



**HAL**  
open science

## Botany of irreducible automorphisms of free groups

Thierry Coulbois, Arnaud Hilion

► **To cite this version:**

Thierry Coulbois, Arnaud Hilion. Botany of irreducible automorphisms of free groups. Pacific Journal of Mathematics, 2012, 256 (2), pp.291-307. 10.2140/pjm.2012.256.291 . hal-01066578

**HAL Id: hal-01066578**

**<https://hal.science/hal-01066578>**

Submitted on 21 Oct 2015

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# BOTANY OF IRREDUCIBLE AUTOMORPHISMS OF FREE GROUPS

THIERRY COULBOIS, ARNAUD HILION

ABSTRACT. We give a classification of iwip outer automorphisms of the free group, by discussing the properties of their attracting and repelling trees.

## 1. INTRODUCTION

An outer automorphism  $\Phi$  of the free group  $F_N$  is **fully irreducible** (abbreviated as **iwip**) if no positive power  $\Phi^n$  fixes a proper free factor of  $F_N$ . Being an iwip is one (in fact the most important) of the analogs for free groups of being pseudo-Anosov for mapping classes of hyperbolic surfaces. Another analog of pseudo-Anosov is the notion of an atoroidal automorphism: an element  $\Phi \in \text{Out}(F_N)$  is **atoroidal** or **hyperbolic** if no positive power  $\Phi^n$  fixes a nontrivial conjugacy class. Bestvina and Feighn [BF92] and Brinkmann [Bri00] proved that  $\Phi$  is atoroidal if and only if the mapping torus  $F_N \rtimes_{\Phi} \mathbb{Z}$  is Gromov-hyperbolic.

Pseudo-Anosov mapping classes are known to be “generic” elements of the mapping class group (in various senses). Rivin [Riv08] and Sisto [Sis11] recently proved that, in the sense of random walks, generic elements of  $\text{Out}(F_N)$  are atoroidal iwip automorphisms.

Bestvina and Handel [BH92] proved that iwip automorphisms have the key property of being represented by (absolute) train-track maps.

A pseudo-Anosov element  $f$  fixes two projective classes of measured foliations  $[(\mathcal{F}^+, \mu^+)]$  and  $[(\mathcal{F}^-, \mu^-)]$ :

$$(\mathcal{F}^+, \mu^+) \cdot f = (\mathcal{F}^+, \lambda\mu^+) \text{ and } (\mathcal{F}^-, \mu^-) \cdot f = (\mathcal{F}^-, \lambda^{-1}\mu^-)$$

where  $\lambda > 1$  is the expansion factor of  $f$ . Alternatively, considering the dual  $\mathbb{R}$ -trees  $T^+$  and  $T^-$ , we get:

$$T^+ \cdot f = \lambda T^+ \text{ and } T^- \cdot f = \lambda^{-1} T^-.$$

We now discuss the analogous situation for iwip automorphisms. The group of outer automorphisms  $\text{Out}(F_N)$  acts on the **outer space**  $\text{CV}_N$  and its boundary  $\partial\text{CV}_N$ . Recall that the compactified outer space  $\overline{\text{CV}}_N = \text{CV}_N \cup \partial\text{CV}_N$  is made up of (projective classes of)  $\mathbb{R}$ -trees with an action of  $F_N$  by isometries which is minimal and very small. See [Vog02] for a survey on outer space. An iwip outer automorphism  $\Phi$  has North-South dynamics on  $\overline{\text{CV}}_N$ : it has a unique attracting fixed tree  $[T_{\Phi}]$  and a unique repelling fixed tree  $[T_{\Phi^{-1}}]$  in the boundary of outer space (see [LL03]):

$$T_{\Phi} \cdot \Phi = \lambda_{\Phi} T_{\Phi} \text{ and } T_{\Phi^{-1}} \cdot \Phi = \frac{1}{\lambda_{\Phi^{-1}}} T_{\Phi^{-1}},$$

where  $\lambda_{\Phi} > 1$  is the **expansion factor** of  $\Phi$  (i.e. the exponential growth rate of non-periodic conjugacy classes).

---

*Date:* August 13, 2012.

Contrary to the pseudo-Anosov setting, the expansion factor  $\lambda_\Phi$  of  $\Phi$  is typically different from the expansion factor  $\lambda_{\Phi^{-1}}$  of  $\Phi^{-1}$ . More generally, qualitative properties of the fixed trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  can be fairly different. This is the purpose of this paper to discuss and compare the properties of  $\Phi$ ,  $T_\Phi$  and  $T_{\Phi^{-1}}$ .

First, the free group,  $F_N$ , may be realized as the fundamental group of a surface  $S$  with boundary. It is part of folklore that, if  $\Phi$  comes from a pseudo-Anosov mapping class on  $S$ , then its limit trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  live in the Thurston boundary of Teichmüller space: they are dual to a measured foliation on the surface. Such trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are called **surface trees** and such an iwip outer automorphism  $\Phi$  is called **geometric** (in this case  $S$  has exactly one boundary component).

The notion of surface trees has been generalized (see for instance [Bes02]). An  $\mathbb{R}$ -tree which is transverse to measured foliations on a finite CW-complex is called **geometric**. It may fail to be a surface tree if the complex fails to be a surface.

If  $\Phi$  does not come from a pseudo-Anosov mapping class and if  $T_\Phi$  is geometric then  $\Phi$  is called **parageometric**. For a parageometric iwip  $\Phi$ , Guirardel [Gui05] and Handel and Mosher [HM07] proved that the repelling tree  $T_{\Phi^{-1}}$  is not geometric. So we have that,  $\Phi$  comes from a pseudo-Anosov mapping class on a surface with boundary if and only if both trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are geometric. Moreover in this case both trees are indeed surface trees.

In our paper [CH10] we introduced a second dichotomy for trees in the boundary of Outer space with dense orbits. For a tree  $T$ , we consider its **limit set**  $\Omega \subseteq \overline{T}$  (where  $\overline{T}$  is the metric completion of  $T$ ). The limit set  $\Omega$  consists of points of  $\overline{T}$  with at least two pre-images by the map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T} = \overline{T} \cup \partial T$  introduced by Levitt and Lustig [LL03], see Section 4A. We are interested in the two extremal cases: A tree  $T$  in the boundary of Outer space with dense orbits is of **surface type** if  $T \subseteq \Omega$  and  $T$  is of **Levitt type** if  $\Omega$  is totally disconnected. As the terminology suggests, a surface tree is of surface type. Trees of Levitt type were discovered by Levitt [Lev93].

Combining together the two sets of properties, we introduced in [CH10] the following definitions. A tree  $T$  in  $\partial CV_N$  with dense orbits is

- a **surface tree** if it is both geometric and of surface type;
- **Levitt** if it is geometric and of Levitt type;
- **pseudo-surface** if it is not geometric and of surface type;
- **pseudo-Levitt** if it is not geometric and of Levitt type

The following Theorem is the main result of this paper.

**Theorem 5.2.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Then exactly one of the following occurs*

- (1) *The trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are surface trees. Equivalently  $\Phi$  is geometric.*
- (2) *The tree  $T_\Phi$  is Levitt (i.e. geometric and of Levitt type), and the tree  $T_{\Phi^{-1}}$  is pseudo-surface (i.e. non-geometric and of surface type). Equivalently  $\Phi$  is parageometric.*
- (3) *The tree  $T_{\Phi^{-1}}$  is Levitt (i.e. geometric and of Levitt type), and the tree  $T_\Phi$  is pseudo-surface (i.e. non-geometric and of surface type). Equivalently  $\Phi^{-1}$  is parageometric.*
- (4) *The trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are pseudo-Levitt (non-geometric and of Levitt type).*

*Case (1) corresponds to toroidal iwips whereas cases (2), (3) and (4) corresponds to atoroidal iwips. In case (4) the automorphism  $\Phi$  is called **pseudo-Levitt**.*

Gaboriau, Jaeger, Levitt and Lustig [GJLL98] introduced the notion of an **index**  $\text{ind}(\Phi)$ , computed from the rank of the fixed subgroup and from the number of attracting fixed points of the automorphisms  $\varphi$  in the outer class  $\Phi$ . Another index for a tree  $T$  in  $\overline{\text{CV}}_N$  has been defined and studied by Gaboriau and Levitt [GL95], we call it the **geometric index**  $\text{ind}_{\text{geo}}(T)$ . Finally in our paper [CH10] we introduced and studied the **Q-index**  $\text{ind}_{\mathcal{Q}}(T)$  of an  $\mathbb{R}$ -tree  $T$  in the boundary of outer space with dense orbits. The two indices  $\text{ind}_{\text{geo}}(T)$  and  $\text{ind}_{\mathcal{Q}}(T)$  describe qualitative properties of the tree  $T$  [CH10]. We define these indices and recall our botanical classification of trees in Section 4A.

The key to prove Theorem 5.2 is:

**Propositions 4.2 and 4.4.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_{\Phi}$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Replacing  $\Phi$  by a suitable power, we have*

$$2 \text{ind}(\Phi) = \text{ind}_{\text{geo}}(T_{\Phi}) = \text{ind}_{\mathcal{Q}}(T_{\Phi^{-1}}).$$

We prove this Proposition in Sections 4B and 4C.

To study limit trees of iwip automorphisms, we need to state that they have the strongest mixing dynamical property, which is called **indecomposability**.

**Theorem 2.1.** *Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism. The attracting tree  $T_{\Phi}$  of  $\Phi$  is indecomposable.*

The proof of this Theorem is quite independent of the rest of the paper and is the purpose of Section 2. The proof relies on a key property of iwip automorphisms: they can be represented by (absolute) train-track maps.

## 2. INDECOMPOSABILITY OF THE ATTRACTING TREE OF AN IWIP AUTOMORPHISM

Following Guirardel [Gui08], a (projective class of)  $\mathbb{R}$ -tree  $T \in \overline{\text{CV}}_N$  is **indecomposable** if for all non degenerate arcs  $I$  and  $J$  in  $T$ , there exists finitely many elements  $u_1, \dots, u_n$  in  $F_N$  such that

$$(2.1) \quad J \subseteq \bigcup_{i=1}^n u_i I$$

and

$$(2.2) \quad \forall i = 1, \dots, n-1, \quad u_i I \cap u_{i+1} I \text{ is a non degenerate arc.}$$

The main purpose of this Section is to prove

**Theorem 2.1.** *Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism. The attracting tree  $T_{\Phi}$  of  $\Phi$  is indecomposable.*

Before proving this Theorem in Section 2C, we collect the results we need from [BH92] and [GJLL98].

**2A. Train-track representative of  $\Phi$ .** The rose  $R_N$  is the graph with one vertex  $*$  and  $N$  edges. Its fundamental group  $\pi_1(R_N, *)$  is naturally identified with the free group  $F_N$ . A **marked graph** is a finite graph  $G$  with a homotopy equivalence  $\tau : R_N \rightarrow G$ . The marking  $\tau$  induces an isomorphism  $\tau_* : F_N = \pi_1(R_N, *) \xrightarrow{\cong} \pi_1(G, v_0)$ , where  $v_0 = \tau(*)$ .

A homotopy equivalence  $f : G \rightarrow G$  defines an outer automorphism of  $F_N$ . Indeed, if a path  $m$  from  $v_0$  to  $f(v_0)$  is given,  $a \mapsto mf(a)m^{-1}$  induces an automorphism  $\varphi$  of  $\pi_1(G, v_0)$ ,

and thus of  $F_N$  through the marking. Another path  $m'$  from  $v_0$  to  $f(v_0)$  gives rise to another automorphism  $\varphi'$  of  $F_N$  in the same outer class  $\Phi$ .

A **topological representative** of  $\Phi \in \text{Out}(F_N)$  is an homotopy equivalence  $f : G \rightarrow G$  of a marked graph  $G$ , such that:

- (i)  $f$  maps vertices to vertices,
- (ii)  $f$  is locally injective on any edge,
- (iii)  $f$  induces  $\Phi$  on  $F_N \cong \pi_1(G, v_0)$ .

Let  $e_1, \dots, e_p$  be the edges of  $G$  (an orientation is arbitrarily given on each edge, and  $e^{-1}$  denotes the edge  $e$  with the reverse orientation). The **transition matrix** of the map  $f$  is the  $p \times p$  non-negative matrix  $M$  with  $(i, j)$ -entry equal to the number of times the edge  $e_i$  occurs in  $f(e_j)$  (we say that a path (or an edge)  $w$  of a graph  $G$  **occurs** in a path  $u$  of  $G$  if it is  $w$  or its inverse  $w^{-1}$  is a subpath of  $u$ ).

A topological representative  $f : G \rightarrow G$  of  $\Phi$  is a **train-track map** if moreover:

- (iv) for all  $k \in \mathbb{N}$ , the restriction of  $f^k$  on any edge of  $G$  is locally injective,
- (v) any vertex of  $G$  has valence at least 3.

According to [BH92, Theorem 1.7], an iwip outer automorphism  $\Phi$  can be represented by a train-track map, with a primitive transition matrix  $M$  (i.e. there exists some  $k \in \mathbb{N}$  such all the entries of  $M^k$  are strictly positive). Thus the Perron-Frobenius Theorem applies. In particular,  $M$  has a real dominant eigenvalue  $\lambda > 1$  associated to a strictly positive eigenvector  $u = (u_1, \dots, u_p)$ . Indeed,  $\lambda$  is the expansion factor of  $\Phi$ :  $\lambda = \lambda_\Phi$ . We turn the graph  $G$  to a metric space by assigning the length  $u_i$  to the edge  $e_i$  (for  $i = 1, \dots, p$ ). Since, with respect to this metric, the length of  $f(e_i)$  is  $\lambda$  times the length of  $e_i$ , we can assume that, on each edge,  $f$  is linear of ratio  $\lambda$ .

We define the set  $\mathcal{L}_2(f)$  of paths  $w$  of combinatorial length 2 (i.e.  $w = ee'$ , where  $e, e'$  are edges of  $G$ ,  $e^{-1} \neq e'$ ) which occurs in some  $f^k(e_i)$  for some  $k \in \mathbb{N}$  and some edge  $e_i$  of  $G$ :

$$\mathcal{L}_2(f) = \{ee' : \exists e_i \text{ edge of } G, \exists k \in \mathbb{N} \text{ such that } ee' \text{ is a subpath of } f^k(e_i^{\pm 1})\}.$$

Since the transition matrix  $M$  is primitive, there exists  $k \in \mathbb{N}$  such that for any edge  $e$  of  $G$ , for any  $w \in \mathcal{L}_2(f)$ ,  $w$  occurs in  $f^k(e)$ .

Let  $v$  be a vertex of  $G$ . The **Whitehead graph**  $\mathcal{W}_v$  of  $v$  is the unoriented graph defined by:

- the vertices of  $\mathcal{W}_v$  are the edges of  $G$  with  $v$  as terminal vertex,
- there is an edge in  $\mathcal{W}_v$  between  $e$  and  $e'$  if  $e'e^{-1} \in \mathcal{L}_2(f)$ .

As remarked in [BFH97, Section 2], if  $f : G \rightarrow G$  is a train-track representative of an iwip outer automorphism  $\Phi$ , any vertex of  $G$  has a connected Whitehead graph. We summarize the previous discussion in:

**Proposition 2.2.** *Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism. There exists a train-track representative  $f : G \rightarrow G$  of  $\Phi$ , with primitive transition matrix  $M$  and connected Whitehead graphs of vertices. The edge  $e_i$  of  $G$  is isometric to the segment  $[0, u_i]$ , where  $u = (u_1, \dots, u_p)$  is a Perron-Frobenius eigenvector of  $M$ . The map  $f$  is linear of ratio  $\lambda$  on each edge  $e_i$  of  $G$ .*

**Remark 2.3.** *Let  $f : G \rightarrow G$  be a train-track map, with primitive transition matrix  $M$  and connected Whitehead graphs of vertices. Then for any path  $w = ab$  in  $G$  of combinatorial length 2, there exist  $w_1 = a_1b_1, \dots, w_q = a_qb_q \in \mathcal{L}_2(f)$  ( $a, b, a_i, b_i$  edges of  $G$ ) such that:*

- $a_{i+1} = b_i^{-1}$ ,  $i \in \{1, \dots, q-1\}$
- $a = a_1$  and  $b = b_q$ .

**2B. Construction of  $T_\Phi$ .** Let  $\Phi \in \text{Out}(F_N)$  be an iwip automorphism, and let  $T_\Phi$  be its attracting tree. Following [GJLL98], we recall a concrete construction of the tree  $T_\Phi$ .

We start with a train-track representative  $f : G \rightarrow G$  of  $\Phi$  as in Proposition 2.2. The universal cover  $\tilde{G}$  of  $G$  is a simplicial tree, equipped with a distance  $d_0$  obtained by lifting the distance on  $G$ . The fundamental group  $F_N$  acts by deck transformations, and thus by isometries, on  $\tilde{G}$ . Let  $\tilde{f}$  be a lift of  $f$  to  $\tilde{G}$ . This lift  $\tilde{f}$  is associated to a unique automorphism  $\varphi$  in the outer class  $\Phi$ , characterized by

$$(2.3) \quad \forall u \in F_N, \forall x \in \tilde{G}, \quad \varphi(u)\tilde{f}(x) = \tilde{f}(ux).$$

For  $x, y \in \tilde{G}$  and  $k \in \mathbb{N}$ , we define:

$$d_k(x, y) = \frac{d_0(\tilde{f}^k(x), \tilde{f}^k(y))}{\lambda^k}.$$

The sequence of distances  $d_k$  is decreasing and converges to a pseudo-distance  $d_\infty$  on  $\tilde{G}$ . Identifying points  $x, y$  in  $\tilde{G}$  which have distance  $d_\infty(x, y)$  equal to 0, we obtain the tree  $T_\Phi$ . The free group  $F_N$  still acts by isometries on  $T_\Phi$ . The quotient map  $p : \tilde{G} \rightarrow T_\Phi$  is  $F_N$ -equivariant and 1-Lipschitz. Moreover, for any edge  $e$  of  $\tilde{G}$ , for any  $k \in \mathbb{N}$ , the restriction of  $p$  to  $f^k(e)$  is an isometry. Through  $p$  the map  $\tilde{f}$  factors to a homothety  $H$  of  $T_\Phi$ , of ratio  $\lambda_\Phi$ :

$$\forall x \in \tilde{G}, \quad H(p(x)) = p(\tilde{f}(x)).$$

Property (2.3) leads to

$$(2.4) \quad \forall u \in F_N, \forall x \in T_\Phi, \quad \varphi(u)H(x) = H(ux).$$

**2C. Indecomposability of  $T_\Phi$ .** We say that a path (or an edge)  $w$  of the graph  $G$  occurs in a path  $u$  of the universal cover  $\tilde{G}$  of  $G$  if  $w$  has a lift  $\tilde{w}$  which occurs in  $u$ .

**Lemma 2.4.** *Let  $I$  be a non degenerate arc in  $T_\Phi$ . There exists an arc  $I'$  in  $\tilde{G}$  and an integer  $k$  such that*

- $p(I') \subseteq I$
- any element of  $\mathcal{L}_2(f)$  occurs in  $H^k(I')$ .

*Proof.* Let  $I \subset T_\Phi$  be a non-degenerate arc. There exists an edge  $e$  of  $\tilde{G}$  such that  $I_0 = p(e) \cap I$  is a non-degenerate arc:  $I_0 = [x, y]$ . We choose  $k_1 \in \mathbb{N}$  such that  $d_\infty(H^{k_1}(x), H^{k_1}(y)) > L$  where

$$L = 2 \max\{u_i = |e_i| \mid e_i \text{ edge of } G\}.$$

Let  $x', y'$  be the points in  $e$  such that  $p(x') = x$ ,  $p(y') = y$ , and let  $I'$  be the arc  $[x', y']$ . Since  $p$  maps  $f^{k_1}(e)$  isometrically into  $T_\Phi$ , we obtain that  $d_0(f^{k_1}(x'), f^{k_1}(y')) \geq L$ . Hence there exists an edge  $e'$  of  $\tilde{G}$  contained in  $[f^{k_1}(x'), f^{k_1}(y')]$ . Moreover, for any  $k_2 \in \mathbb{N}$ , the path  $f^{k_2}(e')$  isometrically injects in  $[H^{k_1+k_2}(x), H^{k_1+k_2}(y)]$ . We take  $k_2$  big enough so that any path in  $\mathcal{L}_2(f)$  occurs in  $f^{k_2}(e')$ . Then  $k = k_1 + k_2$  is suitable.  $\square$

*Proof of Theorem 2.1.* Let  $I, J$  be two non-trivial arcs in  $T_\Phi$ . We have to prove that  $I$  and  $J$  satisfy properties (2.1) and (2.2). Since  $H$  is a homeomorphism, and because of (2.4), we can replace  $I$  and  $J$  by  $H^k(I)$  and  $H^k(J)$ , accordingly, for some  $k \in \mathbb{N}$ .

We consider an arc  $I'$  in  $\tilde{G}$  and an integer  $k \in \mathbb{N}$  as given by Lemma 2.4. Let  $x, y$  be the endpoints of the arc  $H^k(J)$ :  $H^k(J) = [x, y]$ . Let  $x', y'$  be points in  $\tilde{G}$  such that  $p(x') = x$ ,  $p(y') = y$ , and let  $J'$  be the arc  $[x', y']$ . According to Remark 2.3, there exist  $w_1, \dots, w_n$  such that:

- $w_i$  is a lift of some path in  $\mathcal{L}_2(f)$ ,
- $J' \subseteq \bigcup_{i=1}^n w_i$ ,
- $w_i \cap w_{i+1}$  is an edge.

Since Lemma 2.4 ensures that any element of  $\mathcal{L}_2(f)$  occurs in  $H^k(I')$ , we deduce that  $H^k(I)$  and  $H^k(J)$  satisfy properties (2.1) and (2.2), concluding the proof of Theorem 2.1.  $\square$

### 3. INDEX OF AN OUTER AUTOMORPHISM

An automorphism  $\varphi$  of the free group  $F_N$  extends to a homeomorphism  $\partial\varphi$  of the boundary at infinity  $\partial F_N$ . We denote by  $\text{Fix}(\varphi)$  the fixed subgroup of  $\varphi$ . It is a finitely generated subgroup of  $F_N$  and thus its boundary  $\partial\text{Fix}(\varphi)$  naturally embeds in  $\partial F_N$ . Elements of  $\partial\text{Fix}(\varphi)$  are fixed by  $\partial\varphi$  and they are called **singular**. Non-singular fixed points of  $\partial\varphi$  are called **regular**. A fixed point  $X$  of  $\partial\varphi$  is **attracting** (resp. **repelling**) if it is regular and if there exists an element  $u$  in  $F_N$  such that  $\varphi^n(u)$  (resp.  $\varphi^{-n}(u)$ ) converges to  $X$ . The set of fixed points of  $\partial\varphi$  is denoted by  $\text{Fix}(\partial\varphi)$ .

Following Nielsen, fixed points of  $\partial\varphi$  have been classified by Gaboriau, Jaeger, Levitt and, Lustig:

**Proposition 3.1** ([GJLL98, Proposition 1.1]). *Let  $\varphi$  be an automorphism of the free group  $F_N$ . Let  $X$  be a fixed point of  $\partial\varphi$ . Then exactly one of the following occurs:*

- (1)  $X$  is in the boundary of the fixed subgroup of  $\varphi$ ;
- (2)  $X$  is attracting;
- (3)  $X$  is repelling.  $\square$

We denote by  $\text{Att}(\varphi)$  the set of attracting fixed points of  $\partial\varphi$ . The fixed subgroup  $\text{Fix}(\varphi)$  acts on the set  $\text{Att}(\varphi)$  of attracting fixed points.

In [GJLL98] the following **index** of the automorphism  $\varphi$  is defined:

$$\text{ind}(\varphi) = \frac{1}{2} \#(\text{Att}(\varphi)/\text{Fix}(\varphi)) + \text{rank}(\text{Fix}(\varphi)) - 1$$

If  $\varphi$  has a trivial fixed subgroup, the above definition is simpler:

$$\text{ind}(\varphi) = \frac{1}{2} \#\text{Att}(\varphi) - 1.$$

Let  $u$  be an element of  $F_N$  and let  $i_u$  be the corresponding inner automorphism of  $F_N$ :

$$\forall w \in F_N, i_u(w) = u w u^{-1}.$$

The inner automorphism  $i_u$  extends to the boundary of  $F_N$  as left multiplication by  $u$ :

$$\forall X \in \partial F_N, \partial i_u(X) = uX.$$

The group  $\text{Inn}(F_N)$  of inner automorphisms of  $F_N$  acts by conjugacy on the automorphisms in an outer class  $\Phi$ . Following Nielsen, two automorphisms,  $\varphi, \varphi' \in \Phi$  are **isogredient** if they are conjugated by some inner automorphism  $i_u$ :

$$\varphi' = i_u \circ \varphi \circ i_{u^{-1}} = i_{u\varphi(u)^{-1}} \circ \varphi.$$

In this case, the actions of  $\partial\varphi$  and  $\partial\varphi'$  on  $\partial F_N$  are conjugate by the left multiplication by  $u$ . In particular, a fixed point  $X'$  of  $\partial\varphi'$  is a translate  $X' = uX$  of a fixed point  $X$  of  $\partial\varphi$ . Two isogredient automorphisms have the same index: this is the index of the isogrediency class. An isogrediency class  $[\varphi]$  is **essential** if it has positive index:  $\text{ind}([\varphi]) > 0$ . We note that essential isogrediency classes are principal in the sense of [FH06], but the converse is not true.

The **index** of the outer automorphism  $\Phi$  is the sum, over all essential isogrediency classes of automorphisms  $\varphi$  in the outer class  $\Phi$ , of their indices, or alternatively:

$$\text{ind}(\Phi) = \sum_{[\varphi] \in \Phi / \text{Inn}(F_N)} \max(0; \text{ind}(\varphi)).$$

We adapt the notion of *forward rotationless outer automorphism* of Feighn and Handel [FH06] to our purpose. We denote by  $\text{Per}(\varphi)$  the set of elements of  $F_N$  fixed by some positive power of  $\varphi$ :

$$\text{Per}(\varphi) = \bigcup_{n \in \mathbb{N}^*} \text{Fix}(\varphi^n);$$

and by  $\text{Per}(\partial\varphi)$  the set of elements of  $\partial F_N$  fixed by some positive power of  $\partial\varphi$ :

$$\text{Per}(\partial\varphi) = \bigcup_{n \in \mathbb{N}^*} \text{Fix}(\partial\varphi^n).$$

**Definition 3.2.** *An outer automorphism  $\Phi \in \text{Out}(F_N)$  is FR if:*

- (FR1) *for any automorphism  $\varphi \in \Phi$ ,  $\text{Per}(\varphi) = \text{Fix}(\varphi)$  and  $\text{Per}(\partial\varphi) = \text{Fix}(\partial\varphi)$ ;*
- (FR2) *if  $\psi$  is an automorphism in the outer class  $\Phi^n$  for some  $n > 0$ , with  $\text{ind}(\psi)$  positive, then there exists an automorphism  $\varphi$  in  $\Phi$  such that  $\psi = \varphi^n$ .*

**Proposition 3.3.** *Let  $\Phi \in \text{Out}(F_N)$ . There exists  $k \in \mathbb{N}^*$  such that  $\Phi^k$  is FR.*

*Proof.* By [LL00, Theorem 1] there exists a power  $\Phi^k$  with (FR1). An automorphism  $\varphi \in \text{Aut}(F_N)$  with positive index  $\text{ind}(\varphi) > 0$  is principal in the sense of [FH06, Definition 3.1]. Thus our property (FR2) is a consequence of the forward rotationless property of [FH06, Definition 3.13]. By [FH06, Lemma 4.43] there exists a power  $\Phi^{k\ell}$  which is forward rotationless and thus which satisfies (FR2).  $\square$

## 4. INDICES

**4A. Botany of trees.** We recall in this Section the classification of trees in the boundary of outer space of our paper [CH10].

Gaboriau and Levitt [GL95] introduced an index for a tree  $T$  in  $\overline{\text{CV}}_N$ , we call it the **geometric index** and denote it by  $\text{ind}_{\text{geo}}(T)$ . It is defined using the valence of the branch points, of the  $\mathbb{R}$ -tree  $T$ , with an action of the free group by isometries:

$$\text{ind}_{\text{geo}}(T) = \sum_{[P] \in T/F_N} \text{ind}_{\text{geo}}(P).$$



where the local index of a point  $P$  in  $T$  is

$$\text{ind}_{\text{geo}}(P) = \#(\pi_0(T \setminus \{P\})/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2.$$

Gaboriau and Levitt [GL95] proved that the geometric index of a geometric tree is equal to  $2N - 2$  and that for any tree in the compactification of outer space  $\overline{\text{CV}}_N$  the geometric index is bounded above by  $2N - 2$ . Moreover, they proved that the trees in  $\overline{\text{CV}}_N$  with geometric index equal to  $2N - 2$  are precisely the geometric trees.

If, moreover,  $T$  has dense orbits, Levitt and Lustig [LL03, LL08] defined the map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$  which is characterized by

**Proposition 4.1.** *Let  $T$  be an  $\mathbb{R}$ -tree in  $\overline{\text{CV}}_N$  with dense orbits. There exists a unique map  $\mathcal{Q} : \partial F_N \rightarrow \widehat{T}$  such that for any sequence  $(u_n)_{n \in \mathbb{N}}$  of elements of  $F_N$  which converges to  $X \in \partial F_N$ , and any point  $P \in T$ , if the sequence of points  $(u_n P)_{n \in \mathbb{N}}$  converges to a point  $Q \in \widehat{T}$ , then  $\mathcal{Q}(X) = Q$ . Moreover,  $\mathcal{Q}$  is onto.*

Let us consider the case of a tree  $T$  dual to a measured foliation  $(\mathcal{F}, \mu)$  on a hyperbolic surface  $S$  with boundary ( $T$  is a surface tree). Let  $\widetilde{\mathcal{F}}$  be the lift of  $\mathcal{F}$  to the universal cover  $\widetilde{S}$  of  $S$ . The boundary at infinity of  $\widetilde{S}$  is homeomorphic to  $\partial F_N$ . On the one hand, a leaf  $\ell$  of  $\widetilde{\mathcal{F}}$  defines a point in  $T$ . On the other hand, the ends of  $\ell$  define points in  $\partial F_N$ . The map  $\mathcal{Q}$  precisely sends the ends of  $\ell$  to the point in  $T$ . The Poincaré-Lefschetz index of the foliation  $\mathcal{F}$  can be computed from the cardinal of the fibers of the map  $\mathcal{Q}$ . This leads to the following definition of the  $\mathcal{Q}$ -index of an  $\mathbb{R}$ -tree  $T$  in a more general context.

Let  $T$  be an  $\mathbb{R}$ -tree in  $\overline{\text{CV}}_N$  with dense orbits. The  $\mathcal{Q}$ -index of the tree  $T$  is defined as follows:

$$\text{ind}_{\mathcal{Q}}(T) = \sum_{[P] \in \widehat{T}/F_N} \max(0; \text{ind}_{\mathcal{Q}}(P)).$$

where the local index of a point  $P$  in  $T$  is:

$$\text{ind}_{\mathcal{Q}}(P) = \#(\mathcal{Q}_r^{-1}(P)/\text{Stab}(P)) + 2 \text{rank}(\text{Stab}(P)) - 2$$

with  $\mathcal{Q}_r^{-1}(P) = \mathcal{Q}^{-1}(P) \setminus \partial \text{Stab}(P)$  the regular fiber of  $P$ .

Levitt and Lustig [LL03] proved that points in  $\partial T$  have exactly one pre-image by  $\mathcal{Q}$ . Thus, only points in  $\overline{T}$  contribute to the  $\mathcal{Q}$ -index of  $T$ .

We proved [CH10] that the  $\mathcal{Q}$ -index of an  $\mathbb{R}$ -tree in the boundary of outer space with dense orbits is bounded above by  $2N - 2$ . And it is equal to  $2N - 2$  if and only if it is of surface type.

Our botanical classification [CH10] of a tree  $T$  with a minimal very small indecomposable action of  $F_N$  by isometries is as follows

		geometric	not geometric
		$\text{ind}_{\text{geo}}(T) = 2N - 2$	$\text{ind}_{\text{geo}}(T) < 2N - 2$
Surface type	$\text{ind}_{\mathcal{Q}}(T) = 2N - 2$	surface	pseudo-surface
Levitt type	$\text{ind}_{\mathcal{Q}}(T) < 2N - 2$	Levitt	pseudo-Levitt

The following remark is not necessary for the sequel of the paper, but may help the reader's intuition.

**Remark.** In [CHL08a, CHL08], in collaboration with Lustig, we defined and studied the dual lamination of an  $\mathbb{R}$ -tree  $T$  with dense orbits:

$$L(T) = \{(X, Y) \in \partial^2 F_N \mid \mathcal{Q}(X) = \mathcal{Q}(Y)\}.$$

The  $\mathcal{Q}$ -index of  $T$  can be interpreted as the index of this dual lamination.

Using the dual lamination, with Lustig [CHL09], we defined the compact heart  $K_A \subseteq \overline{T}$  (for a basis  $A$  of  $F_N$ ). We proved that the tree  $T$  is completely encoded by a system of partial isometries  $S_A = (K_A, A)$ . We also proved that the tree  $T$  is geometric if and only if the compact heart  $K_A$  is a finite tree (that is to say the convex hull of finitely many points). In our previous work [CH10] we used the Rips machine on the system of isometries  $S_A$  to get the bound on the  $\mathcal{Q}$ -index of  $T$ . In particular, an indecomposable tree  $T$  is of Levitt type if and only if the Rips machine never halts.

**4B. Geometric index.** As in Section 2B, an iwip outer automorphism  $\Phi$  has an expansion factor  $\lambda_\Phi > 1$ , an attracting  $\mathbb{R}$ -tree  $T_\Phi$  in  $\partial CV_N$ . For each automorphism  $\varphi$  in the outer class  $\Phi$  there is a homothety  $H$  of the metric completion  $\overline{T}_\Phi$ , of ratio  $\lambda_\Phi$ , such that

$$(4.1) \quad \forall P \in \overline{T}_\Phi, \quad \forall u \in F_N, \quad H(uP) = \varphi(u)H(P)$$

In addition, the action of  $\Phi$  on the compactification of Culler and Vogtmann's Outer space has a North-South dynamic and the projective class of  $T_\Phi$  is the attracting fixed point [LL03]. Of course the attracting trees of  $\Phi$  and  $\Phi^n$  ( $n > 0$ ) are equal.

For the attracting tree  $T_\Phi$  of the iwip outer automorphism  $\Phi$ , the geometric index is well understood.

**Proposition 4.2** ([GJLL98, Section 4]). *Let  $\Psi$  be an iwip outer automorphism. There exists a power  $\Phi = \Psi^k$  ( $k > 0$ ) of  $\Psi$  such that:*

$$2 \operatorname{ind}(\Phi) = \operatorname{ind}_{\text{geo}}(T_\Phi),$$

where  $T_\Phi$  is the attracting tree of  $\Phi$  (and of  $\Psi$ ). □

**4C.  $\mathcal{Q}$ -index.** Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  be its attracting tree. The action of  $F_N$  on  $T_\Phi$  has dense orbits.

Let  $\varphi$  an automorphism in the outer class  $\Phi$ . The homothety  $H$  associated to  $\varphi$  extends continuously to an homeomorphism of the boundary at infinity of  $T_\Phi$  which we still denote by  $H$ . We get from Proposition 4.1 and identity 4.1:

$$(4.2) \quad \forall X \in \partial F_N, \quad \mathcal{Q}(\partial\varphi(X)) = H(\mathcal{Q}(X)).$$

We are going to prove that the  $\mathcal{Q}$ -index of  $T_\Phi$  is twice the index of  $\Phi^{-1}$ . As mentioned in the introduction for geometric automorphisms both these numbers are equal to  $2N - 2$  and thus we restrict to the study of non-geometric automorphisms. For the rest of this section we assume that  $\Phi$  is non-geometric. This will be used in two ways:

- the action of  $F_N$  on  $T_\Phi$  is free;
- for any  $\varphi$  in the outer class  $\Phi$ , all the fixed points of  $\varphi$  in  $\partial F_N$  are regular.

Let  $C_H$  be the center of the homothety  $H$ . The following Lemma is essentially contained in [GJLL98], although the map  $\mathcal{Q}$  is not used there.

**Lemma 4.3.** *Let  $\Phi \in \operatorname{Out}(F_N)$  be a FR non-geometric iwip outer automorphism. Let  $T_\Phi$  be the attracting tree of  $\Phi$ . Let  $\varphi \in \Phi$  be an automorphism in the outer class  $\Phi$ , and let  $H$  be the homothety of  $T_\Phi$  associated to  $\varphi$ , with  $C_H$  its center. The  $\mathcal{Q}$ -fiber of  $C_H$  is the set of repelling points of  $\varphi$ .*

*Proof.* Let  $X \in \partial F_N$  be a repelling point of  $\partial\varphi$ . By definition there exists an element  $u \in F_N$  such that the sequence  $(\varphi^{-n}(u))_n$  converges towards  $X$ . By Equation 4.1,

$$\varphi^{-n}(u)C_H = \varphi^{-n}(u)H^{-n}(C_H) = H^{-n}(uC_H).$$

The homothety  $H^{-1}$  is strictly contracting and thus the sequence of points  $(\varphi^{-n}(u)C_H)_n$  converges towards  $C_H$ . By Proposition 4.1 we get that  $\mathcal{Q}(X) = C_H$ .

Conversely let  $X \in \mathcal{Q}^{-1}(C_H)$  be a point in the  $\mathcal{Q}$ -fiber of  $C_H$ . Using the identity 4.2,  $\partial\varphi(X)$  is also in the  $\mathcal{Q}$ -fiber. The  $\mathcal{Q}$ -fiber is finite by [CH10, Corollary 5.4],  $X$  is a periodic point of  $\partial\varphi$ . Since  $\Phi$  satisfies property (FR1),  $X$  is a fixed point of  $\partial\varphi$ . From [GJLL98, Lemma 3.5], attracting fixed points of  $\partial\varphi$  are mapped by  $\mathcal{Q}$  to points in the boundary at infinity  $\partial T_\Phi$ . Thus  $X$  has to be a repelling fixed point of  $\partial\varphi$ .  $\square$

**Proposition 4.4.** *Let  $\Phi \in \text{Out}(F_N)$  be a FR non-geometric iwip outer automorphism. Let  $T_\Phi$  be the attracting tree of  $\Phi$ . Then*

$$2 \text{ind}(\Phi^{-1}) = \text{ind}_{\mathcal{Q}}(T_\Phi).$$

*Proof.* To each automorphism  $\varphi$  in the outer class  $\Phi$  is associated a homothety  $H$  of  $T_\Phi$  and the center  $C_H$  of this homothety. As the action of  $F_N$  on  $T_\Phi$  is free, two automorphisms are isogredient if and only if the corresponding centers are in the same  $F_N$ -orbit.

The index of  $\Phi^{-1}$  is the sum over all essential isogrediency classes of automorphism  $\varphi^{-1}$  in  $\Phi^{-1}$  of the index of  $\varphi^{-1}$ . For each of these automorphisms the index  $2 \text{ind}(\varphi^{-1})$  is equal by Proposition 4.3 to the contribution  $\#\mathcal{Q}^{-1}(C_H)$  of the orbit of  $C_H$  to the  $\mathcal{Q}$  index of  $T_\Phi$ .

Conversely, let now  $P$  be a point in  $\overline{T_\Phi}$  with at least three elements in its  $\mathcal{Q}$ -fiber. Let  $\varphi$  be an automorphism in  $\Phi$  and let  $H$  be the homothety of  $T_\Phi$  associated to  $\varphi$ . For any integer  $n$ , the  $\mathcal{Q}$ -fiber  $\mathcal{Q}^{-1}(H^n(P)) = \partial\varphi^n(\mathcal{Q}^{-1}(P))$  of  $H^n(P)$  also has at least three elements. By [CH10, Theorem 5.3] there are finitely many orbits of such points in  $T_\Phi$  and thus we can assume that  $H^n(P) = wP$  for some  $w \in F_N$  and some integer  $n > 0$ . Then  $P$  is the center of the homothety  $w^{-1}H^n$  associated to  $i_{w^{-1}} \circ \varphi^n$ . Since  $\Phi$  satisfies property (FR2),  $P$  is the center of a homothety  $uH$  associated to  $i_u \circ \varphi$  for some  $u \in F_N$ . This concludes the proof of the equality of the indices.  $\square$

This Proposition can alternatively be deduced from the techniques of Handel and Mosher [HM06].

## 5. BOTANICAL CLASSIFICATION OF IRREDUCIBLE AUTOMORPHISMS

**Theorem 5.1.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Then, the  $\mathcal{Q}$ -index of the attracting tree is equal to the geometric index of the repelling tree:*

$$\text{ind}_{\mathcal{Q}}(T_\Phi) = \text{ind}_{\text{geo}}(T_{\Phi^{-1}}).$$

*Proof.* First, if  $\Phi$  is geometric, then the trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  have maximal geometric indices  $2N - 2$ . On the other hand the trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are surface trees and thus their  $\mathcal{Q}$ -indices are also maximal:

$$\text{ind}_{\text{geo}}(T_\Phi) = \text{ind}_{\mathcal{Q}}(T_\Phi) = \text{ind}_{\text{geo}}(T_{\Phi^{-1}}) = \text{ind}_{\mathcal{Q}}(T_{\Phi^{-1}}) = 2N - 2$$

We now assume that  $\Phi$  is not geometric and we can apply Propositions 4.2 and 4.4 to get the desired equality.  $\square$

From Theorem 5.1 and from the characterization of geometric and surface-type trees by the maximality of the indices we get

**Theorem 5.2.** *Let  $\Phi$  be an iwip outer automorphism of  $F_N$ . Let  $T_\Phi$  and  $T_{\Phi^{-1}}$  be its attracting and repelling trees. Then exactly one of the following occurs*

- (1)  $T_\Phi$  and  $T_{\Phi^{-1}}$  are surface trees;
- (2)  $T_\Phi$  is Levitt and  $T_{\Phi^{-1}}$  is pseudo-surface;
- (3)  $T_{\Phi^{-1}}$  is Levitt and  $T_\Phi$  is pseudo-surface;
- (4)  $T_\Phi$  and  $T_{\Phi^{-1}}$  are pseudo-Levitt.

*Proof.* The trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are indecomposable by Theorem 2.1 and thus they are either of surface type or of Levitt type by [CH10, Proposition 5.14]. Recall, from [GL95] (see also [CH10, Theorem 5.9] or [CHL09, Corollary 6.1]) that  $T_\Phi$  is geometric if and only if its geometric index is maximal:

$$\text{ind}_{\text{geo}}(T_\Phi) = 2N - 2.$$

From [CH10, Theorem 5.10],  $T_\Phi$  is of surface type if and only if its  $\mathcal{Q}$ -index is maximal:

$$\text{ind}_{\mathcal{Q}}(T_\Phi) = 2N - 2.$$

The Theorem now follows from Theorem 5.1. □

Let  $\Phi \in \text{Out}(F_N)$  be an iwip outer automorphism.

The outer automorphism  $\Phi$  is **geometric** if both its attracting and repelling trees  $T_\Phi$  and  $T_{\Phi^{-1}}$  are geometric. This is equivalent to saying that  $\Phi$  is induced by a pseudo-Anosov homeomorphism of a surface with boundary, see [Gui05] and [HM07]. This is case 1 of Theorem 5.2.

The outer automorphism  $\Phi$  is **parageometric** if its attracting tree  $T_\Phi$  is geometric but its repelling tree  $T_{\Phi^{-1}}$  is not. This is case 2 of Theorem 5.2.

The outer automorphism  $\Phi$  is **pseudo-Levitt** if both its attracting and repelling trees are not geometric. This is case 4 of Theorem 5.2

We now bring expansion factors into play. An iwip outer automorphism  $\Phi$  of  $F_N$  has an expansion factor  $\lambda_\Phi > 1$ : it is the exponential growth rate of (non fixed) conjugacy classes under iteration of  $\Phi$ .

If  $\Phi$  is geometric, the expansion factor of  $\Phi$  is equal to the expansion factor of the associated pseudo-Anosov mapping class and thus  $\lambda_\Phi = \lambda_{\Phi^{-1}}$ .

Handel and Mosher [HM07] proved that if  $\Phi$  is a parageometric outer automorphism of  $F_N$  then  $\lambda_\Phi > \lambda_{\Phi^{-1}}$  (see also [BBC08]). Examples are also given by Gautero [Gau07].

For pseudo-Levitt outer automorphisms of  $F_N$  nothing can be said on the comparison of the expansion factors of the automorphism and its inverse. On one hand, Handel and Mosher give in the introduction of [HM07] an explicit example of a non geometric automorphism with  $\lambda_\Phi = \lambda_{\Phi^{-1}}$ : thus this automorphism is pseudo-Levitt. On the other hand, there are examples of pseudo-Levitt automorphisms with  $\lambda_\Phi > \lambda_{\Phi^{-1}}$ . Let  $\varphi \in \text{Aut}(F_3)$  be the automorphism such that

$$\begin{array}{ll} \varphi : a \mapsto b & \text{and} \quad \varphi^{-1} : a \mapsto c \\ b \mapsto ac & b \mapsto a \\ c \mapsto a & c \mapsto c^{-1}b \end{array}$$

Let  $\Phi$  be its outer class. Then  $\Phi^6$  is FR, has index  $\text{ind}(\Phi^6) = 3/2 < 2$ . The expansion factor is  $\lambda_\Phi \simeq 1,3247$ . The outer automorphism  $\Phi^{-3}$  is FR, has index  $\text{ind}(\Phi^{-3}) = 1/2 < 2$ .

The expansion factor is  $\lambda_{\Phi^{-1}} \simeq 1,4655 > \lambda_{\Phi}$ . The computation of these two indices can be achieved using the algorithm of [Jul09].

Now that we have classified outer automorphisms of  $F_N$  into four categories, questions of genericity naturally arise. In particular, is a generic outer automorphism of  $F_N$  iwip, pseudo-Levitt and with distinct expansion factors? This is suggested by Handel and Mosher [HM07], in particular for statistical genericity: given a set of generators of  $\text{Out}(F_N)$  and considering the word-metric associated to it, is it the case that

$$\lim_{k \rightarrow \infty} \frac{\#(\text{pseudo-Levitt iwip with } \lambda_{\Phi} \neq \lambda_{\Phi^{-1}}) \cap B(k)}{\#B(k)} = 1$$

where  $B(k)$  is the ball of radius  $k$ , centered at 1, in  $\text{Out}(F_N)$ ?

**5A. Botanical memo.** In this Section we give a glossary of our classification of automorphisms for the working mathematician.

For a FR iwip outer automorphism  $\Phi$  of  $F_N$ , we used 6 indices which are related in the following way:

$2 \text{ind}(\Phi) = \text{ind}_{\text{geo}}(T_{\Phi}) = \text{ind}_{\mathcal{Q}}(T_{\Phi^{-1}})$
$2 \text{ind}(\Phi^{-1}) = \text{ind}_{\text{geo}}(T_{\Phi^{-1}}) = \text{ind}_{\mathcal{Q}}(T_{\Phi})$

All these indices are bounded above by  $2N - 2$ . We sum up our Theorem 5.2 in the following table.

Automorphisms	Trees	Indices
$\Phi$ geometric $\Updownarrow$ $\Phi^{-1}$ geometric	$T_{\Phi}$ and $T_{\Phi^{-1}}$ geometric $\Updownarrow$ $T_{\Phi}$ surface $\Updownarrow$ $T_{\Phi^{-1}}$ surface	$\Leftrightarrow \text{ind}(\Phi) = \text{ind}(\Phi^{-1}) = N - 1$
$\Phi$ parageometric	$\left\{ \begin{array}{l} T_{\Phi} \text{ geometric} \\ \text{and} \\ T_{\Phi^{-1}} \text{ non geometric} \end{array} \right.$ $\Updownarrow$ $T_{\Phi}$ Levitt $\Updownarrow$ $T_{\Phi^{-1}}$ pseudo-surface	$\Leftrightarrow \left\{ \begin{array}{l} \text{ind}(\Phi) = N - 1 \\ \text{and} \\ \text{ind}(\Phi^{-1}) < N - 1 \end{array} \right.$
$\Phi$ pseudo-Levitt $\Updownarrow$ $\Phi^{-1}$ pseudo-Levitt	$T_{\Phi}$ and $T_{\Phi^{-1}}$ non geometric $\Updownarrow$ $T_{\Phi}$ pseudo-Levitt $\Updownarrow$ $T_{\Phi^{-1}}$ pseudo-Levitt	$\Leftrightarrow \left\{ \begin{array}{l} \text{ind}(\Phi) < N - 1 \\ \text{and} \\ \text{ind}(\Phi^{-1}) < N - 1 \end{array} \right.$

#### ACKNOWLEDGMENTS

We thank Martin Lustig for his constant interest in our work and the referee for the suggested improvements.

## REFERENCES

- [BBC08] Jason Behrstock, Mladen Bestvina, and Matt Clay. Growth of intersection numbers for free group automorphisms. *J. Topol.*, 3:280–310, 2010.
- [Bes02] Mladen Bestvina.  $\mathbb{R}$ -trees in topology, geometry, and group theory. In *Handbook of geometric topology*, pages 55–91. North-Holland, Amsterdam, 2002.
- [BF92] Mladen Bestvina and Mark Feighn. A combination theorem for negatively curved groups. *J. Differential Geom.*, 35(1):85–101, 1992.
- [BF95] Mladen Bestvina and Mark Feighn. Stable actions of groups on real trees. *Invent. Math.*, 121(2):287–321, 1995.
- [BFH97] Mladen Bestvina, Mark Feighn and Michael Handel. Laminations, trees, and irreducible automorphisms of free groups. *Geom. Funct. Anal.*, 7(2):215–244, 1997.
- [BH92] Mladen Bestvina and Michael Handel. Train tracks and automorphisms of free groups, *Ann. of Math. (2)*, 135(2):1–51, 1992.
- [Bri00] P. Brinkmann. Hyperbolic automorphisms of free groups. *Geom. Funct. Anal.*, 10(5):1071–1089, 2000.
- [CH10] Thierry Coulbois, and Arnaud Hilion. Rips Induction: Index of the dual lamination of an  $\mathbb{R}$ -tree Preprint. arXiv:1002.0972.
- [CHL07] Thierry Coulbois, Arnaud Hilion, and Martin Lustig. Non-unique ergodicity, observers’ topology and the dual algebraic lamination for  $\mathbb{R}$ -trees. *Illinois J. Math.*, 51(3):897–911, 2007.
- [CHL08a] Thierry Coulbois, Arnaud Hilion, and Martin Lustig.  $\mathbb{R}$ -trees and laminations for free groups. I. Algebraic laminations. *J. Lond. Math. Soc. (2)*, 78(3):723–736, 2008.
- [CHL08] Thierry Coulbois, Arnaud Hilion, and Martin Lustig.  $\mathbb{R}$ -trees and laminations for free groups. II. The dual lamination of an  $\mathbb{R}$ -tree. *J. Lond. Math. Soc. (2)*, 78(3):737–754, 2008.
- [CHL09] Thierry Coulbois, Arnaud Hilion, and Martin Lustig.  $\mathbb{R}$ -trees, dual laminations, and compact systems of partial isometries. *Math. Proc. Cambridge Phil. Soc.*, 147:345–368, 2009.
- [CL89] Marshall M. Cohen and Martin Lustig. On the dynamics and the fixed subgroup of a free group automorphism. *Invent. Math.*, 96(3):613–638, 1989.
- [CV86] Marc Culler and Karen Vogtmann. Moduli of graphs and automorphisms of free groups. *Invent. Math.*, 84:91–119, 1986.
- [FH06] Mark Feighn and Michael Handel. The recognition theorem for  $\text{Out}(F_n)$ . *Groups Geom. Dyn.*, 5:39–106, 2011.
- [FLP79] A. Fathi, F. Laudenbach, and V. Poénaru, editors. *Travaux de Thurston sur les surfaces*. Société Mathématique de France, Paris, 1979. Astérisque No. 66–67.
- [Gab96] Damien Gaboriau. Dynamique des systèmes d’isométries: sur les bouts des orbites. *Invent. Math.*, 126(2):297–318, 1996.
- [Gab97] Damien Gaboriau. Générateurs indépendants pour les systèmes d’isométries de dimension un. *Ann. Inst. Fourier (Grenoble)*, 47(1):101–122, 1997.
- [Ger83] S. M. Gersten. Intersections of finitely generated subgroups of free groups and resolutions of graphs. *Invent. Math.*, 71(3):567–591, 1983.
- [GJLL98] Damien Gaboriau, Andre Jaeger, Gilbert Levitt, and Martin Lustig. An index for counting fixed points of automorphisms of free groups. *Duke Math. J.*, 93(3):425–452, 1998.
- [GL95] Damien Gaboriau and Gilbert Levitt. The rank of actions on  $\mathbb{R}$ -trees. *Ann. Sci. École Norm. Sup. (4)*, 28(5):549–570, 1995.
- [GLP94] Damien Gaboriau, Gilbert Levitt, and Frédéric Paulin. Pseudogroups of isometries of  $\mathbb{R}$  and Rips’ theorem on free actions on  $\mathbb{R}$ -trees. *Israel J. Math.*, 87(1-3):403–428, 1994.
- [Gau07] François Gautero. Combinatorial mapping-torus, branched surfaces and free group automorphisms. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 6(3):405–440, 2007.
- [Gui05] Vincent Guirardel. Cœur et nombre d’intersection pour les actions de groupes sur les arbres. *Ann. Sci. École Norm. Sup. (4)*, 38(6):847–888, 2005.
- [Gui08] Vincent Guirardel. Actions of finitely generated groups on  $\mathbb{R}$ -trees. *Ann. Inst. Fourier (Grenoble)*, 58(1):159–211, 2008.

- [HM06] Michael Handel and Lee Mosher. Axes in outer space. *Mem. Amer. Math. Soc.*, 1004, 2011. arXiv:math/0605355, 2006.
- [HM07] Michael Handel and Lee Mosher. Parageometric outer automorphisms of free groups. *Trans. Amer. Math. Soc.*, 359(7):3153–3183 (electronic), 2007.
- [Jul09] Yann Jullian. *Représentations géométriques des systèmes dynamiques substitutifs par substitutions d'arbre*. PhD thesis, Université Aix-Marseille II, 2009.
- [Kap01] Michael Kapovich. *Hyperbolic manifolds and discrete groups*, volume 183 of *Progress in Mathematics*. Birkhäuser Boston Inc., Boston, MA, 2001.
- [Lev93] Gilbert Levitt. La dynamique des pseudogroupes de rotations. *Invent. Math.*, 113(3):633–670, 1993.
- [LL00] Gilbert Levitt and Martin Lustig. Periodic ends, growth rates, Hölder dynamics for automorphisms of free groups. *Comment. Math. Helv.*, 75(3):415–429, 2000.
- [LL03] Gilbert Levitt and Martin Lustig. Irreducible automorphisms of  $F_n$  have north-south dynamics on compactified outer space. *J. Inst. Math. Jussieu*, 2(1):59–72, 2003.
- [LL08] Gilbert Levitt and Martin Lustig. Automorphisms of free groups have asymptotically periodic dynamics. *J. Reine Angew. Math.*, 619:1–36, 2008.
- [Mor88] John W. Morgan. Ergodic theory and free actions of groups on  $\mathbf{R}$ -trees. *Invent. Math.*, 94(3):605–622, 1988.
- [Rey10] Patrick Reynolds. On indecomposable trees in outer space. *Geom. Dedicata*, 153:59–71, 2011.
- [Riv08] Igor Rivin. Walks on groups, counting reducible matrices, polynomials, and surface and free group automorphisms. *Duke Math. J.*, 142(2):353–379, 2008.
- [Sis11] Alessandro Sisto. Contracting elements and random walks. arXiv:1112.2666, 2011.
- [Sta83] John R. Stallings. Topology of finite graphs. *Invent. Math.*, 71(3):551–565, 1983.
- [Vog02] Karen Vogtmann. Automorphisms of free groups and outer space. *Geom. Dedicata*, 94:1–31, 2002.

THIERRY COULBOIS

LATP

AVENUE DE L'ESCADRILLE NORMANDIE-NIÉMEN

13013 MARSEILLE

FRANCE

thierry.coulbois@univ-amu.fr

ARNAUD HILION

LATP

AVENUE DE L'ESCADRILLE NORMANDIE-NIÉMEN

13013 MARSEILLE

FRANCE

arnaud.hilion@univ-amu.fr