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► **To cite this version:**

Federico Pellarin, Rudolph Perkins. On certain generating functions in positive characteristic. 2014.
hal-01066478v1

HAL Id: hal-01066478

<https://hal.science/hal-01066478v1>

Preprint submitted on 20 Sep 2014 (v1), last revised 14 Jan 2016 (v2)

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ON CERTAIN GENERATING FUNCTIONS IN POSITIVE CHARACTERISTIC

F. PELLARIN AND R. PERKINS

ABSTRACT. We introduce new methods into the study of a class of generating functions introduced by the second author which carry some formal similarities with the Hurwitz zeta function. We prove functional identities which establish an explicit connection with certain deformations of the Carlitz logarithms introduced by M. Papanikolas and involve the Anderson-Thakur function and the Carlitz exponential function. They collect certain functional identities in families for a new class of L -functions in equal positive characteristic introduced by the first author. This paper also deals with specializations at roots of unity of these functions, a link with Gauss-Thakur sums, and an analogue of the classical Carlson theorem from complex analysis.

1. INTRODUCTION

Let A be the ring $\mathbb{F}_q[\theta]$ of polynomials in an indeterminate θ with coefficients in the finite field of q elements \mathbb{F}_q , and let $A^+ \subset A$ denote the subset of monic elements in θ . We denote by K the fraction field of A and by K_∞ the completion of K at the infinite place. We also denote by \mathbb{C}_∞ the completion of an algebraic closure K_∞^{ac} of K_∞ ; we endow it with the norm $|\cdot|$ defined by $|\theta| = q$.

Let \mathbb{T}_s be the standard Tate algebra in the indeterminates t_1, \dots, t_s with coefficients in \mathbb{C}_∞ . Explicitly, it is the ring of formal series

$$\sum_{i_1, \dots, i_s \geq 0} c_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s} \in \mathbb{C}_\infty[[t_1, \dots, t_s]]$$

such that $c_{i_1, \dots, i_s} \rightarrow 0$ as $i_1 + \dots + i_s \rightarrow \infty$. It can also be viewed as the completion of $\mathbb{C}_\infty[t_1, \dots, t_s]$ for the Gauss valuation associated to the absolute value $|\cdot|$ of \mathbb{C}_∞ . The Tate algebra \mathbb{T}_s is endowed with the open and continuous $\mathbb{F}_q[t_1, \dots, t_s]$ -linear automorphism $\tau : \mathbb{T}_s \rightarrow \mathbb{T}_s$ defined by $\tau(c) = c^q$, for $c \in \mathbb{C}_\infty$. For each $i \in \Sigma_s := \{1, 2, \dots, s\}$, let $\chi_{t_i} : A \rightarrow \mathbb{F}_q[t_i] \subset \mathbb{T}_s$ be the \mathbb{F}_q -algebra morphism determined by $\theta \mapsto t_i$, and note that much of what we do to follow depends on these choices of algebra generators. Finally, we denote by \mathbb{E}_s the sub-algebra of \mathbb{T}_s whose elements are entire in the all variables t_1, \dots, t_s . When $s = 1$, we will write $t_1 = t, \mathbb{T}_1 = \mathbb{T}$ and $\mathbb{E}_1 = \mathbb{E}$.

The purpose of this note is to introduce new methods into the study of the properties of the \mathbb{T}_s -valued functions

$$\psi_s(z) := \sum_{a \in A} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z - a} = \sum'_{a \in A} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z - a};$$

2010 *Mathematics Subject Classification.* Primary 11M38.

Key words and phrases. Positive characteristic, Carlitz module, Anderson generating functions, L -series, special values, periodic functions.

The first author is supported by the ANR HAMOT.

here the primed summation indicates the exclusion of $0 \in A$ from the sum, and second equality follows from the fact that $\chi_t(0) = 0$. For all positive integers s , the series ψ_s converges in \mathbb{T}_s for all $z \in \mathbb{C}_\infty \setminus (A \setminus \{0\})$, and, in particular, ψ_s converges in an open neighborhood of $z = 0$. These functions are a positive characteristic analog of the Hurwitz zeta function and were introduced by the second author as generating functions, when s is fixed and $n \equiv s \pmod{q-1}$, for the following multivariate L -functions

$$L(\chi_{t_1} \cdots \chi_{t_s}, n) := \sum_{a \in A^+} a^{-n} \chi_{t_1}(a) \cdots \chi_{t_s}(a);$$

see Lemma 18 below. For example, when $s = 1$, for all $z \in \mathbb{C}_\infty$ such that $|z| < 1$, we have

$$\psi_1(z) = \sum_{k \geq 0} z^{k(q-1)} L(\chi_t, 1 + k(q-1)).$$

The L -functions above were introduced by the first author in [12] and converge for all positive integers n and s to non-zero elements of $\mathbb{T}_s(K_\infty)$, the standard Tate algebra in t_1, \dots, t_s with coefficients in the local field $K_\infty = \mathbb{F}_q((1/\theta))$. They may be regarded as complementary objects to the global L -functions in equal positive characteristic introduced by Goss (see [13]).

The authors plan to use the functions ψ_s in a crucial way in a forthcoming work developing the theory of deformations vectorial modular forms introduced by the first author in [12]. Indeed, these functions allow us to introduce a notion of regularity at infinity for weak deformations of vectorial modular forms on the Drinfeld upper-half plane which guarantees the finite rankedness of the \mathbb{T} -modules of such forms of a given weight and type. Additionally, we think that the functions ψ_s carry their own intrinsic interest, and we intend to demonstrate this by cataloging in this paper some of their finer properties.

As a product of our labor, we find a new proof of a basic relation that exists between the L -series value $L(\chi_t, 1)$ evaluated in t at roots of unity and Gauss-Thakur sums. The precise statement is given in Theorem 4 below.

1.1. Further notations. Let $d_0 = 1$ and, for all $i \geq 1$, let $d_i := \prod_{j=0}^{i-1} (\theta^{q^i} - \theta^{q^j})$. Define

$$\exp_C(z) := \sum_{i \geq 0} d_i^{-1} z^{q^i} \quad \text{and} \quad \tilde{\pi} := -\lambda_\theta^q \prod_{i \geq 1} (1 - \theta^{1-q^i})^{-1} \in K_\infty(\lambda_\theta),$$

respectively, the *Carlitz exponential function* and a *fundamental period of the Carlitz exponential*, the latter being unique up to the choice of $(q-1)$ -th root λ_θ of $-\theta$ in K^{ac} . The kernel of \exp_C is $\tilde{\pi}A$, and one readily shows that \exp_C is entire in z . Finally, set $e_C(z) := \exp_C(\tilde{\pi}z)$, and note the difference in notation from [9].

We consider the following special case of an *Anderson generating function* for the Carlitz module introduced by the first author in [11] and studied in [7],

$$f_t(z) := \sum_{j \geq 0} e_C(z\theta^{-j-1})t^j = \sum_{n \geq 0} \frac{(\tilde{\pi}z)^{q^n}}{d_n(\theta^{q^n} - t)} \in \mathbb{T};$$

it is the notation introduced in [7] which we follow here. From the second description of f_t above we see that it is an entire function of z and a meromorphic function

of t with simple poles at $t = \theta, \theta^q, \dots$. From the first description, we see that, for each $z \in \mathbb{C}_\infty$, $f_t(z)$ satisfies the τ -difference equation

$$(1) \quad \tau(f_t(z)) = e_C(z) + (t - \theta)f_t(z).$$

This allows us to take $\omega(t) := f_t(1)$ as the definition of the *Anderson-Thakur function*, which is a generator of the rank one $\mathbb{F}_q[t]$ -submodule of \mathbb{T} of solutions to the equation $\tau(X) = (t - \theta)X$. With this, one readily shows that

$$\omega(t) = \lambda_\theta \prod_{i \geq 0} \left(1 - \frac{t}{\theta^{q^i}}\right)^{-1} \in \mathbb{T},$$

as introduced in [1, Lem. 2.5.4]. The function ω is in many ways a cousin of Euler's gamma function, see *e.g.* the discussion in [12].

The difference equation (1) also allows for a connection with the following function introduced by Papanikolas in [10]

$$\mathbf{L}_\alpha(t) := \alpha + \sum_{j \geq 1} \frac{\alpha^{q^j}}{(t - \theta^q)(t - \theta^{q^2}) \dots (t - \theta^{q^j})}.$$

This function converges in \mathbb{T} for each $\alpha \in \mathbb{C}_\infty$ such that $|\alpha| < q^{\frac{q}{q-1}}$. The function $\mathbf{L}_\alpha(t)$ is a deformation of the Carlitz logarithm function; indeed, one may evaluate the variable t at θ , and we recover the Carlitz logarithm $\log_C(\alpha) = \mathbf{L}_\alpha(\theta)$ in this way. The function $\mathbf{L}_\alpha(t)$ plays a crucial role in Papanikolas' proof of a "folklore conjecture:" *a K -linearly independent set of Carlitz logarithms of non-zero elements of K^{ac} is algebraically independent over K* . Indeed, $\mathbf{L}_\alpha(t)$ is the main ingredient in the construction of a rigid analytic trivialization of Papanikolas' *logarithm motive*. The reader may consult [10] for the exact details of its use.

For us, the following connection made by El-Guindy and Papanikolas [7] between $\mathbf{L}_\alpha(t)$ and $f_t(z)$ will be of special interest. Letting $\alpha = e_C(z)$, we have an identity of elements of \mathbb{T} which for all $z \in \mathbb{C}_\infty$ such that $|z| < 1$ is given by,

$$(2) \quad \mathbf{L}_\alpha(t) = (\theta - t)f_t(z);$$

note that the right side above converges for all $z \in \mathbb{C}_\infty$.

1.1.1. *A zeta function realization of Papanikolas' functions.* The next result was first obtained in [14] by the second author using interpolation polynomials and τ -difference formalism, and we shall give a new proof of this result using a positive characteristic analog, Theorem 9 below, of the so-called Carlson's theorem first appearing in [5]. We wish to stress here the connection with the function of Papanikolas described above.

Theorem 1. *Let $\alpha = e_C(z)$. For each $z \in \mathbb{C}_\infty$ such that $|z| < 1$, the following identity holds in \mathbb{T} ,*

$$(3) \quad e_C(z)\psi_1(z) = \frac{\tilde{\pi}\mathbf{L}_\alpha(t)}{(\theta - t)\omega(t)}.$$

As the title of this section suggests, the identity of the previous theorem gives a zeta function realization of the function \mathbf{L}_α akin to the identity

$$(4) \quad L(\chi_t, 1) = \frac{\tilde{\pi}}{(\theta - t)\omega(t)},$$

discovered by the first author in [12], which gives a zeta function realization to the rigid analytic trivialization

$$\Omega := \tau(\omega)^{-1} = \frac{1}{(t - \theta)\omega}$$

of the dual Carlitz motive. The identity (4) immediately follows from (3), and we will provide these details after the proof of Theorem 1. Further, (3) provides a curious intertwining of objects coming from both Anderson's theory of t -motives and of t -modules. This leads us to suspect that a similar phenomenon may exist for the functions ψ_s , and though we do not investigate these questions here, we hope to return to them at a future time.

1.2. Main Results.

1.2.1. *Functional identities for ψ_s with s arbitrary.* Without the restriction on s necessary in [14], we shall prove a qualitative generalization of the explicit identities of the second author [14, Theorem 1.1], and, in particular, Theorem 1. Clearly, the following result may also be stated in terms of Papanikolas' functions, but we leave the explicit details to the reader.

Theorem 2. *Let s be a positive integer. For every subset $I \subset \Sigma_s$, $I \neq \Sigma_s$, there exists a polynomial $g_I \in XK(t_1, \dots, t_s)[X]$ of degree at most s/q , such that, for all $z \in \mathbb{C}_\infty$,*

$$\tilde{\pi}^{-1}\omega(t_1) \cdots \omega(t_s)e_C(z)\psi_s(z) = \prod_{i \in \Sigma_s} f_{t_i}(z) + \sum_{I \subsetneq \Sigma_s} g_I(e_C(z)) \prod_{i \in I} f_{t_i}(z).$$

We do not yet know what is the exact connection of the last identity above with the theory of t -motives. Rather, our primary interest in proving the theorem above is that it demonstrates how the functional identities of [2, Theorem 1] assemble in families when s is fixed and α varies in the class of $s \pmod{q-1}$. We shall also discuss the growth of the functions ψ_s and f_t as $|z|_{\mathfrak{S}} := \inf_{\kappa \in K_\infty} |z - \kappa| \rightarrow \infty$ and show that ψ_s tends toward zero, while f_t is unbounded. Thus the identity of the theorem above demonstrates the rich interplay between the functions $e_C, f_{t_1}, \dots, f_{t_s}$ when z is far away (in the sense of the distance $|\cdot|_{\mathfrak{S}}$) from K_∞ .

We must also mention a paper of Gekeler [6] where he uses functions akin to ψ_1 to establish surjectivity of the de Rham morphism for the cohomology of Drinfeld modules. Given a positive integer h , a lattice $\Lambda \subset \mathbb{C}_\infty$ with exponential function e_Λ , and an A -linear function $\mathcal{X} : \Lambda \rightarrow \mathbb{C}_\infty$, Gekeler considers the functions

$$F_{h,\mathcal{X}}(z) := \sum_{\lambda \in \Lambda} \mathcal{X}(\lambda)(e_\Lambda(z)/(z - \lambda))^{q^h},$$

which can be shown to converge uniformly for all $z \in \mathbb{C}_\infty$ to \mathbb{F}_q -linear functions. Further, he interpolates the values of \mathcal{X} on Λ and satisfy, for each $a \in A$, the functional identities

$$F_{h,\mathcal{X}}(az) - aF_{h,\mathcal{X}}(z) = \eta_a(e_\Lambda(z))$$

for some η_a in the ring of twisted polynomials $\tau\mathbb{C}_\infty\{\tau\}$, giving the functions A -linearity upon restriction of z to Λ . Thus, Theorem 2 goes in the direction of an extension of Gekeler's results for the multilinear pointwise products $z \mapsto \mathcal{X}_1(z)\mathcal{X}_2(z) \cdots \mathcal{X}_s(z)$ of the A -linear functions $\mathcal{X}_1, \mathcal{X}_2, \dots, \mathcal{X}_s : \Lambda \rightarrow \mathbb{C}_\infty$.

We are unable to explicitly describe the polynomials g_I guaranteed by Theorem 2 above, in contrast with the explicit results of [14] which we state now for comparison. If $1 \leq s \leq 2(q-1)$, then for all $z \in \mathbb{C}_\infty$ it is shown in [14, Thm. 1.1] that

$$\tilde{\pi}^{-1}\omega(t_1)\cdots\omega(t_s)e_C(z)\psi_s(z) = \prod_{i \in \Sigma_s} f_{t_i}(z) - e_C(z) \sum^* \prod_{i_1, \dots, i_s} f_{t_{i_j}}^{i_j},$$

where the sum \sum^* runs over indices $0 \leq i_1, \dots, i_s \leq 1$ such that $i_1 + \dots + i_s = s - q$.

We are lead to pose the following question.

Question 3. *Are the polynomials $g_I(X)$ of Theorem 2 actually elements of $XK(t_1, \dots, t_s)$?*

1.2.2. *Evaluation at roots of unity.* Another important feature of these multivariate L -series, and hence the functions ψ_s , is the possibility of evaluating the variables t_1, \dots, t_s at roots of unity; see e.g. [12, 2, 3, 4]. As a highlight of such considerations, detailed below, we recover the following connection, due originally to Anglès and the first author [3], between the Anderson-Thakur function ω and Thakur's Gauss sums, from [16],

$$g(\chi) := \sum_{a \in (A/\mathfrak{p}A)^\times} \chi(a)^{-1} e_C(a/\mathfrak{p}),$$

which are defined and non-zero for all monic irreducible polynomials $\mathfrak{p} \in A$ with roots $\zeta \in \mathbb{C}_\infty$; here $\chi(a)$ denotes the image of a under the \mathbb{F}_q -algebra map χ determined by $\theta \mapsto \zeta$. Letting $\ell_0 = 1$ and $\ell_i = (\theta - \theta^q)\ell_{i-1} \in A$, for all positive integers i , we have the following result whose proof appears in §5.1 below.

Theorem 4. *For all $\zeta \in \mathbb{F}_q^{ac}$ with minimal polynomial in A of degree d , we have*

$$\omega|_{t=\zeta} = -\chi(\ell_{d-1})g(\chi).$$

It is worth noting that the sign of $g(\chi)$ is quickly obtained from the previous theorem, and the absolute value of $g(\chi)$ is readily seen from the definition above and will be clear from Cor. 30 below. See [17] where such questions are handled in full generality.

In the case of more indeterminates, we let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be primes (that is, irreducible monic polynomials) of A ; we also set $\mathfrak{m} = \mathfrak{p}_1 \cdots \mathfrak{p}_s$. Let us choose, for all $i = 1, \dots, s$, a root ζ_i of \mathfrak{p}_i in \mathbb{F}_q^{ac} , the algebraic closure of \mathbb{F}_q in \mathbb{C}_∞ . Let

$$\text{ev}_\mathfrak{m} : \mathbb{T}_s \rightarrow \mathbb{C}_\infty$$

be the evaluation map sending a formal series $\sum_{i_1, \dots, i_s} c_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s} \in \mathbb{T}_s$ to $\sum_{i_1, \dots, i_s} c_{i_1, \dots, i_s} \zeta_1^{i_1} \cdots \zeta_s^{i_s}$. We emphasize the dependence on ζ_1, \dots, ζ_s , rather than on $\mathfrak{p}_1, \dots, \mathfrak{p}_s$.

By Lemma 22, for each $a \in A$, the functions ψ_s satisfy the following simple transformation rule,

$$\psi_s(z+a) = \sum_{I \sqcup J \subset \Sigma_s} \sum_{b \in A} (z-b)^{-1} \prod_{i \in I} \chi_{t_i}(b) \prod_{j \in J} \chi_{t_j}(a),$$

where the first sum is over all partitions in two subsets of the set Σ_s . Now, while the ψ_s themselves are not A -periodic, for any such evaluation map $\text{ev}_\mathfrak{m}$, we have

$$\text{ev}_\mathfrak{m}(\psi_s(z + \mathfrak{m}a)) = \text{ev}_\mathfrak{m}(\psi_s(z)), \quad \text{for all } a \in A.$$

Furthermore, letting

$$u_{\mathfrak{m}}(z) := \tilde{\pi}^{-1} \sum_{a \in A} \frac{1}{z - \mathfrak{m}a} = \frac{1}{\mathfrak{m} \exp_C(\tilde{\pi}z/\mathfrak{m})},$$

Proposition 26 below gives

$$\tilde{\pi}^{-1} \operatorname{ev}_{\mathfrak{m}}(\psi_s) \in K^{ac}(u_{\mathfrak{m}}) \cap K^{ac}[[u_{\mathfrak{m}}]],$$

and convergence of the formal series holds for $z \in \mathbb{C}_{\infty}$ such that $|z|_{\mathfrak{G}}$ is big enough. Explicitly computing the rational functions above appears to be a difficult task. However, we can say something more when $s = 1$.

We define, *ad hoc*, $g(\chi^{-1}) := (-1)^d \mathfrak{p}/g(\chi)$. This is the Gauss-Thakur sum for the character χ^{-1} and is an element of $A[\zeta, e_C(1/\mathfrak{p})]$ by [2, Prop. 15.2]. For more precision, in the case of a single prime \mathfrak{p} with root ζ we now write $\operatorname{ev}_{\zeta}$ for the map $\operatorname{ev}_{\mathfrak{p}}$ described above.

Theorem 5. *Writing $\operatorname{ev}_{\zeta}(\psi_1(z)) = \sum a_i u_{\mathfrak{p}}(z)^i$ in $\mathbb{C}_{\infty}[[u_{\mathfrak{p}}(z)]]$, we have*

$$a_{|\mathfrak{p}|(1-q^{-1})} = (-1)^{d+1} g(\chi^{-1}) \chi(\ell_{d-1})^{-1} \quad \text{and} \quad a_j = 0, \quad \text{for } j < |\mathfrak{p}|(1 - q^{-1}).$$

The proofs of Theorems 4 and 5 will appear in §5.1 below.

1.2.3. *Acknowledgements.* Both authors wish to thank D. Goss for his continued interest in our project and helpful comments on previous versions of this work.

2. ENTIRE FUNCTIONS WITH VALUES IN BANACH ALGEBRAS

Let \mathcal{R} be a \mathbb{C}_{∞} -Banach algebra with norm $|\cdot|_{\mathcal{R}}$ extending the norm $|\cdot|$ of \mathbb{C}_{∞} .

Definition 6. A function $f : \mathbb{C}_{\infty} \rightarrow \mathcal{R}$ is called \mathcal{R} -entire (or simply entire) if, on every bounded subset B of \mathbb{C}_{∞} , f can be obtained as a uniform limit of polynomial functions $f_i \in \mathcal{R}[z]$, $f_i : B \rightarrow \mathcal{R}$.

In particular, we can write, for all $z \in \mathbb{C}_{\infty}$,

$$f(z) = \sum_{i \geq 0} c_i z^i, \quad c_i \in \mathcal{R},$$

and $\lim_{i \rightarrow \infty} |c_i|_{\mathcal{R}}^{1/i} = 0$. Identifying \mathbb{C}_{∞} with the sub-algebra $\mathbb{C}_{\infty} \cdot 1$ of \mathcal{R} , we note that every \mathbb{C}_{∞} -entire function can be identified with an \mathcal{R} -entire function.

Remark 7. Similarly, a natural notion of a \mathcal{R} -rigid analytic function exists but will be marginal for the purposes of the present paper.

2.1. **Bounded entire functions.** We assume that $|\mathcal{R}|_{\mathcal{R}} = |\mathbb{C}_{\infty}| = q^{\mathbb{Q}}$, that is, the set of norms $|f|_{\mathcal{R}}$ for $f \in \mathcal{R}$ is equal to the set of norms $|x|$, $x \in \mathbb{C}_{\infty}$.

Proposition 8. *A bounded \mathcal{R} -entire function is constant.*

Proof. We follow Schikhof's [15, Theorems 42.2 and 42.6]. We denote by $B_0(1)$ the set of the $z \in \mathbb{C}_{\infty}$ such that $|z| < 1$. We note that if

$$f : B_0(1) \rightarrow \mathcal{R}$$

is defined by $f(x) = \sum_{i \geq 0} a_i x^i$ for $a_i \in \mathcal{R}$, then

$$(5) \quad \sup\{|f(x)|_{\mathcal{R}}; |x| \leq 1\} = \max\{|a_n|_{\mathcal{R}}; n \geq 0\}$$

(cf. loc. cit. Lemma 42.1). Already, taking $x = 1$, we see that $\max\{|a_n|_{\mathcal{R}}; n \geq 0\}$ is well defined and equals the supremum of the same set. We may assume that this

maximum is one by rescaling f by an element $c \in \mathbb{C}_\infty^\times$ (because of our assumption that $|\mathcal{R}|_{\mathcal{R}} = |\mathbb{C}_\infty|$). We then have:

$$\sup\{|f(x)|_{\mathcal{R}}; |x| < 1\} \leq \sup\{|f(x)|_{\mathcal{R}}; |x| \leq 1\} \leq \max\{|a_n|_{\mathcal{R}}; n \geq 0\} = 1$$

and we need to show that we have equalities everywhere. If $|a_0|_{\mathcal{R}} = 1$, we have $|f(0)|_{\mathcal{R}} = 1$ and we are done. Otherwise, let N be the smallest integer j such that $|a_j|_{\mathcal{R}} = 1$. We have that $N > 0$. Let us choose $\epsilon > 0$ such that $\epsilon < 1 - \max\{|a_0|_{\mathcal{R}}, |a_1|_{\mathcal{R}}, \dots, |a_{N-1}|_{\mathcal{R}}\}$. Since $|\mathbb{C}_\infty^\times| = q^{\mathbb{Q}}$ is dense in \mathbb{R}^\times , there exists $x \in \mathbb{C}_\infty^\times$ such that $1 - \epsilon < |x^N| < 1$. Then, $|f(x)|_{\mathcal{R}} = |x^N| \geq 1 - \epsilon$.

Now let f be \mathcal{R} -entire and let us consider $r \in q^{\mathbb{Q}}$. We deduce from (5) with f replaced by $f(ax)$ with $|a| = r$ that

$$\sup\{|f(x)|_{\mathcal{R}}; |x| \leq r\} = \max\{|a_n|_{\mathcal{R}} r^n; n \geq 0\}.$$

If there exists $M \in \mathbb{R}_{\geq 0}$ such that $|f(x)|_{\mathcal{R}} \leq M$ for all $x \in \mathbb{C}_\infty$, then, for all r as above, $\max\{|a_n|_{\mathcal{R}} r^n; n \geq 0\} \leq M$ which implies $a_1 = a_2 = \dots = 0$ and $f(x) = a_0$ for all $x \in \mathbb{C}_\infty$. \square

2.2. An analog of Carlson's Theorem. On \mathbb{C}_∞ there is a notion of distance away from K_∞ , or *imaginary part*, due to D. Goss (it appears in his Thesis) and already used in the introduction as

$$|z|_{\mathfrak{S}} := \inf_{\kappa \in K_\infty} |z - \kappa|.$$

It is a good analog of the absolute value of the imaginary part of a classical complex number and is used frequently in the theory of Drinfeld modular forms.

For the convenience of the reader and to draw analogies between the classical complex exponential function $e^z := \sum_{n \geq 0} \frac{z^n}{n!}$ which will help develop the analogy in this section, we briefly recall some other basic properties of the Carlitz exponential \exp_C , defined in the introduction.

The function \exp_C is the unique \mathbb{F}_q -linear power series $\mathbb{C}_\infty \rightarrow \mathbb{C}_\infty$ such that $\frac{d}{dz} \exp_C(z) = 1$ and $\exp_C(az) = C_a(\exp_C(z))$ for all $a \in A$ and $z \in \mathbb{C}_\infty$, where C_a is a polynomial function denoting the multiplication by a in \mathbb{C}_∞ for the Carlitz module structure (see Goss' book [9, Chapter 3]). The kernel of \exp_C is $\tilde{\pi}A$.

With these analogies in mind, we give the following version of Carlson's theorem [5] for \mathcal{R} -entire functions.

Theorem 9. *Let f be an \mathcal{R} -entire function, and suppose that there exists a positive real number c such that, for all $z \in \mathbb{C}_\infty$, $|f(z)| \leq c|e_C(z)|$. Suppose further that $|f(z)/e_C(z)|$ tends to 0 as $|z|_{\mathfrak{S}}$ tends to ∞ . If $f(a) = 0$ for all $a \in A$, then f is identically zero.*

Proof. By our assumptions on f , the function $g := f/e_C$ is \mathcal{R} -entire and bounded, hence constant by Proposition 8. This constant equals zero because $g(z) \rightarrow 0$ as $|z|_{\mathfrak{S}} \rightarrow \infty$. \square

2.3. A -periodic \mathcal{R} -entire functions. Let $\mathcal{P}^1(\mathcal{R})$ be the \mathcal{R} -algebra of \mathcal{R} -entire functions which are A -periodic, that is, the \mathcal{R} -algebra of \mathcal{R} -entire functions f such that $f(z+a) = f(z)$ for all $z \in \mathbb{C}_\infty$ and $a \in A$. The following corollary of Theorem 9 will be used in the proof of Theorem 2 below.

Corollary 10. *Let $f \in \mathcal{P}^1(\mathcal{R})$ and let us assume that there exist an integer N and a real number $C > 0$ with $|f(z)|_{\mathcal{R}} \leq C \max\{1, |e_C(z)|^N\}$ for all $z \in \mathbb{C}_\infty$. Further,*

we suppose that $|\mathcal{R}|_{\mathcal{R}} = |\mathbb{C}_{\infty}|$. Then, f is a polynomial of $\mathcal{R}[e_C]$ of degree at most N .

Proof. The function $e_C : \mathbb{C}_{\infty} \rightarrow \mathcal{R}$, is \mathcal{R} -entire, A -periodic, and vanishes at $z = a$ with order one for all $a \in A$ and has no other zeroes. This implies that there exist elements c_0, \dots, c_N of \mathcal{R} such that the function $g : \mathbb{C}_{\infty} \rightarrow \mathcal{R}$

$$g = \frac{f - \sum_{i=0}^N c_i e_C^i}{e_C^N}$$

is \mathcal{R} -entire, bounded, vanishes for all $z \in A$, and $g(z)/e_C(z)$ vanishes as $|z|_{\mathbb{Q}} \rightarrow \infty$ by our assumption. By Carlson's Theorem, g must be the zero function, proving the claim. \square

Definition 11. An \mathcal{R} -entire function $f : \mathbb{C}_{\infty} \rightarrow \mathcal{R}$ is called *tempered* if there exists a real number $C > 0$ and an integer $N \geq 0$ depending on f such that for all $z \in \mathbb{C}_{\infty}$, $|f(z)|_{\mathcal{R}} \leq C \max\{1, |e_C(z)|\}^N$.

We denote by $\mathcal{T}(\mathcal{R})$ the \mathcal{R} -algebra of tempered \mathcal{R} -entire functions. We also set $\mathcal{P}(\mathcal{R}) = \mathcal{P}^1(\mathcal{R}) \cap \mathcal{T}(\mathcal{R})$.

Remark 12. Note that by Corollary 10, if $|\mathcal{R}|_{\mathcal{R}} = |\mathbb{C}_{\infty}|$, then

$$\mathcal{P}(\mathcal{R}) = \mathcal{R}[e_C].$$

2.3.1. *Convention.* For the remainder of this paper, \mathcal{R} will either be \mathbb{T}_s or \mathbb{E}_s equipped with the Gauss norm, both satisfying $|\mathcal{R}|_{\mathcal{R}} = |\mathbb{C}_{\infty}|$.

3. ANDERSON GENERATING FUNCTIONS

For $z \in \mathbb{C}_{\infty}$, we observe that the function $\omega(t)^{-1} f_t(z)$ extends the map $\chi_t : A \rightarrow \mathbb{F}_q[t] \subset \mathbb{T}$ of the introduction to a \mathbb{T} -entire function $\mathbb{C}_{\infty} \rightarrow \mathbb{T}$. Thus, to follow we shall write

$$\chi_t(z) := \omega(t)^{-1} f_t(z).$$

For all $z \in \mathbb{C}_{\infty} \setminus A$ we also set

$$u(z) := \tilde{\pi}^{-1} \sum_{a \in A} \frac{1}{z - a} = \exp_C(\tilde{\pi}z)^{-1} = e_C(z)^{-1}.$$

Lemma 13. *The map $\chi_t : \mathbb{C}_{\infty} \rightarrow \mathbb{T}$ is the unique \mathbb{F}_q -linear, \mathbb{E} -entire function satisfying*

$$\|\chi_t(z)\| \leq \max\{1, |e_C(z)|^{q^{-1}}\} \quad \text{for all } z \in \mathbb{C}_{\infty}.$$

In particular, $u(z)\chi_t(z) \rightarrow 0$ as $|z|_{\mathbb{Q}} \rightarrow \infty$.

Proof. That $\chi_t(z)$ is \mathbb{F}_q -linear and has image in \mathbb{E} for all $z \in \mathbb{C}_{\infty}$ follows easily from the definitions, and uniqueness follows readily from Theorem 9 above.

We recall from [14, Theorem 2.19] that $\chi_t(z)$ satisfies the τ -difference equation

$$(6) \quad \tau(\chi_t(z)) = \chi_t(z) + (u(z)(t - \theta)\omega(t))^{-1}.$$

Thus, if $\|\chi_t(z)\| > 1$, then $\|\tau(\chi_t(z))\| = \|\chi_t(z)\|^q > \|\chi_t(z)\|$, and from (6) we obtain

$$\|\chi_t(z)\| = \|(t - \theta)\omega(t)\|^{-q^{-1}} |e_C(z)|^{q^{-1}}.$$

In all cases, we have

$$\|\chi_t(z)\| \leq \max\{1, \|(t - \theta)\omega(t)\|^{-q^{-1}} |e_C(z)|^{q^{-1}}\} \leq \max\{1, |e_C(z)|^{q^{-1}}\},$$

and the final claim is obvious. \square

Remark 14. One readily shows that when $|z|_{\mathfrak{S}}$ is sufficiently large $\|\chi_t(z)\| > 1$, and we obtain the equality $\|\chi_t(z)\| = \|(t - \theta)\omega(t)\|^{-q^{-1}} |e_C(z)|^{q^{-1}}$.

3.1. Proof of Theorem 1. The following details are given for the convenience of the reader; they also have the advantage of introducing the arguments of §4.

We observe that, for all $a \in A$,

$$\psi_1(z + a) = \psi_1(z) + \chi_t(a)\tilde{\pi}u(z).$$

Similarly, we have

$$\chi_t(z + a) = \chi_t(z) + \chi_t(a).$$

This means that the function $F_1(z) = \tilde{\pi}u(z)\chi_t(z) - \psi_1(z)$ is A -periodic. It is a rigid analytic function $\mathbb{C}_{\infty} \setminus A \rightarrow \mathbb{T}$ with only simple poles. The residue at $z = a \in A$ of ψ_1 is equal to $\chi_t(a)$. But the residue of $u(z)$ at $z = a$ is $\tilde{\pi}^{-1}$ which implies that the function F_1 is \mathbb{T} -entire, and, hence, the function $e_C F_1$ is \mathbb{T} -entire and vanishes on A .

Now, we observe the identity:

$$F_1(z) := \frac{\chi_t(z)}{z} + \sum_{b \in A \setminus \{0\}} \frac{\chi_t(z) - \chi_t(b)}{z - b}.$$

the function $\frac{\chi_t(z)}{z}$ is clearly \mathbb{E} -entire hence bounded for $|z| < 1$. Let b be a non-zero element of A and let us suppose that $|z| < 1$. then, $|z - b|^{-1} = |b|^{-1}$. By Lemma 13, we deduce that the series

$$F^*(z) := \sum_{b \in A \setminus \{0\}} \frac{\chi_t(z) - \chi_t(b)}{z - b}$$

is bounded for $|z| < 1$ and $\|F_1(z)\| \leq c_1$ for an absolute constant c_1 , for $|z| < 1$. Let us choose now $z \in \mathbb{C}_{\infty}$ such that $|z| \geq 1$. If there exists $a \in A$ such that $|z - a| < 1$, then we can deduce, by A -periodicity, that $\|F_1(z)\| \leq c_1$. Otherwise, for all $a \in A$, $|z - a| \geq 1$ and now, by Lemma 13, there exists an absolute constant c_2 such that $\|F^*(z)\| \leq c_2$. This means that after one final application of Lemma 13 the function $e_C F_1$ satisfies the hypotheses of Theorem 9 and must be the zero function, completing the proof of Theorem 1. \square

Remark 15. The above proof implies the identity $L(\chi_t, 1) = -\tilde{\pi}\tau(\omega)(t)^{-1}$. Indeed, we have

$$\lim_{z \rightarrow 0} \frac{\chi_t(z)}{z} = \frac{d}{dz} \frac{\chi_t(z)}{z} = \frac{\tilde{\pi}}{(\theta - t)\omega(t)}.$$

On the other hand,

$$\lim_{z \rightarrow 0} \sum_{b \in A \setminus \{0\}} \frac{\chi_t(z) - \chi_t(b)}{z - b} = - \sum_{b \in A^+} \frac{\chi_t(b)}{b} = -L(\chi_t, 1).$$

But since $F_1 = 0$, we are done.

Remark 16. One classical use of Carlson's theorem is to prove equalities between an entire function on the complex numbers and its binomial (or Newton) series, when it exists. Carlitz' analog of the binomial coefficient functions, defined in the introduction as,

$$E_0(z) := z \text{ and } E_k(z) := d_k^{-1} \prod_{\substack{a \in A \\ \deg(a) < k}} (z - a), \text{ for } k \geq 1,$$

leads us to mention a third description for ψ_1 .

The following identity of \mathbb{T} -valued entire functions on \mathbb{C}_∞ , proved in [14] using interpolation polynomials, may now be deduced as a consequence of Theorem 9,

$$(7) \quad \tilde{\pi}^{-1} e_C(z) \psi_1(z) = \sum_{k \geq 0} E_k(z) \prod_{j=0}^{k-1} (t - \theta^{q^j}),$$

valid for all $z \in \mathbb{C}_\infty$.

Thus Theorem 9 appears to be a useful addition to the theory of Wagner's analog of Mahler series as functions on \mathbb{C}_∞ developed in [8].

4. THE FUNCTIONS ψ_s , F_s AND THE PROOF OF THEOREM 2

The proof of Theorem 2 is a simple generalization of the arguments of §3.1. We recall from the introduction the definition of the series ψ_s :

$$\psi_s(z) = \sum_{a \in A} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z - a}.$$

We also recall that, for all $z \in \mathbb{C}_\infty \setminus (A \setminus \{0\})$, $\psi_s(z)$ converges in \mathbb{T}_s . The proof of Theorem 2 appears later in this section, but before giving it we wish to record some properties of the functions ψ_s and related functions.

For positive integers s , let $F_s := \tilde{\pi} u \prod_{i=1}^s \chi_{t_i} - \psi_s$. Since $\tilde{\pi} u(z) = \sum_{a \in A} (z - a)^{-1}$, we obtain the identity

$$(8) \quad F_s(z) = \sum_{a \in A} \frac{\chi_{t_1}(z) \cdots \chi_{t_s}(z) - \chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z - a}.$$

Lemma 17. *For all s , F_s is \mathbb{E}_s -entire and tempered.*

Proof. Let $z \in \mathbb{C}_\infty$ be such that $|a - z| \geq q$ for all $a \in A$. Then, $|a - z|^{-1} \leq q^{-1}$ and the series $\psi_s(z)$ is well defined, such that $\|\psi_s(z)\| \leq q^{-1}$. Similarly, $|\tilde{\pi} u(z)| = |\sum_{a \in A} (a - z)^{-1}| \leq q^{-1}$ and by Lemma 13, we deduce that $\|F_s(z)\| \leq q^{-1} \max\{1, |e_C(z)|\}^s$.

Now, let $z \in \mathbb{C}_\infty$ be such that there exists $a \in A$ with $z' = z - a$ satisfying $|z'| < q$. We estimate the Gauss norm of the term in (8) corresponding to such an $a \in A$. We observe that

$$\begin{aligned} & \chi_{t_1}(z) \cdots \chi_{t_s}(z) - \chi_{t_1}(a) \cdots \chi_{t_s}(a) = \\ &= \chi_{t_1}(z - a + a) \cdots \chi_{t_s}(z - a + a) - \chi_{t_1}(a) \cdots \chi_{t_s}(a) \\ &= \sum_{\emptyset \neq \{i_1, \dots, i_r\} \subset \Sigma_s} \chi_{t_{i_1}}(z') \cdots \chi_{t_{i_r}}(z') \prod_{j \in \Sigma_s \setminus \{i_1, \dots, i_r\}} \chi_{t_j}(a). \end{aligned}$$

We claim that if $z \in \mathbb{C}_\infty$ is such that $|z| < q$, then

$$(9) \quad \|\chi_t(z)/z\| = 1.$$

Indeed, under these hypotheses, $\|\tilde{\pi} z / (\theta - t)\| < q^{\frac{q}{q-1}}$ so that $\|\chi_t(z)\| = \|\tilde{\pi} z / (\theta - t)\|$. The result follows from the obvious estimate $\|\omega\| = q^{\frac{1}{q-1}}$.

By (9) and Proposition 13, $\|z'^{-1} \prod_i \chi_{t_i}(z')\| \leq \max\{1, |e_C(z')|\}^{s-1} \leq C$ for a constant $C > 0$.

We now observe:

$$F_s(z) = \sum_{|z-a|<q} \frac{\chi_{t_1}(z) \cdots \chi_{t_s}(z) - \chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z-a} + \sum_{|z-a|\geq q} \frac{\chi_{t_1}(z) \cdots \chi_{t_s}(z) - \chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z-a},$$

where the first sum is finite, well defined, and has Gauss norm $\leq C$. The second term has Gauss norm $\leq q^{-1} \max\{1, |e_C(z)|\}^s$. In all cases, we have $\|F_s(z)\| \leq C \max\{1, |e_C(z)|\}^s$ and F_s is \mathbb{E}_s -entire and tempered. \square

We move toward the power series development for F_s about $z = 0$.

Lemma 18. *Let $z \in \mathbb{C}_\infty$ be such that $|z| < 1$. Then:*

$$z\psi_s(z) = \sum_{k \geq 0} z^{m+k(q-1)} L(\chi_{t_1} \cdots \chi_{t_s}, m + k(q-1)),$$

where m is the unique integer $0 < m \leq q-1$ such that $m \equiv s \pmod{q-1}$.

Proof. We observe that $\psi_s(z) = \sum_{a \in A \setminus \{0\}} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{z-a}$. Since for $a \in A \setminus \{0\}$, $(z-a)^{-1} = -\frac{1}{a} \sum_{i \geq 0} \left(\frac{z}{a}\right)^i$, we deduce

$$\begin{aligned} \psi_s(z) &= - \sum_{a \in A \setminus \{0\}} a^{-1} \chi_{t_1}(a) \cdots \chi_{t_s}(a) \sum_{i \geq 0} \left(\frac{z}{a}\right)^i \\ &= - \sum_{i \geq 0} z^i \sum_{\lambda \in \mathbb{F}_q^\times} \lambda^{s-i-1} \sum_{a \in A^+} \frac{\chi_{t_1}(a) \cdots \chi_{t_s}(a)}{a^{i+1}} \\ &= \sum_{k \geq 0} z^{m-1+k(q-1)} L(\chi_{t_1} \cdots \chi_{t_s}, m + k(q-1)). \end{aligned}$$

\square

Remark 19. We observe that for $s = 0$, the previous Lemma reduces to the well-known identity of Carlitz $z \sum_{a \in A} (z-a)^{-1} = \sum_{k \geq 0} z^{k(q-1)} \zeta_C(k(q-1))$, where, for positive integers n , $\zeta_C(n) := \sum_{a \in A^+} a^{-n}$ are the Carlitz zeta values.

For non-negative integers k , let $\Pi(k)$ and $BC(k)$ be the Carlitz factorials and Bernoulli-Carlitz numbers, respectively, as defined in [9].

Proposition 20. *Let m be the unique integer such that $0 < m \leq q-1$ and such that $s \equiv m \pmod{q-1}$. We have*

$$zF_s(z)\omega(t_1) \cdots \omega(t_s) = \sum_{k \geq 0} (\tilde{\pi}z)^{m+k(q-1)} \mu_{s,k} \in (\tilde{\pi}z)^m K(t_1, \dots, t_s)[[(\tilde{\pi}z)^{q-1}]],$$

where $\mu_{s,k} = \nu_{s,k} - \lambda_{s,k}$ and where $\nu_{s,k}, \lambda_{s,k}$ are the elements of \mathbb{T}_s defined by

$$\begin{aligned} \lambda_{s,k} &= \frac{L(\chi_{t_1} \cdots \chi_{t_s}, m + k(q-1))\omega(t_1) \cdots \omega(t_s)}{\tilde{\pi}^{m+k(q-1)}}, \\ \nu_{s,k} &= \sum_{m+k(q-1)=q^{i_1}+\cdots+q^{i_s}+l(q-1)} \frac{BC(l(q-1))}{\Pi(l(q-1))d_{i_1} \cdots d_{i_s}(\theta^{q^{i_1}} - t_1) \cdots (\theta^{q^{i_s}} - t_s)}. \end{aligned}$$

Proof. This is easily deduced from Lemma 18, [2, Theorem 1], the definition of the functions f_{t_i} , and the fact that

$$\tilde{\pi}zu(z) = \sum_{l \geq 0} \frac{BC(l(q-1))}{\Pi(l(q-1))} (\tilde{\pi}z)^{l(q-1)}.$$

□

4.1. Proof of Theorem 2. Recall that we have set $\Sigma_s := \{1, 2, \dots, s\}$. For subsets $I \subset \Sigma_s$ we define

$$\chi^I(z) := \prod_{i \in I} \chi_{t_i}(z) \text{ and } \psi_I(z) := \sum_{a \in A} (z-a)^{-1} \chi^I(a).$$

Consider the Kronecker product of matrices

$$\Xi_s := \begin{pmatrix} 1 & 0 \\ \chi_{t_1} & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ \chi_{t_2} & 1 \end{pmatrix} \otimes \cdots \otimes \begin{pmatrix} 1 & 0 \\ \chi_{t_s} & 1 \end{pmatrix}.$$

The matrix Ξ_s has 2^s rows and 2^s columns and may be decomposed as the sum of the $2^s \times 2^s$ identity matrix plus a strictly lower triangular matrix whose entries consist of the functions χ^I with I ranging over all subsets of Σ_s . We shall not need an explicit description of the coordinates; we note only that the last entry of the first column equals χ^{Σ_s} .

The following result explains the arithmetic interest of this matrix function with values in $\text{GL}_{2^s}(\mathbb{E}_s)$.

Lemma 21. *For all $z \in \mathbb{C}_\infty$ and $a \in A$ we have,*

$$\Xi_s(z+a) = \Xi_s(z)\Xi_s(a) = \Xi_s(a)\Xi_s(z).$$

Further, for all $z \in \mathbb{C}_\infty$, the matrix $\Xi_s(z)$ is invertible, and we have $\Xi_s(z)^{-1} = \Xi_s(-z)$.

Proof. The result is obvious when $s = 1$.

When $s \geq 1$, the first claim follows immediately from the following basic property of the Kronecker product of matrices: for matrices $A_1, \dots, A_s, B_1, \dots, B_s$, such that the pairwise products $A_i B_i$ are defined for all $i \in \Sigma_s$ we have

$$(A_1 \otimes \cdots \otimes A_s)(B_1 \otimes \cdots \otimes B_s) = (A_1 B_1) \otimes \cdots \otimes (A_s B_s).$$

The invertibility follows from the fact that $\Xi_s(z)$ is a lower triangular matrix, all of whose diagonal entries equal 1, and the inverse is calculated as follows: from the property of Kronecker products used above and the identity $\Xi_1(z)^{-1} = \Xi_1(-z)$ we obtain $\Xi_s(z)^{-1} = \Xi_s(-z)$, for all positive integers s . □

The matrix Ξ_s also gives a slick way of recording how the functions $\psi_s(z)$ transform under the shifts $z \mapsto z+a$ for $z \in \mathbb{C}_\infty$ and $a \in A$. Consider the lower triangular matrix $\Psi_s := \sum_{a \in A} (z-a)^{-1} \Xi_s(a)$, whose entries are given by the ψ_I as I ranges over all subsets of Σ_s . From the previous lemma, we immediately obtain the following result.

Lemma 22. *For all $z \in \mathbb{C}_\infty$ and $a \in A$ we have,*

$$\Psi_s(z+a) = \Psi_s(z)\Xi_s(a) = \Xi_s(a)\Psi_s(z).$$

Proof. We have

$$\Psi_s(z+b) = \sum_{a \in A} (z+b-a)^{-1} \Xi_s(a-b+b) = \Xi_s(b) \sum_{a \in A} (z+b-a)^{-1} \Xi_s(a-b),$$

and a change of variables finishes the proof. \square

Finally, we shall require two auxiliary results.

Lemma 23. *The functions $e_C, \chi_{t_1}, \dots, \chi_{t_s}$ are algebraically independent over the field $\mathbb{C}_\infty((t_1, \dots, t_s))$.*

Proof. Since this is standard, we only give a sketch of proof. One first observes, by the fact that $\chi_{t_i}(z+b) = \chi_{t_i}(z) + \chi_{t_i}(b)$ for $b \in A$, that $\chi_{t_1}(z), \dots, \chi_{t_s}(z)$ are algebraically independent over the fraction field of $\mathcal{P}(\mathbb{T}_s)$, hence over $\mathbb{T}_s(e_C)$. On the other hand, it is easy to show that e_C is transcendental over $\mathbb{C}_\infty((t_1, \dots, t_s))$. \square

Remark 24. In fact, it can be proved that the functions of the previous lemma are also algebraically independent over $\mathbb{C}_\infty((t_1, \dots, t_s))(z)$.

Proposition 25. *Let K be a field, let us consider formal series $f_0, f_1, \dots, f_s \in K[[Z]]$, let L/K be a field extension and let us suppose that the series f_i are algebraically independent over L . Let P be a polynomial of $L[X_0, \dots, X_s]$ such that $P(f_0, \dots, f_s) \in K[[Z]]$. Then, $P \in K[X_0, \dots, X_s]$.*

Proof. Let $P \in L[X_0, \dots, X_s]$ be such that $P(f_0, \dots, f_s) \in K[[Z]]$, let d be its degree. Since the functions f_0, \dots, f_s are algebraically independent over L , the monic monomials M_0, \dots, M_N in f_0, \dots, f_s of degree $\leq d$ are linearly independent over L and there is a L -rational linear combination $F = \sum_i a_i M_i \in K[[Z]]$. The matrix U whose columns are the coefficients of the M_i (it has $N+1$ columns and infinitely many rows) has maximal rank and the right multiplication by the column matrix with coefficients a_i yields an infinite column of elements of K which cannot be identically zero. Extracting from U a non-singular square matrix of order $N+1$ and inverting it, we deduce that the coefficients a_0, \dots, a_N are in K . \square

4.1.1. *Proof of Theorem 2.* For each subset $I \subset \Sigma_s$, let $F_I := \tilde{\pi}u\chi^I - \psi_I$. From Lemma 17 we observe that F_I is \mathbb{E}_s -entire and tempered. Further, the functions F_I occur as the entries of the lower triangular matrix

$$\Phi_s := \tilde{\pi}u\Xi_s - \Psi_s,$$

and we note that by Theorem 1 the first entry in the first column equals zero. We also note that F_s occurs as the last entry in the first column of Φ_s . By Lemmas 21 and 22 we observe that $\Xi_s^{-1}\Phi_s$ has entire, tempered, periodic entries, and hence by Cor. 10 has entries in $\mathbb{T}_s[e_C]$. This establishes the existence of functions $h_I \in \mathbb{T}_s[e_C]$ such that

$$(10) \quad F_s = \sum_{I \subsetneq \Sigma_s} h_I \chi^I;$$

here we are summing only over the strict subsets of Σ_s precisely because the entry in the first row and first column of Φ_s vanishes, as noted above.

To obtain the desired upper-bound on the degree of the h_I in e_C , we argue by contradiction. Suppose there exists an $I \subsetneq \Sigma_s$ such that

$$\deg_{e_C}(h_I) > s/q - 1.$$

From h_I we obtain a monomial $a_0 e_C^{j_0} \chi^I$ with $j_0 > s/q - 1$ and $a_0 \neq 0$ which appears on the right side of (10). In order to preserve the order of growth in e_C at infinity of F_s , there must exist further monomials $a_1 e_C^{j_1} \chi^{J_1}, \dots, a_n e_C^{j_n} \chi^{J_n}$, coming from the polynomials h_{J_i} such that

$$H = \sum_{k=0}^n a_k e_C^{j_k} \chi^{J_k}$$

vanishes at infinity. The function H is \mathbb{T}_s -entire and bounded, hence constant, but this contradicts the algebraic independence result of Lemma 23, finishing the proof of the desired upper bound in e_C .

It remains to show that $g_I := \tilde{\pi}^{-1} \omega(t_1) \cdots \omega(t_s) h_I \in K(t_1, \dots, t_s)[e_C]$, and this follows easily from Proposition 20, the fact that the functions $e_C, f_{t_1}(z), \dots, f_{t_s}(z)$ are algebraically independent (Lemma 23) and their expansions in powers of $\tilde{\pi}z$ are defined over $K(t_1, \dots, t_s)$, thanks to Proposition 25. \square

4.1.2. Remarks on χ -quasi-periodic functions. A vectorial function $\mathcal{F} : \mathbb{C}_\infty \rightarrow \mathbb{T}_s^{2^s}$ whose coefficients consist of \mathbb{T}_s -entire, tempered functions shall be called χ -quasi-periodic if, for all $z \in \mathbb{C}_\infty$ and $a \in A$, we have

$$\mathcal{F}(z + a) = \Xi_s(a) \mathcal{F}(z).$$

We shall also call the entries of such a vectorial function χ -quasi-periodic, and as the proof of Theorem 2 demonstrates, these entries are in the $\mathbb{T}_s[e_C]$ -linear span of the functions χ^J , for J ranging over the subsets of Σ_s . The two authors are already employing such formalism in a forthcoming work on deformations of vectorial modular forms.

5. EVALUATIONS AT ROOTS OF UNITY

Throughout this section, s will be a fixed positive integer, and we retain the notation $\Sigma_s = \{1, 2, \dots, s\}$. We recall our notation from the introduction: we let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be primes (that is, irreducible monic polynomials) of A ; we also set $\mathfrak{m} = \mathfrak{p}_1 \cdots \mathfrak{p}_s$. Let us choose, for all $i = 1, \dots, s$, a root ζ_i of \mathfrak{p}_i in \mathbb{F}_q^{ac} , the algebraic closure of \mathbb{F}_q in \mathbb{C}_∞ . Let

$$\text{ev}_\mathfrak{m} : \mathbb{T}_s \rightarrow \mathbb{C}_\infty$$

be the evaluation map sending a formal series $\sum_{i_1, \dots, i_s} c_{i_1, \dots, i_s} t_1^{i_1} \cdots t_s^{i_s} \in \mathbb{T}_s$ to $\sum_{i_1, \dots, i_s} c_{i_1, \dots, i_s} \zeta_1^{i_1} \cdots \zeta_s^{i_s}$.

We now give a qualitative result on $\text{ev}_\mathfrak{m}(\psi_s)$, showing that it is actually both a rational function in $u_\mathfrak{m}$ and an element in $\mathbb{C}_\infty[[u_\mathfrak{m}]]$.

Let Z be an indeterminate over \mathbb{C}_∞ and, for each subset $J \subset \Sigma_s$ define

$$M_\mathfrak{m}^J(Z) := \sum_{b \in A/\mathfrak{m}A} \frac{C_\mathfrak{m}(Z - \exp_C(\tilde{\pi}b/\mathfrak{m}))}{(Z - \exp_C(\tilde{\pi}b/\mathfrak{m}))} \prod_{j \in J} b(\zeta_j) \in A[\zeta_1, \dots, \zeta_s, e_C(\mathfrak{m}^{-1})][Z].$$

Observe that the degree in Z of $M_\mathfrak{m}^J(Z)$ is strictly less than $|\mathfrak{m}|$ and that $M_\mathfrak{m}^J$ is the Lagrange interpolation polynomial for the data $\exp_C(\tilde{\pi}a/\mathfrak{m}) \mapsto \mathfrak{m} \prod_{j \in J} a(\zeta_j)$, defined for $a \in A/\mathfrak{m}A$. For $J = \Sigma_s$ we more simply write $M_\mathfrak{m}^{\Sigma_s} = M_\mathfrak{m}$.

Proposition 26. *For all $z \in \mathbb{C}_\infty \setminus A$, square-free monic polynomials \mathfrak{m} and subsets $J \subset \Sigma_s$, the following identity holds,*

$$\text{ev}_\mathfrak{m}(\psi_J(z)) = \tilde{\pi} \frac{M_\mathfrak{m}^J(\exp_C(\tilde{\pi}z/\mathfrak{m}))}{C_\mathfrak{m}(\exp_C(\tilde{\pi}z/\mathfrak{m}))} \in \tilde{\pi} K(\zeta_1, \dots, \zeta_s, e_C(\mathfrak{m}^{-1}))(u_\mathfrak{m}(z)),$$

from which we deduce $\text{ev}_{\mathfrak{m}}(\psi_J) \in \frac{\tilde{\pi}}{\mathfrak{m}} A[\zeta_1, \dots, \zeta_s, e_C(\mathfrak{m}^{-1})][[u_{\mathfrak{m}}]]$.

Proof. We have,

$$\begin{aligned} \text{ev}_{\mathfrak{m}}(\psi_J) &= \sum_{b \in A} (z - b)^{-1} \prod_{j \in J} b(\zeta_j) \\ &= \sum_{b \in A/\mathfrak{m}A} \prod_{j \in J} b(\zeta_j) \sum_{a \in \mathfrak{m}A} (z - b - a)^{-1} \\ &= \tilde{\pi} \sum_{b \in A/\mathfrak{m}A} u_{\mathfrak{m}}(z - b) \prod_{j \in J} b(\zeta_j) \\ &= \frac{\tilde{\pi}}{\mathfrak{m}} \sum_{b \in A/\mathfrak{m}A} \frac{\prod_{j \in J} b(\zeta_j)}{\exp_C(\tilde{\pi}z/\mathfrak{m}) - \exp_C(\tilde{\pi}b/\mathfrak{m})}. \end{aligned}$$

From the third equality to the fourth, we have used the identity

$$u_{\mathfrak{m}}(z) = \frac{1}{\mathfrak{m} \exp_C(\tilde{\pi}z/\mathfrak{m})},$$

and the final identity comes from direct comparison with the definition of $M_{\mathfrak{m}}$ combined with the observation that $C_{\mathfrak{m}}(Z - \exp_C(\tilde{\pi}b/\mathfrak{m})) = C_{\mathfrak{m}}(Z)$, for all $b \in A/\mathfrak{m}A$.

The second claim follows from the definition of $M_{\mathfrak{m}}^J$ and the observation that

$$\frac{1}{C_{\mathfrak{m}}(\exp_C(\tilde{\pi}z/\mathfrak{m}))} \in u_{\mathfrak{m}}^{|\mathfrak{m}|} A[[u_{\mathfrak{m}}]],$$

and the strict upper-bound of $|\mathfrak{m}|$ on the degree in Z of $M_{\mathfrak{m}}^J$, for all $J \subset \Sigma_s$. \square

Corollary 27. *We suppose that $s = 1$. For every prime $\mathfrak{m} = \mathfrak{p}$ and all $z \in \mathbb{C}_{\infty}$, we have*

$$\text{ev}_{\mathfrak{p}}(\chi_t(z)) = \mathfrak{p}^{-1} M_{\mathfrak{p}}(\exp_C(\tilde{\pi}z/\mathfrak{p})).$$

Proof. This is immediate from Theorem 1 and the previous proposition. \square

One would like to describe the precise order of vanishing in $u_{\mathfrak{m}}$ of the evaluations $\text{ev}_{\mathfrak{m}}(\psi_s)$ for all square-free conductors \mathfrak{m} and positive integers s , or equivalently to determine the degree in Z of $M_{\mathfrak{m}}(Z)$. This appears to be difficult in general, but it is possible in the case of $s = 1$. We tackle this problem in the next section.

5.1. Gauss-Thakur sums and the polynomial $M_{\mathfrak{p}}$. We now focus our attention completely on the interpolation polynomials $M_{\mathfrak{p}}$ in order to prove Theorems 4 and 5 of the introduction.

For $j \geq 0$, let $A(j)$ be the \mathbb{F}_q -vector subspace of A consisting of all polynomials of degree strictly less than j . Let \mathfrak{p} be a monic irreducible of degree d , and let $\lambda = e_C(\mathfrak{p}^{-1})$ be a fixed element of Carlitz \mathfrak{p} -torsion. Choose a root ζ of \mathfrak{p} , and let χ be the Teichmüller character determined by $\theta \mapsto \zeta$. The following Gauss sum was first introduced by Thakur in [16] and already appeared in the introduction,

$$g(\chi) := \sum_{a \in A(d)} \chi(a)^{-1} C_a(\lambda).$$

The sum $g(\chi)$ is a non-zero element of $A[\zeta, \lambda]$. We recall setting $g(\chi^{-1}) := (-1)^d \mathfrak{p}/g(\chi)$, which is the Gauss-Thakur sum for the character χ^{-1} and is an element of $A[\zeta, \lambda]$ by [2, Prop. 15.2]. We also recall that for each $a \in (A/\mathfrak{p}A)^{\times}$,

there is a unique element σ_a of the Galois group of $K(\zeta, \lambda)$ over $K(\zeta)$ satisfying $\sigma_a(\lambda) = C_a(\lambda) = e_C(a/\mathfrak{p})$ and that $g(\chi)$ satisfies

$$(11) \quad \sigma_a(g(\chi)) = a(\zeta)g(\chi)$$

for all $a \in (A/\mathfrak{p}A)^\times$.

Lemma 28. *We have the following identity,*

$$(12) \quad M_{\mathfrak{p}}(Z) = (-1)^d g(\chi^{-1}) \sum_{j=0}^{d-1} Z^{q^j} \sum_{a \in A(d) \setminus A(j)} a(\zeta)^{-1} E_j(a) \in g(\chi^{-1}) A[\zeta][Z].$$

Proof. Define the coefficients $a_i \in A[\zeta]$ via $g(\chi) := \sum_{i=0}^{|\mathfrak{p}|-2} a_i \lambda^i$; one easily sees here that only those indices $i \equiv 1 \pmod{q-1}$ can appear. We observe immediately from (11) that the polynomial $g(\chi)^{-1} \sum_{i \geq 0} a_i Z^i$ takes the value $a(\zeta)$ for $Z = C_a(\lambda)$ for all $a \in A(d)$ and takes the value zero at $Z = 0$. Further its degree is strictly bounded above by $|A(d)| = |\mathfrak{p}|$. Since $\mathfrak{p}^{-1} M_{\mathfrak{p}}(Z)$ is the Legendre interpolation polynomial of the data $C_a(\lambda) \mapsto a(\zeta)$ for $a \in A(d)$, we must have $\mathfrak{p}^{-1} M_{\mathfrak{p}}(Z) = g(\chi)^{-1} \sum_{i=0}^{|\mathfrak{p}|-1} a_i Z^i$, by consideration on the degrees in Z and uniqueness of the Legendre interpolating polynomial. Replacing $g(\chi)^{-1}$ with $(-1)^d \mathfrak{p}^{-1} g(\chi^{-1})$, we obtain $M_{\mathfrak{p}}(Z) = (-1)^d g(\chi^{-1}) \sum_{i=0}^{|\mathfrak{p}|-2} a_i Z^i$.

Now we derive an expression for the coefficients a_i appearing above. From the definition of $g(\chi)$, we immediately obtain

$$g(\chi) = \sum_{a \in A(d)} a(\zeta)^{-1} \sum_{j=0}^{d-1} E_j(a) \lambda^{q^j} = \sum_{j=0}^{d-1} \lambda^{q^j} \sum_{a \in A(d)} a(\zeta)^{-1} E_j(a).$$

□

Remark 29. It is an interesting question to determine the \mathbb{F}_q -vector subspace of \mathbb{C}_∞ of roots to the polynomial $M_{\mathfrak{p}}(Z)$.

We can give a more compact expression for the coefficients of Z and $Z^{|\mathfrak{p}|/q}$ in $M_{\mathfrak{p}}$. For non-negative integers j , let $A^+(j)$ denote the set of all elements of A^+ of degree j .

Corollary 30. *The coefficient of $Z^{|\mathfrak{p}|/q}$ in $M_{\mathfrak{p}}$ equals $(-1)^{d+1} g(\chi^{-1}) \chi(\ell_{d-1})^{-1}$, and the coefficient of Z equals $(-1)^{d+1} \mathfrak{p} \chi(\ell_{d-1})^{-1} (\theta - \zeta)^{-1} g(\chi^{-1})$; both are elements of $A[\zeta, \lambda]$.*

Proof. From (12), the coefficient of $Z^{|\mathfrak{p}|/q}$ equals

$$(-1)^{d+1} g(\chi^{-1}) \sum_{a \in A^+(d-1)} a(\zeta)^{-1} = (-1)^{d+1} g(\chi^{-1}) \chi(\ell_{d-1})^{-1};$$

the latter equality follows from the work of Carlitz.

Similarly, the coefficient of Z equals

$$(-1)^{d+1} g(\chi^{-1}) \sum_{j=0}^{d-1} \sum_{a \in A^+(j)} a(\zeta)^{-1} a.$$

Letting x, y be indeterminates over \mathbb{F}_q , one may easily calculate that

$$\sum_{a \in A^+(j)} a(x)^{-1} a(y) = \ell_j(x)^{-1} \prod_{k=0}^{j-1} (y - x^{q^k}),$$

and that the sum $\sum_{j=0}^{d-1} \ell_j(x)^{-1} \prod_{k=0}^{j-1} (y - x^{q^k})$ telescopes to $\ell_{d-1}(x)^{-1} \prod_{j=1}^{d-1} (y - x^{q^j})$. Replacing x by ζ and y by θ , we obtain the expression above for the coefficient of Z . \square

5.1.1. *Proof of Theorem 4.* It suffices to compare the coefficients of z on both the left and right sides of the identity of Cor. 27. On the left side we have $\tilde{\pi} \operatorname{ev}_\zeta((\theta - t)\omega(t))^{-1}$, while on the right we have $(-1)^{d+1} \tilde{\pi} \mathbf{p}^{-1} (\theta - \zeta)^{-1} \chi(\ell_{d-1})^{-1} g(\chi^{-1})$, by Cor. 30. Hence, using the definition of $g(\chi^{-1}) = (-1)^d \mathbf{p} g(\chi)^{-1}$, we obtain $\operatorname{ev}_\zeta(\omega) = -\chi(\ell_{d-1})g(\chi)$, as desired. \square

5.1.2. *Proof of Theorem 5.* This is an immediate consequence of Prop. 26 and the Corollary 30 just above. \square

Remark 31. The exact order of vanishing of $\operatorname{ev}_\zeta(\psi_1)$ may also be determined analytically using Remark 14 starting from the identity $\psi_1 = \tilde{\pi} u \chi_t(z)$. Indeed, this tells us that, for all z such that $|z|_{\mathfrak{S}}$ is big enough, we have $\|\psi_1\| = |\tilde{\pi}| |u|^{1-1/q}$. Hence, for \mathbf{p} a prime, $|\operatorname{ev}_\zeta(\psi_1)| = |u_{\mathbf{p}}|^{|p|(1-\frac{1}{q})}$ (provided that $|z|_{\mathfrak{S}}$ is big enough). But we know that $\operatorname{ev}_\zeta(\psi_1) \in \mathbb{C}_\infty[[u_{\mathbf{p}}]]$. This implies that $\operatorname{ev}_\zeta(\psi_1) u_{\mathbf{p}}^{|p|(\frac{1}{q}-1)} \in \mathbb{C}_\infty[[u_{\mathbf{p}}]]^\times$.

5.2. **Questions in closing.** We conclude with a speculative subsection. Let us fix an integer $s \geq 1$. As Theorem 2 shows, the functions ψ_s are elements of the \mathbb{T}_s -algebra $\mathcal{R}_s := \mathbb{T}_s[u, u^{-1}, \{\chi^J\}_{J \subset \Sigma_s}]$. It follows from Corollary 27 that for all monic $\mathbf{m} = \mathbf{p}_1 \cdots \mathbf{p}_s$, as above and all $h \in \mathcal{R}_s$ we have $\operatorname{ev}_{\mathbf{m}}(h) \in \mathbb{C}_\infty(u_{\mathbf{m}})$ for \mathbf{m} a product of primes $\mathbf{m} = \mathbf{p}_1 \cdots \mathbf{p}_s$ and for every choice of roots ζ_i of \mathbf{p}_i , as we saw for ψ_s .

We define the sub- \mathbb{T}_s -algebra $\mathcal{R}_s^0 \subset \mathcal{R}_s$ whose elements are the $h \in \mathcal{R}_s$ such that $\operatorname{ev}_{\mathbf{m}}(h) \in \mathbb{C}_\infty[[u_{\mathbf{m}}]]$ for all $\mathbf{p}_1, \dots, \mathbf{p}_s$ and all roots ζ_i of \mathbf{p}_i . As proven in Proposition 26, we have $\psi_J \in \mathcal{R}_s^0$, for all $J \subset \Sigma_s$.

We denote by $\mathbb{T}_{s,i}$ the completion of $\mathbb{C}_\infty[t_1, \dots, \hat{t}_i, \dots, t_s] \subset \mathbb{T}_s$ for the Gauss norm (the hat denotes an omitted term and the completion is canonically identified to a subring of \mathbb{T}_s). We also denote by φ_i the $\mathbb{T}_{s,i}$ -linear endomorphism of \mathbb{T}_s uniquely defined by $\varphi_i(t_i) = t_i^q$.

Lemma 32. *The ring \mathcal{R}_s^0 is φ_i -stable for all i . That is, for all $f \in \mathcal{R}_s^0$ and all $i = 1, \dots, s$, we have $\varphi_i(f) \in \mathcal{R}_s^0$.*

Proof. We know that for any choice of $\zeta_1, \dots, \zeta_s \in \mathbb{F}_q^{ac}$, for $f \in \mathcal{R}_s^0$, we have $f|_{t_i=\zeta_i} \in \mathbb{C}_\infty[[u_{\mathbf{m}}]]$, where \mathbf{p}_i is the prime of A vanishing at ζ_i (for $i = 1, \dots, s$) and $\mathbf{m} = \mathbf{p}_1 \cdots \mathbf{p}_s$. Let us write $g = \varphi_j(f)$ for a fixed $j \in \{1, \dots, s\}$. Since ζ_j^q is conjugate to ζ_j over \mathbb{F}_q , we again get, thanks to the hypothesis on f , that $g|_{t_i=\zeta_i} \in \mathbb{C}_\infty[[u_{\mathbf{m}}]]$. This means that $g \in \mathcal{R}_s^0$. \square

Example 33. We set $s = 1$, case in which we write $t = t_1$, $\mathbb{T} = \mathbb{T}_1$ and $\varphi = \varphi_1$. From (3) and Lemma 13 we deduce (observe that $\varphi(f) = (\tau^{-1}(f))^q$):

$$\tau(\psi_1) = (\tilde{\pi} u)^{q-1} (\psi_1 - L(\chi_t, 1))$$

and

$$\varphi(\psi_1) = (\tilde{\pi}u)^{1-q}\psi_1^q + L(\chi_t^q, 1).$$

Since $\varphi(\psi_1), L(\chi_t^q, 1)$ both belong to \mathcal{R}_1^0 , we deduce that $\psi_1^q u^{1-q}$ belongs to \mathcal{R}_1^0 . More generally, the reader can check that for all $d \geq 0$, and with $L = L(\chi_t, 1)$ and $\mu = \tilde{\pi}u$,

$$\mu^{1-q^d}\psi^{q^{d-1}} + \mu^{1-q^{d-1}}\varphi(L)^{q^{d-1}} + \mu^{1-q^{d-2}}\varphi^2(L)^{q^{d-2}} + \dots + \mu^{1-q}\varphi^{d-1}(L)^q \in \mathcal{R}_1^0.$$

One interesting consequence of the τ and φ difference equations satisfied by ψ_1 is that by comparing the expansions in $\mathbb{T}[[z]]$ of the left and right sides one may obtain recursion relations satisfied by the L -functions $L(\chi_t, n)$ with $n \equiv 1 \pmod{q-1}$.

The structure of the algebras \mathcal{R}_s^0 is likely to be quite intricate.

Question 34. *Determine explicitly a set of generators of \mathcal{R}_s^0 .*

Another natural question is the following.

Question 35. *Is the algebra \mathcal{R}_s^0 a τ -stable algebra, that is, for all $f \in \mathcal{R}_s^0$, $\tau(f) \in \mathcal{R}_s^0$?*

In this direction we observe, if $s = 1$, that the sub- \mathbb{T} -algebra $\mathcal{R}_1^1 = \mathbb{T}[\varphi^k(\psi_1); k \geq 0]$ of \mathcal{R}_1^0 is also τ -stable. Is it true that $\mathcal{R}_1^0 = \mathcal{R}_1^1$?

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