  is a viscosity subsolution to the HJB equation (2.10).

**Step II(i).** *Interior viscosity subsolution property.* Let  $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$ ,  $i \in \mathbb{I}_m$ , and consider a test function  $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$  such that  $v^+(\cdot, \cdot, i) - \varphi(\cdot, \cdot)$  attains a strict global maximum equal to zero at  $(t_0, x_0)$ . Reasoning by contradiction, we assume that

$$\min \left\{ -\frac{\partial \varphi}{\partial t}(t_0, x_0) - \inf_{u \in U} [\mathcal{L}^{i, u} \varphi(t_0, x_0) + f(x_0, i, u)], \right. \\ \left. v^+(t_0, x_0, i) - \max_{j \neq i} [v^+(t_0, x_0, j) - c(x_0, i, j)] \right\} > 0.$$

Then, there exists  $\varepsilon > 0$  and  $\underline{u} \in U$  such that

$$-\frac{\partial \varphi}{\partial t}(t_0, x_0) - \mathcal{L}^{i, \underline{u}} \varphi(t_0, x_0) - f(x_0, i, \underline{u}) > \varepsilon.$$

From the continuity of  $b, \sigma, f$ , it follows that we can find a smaller  $\varepsilon > 0$  such that

$$-\frac{\partial \varphi}{\partial t}(t, x) - \mathcal{L}^{i, \underline{u}} \varphi(t, x) - f(x, i, \underline{u}) > \varepsilon, \quad \forall (t, x) \in B(t_0, x_0, \varepsilon),$$

where  $B(t_0, x_0, \varepsilon)$  is given by (3.8). As  $v^+(\cdot, \cdot, i) - \varphi(\cdot, \cdot)$  is usc and strictly negative on the compact set  $\mathcal{C} := \overline{B(t_0, x_0, \varepsilon)} \setminus B(t_0, x_0, \varepsilon/2)$ , we see that there exists  $\delta > 0$  such that  $\sup_{(t, x) \in \mathcal{C}} (v^+(t, x, i) - \varphi(t, x)) < -\delta$ . Denoting  $F(p, t, x) := p - \varphi(t, x)$ , it follows from Lemma 3.6 that there exists  $v \in \mathcal{V}^+$  such that  $\varphi(t, x) - \delta > v(t, x, i)$  on  $\mathcal{C}$ . Now, define

$$v^\delta(t, x, i) = \begin{cases} (\varphi(t, x) - \delta) \wedge v(t, x, i), & \text{on } \overline{B(t_0, x_0, \varepsilon)}, \\ v(t, x, i), & \text{outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Moreover,  $v^\delta(t, x, j) = v(t, x, j)$  for any  $(t, x, j) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ , with  $j \neq i$ . As  $v^\delta(t_0, x_0, i) < v^+(t_0, x_0, i)$ , we find a contradiction if we are able to prove that  $v^\delta \in \mathcal{V}^+$ . To this end, fix  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ , and  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ . We have to construct an elementary feedback strategy  $\tilde{u} \in \mathcal{U}_{s, \tau}^E$  which works for  $v^\delta$ . Consider the stopping rule  $\rho_1 \in \mathcal{T}^s$  given by (3.9), and let  $\tilde{u}^1 \in \mathcal{U}_{s, \rho_1}^E$  be the elementary feedback strategy for  $v$ , corresponding to  $s, \rho_1, \alpha$ . Then, we define  $\tilde{u}^2 = \underline{u} \otimes_{\rho_1} \tilde{u}^1$ , which belongs to  $\mathcal{U}_{s, \tau}^E$  thanks to Proposition 3.2. Now, let  $\tilde{u}^3 \in \mathcal{U}_{s, \tau}^E$  be the elementary feedback strategy for  $v$ , corresponding to  $s, \tau, \alpha$ . Then, we define

$$\begin{aligned} \tilde{u}(t, y) &= \tilde{u}^2(t, y) \mathbf{1}_{\{(v-\varphi)(\tau(y), y(\tau(y)^+)) > -\delta, y^I(\tau(y)^+) = i\}} \\ &\quad + \tilde{u}^3(t, y) \mathbf{1}_{\{(v-\varphi)(\tau(y), y(\tau(y)^+)) > -\delta, y^I(\tau(y)^+) = i\}^c}. \end{aligned}$$

From Lemma 3.1 we see that  $\tilde{u} \in \mathcal{U}_{s, \tau}^E$ . Moreover,  $\tilde{u}$  is the elementary feedback strategy for the stochastic supersolution property of  $v^\delta$ . Indeed, fix  $u \in \mathcal{U}_{s, s}^E$ ,  $(x, i) \in \mathbb{R}^d \times \mathbb{I}_m$ , and  $\rho \in \mathcal{T}^s$ , with  $\tau \leq \rho \leq T$ . We adopt the shorthands:

$$\begin{aligned} (X, I) &= (X^{s, x, i; \alpha, u \otimes_\tau \tilde{u}}, I^{s, x, i; \alpha, u \otimes_\tau \tilde{u}}), \\ (X^1, I^1) &= (X^{s, x, i; \alpha, u \otimes_\tau \tilde{u}^2}, I^{s, x, i; \alpha, u \otimes_\tau \tilde{u}^2}), \\ (X^2, I^2) &= (X^{s, x, i; \alpha, u \otimes_\tau \tilde{u}^3}, I^{s, x, i; \alpha, u \otimes_\tau \tilde{u}^3}). \end{aligned}$$

We also denote  $\tau' = \tau(X, I_-)$ ,  $\rho'_1 = \rho_1(X, I_-)$ , and  $\rho' = \rho(X, I_-)$ . Notice that

$$(X, I) = (X^1, I^1) \mathbf{1}_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta, I_{\tau'} = i\}} + (X^2, I^2) \mathbf{1}_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta, I_{\tau'} = i\}^c}.$$

Moreover, write  $v^\delta(\tau', X_{\tau'}, I_{\tau'})$  as follows

$$\begin{aligned} v^\delta(\tau', X_{\tau'}, I_{\tau'}) &= (\varphi(\tau', X_{\tau'}^1) - \delta) \mathbf{1}_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta, I_{\tau'} = i\}} \\ &\quad + v(\tau', X_{\tau'}^2, I_{\tau'}^2) \mathbf{1}_{\{(v-\varphi)(\tau', X_{\tau'}, I_{\tau'}) > -\delta, I_{\tau'} = i\}^c}. \end{aligned}$$

Then, applying Itô's formula to  $\varphi$  and using the stochastic supersolution property of  $v$ , we derive the stochastic supersolution property for  $v^\delta$ .

**Step II(ii).** *Terminal condition.* Reasoning by contradiction, we assume that there exist  $x_0 \in \mathbb{R}^d$  and  $i \in \mathbb{I}_m$  such that

$$v^+(T, x_0, i) > g(x_0, i).$$



Since  $g$  is continuous, there exists  $\varepsilon > 0$  such that  $v^+(T, x_0, i) \geq g(x, i) + \varepsilon$  whenever  $|x - x_0| \leq \varepsilon$ . Consider the compact set

$$\mathcal{C} := \left( \overline{B(T, x_0, \varepsilon)} \setminus B(T, x_0, \varepsilon/2) \right) \cap ([0, T] \times \mathbb{R}^d).$$

As  $v^+$  is usc, it is bounded from above on  $\mathcal{C}$ . Therefore, we can find  $\eta > 0$  small enough (possibly depending on  $\varepsilon$ ) such that

$$v^+(T, x_0, i) + \frac{\varepsilon^2}{4\eta} > \varepsilon + \sup_{(t,x) \in \mathcal{C}} v^+(t, x, i).$$

From Lemma 3.6 with  $F(p, t, x) = p$  for any  $(p, t, x) \in \mathbb{R} \times \mathcal{C}$ , we can find a stochastic supersolution  $v \in \mathcal{V}^+$  such that

$$v^+(T, x_0, i) + \frac{\varepsilon^2}{4\eta} > \varepsilon + \sup_{(t,x) \in \mathcal{C}} v(t, x, i). \quad (3.13)$$

For  $k > 0$  define

$$\varphi^{\eta, \varepsilon, k}(t, x) = v^+(T, x_0, i) + \frac{|x - x_0|^2}{\eta} + k(T - t).$$

Since  $b, \sigma, f$  are continuous, we can choose  $k$  large enough and  $\underline{u} \in U$  such that

$$-\frac{\partial \varphi^{\eta, \varepsilon, k}}{\partial t}(t, x) - \mathcal{L}^{i, \underline{u}} \varphi^{\eta, \varepsilon, k}(t, x) - f(x, i, \underline{u}) > 0, \quad \forall (t, x) \in \overline{B(T, x_0, \varepsilon)}.$$

From (3.13) it follows that  $\varphi^{\eta, \varepsilon, k}(t, x) > \varepsilon + v(t, x, i)$  on  $\mathcal{C}$ . Moreover

$$\varphi^{\eta, \varepsilon, k}(T, x) \geq v^+(T, x_0, i) \geq g(x, i) + \varepsilon, \quad \text{whenever } |x - x_0| \leq \varepsilon.$$

Now, for  $\delta \in (0, \varepsilon)$  define

$$v^\delta(t, x, i) = \begin{cases} (\varphi^{\eta, \varepsilon, k}(t, x) - \delta) \wedge v(t, x, i), & \text{on } \overline{B(t_0, x_0, \varepsilon)}, \\ v(t, x, i), & \text{outside } \overline{B(t_0, x_0, \varepsilon)}. \end{cases}$$

Moreover,  $v^\delta(t, x, j) = v(t, x, j)$  for any  $(t, x, j) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ , with  $j \neq i$ . As  $v^\delta(T, x_0, i) < v^+(T, x_0, i)$ , we get a contradiction if we prove that  $v^\delta \in \mathcal{V}^+$ . In particular, for any  $s \in [0, T]$ ,  $\tau \in \mathcal{T}^s$ , and  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s, s}$ , we have to find  $\tilde{u} \in \mathcal{U}_{s, \tau}^E$  for the stochastic supersolution property of  $v^\delta$ . Let  $\tilde{u} \in \mathcal{U}_{s, \tau}^E$  be the elementary feedback strategy defined in Step II(i), with  $\rho_1$  the exit time from  $B(T, x_0, \varepsilon/2)$ . Then, we can prove, as in Step II(i), that  $\tilde{u}$  satisfies the condition in the definition of stochastic supersolution for  $v^\delta$ .  $\square$

## 4 Dynamic programming and viscosity properties of $V$

In the present section, by means of the comparison principle for equation (2.10), we prove that  $V$  satisfies the dynamic programming principle and is a viscosity solution to equation (2.10), which therefore turns out to be the dynamic programming equation of the robust switching control problem.

## 4.1 Comparison principle and viscosity characterization

We need to make an additional assumption on the switching costs in order to get comparison principle.

**(H3)**

The switching cost function  $c$  satisfies the *no free loop property*: for any sequence of indices  $i_1, \dots, i_k \in \mathbb{I}_m$ , with  $k \in \mathbb{N} \setminus \{0, 1, 2\}$ ,  $i_1 = i_k$ , and  $\text{card}\{i_1, \dots, i_k\} = k - 1$ , we have

$$c(x, i_1, i_2) + c(x, i_2, i_3) + \dots + c(x, i_{k-1}, i_k) + c(x, i_k, i_1) > 0, \quad \forall x \in \mathbb{R}^d.$$

We also assume that  $c(x, i, i) = 0$ , for any  $x \in \mathbb{R}^d$  and  $i \in \mathbb{I}_m$ .

**Theorem 4.1 (Comparison principle)** *Let Assumptions (H1), (H2) and (H3) hold and consider a viscosity subsolution  $u$  (resp. supersolution  $v$ ) to equation (2.10). Suppose that*

$$\sup_{(t,x,i) \in [0,T] \times \mathbb{R}^d \times \mathbb{I}_m} \frac{|u(t,x,i)| + |v(t,x,i)|}{1 + |x|^q} < \infty,$$

for some  $q \geq 1$ . Then, we have  $u(t, x, i) \leq v(t, x, i)$  for any  $(t, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ .

**Remark 4.1** The proof can be done along the lines of Proposition 3.1 in [13], apart from minor changes due to the presence of the infimum over  $U$  in (2.10), which are dealt with by the uniform Lipschitz condition in (H1)(ii). More precisely, it is proved, as usual, proceeding by contradiction and then using the doubling variable technique. We simply notice here that equation (2.10) requires a particular step. Indeed, along the sequence of maximum points  $(t_n, x_n)_n$  coming through the doubling of variables, we require

$$u(t_n, x_n, i) > \max_{j \neq i} [u(t_n, x_n, j) - c(x_n, i, j)], \quad (4.1)$$

so that, from the viscosity subsolution property of  $u$ , we can derive an inequality for the PDE part of equation (2.10) (concerning  $v$ , the viscosity supersolution property implies already the nonnegativity of both terms in (2.10)). Condition (4.1) is obtained from a “no-loop” argument presented in Theorem 3.1 of [16] (see also Lemma A.2 in [1] and Proposition 3.1 in [13]), which is based on the no free loop property in (H3).  $\square$

**Corollary 4.1** *Under Assumptions (H1), (H2), and (H3), we have  $v^- = V = \bar{V} = v^+$ . In particular,  $V$  (as  $v^-, \bar{V}, v^+$ ) is continuous. Moreover,  $V$  is the unique viscosity solution to equation (2.10) satisfying a polynomial growth condition. Furthermore,  $V$  satisfies the dynamic programming principle: for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and  $\rho \in \mathcal{T}^s$ ,*

$$\begin{aligned} V(s, x, i) = & \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + V(\rho', X_{\rho'}, I_{\rho'}) \right. \\ & \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right] \end{aligned}$$

$$\begin{aligned}
&= \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}^E} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + V(\rho', X_{\rho'}, I_{\rho'}) \right. \\
&\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right],
\end{aligned}$$

with the shorthands  $X = X^{s,x,i;\alpha,u}$ ,  $I = I^{s,x,i;\alpha,u}$ ,  $\rho' = \rho(X, I, \cdot)$ ,  $\tau'_n = \tau_n(X, I, \cdot)$ , and  $u'_t = u(t, X, I, \cdot)$ .

**Proof.** The equality  $v^- = V = \bar{V} = v^+$  follows from the comparison Theorem 4.1. Since  $v^-$  is lsc and  $v^+$  is usc, we see that  $V$  is continuous. Moreover, from Remark 2.5, Theorem 3.1, and Theorem 4.1 it follows that  $V$  is the unique viscosity solution to equation (2.10) satisfying a polynomial growth condition. Finally, let us prove the dynamic programming principle for  $V$ . We begin noting that  $v^-$  and  $v^+$  satisfy, respectively, the half dynamic programming principles: for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and  $\rho \in \mathcal{T}^s$ ,

$$\begin{aligned}
v^-(s, x, i) &\leq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v^-(\rho', X_{\rho'}, I_{\rho'}) \right. \\
&\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right] \tag{4.2}
\end{aligned}$$

and

$$\begin{aligned}
v^+(s, x, i) &\geq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}^E} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u'_t) dt + v^+(\rho', X_{\rho'}, I_{\rho'}) \right. \\
&\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right], \tag{4.3}
\end{aligned}$$

with the shorthands  $X = X^{s,x,i;\alpha,u}$ ,  $I = I^{s,x,i;\alpha,u}$ ,  $\rho' = \rho(X, I, \cdot)$ ,  $\tau'_n = \tau_n(X, I, \cdot)$ , and  $u'_t = u(t, X, I, \cdot)$ . As a matter of fact, let  $(v_n)_{n \in \mathbb{N}} \subset \mathcal{V}^-$  be the sequence in Lemma 3.5(i). From Lemma 3.2 we know that each  $v_n$  satisfies the half dynamic programming principle: for any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$  and  $\rho \in \mathcal{T}^s$ ,

$$\begin{aligned}
v_n(s, x, i) &\leq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v_n(\rho', X_{\rho'}, I_{\rho'}) \right. \\
&\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right].
\end{aligned}$$

Since  $v_n \leq v^-$ , we get

$$\begin{aligned}
v_n(s, x, i) &\leq \sup_{\alpha \in \mathcal{A}_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} \mathbb{E} \left[ \int_s^{\rho'} f(X_t, I_t, u_t) dt + v^-(\rho', X_{\rho'}, I_{\rho'}) \right. \\
&\quad \left. - \sum_{n \in \mathbb{N}} c(X_{\tau'_n}, I_{(\tau'_n)^-}, I_{\tau'_n}) 1_{\{s \leq \tau'_n < \rho'\}} \right]. \tag{4.4}
\end{aligned}$$

Letting  $n \rightarrow \infty$  in (4.4), we finally obtain the half DPP (4.2) for  $v^-$ . In a similar way we can prove (4.3). Combining (4.2) and (4.3) with the equalities  $v^- = V = v^+$ , we end up with the dynamic programming principle for  $V$ .  $\square$

## 4.2 A counterexample to uniqueness

We now describe the Elliott-Kalton version of the robust switching control problem, and compare its value function to the value function  $V$ . To this end, we need to introduce the concept of non-anticipating strategy for the switcher. Firstly, we define a standard switching control, not necessarily of feedback form.

**Definition 4.1 (Switching controls)** Fix  $s \in [0, T]$ . We say that the double sequence  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}}$  is a switching control starting at  $s$  if:

- $\tau_n$  is an  $\mathbb{F}^s$ -stopping time, for any  $n \in \mathbb{N}$ , and

$$s \leq \tau_0 \leq \dots \leq \tau_n \leq \dots \leq T.$$

Moreover,  $(\tau_n)_{n \in \mathbb{N}}$  satisfies the following property: for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$ ,

$$\tau_n(\omega) = T, \quad \text{for } n \text{ large enough.}$$

- $\iota_n: \Omega \rightarrow \mathbb{I}_m$  is  $\mathcal{F}_{\tau_n}^s$ -measurable, for any  $n \in \mathbb{N}$ .

$\overline{\mathcal{A}}_{s,s}$  denotes the family of all switching controls starting at  $s$ .

When using switching controls as defined above, the well-posedness of equation (2.1) becomes easier. In particular, we have the following result, whose standard proof is omitted.

**Proposition 4.1** Let Assumption **(H1)** hold. For any  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ ,  $\alpha \in \overline{\mathcal{A}}_{s,s}$ ,  $u \in \mathcal{U}_{s,s}$ , there exists a unique (up to indistinguishability)  $\mathbb{F}^s$ -adapted process  $(X^{s,x,i;\alpha,u}, I^{s,i;\alpha}) = (X_t^{s,x,i;\alpha,u}, I_t^{s,i;\alpha})_{s \leq t \leq T}$  to equation (2.1). Moreover, estimate (2.2) holds.

We can now introduce the concept of non-anticipating strategy for the switcher.

**Definition 4.2 (Non-anticipating strategies)** Fix  $s \in [0, T]$ . We say that the map

$$\begin{aligned} \beta: \mathcal{U}_{s,s} &\longrightarrow \overline{\mathcal{A}}_{s,s} \\ u &\longmapsto \beta[u] = (\tau_n[u], \iota_n[u])_{n \in \mathbb{N}} \end{aligned}$$

is a non-anticipating strategy starting at  $s$  if

$$\mathbb{P}[(\tau_n[u^1], \iota_n[u^1])1_{\{\tau_n[u^1] \leq t\}} = (\tau_n[u^2], \iota_n[u^2])1_{\{\tau_n[u^2] \leq t\}}, \forall n \in \mathbb{N}] = 1$$

whenever  $\mathbb{P}(u_r^1 = u_r^2, \forall r \in [s, t]) = 1$ , for any  $t \in [s, T]$  and  $u^1, u^2 \in \mathcal{U}_{s,s}$ .  $\Delta_{s,s}$  denotes the family of all non-anticipating strategies starting at  $s$ .

We can now define the corresponding value function:

$$\hat{V}(s, x, i) := \sup_{\beta \in \Delta_{s,s}} \inf_{u \in \mathcal{U}_{s,s}} J(s, x, i; \beta[u], u),$$

for all  $(s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m$ . Notice that

$$V(s, x, i) \leq \hat{V}(s, x, i), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R}^d \times \mathbb{I}_m. \quad (4.5)$$

Under Assumptions **(H1)** and **(H2)**, we expect that  $\hat{V}$  (as  $V$ ) is a viscosity solution to equation (2.10), so that, by comparison,  $V = \hat{V}$ . However, when comparison does not hold, the above inequality (4.5) might be strict at some point, as the following example shows.

**Example 4.1** Fix  $d = 1$ ,  $m = 2$  so that  $\mathbb{I}_2 = \{1, 2\}$ , and take  $U = \mathbb{I}_2$ . Moreover, set  $b(x, i, u) = -(i - u)^2$  and  $\sigma \equiv 0$ . Notice that  $b \in \{-1, 0\}$ . Since Assumption **(H1)** is satisfied, from Proposition 4.1 it follows that, for any  $(s, x, i) \in [0, T] \times \mathbb{R} \times \mathbb{I}_2$ ,  $\alpha \in \overline{\mathcal{A}}_{s,s}$ ,  $u \in \mathcal{U}_{s,s}$ , there exists a unique solution  $(X^{s,x,i;\alpha,u}, I^{s,i;\alpha}) = (X_t^{s,x,i;\alpha,u}, I_t^{s,i;\alpha})_{s \leq t \leq T}$  to equation (2.1).

Set  $g(x, i) = x$ ,  $f \equiv 0$ , and  $c \equiv 0$ . Our aim is now to determine the explicit form of  $\hat{V}$  and  $V$ . To this end, it is convenient to prove, relying on an approximation result by [17], that  $\hat{V}$  can be defined equivalently taking the infimum over the set of step controls, i.e., controls which are pure jump processes and have a uniformly finite number of jumps. In our particular example, since  $U = \{1, 2\}$  we notice that every control  $u \in \mathcal{U}_{s,s}$  is already a pure jump process, however it is not guaranteed that it has a uniformly finite number of jumps.

**Definition 4.3 (Step controls)** Fix  $s \in [0, T]$ . We say that  $u$  is a step control starting at  $s$  if there exists  $n \in \mathbb{N} \setminus \{0\}$ :

- $\tau_k$  is an  $\mathbb{F}^s$ -stopping time, for any  $k = 1, \dots, n$ , and

$$s =: \tau_0 \leq \dots \leq \tau_k \leq \dots \leq \tau_n = T.$$

- $\xi_k: \Omega \rightarrow U$  is  $\mathcal{F}_{\tau_{k-1}}^s$ -measurable, for any  $k = 1, \dots, n$ .

The control  $u: [s, T] \times \Omega \rightarrow U$  is given by

$$u_t := \xi_1 1_{\{t=s\}} + \sum_{k=1}^n \xi_k 1_{\{\tau_{k-1} < t \leq \tau_k\}}.$$

$\mathcal{U}_{s,s}^S$  denotes the family of all step controls starting at  $s$ .

Now, recalling the approximation result in [17], Lemma 3.2.6, together with the form of the functional  $J(s, x, i; \alpha, u) = \mathbb{E}[X_T^{s,x,i;\alpha,u}]$  in the present particular example, we conclude that

$$\hat{V}(s, x, i) = \sup_{\beta \in \Delta_{s,s}} \inf_{u \in \mathcal{U}_{s,s}^S} J(s, x, i; \beta[u], u), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R} \times \mathbb{I}_2.$$

Let us now determine the form of the function  $\hat{V}$ . Since the terminal payoff  $g$  is strictly increasing and the drift  $b$  is nonpositive, the aim of the switcher is to keep the system still. This can be done adopting the following strategy:  $\beta[u] = (\tau_n[u], \iota_n[u])_{n \in \mathbb{N}}$  where, for any  $u_t := \xi_1 1_{\{t=s\}} + \sum_{k=1}^n \xi_k 1_{\{\sigma_{k-1} < t \leq \sigma_k\}}$  in  $\mathcal{U}_{s,s}^S$ , we have

$$(\tau_k[u], \iota_k[u]) = (\sigma_k, \xi_{k+1}), \quad \forall k = 0, \dots, n-1.$$

With this choice,  $X_t^{s,x,i;\beta[u],u} = x$  for any  $t \in [s, T]$ . Moreover,  $\beta \in \Delta_{s,s}$ . In conclusion, we find

$$\hat{V}(s, x, i) = g(x, i) = x, \quad \forall (s, x, i) \in [0, T] \times \mathbb{R} \times \mathbb{I}_2.$$

Let us now find the expression of  $V$ . Fix  $(s, x, i) \in [0, T] \times \mathbb{R} \times \mathbb{I}_2$  and  $\alpha = (\tau_n, \iota_n)_{n \in \mathbb{N}} \in \mathcal{A}_{s,s}$ . The aim of nature is to minimize the quantity  $J(s, x, i; \alpha, u)$  over  $\mathcal{U}_{s,s}$ , which means to maximize the drift  $b$ , i.e., to keep it at the value  $-1$ . This can be done as follows. Define  $u \in \mathcal{U}_{s,s}$ , depending on  $\alpha$ , by

$$u_t := (3 - i) 1_{\{s \leq t \leq \tau_0\}} + \sum_{n \in \mathbb{N}} (3 - \iota_n) 1_{\{\tau_n < t \leq \tau_{n+1}\}}, \quad \forall t \in [s, T].$$

Observe that, since  $i, \iota_n \in \mathbb{I}_2$  then  $3 - i, 3 - \iota_n \in \mathbb{I}_2$ ; moreover, when  $i = 1$  then  $3 - i = 2$ , while if  $i = 2$  then  $3 - i = 1$ . Notice that, for  $\mathbb{P}$ -a.e.  $\omega \in \Omega$  we have  $I_t^{s,i;\alpha}(\omega) = 3 - u_t(\omega)$ , for all  $t \in [s, T]$  with  $t \neq \tau_n(\omega)$ ,  $n \in \mathbb{N}$ . Therefore,  $\mathbb{P}$ -a.s.,

$$b(X_t^{s,x,i;\alpha,u}, I_t^{s,i;\alpha}, u_t) = -(I_t^{s,i;\alpha} - u_t)^2 = -1,$$

for all  $t \in [s, T]$ , with  $t \neq \tau_n$ ,  $n \in \mathbb{N}$ . It follows that,  $\mathbb{P}$ -a.s. we have  $X_T^{s,x,i;\alpha,u} = x - (T - s)$ . In other words, we obtain

$$V(s, x, i) = x - (T - s), \quad \forall (s, x, i) \in [0, T] \times \mathbb{R} \times \mathbb{I}_2.$$

In conclusion,  $V < \hat{V}$  on  $[0, T] \times \mathbb{R} \times \mathbb{I}_2$ . We finally observe that both  $V$  and  $\hat{V}$  are classical solutions to equation (2.10), so that comparison does not hold. This is due to the fact that while Assumptions **(H1)** and **(H2)** hold, the no free loop property in **(H3)** is not satisfied.  $\square$

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