



HAL
open science

Compact CR-solvmanifolds as Kähler obstructions.

Bruce Gilligan, Karl Oeljeklaus

► **To cite this version:**

Bruce Gilligan, Karl Oeljeklaus. Compact CR-solvmanifolds as Kähler obstructions.. *Mathematische Zeitschrift*, 2011, 269 (1), pp.179-191. 10.1007/s00209-010-0721-6 . hal-01066269

HAL Id: hal-01066269

<https://hal.science/hal-01066269>

Submitted on 19 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

HOMOGENEOUS CR-SOLVMANIFOLDS AS KÄHLER OBSTRUCTIONS

BRUCE GILLIGAN AND KARL OELJEKLAUS

ABSTRACT. We give a precise characterization when a compact homogeneous CR-solvmanifold is CR-embeddable in a complex Kähler manifold. Equivalently this gives a non-Kähler criterion for complex manifolds containing CR-solvmanifolds not satisfying these conditions. This paper is the natural continuation of [OR] and [GOR].

1. INTRODUCTION

There are many results known about the structure of real solvmanifolds. One of these is the conjecture of Mostow [Mos], subsequently proved by L. Auslander, e.g., see [Aus] for the history, that every solvmanifold is a vector bundle over a compact solvmanifold. In the category of complex solvmanifolds one would additionally like to understand the structure of the manifold with respect to complex analytic objects defined on it and, particularly, the role played by the base of the vector bundle noted above - this base determines the topology, so does it also control the complex analysis? Because of the connection with the existence of plurisubharmonic functions and analytic hypersurfaces, one problem of this type concerns the existence of a Kähler metric.

In the fundamental paper [Lo], triples (Γ, G_0, G) are considered, where G is a connected simply-connected solvable complex Lie group, G_0 is a connected (simply-connected) *totally real* subgroup of G , and Γ is discrete and cocompact in G_0 . It is proven that the complex manifold $X := G/\Gamma$ admits a Kähler metric if and only if for every element $x \in \mathfrak{g}_0$ in the Lie algebra of G_0 the operator $\text{ad}x$ has *purely imaginary spectrum* on \mathfrak{g}_0 .

In [OR] the more general situation of triples (Γ, G_0, G) with Lie algebras \mathfrak{g}_0 and \mathfrak{g} like above and $\mathfrak{g}_0 + i\mathfrak{g}_0 = \mathfrak{g}$, but not necessarily totally real \mathfrak{g}_0 , is studied. Note that

$$\mathfrak{m} := \mathfrak{g}_0 \cap i\mathfrak{g}_0$$

Date: November 24, 2013.

This work was partially done during a visit of the first author at the LATP at the Université de Provence in June 2008 and of the second author at the University of Regina in October-November 2008. We would like to thank the respective institutions for their hospitality and financial support. This work was also partially supported by an NSERC Discovery Grant.

is an ideal in \mathfrak{g} and therefore also in \mathfrak{g}_0 . Amongst other things, it is shown in [OR] that G/Γ being Kähler implies that the adjoint action of \mathfrak{g}_0 on \mathfrak{m} has purely imaginary spectrum.

We underline that in both papers [Lo] and [OR] the Kähler assumption on the whole complex homogeneous manifold G/Γ is investigated and characterized.

In contrast to this condition we study in the present paper the weaker "locally Kähler" property. To be precise let $Y := G_0/\Gamma \subset G/\Gamma =: X$ be the natural inclusion of the Levi-flat compact homogeneous CR-solvmanifold Y in the complex-homogeneous solvmanifold X . We characterize all the CR-solvmanifolds Y which admit an embedding in some Kähler manifold. This is equivalent to assuming that there is an open subset $U \subset G/\Gamma$ which is Kähler. The main result that we prove is a characterization of this situation. It turns out that one of the main necessary conditions is that the restriction of the adjoint representation of \mathfrak{g}_0 to \mathfrak{m} has imaginary spectrum and is diagonalizable. Note in passing that because of the set up, the manifold Y is the base of the vector bundle in Mostow's conjecture.

In addition to recalling some details from [OR] that we need and proving the main result, we also present four examples to illustrate the theory, with two of these being locally Kähler and the other two not. The paper closes with a classification of locally Kähler compact CR-solvmanifolds of codimension one or two: a finite covering splits as a direct product of a Cousin group and some appropriate number of \mathbb{C}^* 's. This extends a result in [Ri] for the CR-hypersurface case, which is included due to the scope of the result and the method of its proof.

2. THE NILPOTENT CASE

*For the rest of the paper we call a triple (Γ, G_0, G) consisting of a connected simply-connected solvable complex Lie group G , a real connected (simply-connected) subgroup G_0 of G , and a discrete cocompact subgroup Γ of G_0 , a **CR-solvmanifold**, or **CRS** for short. We also assume throughout that G_0/Γ is a generic CR-manifold of the complex manifold G/Γ , i.e. for the Lie algebras \mathfrak{g}_0 and \mathfrak{g} of G_0 and G one has $\mathfrak{g}_0 + i\mathfrak{g}_0 = \mathfrak{g}$.*

If the CRS (Γ, G_0, G) is *locally Kähler*, i.e., if Y admits an open Kähler neighbourhood in X , which one may assume to be right- G_0 -invariant with right- G_0 -invariant Kähler form ω (this is obviously equivalent to saying that Y is CR-embeddable in some Kähler manifold), then \mathfrak{m} is an abelian ideal in \mathfrak{g} and in \mathfrak{g}_0 (see [OR], Corollary 7, p. 406). Let $N, N_0, \mathfrak{n}, \mathfrak{n}_0$ denote the associated nilradical objects. It is clear that $\mathfrak{m} \subset \mathfrak{n}$. A result of Mostow

([Mos]) assures the existence of the so-called *nilradical fibration* which we recall in the next lemma.

Lemma 2.1. *The N_0 -orbits (respectively the N -orbits) in Y (respectively in X) are closed and there is the following commutative diagram of fiber bundles*

$$\begin{array}{ccc} Y = G_0/\Gamma & \hookrightarrow & X = G/\Gamma \\ \downarrow & & \downarrow \\ G_0/N_0 \cdot \Gamma & \hookrightarrow & G/N \cdot \Gamma \end{array}$$

the right vertical arrow being holomorphic.

It will be convenient if we now recall some results and remarks from section 3 in [OR] that we need later. Since it is not always true that every subalgebra of a solvable Lie algebra has a complementary subalgebra, e.g., the center of a Heisenberg algebra has no complement, we first note the following in the locally Kähler CRS setting. In passing, we recall that Malcev [Mal] showed that every Lie algebra over \mathbb{C} admits a faithful representation into a splittable Lie algebra; see [Reed] for another approach.

Proposition 2.2. *Let (Γ, G_0, G) be a locally Kähler CRS. Then there exists a complementary subalgebra \mathfrak{a}_0 to \mathfrak{m} in \mathfrak{g}_0 , i.e., one has*

$$\mathfrak{g}_0 = \mathfrak{a}_0 \times \mathfrak{m}.$$

Proof. Let

$$\mathfrak{a}_0 = \{X \in \mathfrak{g}_0 \mid \omega(X, \mathfrak{m}) = 0\}.$$

Then \mathfrak{a}_0 is a real subalgebra of \mathfrak{g}_0 and we have $\mathfrak{g}_0 = \mathfrak{a}_0 \times \mathfrak{m}$. To see this consider the hermitian metric h defined by ω and let $(v_1, \dots, v_{2k}, w_1, \dots, w_l)$ be an orthonormal basis of \mathfrak{g}_0 , where (v_1, \dots, v_{2k}) is an orthonormal basis of \mathfrak{m} , such that $v_{j+k} = Jv_j$ for $1 \leq j \leq k$, and J denotes the complex structure.

Using the relation $h(v, w) = \omega(Jv, w)$, it is easy to see that \mathfrak{a}_0 is just the real span of (w_1, \dots, w_l) . Thus $\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m}$ as a vector space. The fact that \mathfrak{a}_0 is a Lie algebra follows from the formula

$$(\diamond) \quad 0 = \omega(X, [Y, Z]) + \omega(Z, [X, Y]) + \omega(Y, [Z, X]),$$

for all $X, Y, Z \in \mathfrak{g}_0$, using the fact that \mathfrak{m} is an ideal in \mathfrak{g}_0 . Note that $[\mathfrak{g}, \mathfrak{m}] = [\mathfrak{g}_0, \mathfrak{m}]$. Furthermore \mathfrak{m} is abelian, so we get $[\mathfrak{g}, \mathfrak{m}] = [\mathfrak{a}_0, \mathfrak{m}]$. \square

Theorem 2.3 (Theorem 2', [OR]). *Let (Γ, G_0, G) be a locally Kähler CRS. Let \mathfrak{z} be the center of \mathfrak{g} and ω a right G_0 -invariant Kähler form in a right G_0 -invariant open neighbourhood of G_0 in G . Then*

$$\mathfrak{z} \cap \mathfrak{m} = \{X \in \mathfrak{m} \mid \omega(X, [\mathfrak{g}, \mathfrak{m}]) = 0\}.$$

In particular, if $\mathfrak{z} = 0$, then $[\mathfrak{g}, \mathfrak{m}] = \mathfrak{m}$.

Proof.

By (\diamond) , it follows now that

$$\mathfrak{z} \cap \mathfrak{m} \subset \{X \in \mathfrak{m} \mid \omega(X, [\mathfrak{g}, \mathfrak{m}]) = 0\}.$$

To verify the opposite inclusion it is enough to show that for $X_0 \in \{X \in \mathfrak{m} \mid \omega(X, [\mathfrak{g}, \mathfrak{m}]) = 0\}$ one has $[X_0, \mathfrak{a}_0] = 0$. For this let $Y \in \mathfrak{a}_0$ and $Z \in \mathfrak{m}$. By (\diamond) we have

$$0 = \omega(X_0, [Y, Z]) + \omega(Z, [X_0, Y]) + \omega(Y, [Z, X_0]).$$

Since $[Y, Z] \in [\mathfrak{g}, \mathfrak{m}]$ and $[Z, X_0] \in \mathfrak{m}$, it follows that

$$\omega(Z, [X_0, Y]) = 0$$

for all $Z \in \mathfrak{m}$, $Y \in \mathfrak{a}$. But $X_0 \in \mathfrak{m}$ implies that $[X_0, Y] \in \mathfrak{m}$. Since \mathfrak{m} is a complex subspace of \mathfrak{g} , we know that $\omega|_{\mathfrak{m}}$ is nondegenerate. Therefore we have $[X_0, Y] = 0$. Since the last remark is clear, the theorem is proved. \square

Corollary 2.4. *Let (Γ, G_0, G) be a locally Kähler CRS where G is a nilpotent complex Lie group. Then $\mathfrak{m} \subset \mathfrak{z}$, $\mathfrak{m} \cap \mathfrak{g}' = \{0\}$ and $\mathfrak{z} = \mathfrak{z}_0 + i\mathfrak{z}_0$, where \mathfrak{z}_0 is the center of \mathfrak{g}_0 .*

Proof. Firstly, it is clear that $[\mathfrak{m}, \mathfrak{g}]$ is an ideal in \mathfrak{g} . The preceding theorem gives an ideal splitting

$$\mathfrak{m} = (\mathfrak{m} \cap \mathfrak{z}) \oplus [\mathfrak{m}, \mathfrak{g}].$$

Consequently $[\mathfrak{g}, [\mathfrak{m}, \mathfrak{g}]] = [\mathfrak{m}, \mathfrak{g}]$. For nilpotent \mathfrak{g} this is only possible if $[\mathfrak{m}, \mathfrak{g}] = 0$, i.e. $\mathfrak{m} \subset \mathfrak{z}$.

This in turn implies that $\mathfrak{g}'_0 = [\mathfrak{g}_0, \mathfrak{g}_0] = [\mathfrak{a}_0, \mathfrak{a}_0] \subset \mathfrak{a}_0$. Therefore $\mathfrak{g}'_0 \cap \mathfrak{m} = \{0\}$, $\mathfrak{g}'_0 \cap i\mathfrak{g}'_0 = 0$ and $\mathfrak{g}' \cap \mathfrak{m} = 0$.

For the last assertion we first remark that \mathfrak{g}_0 being a generic subalgebra of \mathfrak{g} , the center \mathfrak{z}_0 of \mathfrak{g}_0 is given by $\mathfrak{z}_0 = \mathfrak{z} \cap \mathfrak{g}_0$. Hence $\mathfrak{z}_0 + i\mathfrak{z}_0 \subset \mathfrak{z}$.

Furthermore take $Z = X + iY \in \mathfrak{z}$, $X, Y \in \mathfrak{g}_0$. For an arbitrary $X' \in \mathfrak{g}_0$ we get

$$0 = [Z, X'] = [X, X'] + i[Y, X'],$$

which shows that $[X, X'], [Y, X'] \in \mathfrak{g}'_0 \cap i\mathfrak{g}'_0 = 0$. So $X, Y \in \mathfrak{z}_0$ and $\mathfrak{z} \subset \mathfrak{z}_0 + i\mathfrak{z}_0$. \square

Corollary 2.5. *Let (Γ, G_0, G) be a locally Kähler CRS where G is a nilpotent complex Lie group and suppose that there are only constant holomorphic functions on the homogeneous manifold $X = G/\Gamma$. Then G is an abelian complex Lie group.*

Proof. By hypothesis, $\mathcal{O}(G/\Gamma) = \mathbb{C}$. By a result of [BaOt], we have an equivariant holomorphic bundle

$$G/\Gamma \rightarrow G/Z\Gamma,$$

Z being the center of G . Since the complex Lie group M corresponding to the Lie algebra \mathfrak{m} is contained in Z by the previous corollary, the homogeneous manifold $G/Z\Gamma$ is Stein, see [GH1], hence a point and $G = Z$ is abelian. \square

Corollary 2. 6. *Let (Γ, G_0, G) be a CRS where G is a nilpotent complex Lie group. Then the following three conditions are equivalent:*

- 1) (Γ, G_0, G) is a locally Kähler CRS.
- 2) (Γ, G_0, G) is a Kähler CRS.
- 3) There is a Lie algebra splitting

$$\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m},$$

where \mathfrak{m} is abelian and $\mathfrak{a}_0 \cap i\mathfrak{a}_0 = 0$.

Proof. 2) \Rightarrow 1) is evident.

1) \Rightarrow 3): In the proof of theorem 2.3 we have constructed the splitting and corollary 2.4 gives the additional properties.

3) \Rightarrow 2): Assume that there is a splitting

$$\mathfrak{g}_0 = \mathfrak{a}_0 \oplus \mathfrak{m},$$

with \mathfrak{m} abelian and $\mathfrak{a}_0 \cap i\mathfrak{a}_0 = 0$. Let $\mathfrak{a} := \mathfrak{a}_0 \oplus i\mathfrak{a}_0$ and A_0, A the corresponding Lie groups. Then we have

$$G_0 = A_0 \times M, \quad G = A \times M.$$

Moreover there is a right A_0 -invariant Kähler form on A , see [Lo]. Since M is abelian, this yields a right G_0 -invariant Kähler form on G and hence a Kähler form on G/Γ . \square

3. THE CHARACTERIZATION

Before coming to the main theorem, we first need three lemmas.

Lemma 3. 7. *Let $\beta \in \mathbb{C}$ and $G := \mathbb{C} \times \mathbb{C}$ the complex Lie group with multiplication*

$$(z_0, w_0) \cdot (z, w) := (z_0 + z, e^{\beta z_0} w + w_0).$$

Let further $G_0 := \mathbb{R} \times \mathbb{C} \subset G$ be the subgroup of G with $z \in \mathbb{R}$. Let U be any connected left G_0 -invariant open neighbourhood of G_0 . Then there is a left G_0 -invariant Kähler form on U if and only if $\Re \beta = 0$. The same conclusion holds if we replace the left-invariant objects by right-invariant ones.

Proof. We explicitly write the left G_0 -action on U :

$$(t, w_0) \cdot (z, w) := (t + z, e^{\beta t} w + w_0)$$

and let H denote the complex Lie subgroup $0 \times \mathbb{C} \subset G_0 \subset G$.

Let

$$\omega = f_1 dz \wedge d\bar{z} + f_2 dz \wedge d\bar{w} + f_3 d\bar{z} \wedge dw + f_4 dw \wedge d\bar{w}$$

be a left G_0 -invariant Kähler form on U .

For $g_0 = (t, w_0) \in G_0$ we get, using the left H -invariance of ω ,

$$\begin{aligned} g_0^*(\omega) &= f_1(t+z) dz \wedge d\bar{z} + e^{t\bar{\beta}} f_2(t+z) dz \wedge d\bar{w} + \\ &+ e^{t\beta} f_3(t+z) d\bar{z} \wedge dw + e^{t(\beta+\bar{\beta})} f_4(t+z) dw \wedge d\bar{w} = \omega \end{aligned}$$

for all $g_0 \in G_0$. But $d\omega = 0$ now forces f_4 to be constant which is the case if and only if $\Re \beta = 0$.

If $\Re \beta = 0$, one can take $\omega = dz \wedge d\bar{z} + dw \wedge d\bar{w}$.

The composition with the inverse map in G exchanges left and right-invariant object and the last assertion is therefore evident. \square

Lemma 3. 8. *Let \mathfrak{s}_0 be a real abelian Lie algebra, $\mathfrak{s} = \mathfrak{s}_0 \oplus i\mathfrak{s}_0$ and \mathfrak{t} a complex abelian Lie algebra. Suppose that $\mathfrak{g}_0 = \mathfrak{s}_0 \ltimes \mathfrak{t} \subset \mathfrak{g} = \mathfrak{s} \ltimes \mathfrak{t}$ is a semi-direct product such that the adjoint action of \mathfrak{s}_0 on \mathfrak{t} has purely imaginary spectrum and is diagonalisable. Then there is a right G_0 -invariant Kähler form on G .*

Proof. By hypothesis, the group multiplication in the associated simply-connected complex Lie group G is given in suitable coordinates by

$$((s_1, \dots, s_n), (t_1, \dots, t_m)) \circ ((s_1^0, \dots, s_n^0), (t_1^0, \dots, t_m^0)) =$$

$$= ((s_1 + s_1^0, \dots, s_n + s_n^0), (e^{i \sum_{j=1}^n \alpha_{j1} s_j} t_1^0 + t_1, \dots, e^{i \sum_{j=1}^n \alpha_{jm} s_j} t_m^0 + t_m)),$$

with real constants α_{jk} . The Kähler form $\omega := \sum ds_j^0 \wedge d\bar{s}_j^0 + \sum dt_k^0 \wedge d\bar{t}_k^0$ on G is then obviously left G_0 -invariant. By composing with the inverse map, one produces in a standard way a right G_0 -invariant Kähler form on G . \square

Lemma 3. 9. *Suppose (Γ, G_0, G) is a totally real CRS, i.e. $\mathfrak{g}_0 \cap i\mathfrak{g}_0 = \mathfrak{m} = \{0\}$. Then (Γ, G_0, G) is a locally Kähler CRS. Furthermore, for any connected complex Lie subgroup $H \subset G$ set $H_0 := G_0 \cap H$. Then there is a right H_0 -invariant neighborhood U of H_0 in G and a right H_0 -invariant Kähler form on U .*

Proof. Since G_0/Γ is a totally real, real analytic submanifold of G/Γ , there exists an open neighbourhood U of G_0/Γ in G/Γ that is Stein [Gr]. By shrinking and renaming, if necessary, we may assume that U is right G_0 -invariant. It is clear that a Kähler form exists on U . Since G_0/Γ is compact, we may average the Kähler form, as in the proof of Proposition 1 (see p. 166) in [GOR], and so obtain a form that is right G_0 -invariant. The remaining statements follow from this. \square

Definition 3. 10. *We call a pair (H_0, H) as in the preceding lemma a locally Kähler pair.*

In this section we shall prove the following

Theorem 3. 11. *Let (Γ, G_0, G) be a CRS. Then the following statements are equivalent:*

- 1) (Γ, G_0, G) is a locally Kähler CRS.
- 2) (Γ, G_0, G) satisfies the conditions
 - i) $(\Gamma \cap N_0, N_0, N)$ is a locally Kähler CRS
 - ii) there is a Lie algebra splitting $\mathfrak{g}_0 = \mathfrak{a}_0 \ltimes \mathfrak{m}$
 - iii) the adjoint representation $ad(\mathfrak{g}_0)|_{\mathfrak{m}}$ of \mathfrak{g}_0 on \mathfrak{m} is diagonalisable and has purely imaginary spectrum.

Proof. 1) \Rightarrow 2):

Suppose that (Γ, G_0, G) is a locally Kähler CRS. By restriction it follows immediately that $(\Gamma \cap N_0, N_0, N)$ is a locally Kähler CRS, i.e. 2) i). Condition 2) ii) is also necessary as we have seen in proposition 4.2. By lemma 4.7 we get the purely imaginary spectrum of $ad(\mathfrak{g}_0)|_{\mathfrak{m}}$ and by the methods of [GOR], p.p. 166-167 one can prove the diagonalizability in condition 2) iii), as we now point out. Since the Kähler form ω restricted to \mathfrak{g}_0 may be assumed to be right invariant and $d\omega = 0$, one has

$$\omega(X, [Y, Z]) + \omega(Z, [X, Y]) + \omega(Y, [Z, X]) = 0$$

for all $X, Y, Z \in \mathfrak{g}_0$. Now suppose v is an eigenvector for the adjoint action on \mathfrak{m} and set $v^\perp := \{w \in \mathfrak{m} \mid \omega(w, \langle v \rangle) = 0\}$. Then for $w \in v^\perp$ and $X \in \mathfrak{g}_0$ one has

$$\omega(X, [w, v]) + \omega(v, [X, w]) + \omega(w, [v, X]) = 0.$$

Now \mathfrak{m} is abelian, so $[w, v] = 0$. Also $[v, X] = \mu v$ for some $\mu \in \mathbb{C}$, so $\omega(w, [v, X]) = \mu\omega(w, v) = 0$. It follows that $[X, w] \in v^\perp$. Hence, by induction, the adjoint action of \mathfrak{g}_0 restricted to \mathfrak{m} is diagonalizable.

2) \Rightarrow 1):

Now assume the three conditions in 2). We define $\mathfrak{h}_0 := \mathfrak{a}_0 \cap \mathfrak{n}$ and $\mathfrak{h} := \mathfrak{a} \cap \mathfrak{n}$. Then $\mathfrak{h} = \mathfrak{h}_0 \oplus i\mathfrak{h}_0$ as a vector space. Furthermore $\mathfrak{n}_0 = \mathfrak{h}_0 \oplus \mathfrak{m}$ and $\mathfrak{n} = \mathfrak{h} \oplus \mathfrak{m}$ as Lie algebras in view of condition 2) i) and Corollary 2.4, which also gives that $\mathfrak{h}_0 \triangleleft \mathfrak{g}_0$ and $\mathfrak{h} \triangleleft \mathfrak{g}$. Let $H_0 \triangleleft G_0$ and $H \triangleleft G$ denote the subgroups of G corresponding to \mathfrak{h}_0 and \mathfrak{h} and $\pi_1 : G \rightarrow G/H$ the projection. We then have that

$$L_0 := G_0/H_0 \simeq A_0/H_0 \times M =: B_0 \times M$$

and

$$L := G/H \simeq A/H \times M =: B \times M$$

are Lie groups. The new Lie algebras are denoted by the corresponding fracture letters. One has $\mathfrak{l}_0 \cap i\mathfrak{l}_0 = \mathfrak{m}$, $\mathfrak{l}_0 + i\mathfrak{l}_0 = \mathfrak{l}$ and condition 2) iii) now gives that $ad(\mathfrak{b}_0)|_{\mathfrak{m}}$ has purely imaginary spectrum and is diagonalizable. So by lemma 3.8 there is a right L_0 -invariant Kähler form ω_1 on L whose pullback to G will also be denoted by ω_1 and which is right G_0 -invariant.

Define now $\tilde{\mathfrak{g}} := \mathfrak{g}_0 \oplus i\mathfrak{g}_0$ the “formal” complexification of \mathfrak{g}_0 and consider the associated totally real CRS (Γ, G_0, \tilde{G}) which is locally Kähler by lemma 3.9. This lemma shows furthermore that the pair (A_0, A) is locally Kähler. Let ω_2 denote the local right A_0 -invariant Kähler form on A and remark that dividing the pair (G_0, G) by the complex Lie group M gives exactly the pair (A_0, A) . Pulling back ω_2 to G we again get a right G_0 -invariant form.

Finally the sum $\omega := \omega_1 + \omega_2$ is non-degenerate and therefore is a right G_0 -invariant Kähler form on a right G_0 -invariant open neighborhood of G_0 in G . This proves that (Γ, G_0, G) is a locally Kähler CRS. \square

4. EXAMPLES

We now present four examples of CRS's (Γ, G_0, G) that illustrate various aspects of the general theory. The second and third of these are locally Kähler, while the other two are not.

4.1. Non-imaginary Spectrum. This example fibers as a 2-dimensional Cousin group bundle over \mathbb{C}^* . The critical point is the existence of an eigenvalue that is not purely imaginary. This example is modeled on a construction of Inoue surfaces, for example see [BPVdeV].

We choose a matrix $A \in SL_3(\mathbb{Z})$ that has one real eigenvalue $\alpha > 1$ and two nonreal complex conjugate ones β and $\bar{\beta}$. To be precise one may pick

$$A = \begin{pmatrix} 0 & 1 & 0 \\ k & 0 & 1 \\ 1 & 1-k & 0 \end{pmatrix}$$

where k is an integer. Further, we let (a_1, a_2, a_3) be a real eigenvector corresponding to α and (b_1, b_2, b_3) an eigenvector corresponding to β .

Consider the group $G := \mathbb{C} \times \mathbb{C}^2$ with group structure given by

$$(t, x, y) \cdot (t_0, x_0, y_0) := (t + t_0, x + e^{t \log \alpha} x_0, y + e^{t \log \beta} y_0),$$

where \log is the principal branch of the logarithm on $\mathbb{C} \setminus \mathbb{R}^+$. The discrete subgroup Γ is generated by the following elements:

$$\Gamma := \langle (t, 0, 0), (0, a_1, b_1), (0, a_2, b_2), (0, a_3, b_3) \rangle_{\mathbb{Z}}$$

where $t \in \mathbb{Z}$. Clearly, $G' = \{(0, x, y) | x, y \in \mathbb{C}\}$, has closed orbits in G/Γ , and the fibration $G/\Gamma \rightarrow G/G' \cdot \Gamma$ is a Cousin group bundle over \mathbb{C}^* . Note that for any choice of the branch of the logarithm, $\log \beta$ is purely imaginary if and only if $|\beta| = 1$. However, this is not possible, because $\alpha\beta\bar{\beta} = \det A = 1$ and $\alpha > 1$. The homogeneous space G/Γ is not Kähler, and Lemma 4.7 shows that this example is not locally Kähler either.

4.2. A 5-dimensional locally Kähler example. Let H_3 be the 3-dimensional Heisenberg group and let the complex Lie group $GL(2, \mathbb{C})$ act as a group of holomorphic transformations on H_3 , where for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C})$

the action is given by

$$\begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{F_A} \begin{pmatrix} 1 & ax + by & (\det A)[z - xy/2] + (ax + by)(cx + dy)/2 \\ 0 & 1 & cx + dy \\ 0 & 0 & 1 \end{pmatrix}$$

A direct calculation shows that the map $A \mapsto F_A$ is a group homomorphism.

The subgroup

$$\Lambda := \left\{ \left(\begin{array}{ccc} 1 & n & k/2 \\ 0 & 1 & m \\ 0 & 0 & 1 \end{array} \right) \mid n, m, k \in \mathbb{Z} \right\}$$

is discrete in H_3 and note that for $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ one has $F_A(\Lambda) = \Lambda$. We now let

$$B = \begin{pmatrix} 1 & (\sqrt{5}-1)/2 \\ -1 & (\sqrt{5}+1)/2 \end{pmatrix}$$

Then $F_B(\Lambda)$ is a discrete subgroup of the Heisenberg group.

Next consider the representation from \mathbb{C} into the group of automorphisms of H_3 , where $t \in \mathbb{C}$ maps to conjugation by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Explicitly, this conjugation is as follows

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}$$

Now let $K = \mathbb{R}$ or $K = \mathbb{C}$ and $(t, x, y, z) \in K^4$. Define a group structure G_K on K^4 by

$$(t, x, y, z) \circ (t', x', y', z') := (t + t', x + e^t x', y + e^{-t} y', z + z' + xy' e^{-t}).$$

Let $\lambda := \frac{1}{2}(3 + \sqrt{5})$ and $\alpha := \ln \lambda$. Let Γ be the subgroup of $G_{\mathbb{R}}$ generated by the elements

$$(\alpha, 0, 0, 0); (0, 1, -1, -\frac{1}{2}); (0, \frac{1}{2}(\sqrt{5}-1), \frac{1}{2}(\sqrt{5}+1), \frac{1}{2}); (0, 0, 0, \sqrt{5}).$$

The remarks above imply that Γ is discrete. Further Γ is a cocompact subgroup of $G_{\mathbb{R}}$.

Finally, define the 5-dimensional solvable complex Lie group $G := G_{\mathbb{C}} \times \mathbb{C}$, where $G_{\mathbb{C}}$ is embedded as the first component. We add to the embedded Γ the following two generators

$$(0, 0, 0, 0, 1) \quad \text{and} \quad (0, 0, 0, \sqrt{2}, i\sqrt{3})$$

and this gives a Cousin group structure in the last two coordinates. The subgroup G_0 is defined to be the real span of the discrete subgroup.

One should note that in this example the complex ideal \mathfrak{m} is contained in the product of the center of the Heisenberg group and the trivial \mathbb{C} -factor.

Therefore the adjoint action of \mathfrak{g}_0 restricted to \mathfrak{m} is trivial. As a consequence of theorem 5.11 we see that (Γ, G_0, G) is a locally Kähler CRS.

4.3. A 4-dimensional locally Kähler example. In the previous example the adjoint action restricted to \mathfrak{m} is trivial. We would next like to present an example where this action is purely imaginary, but in a nontrivial way, i.e., the action has a nonzero eigenvalue.

Let $p(x) := x^4 - 3x^3 + 3x^2 - 3x + 1$ (one can also take $x^4 - x^3 - x^2 - x + 1$, by the way). Then p is \mathbb{Q} -irreducible and the four roots of p are $\alpha > 1$, $0 < \alpha^{-1} < 1$, $\alpha \in \mathbb{R}$, $\beta = e^{is}$ and $\bar{\beta}$ with $|\beta| = 1$, i.e. β is on the unit circle and is multiplicatively of infinite order, i.e. the group $\{\beta^k \mid k \in \mathbb{Z}\}$ is isomorphic to \mathbb{Z} . In fact, α is a unit in the ring of algebraic integers. The number α is, by definition, a *Salem number*.

Now let $K := \mathbb{Q}[\alpha]$ be the number field of degree 4 and $O_K \simeq \mathbb{Z}^4$ the ring of integers of K . Let $\sigma_1, \sigma_2 : K \rightarrow \mathbb{R}$ be the real imbeddings and $\sigma_3, \sigma_4 = \bar{\sigma}_3 : K \rightarrow \mathbb{C}$ the complex, non-real imbeddings of K . Define $\sigma : K \rightarrow \mathbb{C}^3$ by

$$\sigma(k) := (\sigma_1(k), \sigma_2(k), \sigma_3(k)).$$

It is easy to check that $\Lambda := \sigma(O_K) \simeq \mathbb{Z}^4$ is discrete in \mathbb{C}^3 and that the quotient \mathbb{C}^3/Λ is a Cousin group; see [OT]. Let also

$$D := \begin{pmatrix} \ln \alpha & 0 & 0 \\ 0 & -\ln \alpha & 0 \\ 0 & 0 & is \end{pmatrix}.$$

Define now a solvable G group structure on $\mathbb{C} \times \mathbb{C}^3$ by

$$(z, b)(z_0, b_0) := (z + z_0, e^{zD}(b_0) + b).$$

Let $V_{\mathbb{R}}$ be the real span of Λ in \mathbb{C}^3 and $G_0 := \mathbb{R} \times V_{\mathbb{R}}$ be the corresponding real subgroup of G . Then $\Gamma := \mathbb{Z} \times \Lambda$ is a discrete cocompact subgroup of G_0 and (Γ, G_0, G) is a CRS.

Further note that the subgroup $M := V_{\mathbb{R}} \cap iV_{\mathbb{R}}$ has dense orbits in $V_{\mathbb{R}}/\Lambda \simeq (S^1)^4$ and that the Lie algebra \mathfrak{g}_0 has purely imaginary spectrum on the Lie algebra \mathfrak{m} of the group M . In fact, in our realisation $M = 0 \times 0 \times \mathbb{C} \subset \mathbb{C}^3$! Since \mathfrak{g}_0 does NOT have a purely imaginary spectrum on itself, (Γ, G_0, G) is NOT a Kähler CRS. But the following holds.

Lemma 4.12. (Γ, G_0, G) is a locally Kähler CRS.

Proof. All conditions in 2) of theorem 5.11 are satisfied, therefore (Γ, G_0, G) is a locally Kähler CRS. \square

4.4. Non-diagonalizable Example. We now show by an example that the diagonalizability assumption is needed.

Define the real 7-dimensional solvable Lie group G_0 as the semi-direct product of $(\mathbb{R}, +)$ and $(\mathbb{C}^3, +)$ with multiplication

$$(t, z, \begin{pmatrix} a \\ b \end{pmatrix}) \circ (t_0, z_0, \begin{pmatrix} a_0 \\ b_0 \end{pmatrix}) := (t + t_0, e^{2\pi it} z_0 + z, e^{tD} \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} + \begin{pmatrix} a \\ b \end{pmatrix}),$$

where $D := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Let G be the complex Lie group given by the corresponding semi-direct product of $(\mathbb{C}, +)$ and $(\mathbb{C}^3, +)$. Now let Γ be the discrete subgroup of G_0 given by the semi-direct product of $(\mathbb{Z}, +)$ and $((\mathbb{Z} + i\mathbb{Z})^3, +)$. Then it is easy to see that Γ is cocompact in G_0 . In this example property 2) of Theorem 5.11 is satisfied except for diagonalizability. Furthermore example 6b), p.414 in [OR] shows that Y is not locally Kähler in X .

5. A REMARK ON HOLOMORPHIC REDUCTIONS

We make some simple observations in this short section about holomorphic reductions in the locally Kähler setting. In [OR] it is shown that the fiber of the holomorphic reduction of a Kähler CRS is a Cousin group. This is no longer the case, in general, for the globalization of locally Kähler CR-solvmanifolds, but we can extend Corollary 3.5 in the following way.

Proposition 5.13. *Let (Γ, G_0, G) be a locally Kähler CRS. Then $\mathcal{O}(G/\Gamma) = \mathbb{C}$ if and only if G/Γ is a Cousin group.*

Proof. By lemma 2.1, we may consider the nilradical fibration $G/\Gamma \rightarrow G/N \cdot \Gamma$ of the complex solvmanifold $X = G/\Gamma$. Its base $G/N \cdot \Gamma$ is a Stein abelian Lie group, because $G' \subset N$ and $\mathfrak{m} \subset \mathfrak{n}$. Thus $G = N$ is nilpotent whenever $\mathcal{O}(G/\Gamma) = \mathbb{C}$. Since locally Kähler and Kähler are equivalent in the nilpotent setting, see Corollary 2.6, one direction follows from Corollary 2.5. The other direction is evident. \square

Remark: There are always a finite number of fibrations given by successive holomorphic reductions. However, we would now like to note what happens in the locally Kähler setting. Suppose G_0/H_0 is a locally Kähler CRS with globalization G/H . Let

$$(1) \quad G/H \longrightarrow G/J_1$$

be its holomorphic reduction. Denote by $G_N(H^\circ)$ the normalizer in G of the connected component of the identity H° of H . The base $G/N_G(H^\circ)$ of the normalizer fibration of G/H is biholomorphic to $(\mathbb{C}^*)^k$ for some k , see [GH2]. This implies that J_1 is a subgroup of $G_N(H^\circ)$, the complex ideal $\mathfrak{m} := \mathfrak{g}_0 \cap i\mathfrak{g}_0$ is contained in the Lie algebra of J_1 , and J_1/H is parallelizable.

Now set $\widehat{J}_1 := J/H^\circ$ and $\Gamma := H/H^\circ$ and let $\widehat{\mathfrak{m}} := \mathfrak{m}/\mathfrak{h} \cap \mathfrak{m}$ be the quotient of \mathfrak{m} by the Lie algebra \mathfrak{h} of H° intersected with \mathfrak{m} . Let

$$(2) \quad \widehat{J}_1/\Gamma \longrightarrow \widehat{J}_1/\widehat{J}_2$$

be the holomorphic reduction of the fiber of the fibration given in (1). As observed in the proof of the previous Proposition, the base $\widehat{J}_1/\Gamma \cdot N_{\widehat{J}_1}$ of the nilradical fibration of \widehat{J}_1/Γ is a Stein abelian Lie group isomorphic to $(\mathbb{C}^*)^l$ for some l . This implies that \widehat{J}_2 is a subgroup of $\Gamma \cdot N_{\widehat{J}_1}$ and that the complex ideal $\widehat{\mathfrak{m}}$ is contained in the Lie algebra of the group \widehat{J}_2 . Since the base $\widehat{J}_1/\widehat{J}_2$ is a holomorphically separable solvmanifold, it is Stein, see [HO]. Thus the fiber \widehat{J}_2/Γ of the fibration in (2) is connected by a theorem of K. Stein. Now the nilpotent group \widehat{J}_2° acts transitively on the connected fiber of (2). By Corollary 2.6 the space \widehat{J}_2/Γ is Kähler. Let $\widehat{J}_2/\Gamma \rightarrow \widehat{J}_2/\widehat{J}_3$ be its holomorphic reduction. Its fiber \widehat{J}_3/Γ is a Cousin group by [OR]. Finally we have the following tower of fibrations

$$(3) \quad G/H \xrightarrow{J_3/H} G/J_3 \longrightarrow G/J_2 \longrightarrow G/J_1,$$

because of the fact that holomorphic reductions are independent of the particular group that is acting transitively on the space.

Caveat: The space G/J_3 is not necessarily holomorphically separable, i.e., the fibration $G/H \rightarrow G/J_3$ need not be the holomorphic reduction of G/H . This is illustrated by the Coeuré-Loeb example, [CL], where the holomorphic reduction has $\mathbb{C}^* \times \mathbb{C}^*$ as fiber; i.e., one has the following

$$G/H = G/J_3 \xrightarrow{\mathbb{C}^* \times \mathbb{C}^*} G/J_2 = G/J_1 = \mathbb{C}^*.$$

Remark: In Example 4.3 involving the Salem numbers, one should note that because of the nature of the adjoint action of the Lie algebra \mathfrak{g}_0 on \mathfrak{m} , it follows that neither the first bundle given in (3), namely $G/H \rightarrow G/J_3$, nor any finite covering of it can have the structure of a **principal** Cousin bundle. Thus the locally Kähler case is in sharp contrast with the Kähler case; in [GO] we showed that every Kähler solvmanifold has a finite covering whose holomorphic reduction is a principal Cousin bundle.

6. CLASSIFICATION IN LOW CODIMENSIONS

Some of the first results concerning compact, locally Kähler, homogeneous hypersurfaces appear in the dissertation of W. Richthofer [Ri]. Using the tools at hand we now present a classification of locally Kähler CRS (Γ, G_0, G) in codimensions one and two.

Theorem 6.14. *Suppose (Γ, G_0, G) is a locally Kähler CRS of codimension at most two.*

- (1) If G_0/Γ is a hypersurface, then one of the following occurs:
- (i) G/Γ is a Cousin group, or,
 - (ii) A finite covering of G/Γ is a product of a torus and \mathbb{C}^* .
- (2) If G_0/Γ has codimension two, then one of the following occurs:
- (i) G/Γ is a Cousin group, or,
 - (ii) A finite covering of G/Γ is a product of a Cousin group (of hypersurface type) and \mathbb{C}^* , or,
 - (iii) A finite covering of G/Γ is a product of a torus and $\mathbb{C}^* \times \mathbb{C}^*$.

Proof . Codimension 1: Proposition 4.13 handles the case when $\mathcal{O}(G/\Gamma) = \mathbb{C}$. Suppose $\mathcal{O}(G/\Gamma) \neq \mathbb{C}$ and let

$$G/\Gamma \longrightarrow G/J \cdot \Gamma$$

be the holomorphic reduction of G/Γ . For codimension reasons its base is biholomorphic to \mathbb{C}^* and its fiber is a compact complex torus. It then follows from Proposition 1 in [GOR] that a finite covering of G/Γ is biholomorphic to a product.

Codimension 2: The case of no holomorphic functions is again handled by proposition 4.13. Using the same notation we again consider the holomorphic reduction. Its base cannot be \mathbb{C} , since G_0/Γ would then be complex, contrary to the generic assumption. So its base is either \mathbb{C}^* or $\mathbb{C}^* \times \mathbb{C}^*$. In the second case its fiber is compact and a finite covering splits as a product.

One is reduced to considering the case where the base of the holomorphic reduction

$$X = G/\Gamma \longrightarrow G/J \cdot \Gamma = \mathbb{C}^*$$

is \mathbb{C}^* and its fiber $F := J/J \cdot \Gamma = J/(J \cap \Gamma)$ is a codimension one locally Kähler CRS. First we claim that F is a hypersurface Cousin group. By the codimension one case considered above either F is a Cousin group or there is a splitting $J/(J \cap \Gamma) = J/H \times H/(H \cap \Gamma)$, where we have abused the language and assumed that the space itself splits. Assume we are in the second case. If $H/(H \cap \Gamma)$ were biholomorphic to a torus, one would then have an intermediate fibration

$$G/\Gamma \longrightarrow G/H \cdot \Gamma$$

with fiber the torus and base a \mathbb{C}^* -bundle over \mathbb{C}^* . By the classification of two dimensional solvmanifolds a finite covering of $G/H \cdot \Gamma$ would be biholomorphic to a direct product $\mathbb{C}^* \times \mathbb{C}^*$ and this would contradict the fact that the base of the holomorphic reduction of X has dimension one. Thus F is a Cousin group. Now we claim that a finite cover of X is the trivial bundle.

Clearly, $J \subset N \subset G$ and J has complex codimension one in G . So if $N = G$, then we are in the nilpotent setting and this case is easily handled as follows. The complex ideal \mathfrak{m} lies in the center \mathfrak{z} of the Lie algebra \mathfrak{g} of G ,

see Theorem 2' in [OR]. If $\mathfrak{m} = \mathfrak{z}$, then J is the center of G and has closed orbits and the base of the holomorphic reduction of G/Γ has dimension two. (This is a consequence of the construction of the holomorphic reduction of complex nilmanifolds, see [GH1].) Therefore, \mathfrak{m} has codimension at least one in \mathfrak{z} . But then the center of G has codimension at most one and thus G is abelian. The rest is obvious.

So we assume that $J = N$ and the fiber is a Cousin group of CR-hypersurface type. It is clear that $\text{Ad}(\Gamma) \subset \text{GL}(J_0 \cap \Gamma)$, is diagonalisable (as an element of $\text{GL}(J)$ over \mathbb{C}), and its restriction to M is contained in the compact torus acting on M . Since it stabilizes a discrete group and is invertible, it has a generator of determinant 1. Since M is of codimension one in J_0 , there is one remaining eigenvalue for the adjoint action of the Lie algebra of J_0 on itself and this also has absolute value 1. But all this together implies that $\text{Ad}(\Gamma)$ is a finite group. The rest of the proof is identical to the proof of proposition 1 in [GOR], p. 167. \square

Remark: One should note that this proof relies on the codimension two assumption. The example constructed using the Salem number (see 4.3) and the example on p. 413 in [OR] are both of codimension three and in both cases no finite covering of their holomorphic reductions splits as a product.

REFERENCES

- [Aus] L. AUSLANDER, An exposition of the structure of solvmanifolds. Part I: Algebraic Theory. *Bull. Amer. Math. Soc.* **79** (1973), 227 - 261.
- [BaOt] W. BARTH AND M. OTTE, Über fast-uniforme Untergruppen komplexer Liegruppen, *Comm. Math. Helv.* **44** (1969), 269–281.
- [BPVdeV] W. BARTH, C. PETERS, A. VAN DE VEN: Compact Complex Surfaces. *Springer-Verlag*, 1984.
- [CL] G. CÉURÉ, J.-J. LOEB: A counterexample to the Serre problem with a bounded domain of C^2 as fiber. *Ann. of Math. (2)* **122** (1985), no. 2, 329–334.
- [GH1] B. GILLIGAN AND A.T. HUCKLEBERRY, On non-compact complex nil-manifolds, *Math. Ann.* **238** (1978), 39 - 49.
- [GH2] B. GILLIGAN AND A. HUCKLEBERRY, Fibrations and globalizations of compact homogeneous CR-manifolds, *Izv. Math.* **73** (2009), 501 - 553.
- [GO] B. GILLIGAN AND K. OELJEKLAUS, Two remarks on Kähler homogeneous manifolds, *Annales Fac. Sci. Toulouse* **17** (2008), no. 1, 73 - 80.
- [GOR] B. GILLIGAN, K. OELJEKLAUS, AND W. RICHTHOFER, Homogeneous complex manifolds with more than one end, *Canad. J. Math.* **41** (1989), 163 - 177.
- [Gr] GRAUERT, H.: On Levi's problem and the imbedding of real-analytic manifolds. *Ann. of Math. (2)* **68** (1958), 460–472.
- [HO] A. HUCKLEBERRY AND E. OELJEKLAUS, On holomorphically separable complex solv-manifolds, *Ann. Inst. Fourier (Grenoble)* **36** (1986), 57 - 65.
- [Lo] J.-J. LOEB, Action d'une forme réelle d'un groupe de Lie complexe sur les fonctions plurisousharmoniques, *Ann. Inst. Fourier (Grenoble)* **35** (1985), 59 - 97.
- [Mal] A. MAL'CEV, On solvable Lie algebras. (Russian) *Bull. Acad. Sci. URSS. Sér. Math. [Izvestia Akad. Nauk SSSR]* **9**, (1945). 329 - 356.

- [Mos] G.D. MOSTOW, Factor spaces of solvable groups. *Ann. of Math.* (2) **60**, (1954). 1–27.
- [OR] K. OELJEKLAUS AND W. RICHTHOFER, On the structure of complex solvmanifolds, *J. Differential Geom.* **27** (1988), 399 - 421.
- [OT] K. OELJEKLAUS AND M. TOMA, Non-Kähler compact complex manifolds associated to number fields, *Ann. Inst. Fourier (Grenoble)* **55** (2005), 161 - 171.
- [Reed] B.E. REED, Representations of solvable Lie algebras. *Michigan Math. J.* **16** (1969), 227 – 233
- [Ri] W. RICHTHOFER, Homogene *CR*-Mannigfaltigkeiten, Ph.D. Dissertation, Ruhr-Universität Bochum, 1985.

Bruce Gilligan: DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF REGINA, 3737 WASCANA BOULEVARD, REGINA, CANADA S4S 0A2.

E-mail address: gilligan@math.uregina.ca

Karl Oeljeklaus: LATP-UMR(CNRS) 6632, CMI-UNIVERSITÉ D'AIX-MARSEILLE I, 39, RUE JOLIOT-CURIE, F-13453 MARSEILLE CEDEX 13, FRANCE.

E-mail address: karloelj@cmi.univ-mrs.fr