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# SHARP WEIGHTS IN THE CAUCHY PROBLEM FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIAL

RÉMI CARLES

ABSTRACT. We review different properties related to the Cauchy problem for the (non-linear) Schrödinger equation with a smooth potential. For energy-subcritical nonlinearities and at most quadratic potentials, we investigate the necessary decay in space in order for the Cauchy problem to be locally (and globally) well-posed. The characterization of the minimal decay is different in the case of super-quadratic potentials.

## 1. GENERALITIES

We consider the nonlinear Schrödinger equation

$$(1.1) \quad i\partial_t u = Hu + \lambda|u|^{2\sigma}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, \quad H = -\frac{1}{2}\Delta + V(x),$$

with  $\lambda \in \mathbb{R}$ ,  $\sigma > 0$ , for some smooth, real-valued potential  $V$ . In order to work at the level of regularity  $L^2$  or  $H^1$ , we suppose that the nonlinearity is energy-subcritical, that is,  $\sigma < 2/(d-2)$  when  $d \geq 3$  (see e.g. [7]). Such models appear in various fields of Physics, such as laser propagation or Bose-Einstein Condensation (see e.g. [16, 19]): for instance the potential  $V$  can be quadratic (harmonic potential), linear (Stark effect), or super-quadratic to ensure a strong confinement.

**Assumption 1.1.** *We suppose that  $V$  is smooth and real-valued,  $V \in C^\infty(\mathbb{R}^d; \mathbb{R})$ , and,*

- *Either  $V$  is at most quadratic,  $\partial^\alpha V \in L^\infty(\mathbb{R}^d)$  for all  $\alpha \in \mathbb{N}^d$  with  $|\alpha| \geq 2$ ,*
- *Or  $V \geq 0$  is super-quadratic, in the sense that  $V(x) \rightarrow \infty$  as  $|x| \rightarrow \infty$  and there exists  $m > 2$  such that*

$$|\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{m-|\alpha|}, \quad \forall \alpha \in \mathbb{N}^d.$$

Typically, the second case addresses potentials of the form  $V(x) = \langle x \rangle^m$ ,  $m > 2$ . Of course, the second case is formally compatible with the first one, but should be thought of as rather complementary. The borderline case corresponds to quadratic potentials. The fact that quadratic potentials play a special role has been known for many years: as established in [21], the fundamental solution associated to the linear solution is smooth and bounded except at the initial time if  $V(x) = o(|x|^2)$  at infinity, while at least if  $d = 1$ , the fundamental solution associated to super-quadratic potentials is nowhere  $C^1$ . In the limiting exactly quadratic case, the fundamental solution has isolated singularities ([24]). Finally, the linear Schrödinger flow is not uniquely defined for non-positive super-quadratic potentials: if for instance  $d = 1$  and  $V(x) = -x^4$ , then  $H$  is not essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$ , due to infinite speed of propagation in the classical trajectories (see e.g. [10]). From this point of view, for smooth potentials, the assumption that there exist  $a, b > 0$

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such that  $V(x) \geq -a|x|^2 - b$  is sharp in order to ensure that  $H$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^d)$  ([17]).

In this note, we review known results concerning the Cauchy problem for (1.1) with a level of regularity  $H^1(\mathbb{R}^d)$ . The only new result is Theorem 4.2, which shows that the sharp weights in space, at the level of regularity  $H^1$ , have different characterizations for at most quadratic potentials and for super-quadratic potentials. The sharpness of the required decay in space is presented in Proposition 4.3.

## 2. STRICHARTZ ESTIMATES

For at most quadratic potentials, a parametrix for  $e^{-itH}$  has been constructed in [11] (see also [13]). We simply emphasize that as a consequence, the propagator  $e^{-itH}$ , which is unitary on  $L^2(\mathbb{R}^d)$ , satisfies the following local dispersive estimate: there exist  $C, \delta > 0$  such that

$$\|e^{-itH}\|_{L^1(\mathbb{R}^d) \rightarrow L^\infty(\mathbb{R}^d)} \leq \frac{C}{|t|^{d/2}}, \quad |t| \leq \delta.$$

Recall that under the general Assumption 1.1, such an estimate is necessarily local in time, since typically in the case of the harmonic potential, the flow  $e^{-itH}$  is periodic in time.

The case of super-quadratic potentials has been addressed in [22, 23], and in [14] (see also [9, 18, 20]). We summarize the main results on Strichartz estimates for  $e^{-itH}$  in the following statement.

**Proposition 2.1** (From [11, 14]). *Let  $d \geq 1$  and  $V$  satisfying Assumption 1.1. Let  $(q, r)$  be an admissible pair, that is, satisfying*

$$\frac{2}{q} = d \left( \frac{1}{2} - \frac{1}{r} \right),$$

with  $2 \leq q, r \leq \infty$  and  $q > 2$  is  $d = 2$ . Let  $T > 0$ .

- If  $V$  is at most quadratic, there exists  $C = C(q, d, T)$  such that

$$\|e^{-itH} f\|_{L^q([-T, T]; L^r(\mathbb{R}^d))} \leq C \|f\|_{L^2(\mathbb{R}^d)}, \quad \forall f \in L^2(\mathbb{R}^d).$$

- If  $d \geq 3$  and  $V$  is super-quadratic, there exists  $C = C(q, d, T)$  such that

$$\|e^{-itH} f\|_{L^q([-T, T]; L^r(\mathbb{R}^d))} \leq C \|f\|_{B^{\frac{1}{q}(1 - \frac{2}{m})}}, \quad \forall f \in B^{\frac{1}{q}(1 - \frac{2}{m})},$$

where for  $s \geq 0$ ,

$$B^s = \left\{ f \in L^2(\mathbb{R}^d); \|f\|_{B^s} := \|H^{s/2} f\|_{L^2(\mathbb{R}^d)} < \infty \right\}.$$

We refer to [22] and [14] for local in time Strichartz estimates with super-quadratic potentials in the case  $d = 1$  and  $d = 2$ , respectively. We omit them for the sake of concision. Similar estimates are available for retarded terms, which appear in the Duhamel's formula associated to inhomogeneous Schrödinger equations. In the same fashion as above, these estimates are necessarily local in time without extra assumption on  $V$ , since  $H$  may possess eigenvalues (this is the case if  $V \rightarrow +\infty$  as  $|x| \rightarrow \infty$ ).

The following equivalence of norms is established, in [23] when  $V$  grows like  $\langle x \rangle^m$  as  $|x| \rightarrow \infty$ , and in [1] under assumptions on  $V$  which are weaker than in Assumption 1.1,

$$(2.1) \quad \|H^{s/2} f\|_{L^2(\mathbb{R}^d)} \approx \|f\|_{H^s(\mathbb{R}^d)} + \|V^{s/2} f\|_{L^2(\mathbb{R}^d)}, \quad s \geq 0.$$

As shown by the second case of Proposition 2.1, a loss of regularity must be expected for super-quadratic potentials. reminiscent of what happens on compact manifolds without boundary, [2, 3]. Formally, as  $k$  ranges from 2 to  $\infty$ , the loss of regularity varies between

0 and  $1/q$  derivative, this limiting case corresponding to the estimate established in [3] for general compact manifolds. This is in agreement with the property, used in Physics, that for  $V(x) = \langle x \rangle^m$ , the larger the  $m$ , the more confining  $H$ .

### 3. NONLINEAR CAUCHY PROBLEM

Formally, (1.1) enjoys the conservations of mass and energy:

$$\frac{d}{dt} \|u(t)\|_{L^2(\mathbb{R}^d)}^2 = \frac{d}{dt} \left( \langle Hu, u \rangle + \frac{\lambda}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \right) = 0.$$

For  $V \geq 0$  and at most quadratic, a local in time solution to (1.1) was constructed by Oh [15] with data in  $\sqrt{H}$ . We emphasize that the proof there does not rely on a fixed point argument, but on an approximation procedure, as in [12]. In particular, it is not necessary to understand the action of the pseudo-differential operator  $\sqrt{H}$  on the nonlinear term  $|u|^{2\sigma}u$ . The case of the focusing,  $L^2$ -subcritical nonlinearity considered in [15] can easily be generalized in view of the known results for  $V = 0$ .

**Theorem 3.1** (From [15]). *Let  $V \geq 0$  be at most quadratic, and  $u_0 \in \sqrt{H} = B^1$ .*

- *There exists a unique solution  $u \in C([-T, T]; \sqrt{H}) \cap L^{\frac{4\sigma+4}{d\sigma}}([-T, T]; L^{2\sigma+2}(\mathbb{R}^d))$  to (1.1), with initial datum  $u_0$ , for some  $T > 0$  depending on  $\|u_0\|_{B^1}$ .*
- *This solution is global in time,  $u \in C(\mathbb{R}; \sqrt{H}) \cap L^{\frac{4\sigma+4}{d\sigma}}_{\text{loc}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d))$ , in either of the following cases:*
  - $\sigma < 2/d$ ,
  - $\sigma \geq 2/d$  and  $\lambda \geq 0$ .

The assumption on the sign of  $V$  has been removed for initial data in

$$\Sigma = \{f \in H^1(\mathbb{R}^d), \|f\|_{\Sigma} := \|f\|_{H^1(\mathbb{R}^d)} + \|xf\|_{L^2(\mathbb{R}^d)} < \infty\}.$$

In [6], the global Cauchy problem for (1.1) was considered:

**Theorem 3.2** (From [6]). *Let  $V$  be at most quadratic, and  $u_0 \in \Sigma$ .*

- *There exists a unique solution  $u \in C([-T, T]; \Sigma)$  to (1.1), with initial datum  $u_0$ , such that*

$$u, xu, \nabla u \in L^{\frac{4\sigma+4}{d\sigma}}([-T, T]; L^{2\sigma+2}(\mathbb{R}^d)),$$
*for some  $T > 0$  depending on  $\|u_0\|_{\Sigma}$ .*
- *This solution is global in time,  $u \in C(\mathbb{R}; \Sigma)$ ,  $u, xu, \nabla u \in L^{\frac{4\sigma+4}{d\sigma}}_{\text{loc}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d))$ , in either of the following cases:*
  - $\sigma < 2/d$ ,
  - $\sigma \geq 2/d$  and  $\lambda \geq 0$ .

For  $V \geq 0$  at most quadratic, (2.1) shows that  $\Sigma \subset \sqrt{H}$ , and in the case where  $V \geq 0$  is a non-degenerate quadratic form (e.g. isotropic harmonic potential),  $\Sigma = \sqrt{H}$ . Therefore, the above result removes the sign assumption in Theorem 3.1, up to possibly requiring stronger decay in space on the initial datum  $u_0$ . Note also that in the framework of Theorem 3.2, even if  $\lambda \geq 0$ , the energy functional

$$\begin{aligned} E &= \langle Hu, u \rangle + \frac{\lambda}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \\ &= \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \int_{\mathbb{R}^d} V(x) |u(t, x)|^2 dx + \frac{\lambda}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2}, \end{aligned}$$

is not necessarily positive: it turns out that a negative (at most quadratic) potential is not an obstruction to the existence of a solution to (1.1). It may actually prevent finite time blow-up (see e.g. [6] and references therein).

We now turn to the case of a super-quadratic potential. It follows from the analysis in [1] that  $B^s$  is a Banach algebra for  $s > d/2$  (see (2.1)), and the following result is proved:

**Theorem 3.3** (From [1]). *Let  $V$  be super-quadratic,  $s > d/2$  and  $u_0 \in B^s$ . Then for any  $\sigma > 0$  and  $\lambda \in \mathbb{C}$ , there exist  $T > 0$  and a unique solution  $u \in C([-T, T]; B^s)$  to (1.1) with initial datum  $u_0$ .*

To decrease the regularity, Strichartz inequalities make it possible to prove:

**Theorem 3.4** (From [23] and [14]). *Let  $V$  be super-quadratic and  $s \geq 0$  with*

$$s > \frac{d}{2} - \frac{1}{\sigma} \left( \frac{1}{2} + \frac{1}{m} \right).$$

*Let  $u_0 \in B^s$ . There exist  $T > 0$  and a unique solution  $C([-T, T]; B^s)$  to (1.1) with initial datum  $u_0$ . (Uniqueness is actually granted in smaller spaces, involving a mixed time-space norm which we omit to simplify the presentation.)*

Here again, for  $m = 2$ , the above condition on  $s$  is the standard one in the case without potential ([8]), and letting  $m \rightarrow \infty$ , we recover the condition established in [3] on compact manifolds without boundary. To work at the level of  $H^1$ -regularity ( $s = 1$ ), the above condition reads

$$\sigma < \frac{m+2}{m(d-2)_+}.$$

Both Theorem 3.3 and Theorem 3.4 imply, in view of the conservation of the energy:

**Corollary 3.5.** *Let  $d = 1$ , and  $V$  be super-quadratic. If  $s \geq 1$  and  $u_0 \in B^s$ , then (1.1) has a unique, global solution  $u \in C(\mathbb{R}; B^s)$  with initial datum  $u_0$ , in either of the following cases:*

- $\sigma < 2$  and  $\lambda \in \mathbb{R}$ ,
- $\sigma \geq 2$  and  $\lambda \geq 0$ .

#### 4. SHARP WEIGHT FOR AT MOST QUADRATIC POTENTIALS

As noticed in [5], if  $V$  is at most quadratic and  $\nabla V$  is bounded, then Theorem 3.2 remains valid with  $\Sigma$  replaced by  $H^1(\mathbb{R}^d)$  (and no property involving  $xu$ ). Note that typically if  $V(x) = \langle x \rangle$ , this shows that the assumptions on the space decay of the initial datum are sharp neither in Theorem 3.1 nor in Theorem 3.2, when one wants to deal with an  $H^1$ -regularity. More generally, set

$$\tilde{\Sigma} = \{f \in H^1(\mathbb{R}^d) ; \|f\|_{\tilde{\Sigma}} := \|f\|_{H^1(\mathbb{R}^d)} + \|f\nabla V\|_{L^2(\mathbb{R}^d)} < \infty\}.$$

Since  $V$  is at most quadratic, we have  $\Sigma \subset \tilde{\Sigma}$ , and the inclusion is strict unless  $V$  is quadratic (and non-degenerate). Typically, when  $\nabla V \in L^\infty$ , we have  $\tilde{\Sigma} = H^1(\mathbb{R}^d)$ . When  $V \geq 0$ , we also have  $\sqrt{H} \subset \tilde{\Sigma}$ , from the following elementary result.

**Lemma 4.1.** *Let  $f \in C^2(\mathbb{R}; \mathbb{R})$  be such that  $f \geq 0$  and  $f''$  is bounded. Then*

$$f'(x)^2 \leq 2\|f''\|_{L^\infty} f(x), \quad \forall x \in \mathbb{R}.$$

*Proof.* Taylor's formula yields, for  $x, y \in \mathbb{R}$ ,

$$f(x+y) = f(x) + yf'(x) + y^2 \int_0^1 (1-\theta)f''(x+\theta y)d\theta \leq f(x) + yf'(x) + \frac{y^2}{2}\|f''\|_{L^\infty}.$$

Since by assumption  $f(x+y) \geq 0$ , the discriminant of  $f(x) + yf'(x) + \frac{y^2}{2}\|f''\|_{L^\infty}$ , seen as a polynomial in  $y$ , must be non-positive, hence the result.  $\square$

**Theorem 4.2.** *Let  $V$  be at most quadratic, and  $u_0 \in \tilde{\Sigma}$ .*

- *There exists a unique solution  $u \in C([-T, T]; \tilde{\Sigma})$  to (1.1), with initial datum  $u_0$ , such that*

$$u, u\nabla V, \nabla u \in L^{\frac{4\sigma+4}{d\sigma}}([-T, T]; L^{2\sigma+2}(\mathbb{R}^d)),$$

*for some  $T > 0$  depending on  $\|u_0\|_{\tilde{\Sigma}}$ .*

- *This solution is global in time,  $u \in C(\mathbb{R}; \tilde{\Sigma})$ ,  $u, u\nabla V, \nabla u \in L^{\frac{4\sigma+4}{d\sigma}}_{\text{loc}}(\mathbb{R}; L^{2\sigma+2}(\mathbb{R}^d))$ , in either of the following cases:*
  - $\sigma < 2/d$ ,
  - $\sigma \geq 2/d$  and  $\lambda \geq 0$ .

*Proof.* We sketch the main steps of the proof, which follow classical arguments. Duhamel's formula for (1.1) with initial datum  $u_0$  reads

$$u(t) = e^{-itH}u_0 - i\lambda \int_0^t e^{-i(t-s)H} (|u|^{2\sigma}u)(s)ds =: \Phi(u)(t).$$

Local existence stems from a fixed point argument in a ball of the space

$$X_T = \left\{ u \in C([-T, T]; \tilde{\Sigma}) ; u, u\nabla V, \nabla u \in L^{\frac{4\sigma+4}{d\sigma}}([-T, T]; L^{2\sigma+2}(\mathbb{R}^d)) \right\},$$

for  $T > 0$  sufficiently small. Since local in time Strichartz estimates are available for at most quadratic potentials, the only aspect which differs from the usual approach where  $V = 0$  is that  $\nabla$  does not commute with  $H$ , hence does not commute with  $e^{-itH}$  for  $t \neq 0$ . However, we have the following commutator formulas,

$$[i\partial_t - H, \nabla] = \nabla V, \quad [i\partial_t - H, \nabla V] = -\nabla^2 V \cdot \nabla - \frac{1}{2}\nabla \Delta V.$$

Since  $\nabla^2 V$  and  $\nabla \Delta V$  are bounded by assumption, we get a closed system of estimates. In terms of  $\Phi$ , we have:

$$\begin{aligned} \nabla \Phi(u)(t) &= e^{-itH} \nabla u_0 - i\lambda \int_0^t e^{-i(t-s)H} \nabla (|u|^{2\sigma}u)(s)ds \\ &\quad - i \int_0^t e^{-i(t-s)H} (\Phi(u)(s) \nabla V) ds, \\ \Phi(u)(t) \nabla V &= e^{-itH} (u_0 \nabla V) - i\lambda \int_0^t e^{-i(t-s)H} (|u|^{2\sigma}u)(s) \nabla V ds \\ &\quad + i \int_0^t e^{-i(t-s)H} (\nabla^2 V \cdot \nabla \Phi(u)(s)) ds + \frac{i}{2} \int_0^t e^{-i(t-s)H} (\Phi(u)(s) \nabla \Delta V) ds. \end{aligned}$$

We refer to [6, 7] for details on the fixed point argument.

Showing that the solution is global if the nonlinearity is not both focusing and mass critical or super-critical, follows from the following remark, which stems from the formal conservation of the energy, and justified by a regularizing argument:

$$\begin{aligned}
\frac{d}{dt} \left( \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} \right) &= -\frac{d}{dt} \int_{\mathbb{R}^d} V(x) |u(t, x)|^2 dx \\
&= -2 \operatorname{Re} \int_{\mathbb{R}^d} V(x) \bar{u}(t, x) \partial_t u(t, x) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^d} V(x) \bar{u}(t, x) \Delta u(t, x) dx \\
&= -\operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t, x) \nabla V(x) \cdot \nabla u(t, x) dx.
\end{aligned}$$

On the other hand, we compute

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d} |\nabla V(x)|^2 |u(t, x)|^2 dx &= -2 \operatorname{Re} \int_{\mathbb{R}^d} |\nabla V(x)|^2 \bar{u}(t, x) \partial_t u(t, x) dx \\
&= \operatorname{Im} \int_{\mathbb{R}^d} |\nabla V(x)|^2 \bar{u}(t, x) \Delta u(t, x) dx \\
&= -2 \operatorname{Im} \int_{\mathbb{R}^d} \bar{u}(t, x) \nabla^2 V(x) \nabla V(x) \cdot \nabla u(t, x) dx.
\end{aligned}$$

Let

$$\mathcal{E}_\lambda(t) = \frac{1}{2} \|\nabla u(t)\|_{L^2}^2 + \frac{\lambda}{\sigma+1} \|u(t)\|_{L^{2\sigma+2}}^{2\sigma+2} + \int_{\mathbb{R}^d} |\nabla V(x)|^2 |u(t, x)|^2 dx.$$

In view of the above computations, Cauchy-Schwarz and Young inequalities yield

$$\frac{d\mathcal{E}_\lambda}{dt} \leq (1 + 2\|\nabla^2 V\|_{L^\infty}) \|u(t)\nabla V\|_{L^2} \|\nabla u(t)\|_{L^2} \lesssim \mathcal{E}_0(t).$$

Global existence readily follows from the local theory when  $\lambda \geq 0$ . In the mass sub-critical focusing case, one can invoke either Gagliardo-Nirenberg inequality, or global existence at the  $L^2$ -level (see e.g. [7]).  $\square$

To conclude, we outline that if  $\nabla V$  is unbounded, then for (1.1) to possess an  $H^1$  local solution, one has to assume that  $u_0 \in \tilde{\Sigma}$ . This phenomenon is geometrical, in the sense that it is present in the linear case  $\lambda = 0$ . It remains in the nonlinear setting, in the same spirit as in [5]. We shall therefore present the linear result, and refer to [5] for the adaptation to the nonlinear framework of (1.1).

**Proposition 4.3.** *Let  $V$  be at most quadratic, with  $\nabla V \notin L^\infty(\mathbb{R}^d)$ . If  $u_0 \in H^2(\mathbb{R}^d) \setminus \tilde{\Sigma}$ , then for arbitrarily small time  $\tau > 0$ , the solution  $u \in C([0, \tau]; L^2(\mathbb{R}^d))$  to*

$$(4.1) \quad i\partial_t u = Hu \quad ; \quad u|_{t=0} = u_0,$$

satisfies  $\nabla u(\tau, \cdot) \notin L^2(\mathbb{R}^d)$ .

*Proof.* The phenomenon related to Proposition 4.3 is a rotation in phase space: the presence of  $V$  with an unbounded gradient causes the appearance of oscillations. To filter out these oscillations, introduce the eikonal equation

$$(4.2) \quad \partial_t \phi + \frac{1}{2} |\nabla \phi|^2 + V = 0 \quad ; \quad \phi|_{t=0} = 0.$$

It is classically solved locally in time thanks to the Hamiltonian flow

$$\dot{x}(t, y) = \xi(t, y), \quad \dot{\xi}(t, y) = -\nabla V(x(t, y)), \quad x(0, y) = y, \quad \xi(0, y) = 0.$$

For  $V$  at most quadratic, there exists  $T > 0$  such that the map  $y \mapsto x(t, y)$  is invertible for  $(t, y) \in [0, T] \times \mathbb{R}^d$ . Such a time  $T$  must be expected to be necessarily finite in general, due to the formation of caustics. Then (4.2) has a unique local smooth solution  $\phi \in C^\infty([0, T] \times \mathbb{R}^d)$ , which is at most quadratic in space,

$$\partial_x^\alpha \phi \in L^\infty([0, T] \times \mathbb{R}^d), \quad \forall \alpha \in \mathbb{N}^d, |\alpha| \geq 2.$$

Details can be found for instance in [4]. Introduce  $a$  given by  $u(t, x) = a(t, x)e^{i\phi(t, x)}$ . On  $[0, T] \times \mathbb{R}^d$ , (4.1) is equivalent to

$$(4.3) \quad \partial_t a + \nabla \phi \cdot \nabla a + \frac{1}{2}a\Delta\phi = \frac{i}{2}\Delta a \quad ; \quad a|_{t=0} = u_0.$$

This equation is a transport equation (left hand side), plus a skew-symmetric term (right hand side). Since  $\phi$  is at most quadratic in space and  $u_0 \in H^2(\mathbb{R}^d)$ , we check that (4.3) has a unique solution  $a \in C([0, T]; H^2(\mathbb{R}^d))$ . On the other hand, since  $\phi \in C^\infty([0, T] \times \mathbb{R}^d)$ , (4.2) implies

$$|\partial_t \nabla \phi + \nabla V| = |\nabla^2 \phi \cdot \nabla \phi| \lesssim |\nabla \phi| \lesssim |\nabla \phi + t\nabla V| + |t\nabla V|.$$

Gronwall lemma yields

$$|\nabla \phi(t, x) + t\nabla V(x)| \lesssim |\nabla V(x)| \int_0^t se^{Cs} ds \leq C_T t^2 |\nabla V(x)|, \quad t \in [0, T].$$

To conclude, we approximate  $a$  for short time. Introduce  $\tilde{a}$  solution to

$$(4.4) \quad \partial_t \tilde{a} + \nabla \phi \cdot \nabla \tilde{a} + \frac{1}{2}\tilde{a}\Delta\phi = 0 \quad ; \quad \tilde{a}|_{t=0} = u_0.$$

It is given explicitly by the expression (see e.g. [4])

$$\tilde{a}(t, x) = \frac{1}{\sqrt{J_t(y(t, x))}} u_0(y(t, x)),$$

where  $y(t, x)$  is the inverse map of  $y \mapsto x(t, y)$  (well-defined on  $[0, T] \times \mathbb{R}^d$ ), and  $J_t$  is the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t, y),$$

which is bounded away from zero and infinity on  $[0, T] \times \mathbb{R}^d$ . Subtracting (4.4) from (4.3), multiplying by the conjugate of  $a - \tilde{a}$  and integrating by parts, we get

$$\frac{d}{dt} \|a(t) - \tilde{a}(t)\|_{L^2}^2 = \operatorname{Im} \int_{\mathbb{R}^d} (a(t, x) - \tilde{a}(t, x)) \Delta \bar{a}(t, x) dx.$$

Cauchy-Schwarz inequality and Gronwall lemma yield (recall that  $a \in C([0, T]; H^2)$  since we have assumed  $u_0 \in H^2(\mathbb{R}^d)$ )

$$\|a(t) - \tilde{a}(t)\|_{L^2} \leq Ct \|a\|_{L^\infty([0, T]; H^2(\mathbb{R}^d))}.$$

Now

$$\nabla u(t, x) = e^{i\phi(t, x)} \nabla a(t, x) + e^{i\phi(t, x)} a(t, x) \nabla \phi(t, x),$$

with  $\nabla \phi(t, x) = t(1 + \mathcal{O}(t)) \nabla V(x)$  pointwise, and  $a = \tilde{a} + \mathcal{O}(t)$  in  $L^2(\mathbb{R}^d)$ , as  $t \rightarrow 0$ . Therefore, for arbitrarily small time  $\tau > 0$ ,  $\nabla u(\tau, \cdot) \notin L^2(\mathbb{R}^d)$ , hence the result.  $\square$

*Remark 4.4.* In the case  $V(x) = \langle x \rangle^k$ ,  $k > 2$ , (2.1) shows that  $\tilde{\Sigma} \subsetneq B^1 = \sqrt{H} \subsetneq \Sigma$ . In view of Proposition 2.1, and since the proof of Proposition 4.3 heavily relies on the fact that  $V$  is at most quadratic, this suggests that for super-quadratic potentials, the weakest possible weight in space at the  $H^1$  level of regularity corresponds to  $\sqrt{H}$ , that is, the property  $u_0 \sqrt{V} \in L^2(\mathbb{R}^d)$  (as in Theorems 3.3 and 3.4).



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