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# BOTANY OF IRREDUCIBLE AUTOMORPHISMS OF FREE GROUPS

THIERRY COULBOIS, ARNAUD HILION

ABSTRACT. We give a classification of iwip outer automorphisms of the free group, by discussing the properties of their attracting and repelling trees.

## 1. INTRODUCTION

An outer automorphism  $\Phi$  of the free group  $F_N$  is **fully irreducible** (abbreviated as **iwip**) if no positive power  $\Phi^n$  fixes a proper free factor of  $F_N$ . Being an iwip is one (in fact the most important) of the analogs for free groups of being pseudo-Anosov for mapping classes of hyperbolic surfaces. Another analog of pseudo-Anosov is the notion of an atoroidal automorphism: an element  $\Phi \in \text{Out}(F_N)$  is **atoroidal** or **hyperbolic** if no positive power  $\Phi^n$  fixes a nontrivial conjugacy class. Bestvina and Feighn [BF92] and Brinkmann [Bri00] proved that  $\Phi$  is atoroidal if and only if the mapping torus  $F_N \rtimes_{\Phi} \mathbb{Z}$  is Gromov-hyperbolic.

Pseudo-Anosov mapping classes are known to be “generic” elements of the mapping class group (in various senses). Rivin [Riv08] and Sisto [Sis11] recently proved that, in the sense of random walks, generic elements of  $\text{Out}(F_N)$  are atoroidal iwip automorphisms.

Bestvina and Handel [BH92] proved that iwip automorphisms have the key property of being represented by (absolute) train-track maps.

A pseudo-Anosov element  $f$  fixes two projective classes of measured foliations  $[(\mathcal{F}^+, \mu^+)]$  and  $[(\mathcal{F}^-, \mu^-)]$ :

$$(\mathcal{F}^+, \mu^+) \cdot f = (\mathcal{F}^+, \lambda\mu^+) \text{ and } (\mathcal{F}^-, \mu^-) \cdot f = (\mathcal{F}^-, \lambda^{-1}\mu^-)$$

where  $\lambda > 1$  is the expansion factor of  $f$ . Alternatively, considering the dual  $\mathbb{R}$ -trees  $T^+$  and  $T^-$ , we get:

$$T^+ \cdot f = \lambda T^+ \text{ and } T^- \cdot f = \lambda^{-1} T^-.$$

We now discuss the analogous situation for iwip automorphisms. The group of outer automorphisms  $\text{Out}(F_N)$  acts on the **outer space**  $\text{CV}_N$  and its boundary  $\partial\text{CV}_N$ . Recall that the compactified outer space  $\overline{\text{CV}}_N = \text{CV}_N \cup \partial\text{CV}_N$  is made up of (projective classes of)  $\mathbb{R}$ -trees with an action of  $F_N$  by isometries which is minimal and very small. See [Vog02] for a survey on outer space. An iwip outer automorphism  $\Phi$  has North-South dynamics on  $\overline{\text{CV}}_N$ : it has a unique attracting fixed tree  $[T_{\Phi}]$  and a unique repelling fixed tree  $[T_{\Phi^{-1}}]$  in the boundary of outer space (see [LL03]):

$$T_{\Phi} \cdot \Phi = \lambda_{\Phi} T_{\Phi} \text{ and } T_{\Phi^{-1}} \cdot \Phi = \frac{1}{\lambda_{\Phi^{-1}}} T_{\Phi^{-1}},$$

where  $\lambda_{\Phi} > 1$  is the **expansion factor** of  $\Phi$  (i.e. the exponential growth rate of non-periodic conjugacy classes).

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Contrary to the pseudo-Anosov setting, the expansion factor  $\lambda_\Phi$  of  $\Phi$  is typically different from the expansion factor  $\lambda_{\Phi^{-1}}$  of  $\Phi^{-1}$ . More generally, qualitative properties of the fixed trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  can be fairly different. This is the purpose of this paper to discuss and compare the properties of  $\Phi$ ,  $T_\Phi$  and  $T_{\Phi^{-1}}$ .

First, the free group,  $F_N$ , may be realized as the fundamental group of a surface  $S$  with boundary. It is part of folklore that, if  $\Phi$  comes from a pseudo-Anosov mapping class on  $S$ , then its limit trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  live in the Thurston boundary of Teichmüller space: they are dual to a measured foliation on the surface. Such trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are called **surface trees** and such an iwip outer automorphism  $\Phi$  is called **geometric** (in this case  $S$  has exactly one boundary component).

The notion of surface trees has been generalized (see for instance [Bes02]). An  $\mathbb{R}$ -tree which is transverse to measured foliations on a finite CW-complex is called **geometric**. It may fail to be a surface tree if the complex fails to be a surface.

If  $\Phi$  does not come from a pseudo-Anosov mapping class and if  $T_\Phi$  is geometric then  $\Phi$  is called **parageometric**. For a parageometric iwip  $\Phi$ , Guirardel [Gui05] and Handel and Mosher [HM07] proved that the repelling tree  $T_{\Phi^{-1}}$  is not geometric. So we have that,  $\Phi$  comes from a pseudo-Anosov mapping class on a surface with boundary if and only if both trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are geometric. Moreover in this case both trees are indeed surface trees.

In our paper [CH10] we introduced a second dichotomy for trees in the boundary of Outer space with dense orbits. For a tree  $T$ , we consider its **limit set**  $\Omega \subseteq \overline{T}$  (where  $\overline{T}$  is the metric completion of  $T$ ). The limit set  $\Omega$  consists of points of  $\overline{T}$  with at least two pre-images by the map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T} = \overline{T} \cup \partial T$  introduced by Levitt and Lustig [LL03], see Section 4A. We are interested in the two extremal cases: A tree  $T$  in the boundary of Outer space with dense orbits is of **surface type** if  $T \subseteq \Omega$  and  $T$  is of **Levitt type** if  $\Omega$  is totally disconnected. As the terminology suggests, a surface tree is of surface type. Trees of Levitt type were discovered by Levitt [Lev93].

Combining together the two sets of properties, we introduced in [CH10] the following definitions. A tree  $T$  in  $\partial CV_N$  with dense orbits is

- a **surface tree** if it is both geometric and of surface type;
- **Levitt** if it is geometric and of Levitt type;
- **pseudo-surface** if it is not geometric and of surface type;
- **pseudo-Levitt** if it is not geometric and of Levitt type

The following Theorem is the main result of this paper.

**Theorem 5.2.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Then exactly one of the following occurs*

- (1) *The trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are surface trees. Equivalently  $\Phi$  is geometric.*
- (2) *The tree  $T_\Phi$  is Levitt (i.e. geometric and of Levitt type), and the tree  $T_{\Phi^{-1}}$  is pseudo-surface (i.e. non-geometric and of surface type). Equivalently  $\Phi$  is parageometric.*
- (3) *The tree  $T_{\Phi^{-1}}$  is Levitt (i.e. geometric and of Levitt type), and the tree  $T_\Phi$  is pseudo-surface (i.e. non-geometric and of surface type). Equivalently  $\Phi^{-1}$  is parageometric.*
- (4) *The trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are pseudo-Levitt (non-geometric and of Levitt type).*

*Case (1) corresponds to toroidal iwips whereas cases (2), (3) and (4) corresponds to atoroidal iwips. In case (4) the automorphism  $\Phi$  is called **pseudo-Levitt**.*

Gaboriau, Jaeger, Levitt and Lustig [GJLL98] introduced the notion of an **index**  $\text{ind}(\Phi)$ , computed from the rank of the fixed subgroup and from the number of attracting fixed points of the automorphisms  $\varphi$  in the outer class  $\Phi$ . Another index for a tree  $T$  in  $\overline{\text{CV}}_N$  has been defined and studied by Gaboriau and Levitt [GL95], we call it the **geometric index**  $\text{ind}_{\text{geo}}(T)$ . Finally in our paper [CH10] we introduced and studied the **Q-index**  $\text{ind}_{\text{Q}}(T)$  of an  $\mathbb{R}$ -tree  $T$  in the boundary of outer space with dense orbits. The two indices  $\text{ind}_{\text{geo}}(T)$  and  $\text{ind}_{\text{Q}}(T)$  describe qualitative properties of the tree  $T$  [CH10]. We define these indices and recall our botanical classification of trees in Section 4A.

The key to prove Theorem 5.2 is:

**Propositions 4.2 and 4.4.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Replacing  $\Phi$  by a suitable power, we have*

$$2 \text{ind}(\Phi) = \text{ind}_{\text{geo}}(T_\Phi) = \text{ind}_{\text{Q}}(T_{\Phi^{-1}}).$$

We prove this Proposition in Sections 4B and 4C.

To study limit trees of iwip automorphisms, we need to state that they have the strongest mixing dynamical property, which is called **indecomposability**.

**Theorem 2.1.** *Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism. The attracting tree  $T_\Phi$  of  $\Phi$  is indecomposable.*

The proof of this Theorem is quite independent of the rest of the paper and is the purpose of Section 2. The proof relies on a key property of iwip automorphisms: they can be represented by (absolute) train-track maps.

## 2. INDECOMPOSABILITY OF THE ATTRACTING TREE OF AN IWIP AUTOMORPHISM

Following Guirardel [Gui08], a (projective class of)  $\mathbb{R}$ -tree  $T \in \overline{\text{CV}}_N$  is **indecomposable** if for all non degenerate arcs  $I$  and  $J$  in  $T$ , there exists finitely many elements  $u_1, \dots, u_n$  in  $F_N$  such that

$$(2.1) \quad J \subseteq \bigcup_{i=1}^n u_i I$$

and

$$(2.2) \quad \forall i = 1, \dots, n-1, \quad u_i I \cap u_{i+1} I \text{ is a non degenerate arc.}$$

The main purpose of this Section is to prove

**Theorem 2.1.** *Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism. The attracting tree  $T_\Phi$  of  $\Phi$  is indecomposable.*

Before proving this Theorem in Section 2C, we collect the results we need from [BH92] and [GJLL98].

**2A. Train-track representative of  $\Phi$ .** The rose  $R_N$  is the graph with one vertex  $*$  and  $N$  edges. Its fundamental group  $\pi_1(R_N, *)$  is naturally identified with the free group  $F_N$ . A **marked graph** is a finite graph  $G$  with a homotopy equivalence  $\tau : R_N \rightarrow G$ . The marking  $\tau$  induces an isomorphism  $\tau_* : F_N = \pi_1(R_N, *) \xrightarrow{\cong} \pi_1(G, v_0)$ , where  $v_0 = \tau(*)$ .

A homotopy equivalence  $f : G \rightarrow G$  defines an outer automorphism of  $F_N$ . Indeed, if a path  $m$  from  $v_0$  to  $f(v_0)$  is given,  $a \mapsto mf(a)m^{-1}$  induces an automorphism  $\varphi$  of  $\pi_1(G, v_0)$ ,

and thus of  $F_N$  through the marking. Another path  $m'$  from  $v_0$  to  $f(v_0)$  gives rise to another automorphism  $\varphi'$  of  $F_N$  in the same outer class  $\Phi$ .

A **topological representative** of  $\Phi \in \text{Out}(F_N)$  is an homotopy equivalence  $f : G \rightarrow G$  of a marked graph  $G$ , such that:

- (i)  $f$  maps vertices to vertices,
- (ii)  $f$  is locally injective on any edge,
- (iii)  $f$  induces  $\Phi$  on  $F_N \cong \pi_1(G, v_0)$ .

Let  $e_1, \dots, e_p$  be the edges of  $G$  (an orientation is arbitrarily given on each edge, and  $e^{-1}$  denotes the edge  $e$  with the reverse orientation). The **transition matrix** of the map  $f$  is the  $p \times p$  non-negative matrix  $M$  with  $(i, j)$ -entry equal to the number of times the edge  $e_i$  occurs in  $f(e_j)$  (we say that a path (or an edge)  $w$  of a graph  $G$  **occurs** in a path  $u$  of  $G$  if it is  $w$  or its inverse  $w^{-1}$  is a subpath of  $u$ ).

A topological representative  $f : G \rightarrow G$  of  $\Phi$  is a **train-track map** if moreover:

- (iv) for all  $k \in \mathbb{N}$ , the restriction of  $f^k$  on any edge of  $G$  is locally injective,
- (v) any vertex of  $G$  has valence at least 3.

According to [BH92, Theorem 1.7], an iwip outer automorphism  $\Phi$  can be represented by a train-track map, with a primitive transition matrix  $M$  (i.e. there exists some  $k \in \mathbb{N}$  such all the entries of  $M^k$  are strictly positive). Thus the Perron-Frobenius Theorem applies. In particular,  $M$  has a real dominant eigenvalue  $\lambda > 1$  associated to a strictly positive eigenvector  $u = (u_1, \dots, u_p)$ . Indeed,  $\lambda$  is the expansion factor of  $\Phi$ :  $\lambda = \lambda_\Phi$ . We turn the graph  $G$  to a metric space by assigning the length  $u_i$  to the edge  $e_i$  (for  $i = 1, \dots, p$ ). Since, with respect to this metric, the length of  $f(e_i)$  is  $\lambda$  times the length of  $e_i$ , we can assume that, on each edge,  $f$  is linear of ratio  $\lambda$ .

We define the set  $\mathcal{L}_2(f)$  of paths  $w$  of combinatorial length 2 (i.e.  $w = ee'$ , where  $e, e'$  are edges of  $G$ ,  $e^{-1} \neq e'$ ) which occurs in some  $f^k(e_i)$  for some  $k \in \mathbb{N}$  and some edge  $e_i$  of  $G$ :

$$\mathcal{L}_2(f) = \{ee' : \exists e_i \text{ edge of } G, \exists k \in \mathbb{N} \text{ such that } ee' \text{ is a subpath of } f^k(e_i^{\pm 1})\}.$$

Since the transition matrix  $M$  is primitive, there exists  $k \in \mathbb{N}$  such that for any edge  $e$  of  $G$ , for any  $w \in \mathcal{L}_2(f)$ ,  $w$  occurs in  $f^k(e)$ .

Let  $v$  be a vertex of  $G$ . The **Whitehead graph**  $\mathcal{W}_v$  of  $v$  is the unoriented graph defined by:

- the vertices of  $\mathcal{W}_v$  are the edges of  $G$  with  $v$  as terminal vertex,
- there is an edge in  $\mathcal{W}_v$  between  $e$  and  $e'$  if  $e'e^{-1} \in \mathcal{L}_2(f)$ .

As remarked in [BFH97, Section 2], if  $f : G \rightarrow G$  is a train-track representative of an iwip outer automorphism  $\Phi$ , any vertex of  $G$  has a connected Whitehead graph. We summarize the previous discussion in:

**Proposition 2.2.** *Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism. There exists a train-track representative  $f : G \rightarrow G$  of  $\Phi$ , with primitive transition matrix  $M$  and connected Whitehead graphs of vertices. The edge  $e_i$  of  $G$  is isometric to the segment  $[0, u_i]$ , where  $u = (u_1, \dots, u_p)$  is a Perron-Frobenius eigenvector of  $M$ . The map  $f$  is linear of ratio  $\lambda$  on each edge  $e_i$  of  $G$ .*

**Remark 2.3.** *Let  $f : G \rightarrow G$  be a train-track map, with primitive transition matrix  $M$  and connected Whitehead graphs of vertices. Then for any path  $w = ab$  in  $G$  of combinatorial length 2, there exist  $w_1 = a_1b_1, \dots, w_q = a_qb_q \in \mathcal{L}_2(f)$  ( $a, b, a_i, b_i$  edges of  $G$ ) such that:*

- $a_{i+1} = b_i^{-1}$ ,  $i \in \{1, \dots, q-1\}$
- $a = a_1$  and  $b = b_q$ .

**2B. Construction of  $T_\Phi$ .** Let  $\Phi \in \text{Out}(F_N)$  be an iwip automorphism, and let  $T_\Phi$  be its attracting tree. Following [GJLL98], we recall a concrete construction of the tree  $T_\Phi$ .

We start with a train-track representative  $f : G \rightarrow G$  of  $\Phi$  as in Proposition 2.2. The universal cover  $\tilde{G}$  of  $G$  is a simplicial tree, equipped with a distance  $d_0$  obtained by lifting the distance on  $G$ . The fundamental group  $F_N$  acts by deck transformations, and thus by isometries, on  $\tilde{G}$ . Let  $\tilde{f}$  be a lift of  $f$  to  $\tilde{G}$ . This lift  $\tilde{f}$  is associated to a unique automorphism  $\varphi$  in the outer class  $\Phi$ , characterized by

$$(2.3) \quad \forall u \in F_N, \forall x \in \tilde{G}, \quad \varphi(u)\tilde{f}(x) = \tilde{f}(ux).$$

For  $x, y \in \tilde{G}$  and  $k \in \mathbb{N}$ , we define:

$$d_k(x, y) = \frac{d_0(\tilde{f}^k(x), \tilde{f}^k(y))}{\lambda^k}.$$

The sequence of distances  $d_k$  is decreasing and converges to a pseudo-distance  $d_\infty$  on  $\tilde{G}$ . Identifying points  $x, y$  in  $\tilde{G}$  which have distance  $d_\infty(x, y)$  equal to 0, we obtain the tree  $T_\Phi$ . The free group  $F_N$  still acts by isometries on  $T_\Phi$ . The quotient map  $p : \tilde{G} \rightarrow T_\Phi$  is  $F_N$ -equivariant and 1-Lipschitz. Moreover, for any edge  $e$  of  $\tilde{G}$ , for any  $k \in \mathbb{N}$ , the restriction of  $p$  to  $f^k(e)$  is an isometry. Through  $p$  the map  $\tilde{f}$  factors to a homothety  $H$  of  $T_\Phi$ , of ratio  $\lambda_\Phi$ :

$$\forall x \in \tilde{G}, \quad H(p(x)) = p(\tilde{f}(x)).$$

Property (2.3) leads to

$$(2.4) \quad \forall u \in F_N, \forall x \in T_\Phi, \quad \varphi(u)H(x) = H(ux).$$

**2C. Indecomposability of  $T_\Phi$ .** We say that a path (or an edge)  $w$  of the graph  $G$  occurs in a path  $u$  of the universal cover  $\tilde{G}$  of  $G$  if  $w$  has a lift  $\tilde{w}$  which occurs in  $u$ .

**Lemma 2.4.** *Let  $I$  be a non degenerate arc in  $T_\Phi$ . There exists an arc  $I'$  in  $\tilde{G}$  and an integer  $k$  such that*

- $p(I') \subseteq I$
- any element of  $\mathcal{L}_2(f)$  occurs in  $H^k(I')$ .

*Proof.* Let  $I \subset T_\Phi$  be a non-degenerate arc. There exists an edge  $e$  of  $\tilde{G}$  such that  $I_0 = p(e) \cap I$  is a non-degenerate arc:  $I_0 = [x, y]$ . We choose  $k_1 \in \mathbb{N}$  such that  $d_\infty(H^{k_1}(x), H^{k_1}(y)) > L$  where

$$L = 2 \max\{u_i = |e_i| \mid e_i \text{ edge of } G\}.$$

Let  $x', y'$  be the points in  $e$  such that  $p(x') = x$ ,  $p(y') = y$ , and let  $I'$  be the arc  $[x', y']$ . Since  $p$  maps  $f^{k_1}(e)$  isometrically into  $T_\Phi$ , we obtain that  $d_0(f^{k_1}(x'), f^{k_1}(y')) \geq L$ . Hence there exists an edge  $e'$  of  $\tilde{G}$  contained in  $[f^{k_1}(x'), f^{k_1}(y')]$ . Moreover, for any  $k_2 \in \mathbb{N}$ , the path  $f^{k_2}(e')$  isometrically injects in  $[H^{k_1+k_2}(x), H^{k_1+k_2}(y)]$ . We take  $k_2$  big enough so that any path in  $\mathcal{L}_2(f)$  occurs in  $f^{k_2}(e')$ . Then  $k = k_1 + k_2$  is suitable.  $\square$

*Proof of Theorem 2.1.* Let  $I, J$  be two non-trivial arcs in  $T_\Phi$ . We have to prove that  $I$  and  $J$  satisfy properties (2.1) and (2.2). Since  $H$  is a homeomorphism, and because of (2.4), we can replace  $I$  and  $J$  by  $H^k(I)$  and  $H^k(J)$ , accordingly, for some  $k \in \mathbb{N}$ .

We consider an arc  $I'$  in  $\tilde{G}$  and an integer  $k \in \mathbb{N}$  as given by Lemma 2.4. Let  $x, y$  be the endpoints of the arc  $H^k(J)$ :  $H^k(J) = [x, y]$ . Let  $x', y'$  be points in  $\tilde{G}$  such that  $p(x') = x$ ,  $p(y') = y$ , and let  $J'$  be the arc  $[x', y']$ . According to Remark 2.3, there exist  $w_1, \dots, w_n$  such that:

- $w_i$  is a lift of some path in  $\mathcal{L}_2(f)$ ,
- $J' \subseteq \bigcup_{i=1}^n w_i$ ,
- $w_i \cap w_{i+1}$  is an edge.

Since Lemma 2.4 ensures that any element of  $\mathcal{L}_2(f)$  occurs in  $H^k(I')$ , we deduce that  $H^k(I)$  and  $H^k(J)$  satisfy properties (2.1) and (2.2), concluding the proof of Theorem 2.1.  $\square$

### 3. INDEX OF AN OUTER AUTOMORPHISM

An automorphism  $\varphi$  of the free group  $F_N$  extends to a homeomorphism  $\partial\varphi$  of the boundary at infinity  $\partial F_N$ . We denote by  $\text{Fix}(\varphi)$  the fixed subgroup of  $\varphi$ . It is a finitely generated subgroup of  $F_N$  and thus its boundary  $\partial\text{Fix}(\varphi)$  naturally embeds in  $\partial F_N$ . Elements of  $\partial\text{Fix}(\varphi)$  are fixed by  $\partial\varphi$  and they are called **singular**. Non-singular fixed points of  $\partial\varphi$  are called **regular**. A fixed point  $X$  of  $\partial\varphi$  is **attracting** (resp. **repelling**) if it is regular and if there exists an element  $u$  in  $F_N$  such that  $\varphi^n(u)$  (resp.  $\varphi^{-n}(u)$ ) converges to  $X$ . The set of fixed points of  $\partial\varphi$  is denoted by  $\text{Fix}(\partial\varphi)$ .

Following Nielsen, fixed points of  $\partial\varphi$  have been classified by Gaboriau, Jaeger, Levitt and, Lustig:

**Proposition 3.1** ([GJLL98, Proposition 1.1]). *Let  $\varphi$  be an automorphism of the free group  $F_N$ . Let  $X$  be a fixed point of  $\partial\varphi$ . Then exactly one of the following occurs:*

- (1)  $X$  is in the boundary of the fixed subgroup of  $\varphi$ ;
- (2)  $X$  is attracting;
- (3)  $X$  is repelling.  $\square$

We denote by  $\text{Att}(\varphi)$  the set of attracting fixed points of  $\partial\varphi$ . The fixed subgroup  $\text{Fix}(\varphi)$  acts on the set  $\text{Att}(\varphi)$  of attracting fixed points.

In [GJLL98] the following **index** of the automorphism  $\varphi$  is defined:

$$\text{ind}(\varphi) = \frac{1}{2} \#(\text{Att}(\varphi)/\text{Fix}(\varphi)) + \text{rank}(\text{Fix}(\varphi)) - 1$$

If  $\varphi$  has a trivial fixed subgroup, the above definition is simpler:

$$\text{ind}(\varphi) = \frac{1}{2} \#\text{Att}(\varphi) - 1.$$

Let  $u$  be an element of  $F_N$  and let  $i_u$  be the corresponding inner automorphism of  $F_N$ :

$$\forall w \in F_N, i_u(w) = u w u^{-1}.$$

The inner automorphism  $i_u$  extends to the boundary of  $F_N$  as left multiplication by  $u$ :

$$\forall X \in \partial F_N, \partial i_u(X) = uX.$$

The group  $\text{Inn}(F_N)$  of inner automorphisms of  $F_N$  acts by conjugacy on the automorphisms in an outer class  $\Phi$ . Following Nielsen, two automorphisms,  $\varphi, \varphi' \in \Phi$  are **isogredient** if they are conjugated by some inner automorphism  $i_u$ :

$$\varphi' = i_u \circ \varphi \circ i_{u^{-1}} = i_{u\varphi(u)^{-1}} \circ \varphi.$$

In this case, the actions of  $\partial\varphi$  and  $\partial\varphi'$  on  $\partial F_N$  are conjugate by the left multiplication by  $u$ . In particular, a fixed point  $X'$  of  $\partial\varphi'$  is a translate  $X' = uX$  of a fixed point  $X$  of  $\partial\varphi$ . Two isogredient automorphisms have the same index: this is the index of the isogrediency class. An isogrediency class  $[\varphi]$  is **essential** if it has positive index:  $\text{ind}([\varphi]) > 0$ . We note that essential isogrediency classes are principal in the sense of [FH06], but the converse is not true.

The **index** of the outer automorphism  $\Phi$  is the sum, over all essential isogrediency classes of automorphisms  $\varphi$  in the outer class  $\Phi$ , of their indices, or alternatively:

$$\text{ind}(\Phi) = \sum_{[\varphi] \in \Phi/\text{Inn}(F_N)} \max(0; \text{ind}(\varphi)).$$

We adapt the notion of *forward rotationless outer automorphism* of Feighn and Handel [FH06] to our purpose. We denote by  $\text{Per}(\varphi)$  the set of elements of  $F_N$  fixed by some positive power of  $\varphi$ :

$$\text{Per}(\varphi) = \bigcup_{n \in \mathbb{N}^*} \text{Fix}(\varphi^n);$$

and by  $\text{Per}(\partial\varphi)$  the set of elements of  $\partial F_N$  fixed by some positive power of  $\partial\varphi$ :

$$\text{Per}(\partial\varphi) = \bigcup_{n \in \mathbb{N}^*} \text{Fix}(\partial\varphi^n).$$

**Definition 3.2.** *An outer automorphism  $\Phi \in \text{Out}(F_N)$  is FR if:*

- (FR1) *for any automorphism  $\varphi \in \Phi$ ,  $\text{Per}(\varphi) = \text{Fix}(\varphi)$  and  $\text{Per}(\partial\varphi) = \text{Fix}(\partial\varphi)$ ;*
- (FR2) *if  $\psi$  is an automorphism in the outer class  $\Phi^n$  for some  $n > 0$ , with  $\text{ind}(\psi)$  positive, then there exists an automorphism  $\varphi$  in  $\Phi$  such that  $\psi = \varphi^n$ .*

**Proposition 3.3.** *Let  $\Phi \in \text{Out}(F_N)$ . There exists  $k \in \mathbb{N}^*$  such that  $\Phi^k$  is FR.*

*Proof.* By [LL00, Theorem 1] there exists a power  $\Phi^k$  with (FR1). An automorphism  $\varphi \in \text{Aut}(F_N)$  with positive index  $\text{ind}(\varphi) > 0$  is principal in the sense of [FH06, Definition 3.1]. Thus our property (FR2) is a consequence of the forward rotationless property of [FH06, Definition 3.13]. By [FH06, Lemma 4.43] there exists a power  $\Phi^{k\ell}$  which is forward rotationless and thus which satisfies (FR2).  $\square$

## 4. INDICES

**4A. Botany of trees.** We recall in this Section the classification of trees in the boundary of outer space of our paper [CH10].

Gaboriau and Levitt [GL95] introduced an index for a tree  $T$  in  $\overline{\text{CV}}_N$ , we call it the **geometric index** and denote it by  $\text{ind}_{\text{geo}}(T)$ . It is defined using the valence of the branch points, of the  $\mathbb{R}$ -tree  $T$ , with an action of the free group by isometries:

$$\text{ind}_{\text{geo}}(T) = \sum_{[P] \in T/F_N} \text{ind}_{\text{geo}}(P).$$



where the local index of a point  $P$  in  $T$  is

$$\text{ind}_{\text{geo}}(P) = \#(\pi_0(T \setminus \{P\})/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2.$$

Gaboriau and Levitt [GL95] proved that the geometric index of a geometric tree is equal to  $2N - 2$  and that for any tree in the compactification of outer space  $\overline{\text{CV}}_N$  the geometric index is bounded above by  $2N - 2$ . Moreover, they proved that the trees in  $\overline{\text{CV}}_N$  with geometric index equal to  $2N - 2$  are precisely the geometric trees.

If, moreover,  $T$  has dense orbits, Levitt and Lustig [LL03, LL08] defined the map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$  which is characterized by

**Proposition 4.1.** *Let  $T$  be an  $\mathbb{R}$ -tree in  $\overline{\text{CV}}_N$  with dense orbits. There exists a unique map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$  such that for any sequence  $(u_n)_{n \in \mathbb{N}}$  of elements of  $F_N$  which converges to  $X \in \partial F_N$ , and any point  $P \in T$ , if the sequence of points  $(u_n P)_{n \in \mathbb{N}}$  converges to a point  $Q \in \widehat{T}$ , then  $\mathcal{Q}(X) = Q$ . Moreover,  $\mathcal{Q}$  is onto.*

Let us consider the case of a tree  $T$  dual to a measured foliation  $(\mathcal{F}, \mu)$  on a hyperbolic surface  $S$  with boundary ( $T$  is a surface tree). Let  $\widetilde{\mathcal{F}}$  be the lift of  $\mathcal{F}$  to the universal cover  $\widetilde{S}$  of  $S$ . The boundary at infinity of  $\widetilde{S}$  is homeomorphic to  $\partial F_N$ . On the one hand, a leaf  $\ell$  of  $\widetilde{\mathcal{F}}$  defines a point in  $T$ . On the other hand, the ends of  $\ell$  define points in  $\partial F_N$ . The map  $\mathcal{Q}$  precisely sends the ends of  $\ell$  to the point in  $T$ . The Poincaré-Lefschetz index of the foliation  $\mathcal{F}$  can be computed from the cardinal of the fibers of the map  $\mathcal{Q}$ . This leads to the following definition of the  $\mathcal{Q}$ -index of an  $\mathbb{R}$ -tree  $T$  in a more general context.

Let  $T$  be an  $\mathbb{R}$ -tree in  $\overline{\text{CV}}_N$  with dense orbits. The  $\mathcal{Q}$ -index of the tree  $T$  is defined as follows:

$$\text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in \widehat{T}/F_N} \max(0; \text{ind}_{\mathcal{Q}}(P)).$$

where the local index of a point  $P$  in  $T$  is:

$$\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}_r^{-1}(P)/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2$$

with  $\mathcal{Q}_r^{-1}(P) = \mathcal{Q}^{-1}(P) \setminus \partial \text{Stab}(P)$  the regular fiber of  $P$ .

Levitt and Lustig [LL03] proved that points in  $\partial T$  have exactly one pre-image by  $\mathcal{Q}$ . Thus, only points in  $\overline{T}$  contribute to the  $\mathcal{Q}$ -index of  $T$ .

We proved [CH10] that the  $\mathcal{Q}$ -index of an  $\mathbb{R}$ -tree in the boundary of outer space with dense orbits is bounded above by  $2N - 2$ . And it is equal to  $2N - 2$  if and only if it is of surface type.

Our botanical classification [CH10] of a tree  $T$  with a minimal very small indecomposable action of  $F_N$  by isometries is as follows

		<b>geometric</b>	<b>not geometric</b>
		$\text{ind}_{\text{geo}}(T) = 2N - 2$	$\text{ind}_{\text{geo}}(T) < 2N - 2$
<b>Surface type</b>	$\text{ind}_{\mathcal{Q}}(T) = 2N - 2$	<b>surface</b>	<b>pseudo-surface</b>
<b>Levitt type</b>	$\text{ind}_{\mathcal{Q}}(T) < 2N - 2$	<b>Levitt</b>	<b>pseudo-Levitt</b>

The following remark is not necessary for the sequel of the paper, but may help the reader's intuition.

**Remark.** In [CHL08a, CHL08], in collaboration with Lustig, we defined and studied the dual lamination of an  $\mathbb{R}$ -tree  $T$  with dense orbits:

$$L(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

The  $\mathcal{Q}$ -index of  $T$  can be interpreted as the index of this dual lamination.

Using the dual lamination, with Lustig [CHL09], we defined the compact heart  $K_A \subseteq \overline{T}$  (for a basis  $A$  of  $F_N$ ). We proved that the tree  $T$  is completely encoded by a system of partial isometries  $S_A = (K_A, A)$ . We also proved that the tree  $T$  is geometric if and only if the compact heart  $K_A$  is a finite tree (that is to say the convex hull of finitely many points). In our previous work [CH10] we used the Rips machine on the system of isometries  $S_A$  to get the bound on the  $\mathcal{Q}$ -index of  $T$ . In particular, an indecomposable tree  $T$  is of Levitt type if and only if the Rips machine never halts.

**4B. Geometric index.** As in Section 2B, an iwip outer automorphism  $\Phi$  has an expansion factor  $\lambda_\Phi > 1$ , an attracting  $\mathbb{R}$ -tree  $T_\Phi$  in  $\partial CV_N$ . For each automorphism  $\varphi$  in the outer class  $\Phi$  there is a homothety  $H$  of the metric completion  $\overline{T}_\Phi$ , of ratio  $\lambda_\Phi$ , such that

$$(4.1) \quad \forall P \in \overline{T}_\Phi, \quad \forall u \in F_N, \quad H(uP) = \varphi(u)H(P)$$

In addition, the action of  $\Phi$  on the compactification of Culler and Vogtmann's Outer space has a North-South dynamic and the projective class of  $T_\Phi$  is the attracting fixed point [LL03]. Of course the attracting trees of  $\Phi$  and  $\Phi^n$  ( $n > 0$ ) are equal.

For the attracting tree  $T_\Phi$  of the iwip outer automorphism  $\Phi$ , the geometric index is well understood.

**Proposition 4.2** ([GJLL98, Section 4]). *Let  $\Psi$  be an iwip outer automorphism. There exists a power  $\Phi = \Psi^k$  ( $k > 0$ ) of  $\Psi$  such that:*

$$2 \operatorname{ind}(\Phi) = \operatorname{ind}_{\text{geo}}(T_\Phi),$$

where  $T_\Phi$  is the attracting tree of  $\Phi$  (and of  $\Psi$ ). □

**4C.  $\mathcal{Q}$ -index.** Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  be its attracting tree. The action of  $F_N$  on  $T_\Phi$  has dense orbits.

Let  $\varphi$  an automorphism in the outer class  $\Phi$ . The homothety  $H$  associated to  $\varphi$  extends continuously to an homeomorphism of the boundary at infinity of  $T_\Phi$  which we still denote by  $H$ . We get from Proposition 4.1 and identity 4.1:

$$(4.2) \quad \forall X \in \partial F_N, \quad \mathcal{Q}(\partial\varphi(X)) = H(\mathcal{Q}(X)).$$

We are going to prove that the  $\mathcal{Q}$ -index of  $T_\Phi$  is twice the index of  $\Phi^{-1}$ . As mentioned in the introduction for geometric automorphisms both these numbers are equal to  $2N - 2$  and thus we restrict to the study of non-geometric automorphisms. For the rest of this section we assume that  $\Phi$  is non-geometric. This will be used in two ways:

- the action of  $F_N$  on  $T_\Phi$  is free;
- for any  $\varphi$  in the outer class  $\Phi$ , all the fixed points of  $\varphi$  in  $\partial F_N$  are regular.

Let  $C_H$  be the center of the homothety  $H$ . The following Lemma is essentially contained in [GJLL98], although the map  $\mathcal{Q}$  is not used there.

**Lemma 4.3.** *Let  $\Phi \in \operatorname{Out}(F_N)$  be a FR non-geometric iwip outer automorphism. Let  $T_\Phi$  be the attracting tree of  $\Phi$ . Let  $\varphi \in \Phi$  be an automorphism in the outer class  $\Phi$ , and let  $H$  be the homothety of  $T_\Phi$  associated to  $\varphi$ , with  $C_H$  its center. The  $\mathcal{Q}$ -fiber of  $C_H$  is the set of repelling points of  $\varphi$ .*

*Proof.* Let  $X \in \partial F_N$  be a repelling point of  $\partial\varphi$ . By definition there exists an element  $u \in F_N$  such that the sequence  $(\varphi^{-n}(u))_n$  converges towards  $X$ . By Equation 4.1,

$$\varphi^{-n}(u)C_H = \varphi^{-n}(u)H^{-n}(C_H) = H^{-n}(uC_H).$$

The homothety  $H^{-1}$  is strictly contracting and thus the sequence of points  $(\varphi^{-n}(u)C_H)_n$  converges towards  $C_H$ . By Proposition 4.1 we get that  $\mathcal{Q}(X) = C_H$ .

Conversely let  $X \in \mathcal{Q}^{-1}(C_H)$  be a point in the  $\mathcal{Q}$ -fiber of  $C_H$ . Using the identity 4.2,  $\partial\varphi(X)$  is also in the  $\mathcal{Q}$ -fiber. The  $\mathcal{Q}$ -fiber is finite by [CH10, Corollary 5.4],  $X$  is a periodic point of  $\partial\varphi$ . Since  $\Phi$  satisfies property (FR1),  $X$  is a fixed point of  $\partial\varphi$ . From [GJLL98, Lemma 3.5], attracting fixed points of  $\partial\varphi$  are mapped by  $\mathcal{Q}$  to points in the boundary at infinity  $\partial T_\Phi$ . Thus  $X$  has to be a repelling fixed point of  $\partial\varphi$ .  $\square$

**Proposition 4.4.** *Let  $\Phi \in \text{Out}(F_N)$  be a FR non-geometric iwip outer automorphism. Let  $T_\Phi$  be the attracting tree of  $\Phi$ . Then*

$$2 \text{ind}(\Phi^{-1}) = \text{ind}_{\mathcal{Q}}(T_\Phi).$$

*Proof.* To each automorphism  $\varphi$  in the outer class  $\Phi$  is associated a homothety  $H$  of  $T_\Phi$  and the center  $C_H$  of this homothety. As the action of  $F_N$  on  $T_\Phi$  is free, two automorphisms are isogredient if and only if the corresponding centers are in the same  $F_N$ -orbit.

The index of  $\Phi^{-1}$  is the sum over all essential isogrediency classes of automorphism  $\varphi^{-1}$  in  $\Phi^{-1}$  of the index of  $\varphi^{-1}$ . For each of these automorphisms the index  $2 \text{ind}(\varphi^{-1})$  is equal by Proposition 4.3 to the contribution  $\#\mathcal{Q}^{-1}(C_H)$  of the orbit of  $C_H$  to the  $\mathcal{Q}$  index of  $T_\Phi$ .

Conversely, let now  $P$  be a point in  $\overline{T_\Phi}$  with at least three elements in its  $\mathcal{Q}$ -fiber. Let  $\varphi$  be an automorphism in  $\Phi$  and let  $H$  be the homothety of  $T_\Phi$  associated to  $\varphi$ . For any integer  $n$ , the  $\mathcal{Q}$ -fiber  $\mathcal{Q}^{-1}(H^n(P)) = \partial\varphi^n(\mathcal{Q}^{-1}(P))$  of  $H^n(P)$  also has at least three elements. By [CH10, Theorem 5.3] there are finitely many orbits of such points in  $T_\Phi$  and thus we can assume that  $H^n(P) = wP$  for some  $w \in F_N$  and some integer  $n > 0$ . Then  $P$  is the center of the homothety  $w^{-1}H^n$  associated to  $i_{w^{-1}} \circ \varphi^n$ . Since  $\Phi$  satisfies property (FR2),  $P$  is the center of a homothety  $uH$  associated to  $i_u \circ \varphi$  for some  $u \in F_N$ . This concludes the proof of the equality of the indices.  $\square$

This Proposition can alternatively be deduced from the techniques of Handel and Mosher [HM06].

## 5. BOTANICAL CLASSIFICATION OF IRREDUCIBLE AUTOMORPHISMS

**Theorem 5.1.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Then, the  $\mathcal{Q}$ -index of the attracting tree is equal to the geometric index of the repelling tree:*

$$\text{ind}_{\mathcal{Q}}(T_\Phi) = \text{ind}_{\text{geo}}(T_{\Phi^{-1}}).$$

*Proof.* First, if  $\Phi$  is geometric, then the trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  have maximal geometric indices  $2N - 2$ . On the other hand the trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are surface trees and thus their  $\mathcal{Q}$ -indices are also maximal:

$$\text{ind}_{\text{geo}}(T_\Phi) = \text{ind}_{\mathcal{Q}}(T_\Phi) = \text{ind}_{\text{geo}}(T_{\Phi^{-1}}) = \text{ind}_{\mathcal{Q}}(T_{\Phi^{-1}}) = 2N - 2$$

We now assume that  $\Phi$  is not geometric and we can apply Propositions 4.2 and 4.4 to get the desired equality.  $\square$

From Theorem 5.1 and from the characterization of geometric and surface-type trees by the maximality of the indices we get

**Theorem 5.2.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Then exactly one of the following occurs*

- (1)  $T_\Phi$  and  $T_{\Phi^{-1}}$  are surface trees;
- (2)  $T_\Phi$  is Levitt and  $T_{\Phi^{-1}}$  is pseudo-surface;
- (3)  $T_{\Phi^{-1}}$  is Levitt and  $T_\Phi$  is pseudo-surface;
- (4)  $T_\Phi$  and  $T_{\Phi^{-1}}$  are pseudo-Levitt.

*Proof.* The trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are indecomposable by Theorem 2.1 and thus they are either of surface type or of Levitt type by [CH10, Proposition 5.14]. Recall, from [GL95] (see also [CH10, Theorem 5.9] or [CHL09, Corollary 6.1]) that  $T_\Phi$  is geometric if and only if its geometric index is maximal:

$$\text{ind}_{\text{geo}}(T_\Phi) = 2N - 2.$$

From [CH10, Theorem 5.10],  $T_\Phi$  is of surface type if and only if its  $\mathcal{Q}$ -index is maximal:

$$\text{ind}_{\mathcal{Q}}(T_\Phi) = 2N - 2.$$

The Theorem now follows from Theorem 5.1. □

Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism.

The outer automorphism  $\Phi$  is **geometric** if both its attracting and repelling trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are geometric. This is equivalent to saying that  $\Phi$  is induced by a pseudo-Anosov homeomorphism of a surface with boundary, see [Gui05] and [HM07]. This is case 1 of Theorem 5.2.

The outer automorphism  $\Phi$  is **parageometric** if its attracting tree  $T_\Phi$  is geometric but its repelling tree  $T_{\Phi^{-1}}$  is not. This is case 2 of Theorem 5.2.

The outer automorphism  $\Phi$  is **pseudo-Levitt** if both its attracting and repelling trees are not geometric. This is case 4 of Theorem 5.2

We now bring expansion factors into play. An iwip outer automorphism  $\Phi$  of  $F_N$  has an expansion factor  $\lambda_\Phi > 1$ : it is the exponential growth rate of (non fixed) conjugacy classes under iteration of  $\Phi$ .

If  $\Phi$  is geometric, the expansion factor of  $\Phi$  is equal to the expansion factor of the associated pseudo-Anosov mapping class and thus  $\lambda_\Phi = \lambda_{\Phi^{-1}}$ .

Handel and Mosher [HM07] proved that if  $\Phi$  is a parageometric outer automorphism of  $F_N$  then  $\lambda_\Phi > \lambda_{\Phi^{-1}}$  (see also [BBC08]). Examples are also given by Gautero [Gau07].

For pseudo-Levitt outer automorphisms of  $F_N$  nothing can be said on the comparison of the expansion factors of the automorphism and its inverse. On one hand, Handel and Mosher give in the introduction of [HM07] an explicit example of a non geometric automorphism with  $\lambda_\Phi = \lambda_{\Phi^{-1}}$ : thus this automorphism is pseudo-Levitt. On the other hand, there are examples of pseudo-Levitt automorphisms with  $\lambda_\Phi > \lambda_{\Phi^{-1}}$ . Let  $\varphi \in \text{Aut}(F_3)$  be the automorphism such that

$$\begin{array}{lcl} \varphi : a & \mapsto & b \quad \text{and} \quad \varphi^{-1} : a & \mapsto & c \\ & b & \mapsto & ac & b & \mapsto & a \\ & c & \mapsto & a & c & \mapsto & c^{-1}b \end{array}$$

Let  $\Phi$  be its outer class. Then  $\Phi^6$  is FR, has index  $\text{ind}(\Phi^6) = 3/2 < 2$ . The expansion factor is  $\lambda_\Phi \simeq 1,3247$ . The outer automorphism  $\Phi^{-3}$  is FR, has index  $\text{ind}(\Phi^{-3}) = 1/2 < 2$ .

The expansion factor is  $\lambda_{\Phi^{-1}} \simeq 1,4655 > \lambda_{\Phi}$ . The computation of these two indices can be achieved using the algorithm of [Jul09].

Now that we have classified outer automorphisms of  $F_N$  into four categories, questions of genericity naturally arise. In particular, is a generic outer automorphism of  $F_N$  iwip, pseudo-Levitt and with distinct expansion factors? This is suggested by Handel and Mosher [HM07], in particular for statistical genericity: given a set of generators of  $\text{Out}(F_N)$  and considering the word-metric associated to it, is it the case that

$$\lim_{k \rightarrow \infty} \frac{\#(\text{pseudo-Levitt iwip with } \lambda_{\Phi} \neq \lambda_{\Phi^{-1}}) \cap B(k)}{\#B(k)} = 1$$

where  $B(k)$  is the ball of radius  $k$ , centered at 1, in  $\text{Out}(F_N)$ ?

**5A. Botanical memo.** In this Section we give a glossary of our classification of automorphisms for the working mathematician.

For a FR iwip outer automorphism  $\Phi$  of  $F_N$ , we used 6 indices which are related in the following way:

$2 \text{ind}(\Phi) = \text{ind}_{\text{geo}}(T_{\Phi}) = \text{ind}_{\mathcal{Q}}(T_{\Phi^{-1}})$
$2 \text{ind}(\Phi^{-1}) = \text{ind}_{\text{geo}}(T_{\Phi^{-1}}) = \text{ind}_{\mathcal{Q}}(T_{\Phi})$

All these indices are bounded above by  $2N - 2$ . We sum up our Theorem 5.2 in the following table.

Automorphisms	Trees	Indices
$\Phi$ geometric $\Updownarrow$ $\Phi^{-1}$ geometric	$T_{\Phi}$ and $T_{\Phi^{-1}}$ geometric $\Updownarrow$ $T_{\Phi}$ surface $\Updownarrow$ $T_{\Phi^{-1}}$ surface	$\Leftrightarrow \text{ind}(\Phi) = \text{ind}(\Phi^{-1}) = N - 1$
$\Phi$ parageometric	$\Leftrightarrow \begin{cases} T_{\Phi} \text{ geometric} \\ \text{and} \\ T_{\Phi^{-1}} \text{ non geometric} \end{cases}$ $\Updownarrow$ $T_{\Phi}$ Levitt $\Updownarrow$ $T_{\Phi^{-1}}$ pseudo-surface	$\Leftrightarrow \begin{cases} \text{ind}(\Phi) = N - 1 \\ \text{and} \\ \text{ind}(\Phi^{-1}) < N - 1 \end{cases}$
$\Phi$ pseudo-Levitt $\Updownarrow$ $\Phi^{-1}$ pseudo-Levitt	$\Leftrightarrow T_{\Phi}$ and $T_{\Phi^{-1}}$ non geometric $\Updownarrow$ $T_{\Phi}$ pseudo-Levitt $\Updownarrow$ $T_{\Phi^{-1}}$ pseudo-Levitt	$\Leftrightarrow \begin{cases} \text{ind}(\Phi) < N - 1 \\ \text{and} \\ \text{ind}(\Phi^{-1}) < N - 1 \end{cases}$

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