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PSEUDOCONVEX DOMAINS SPREAD OVER COMPLEX HOMOGENEOUS MANIFOLDS

BRUCE GILLIGAN, CHRISTIAN MIEBACH, AND KARL OELJEKLAUS

ABSTRACT. Using the concept of inner integral curves defined by Hirschowitz we generalize a recent result by Kim, Levenberg and Yamaguchi concerning the obstruction of a pseudoconvex domain spread over a complex homogeneous manifold to be Stein. This is then applied to study the holomorphic reduction of pseudoconvex complex homogeneous manifolds $X = G/H$. Under the assumption that G is solvable or reductive we prove that X is the total space of a G -equivariant holomorphic fiber bundle over a Stein manifold such that all holomorphic functions on the fiber are constant.

1. INTRODUCTION

Let G be a connected complex Lie group and H a closed complex subgroup of G . The complex homogeneous manifold $X = G/H$ admits a Lie theoretic holomorphic reduction $\pi: G/H \rightarrow G/J$ where G/J is holomorphically separable and $\mathcal{O}(G/H) \simeq \pi^*\mathcal{O}(G/J)$. In general, the base G/J is not Stein nor do we have $\mathcal{O}(J/H) = \mathbb{C}$. In some cases one can say more. If G is solvable, then G/J is always a Stein manifold, although $\mathcal{O}(J/H) = \mathbb{C}$ is not true in general (see [HO86]). If G is complex reductive, then due to [BO73] there is a factorization $G/H \rightarrow G/\overline{H} \rightarrow G/J$ where \overline{H} denotes the Zariski closure of H in G ; moreover, G/J is a quasi-affine variety. In general, $\mathcal{O}(\overline{H}/H) \neq \mathbb{C}$ and \overline{H}/H can even be Stein.

As the main result of this paper we prove the following theorem about the holomorphic reduction of *pseudoconvex* complex homogeneous manifolds $X = G/H$ where G is solvable or reductive.

Main Theorem. *Suppose that the complex homogeneous manifold $X = G/H$ is pseudoconvex and let $X = G/H \rightarrow G/J$ be its holomorphic reduction.*

- (1) *If G is a complex reductive Lie group, then the base G/J is Stein and the fiber J/H satisfies $\mathcal{O}(J/H) = \mathbb{C}$. If X is Kähler as well, then J/H is a product of the Cousin group \overline{H}/H with the homogeneous rational manifold J/\overline{H} .*
- (2) *If G is solvable, then the fiber J/H is a Cousin group tower and thus $\mathcal{O}(J/H) = \mathbb{C}$.*

An open question is whether the holomorphic function algebra $\mathcal{O}(G/H)$ is a Stein algebra for G/H pseudoconvex and G general. This lies beyond the scope of the present paper and is not addressed here.

As a first step towards the proof of this theorem we discuss the Levi problem for pseudoconvex domains spread over complex homogeneous spaces and present a Lie theoretic description of the obstruction to their being Stein. A characterization of relatively compact, smoothly bounded, pseudoconvex domains D in complex homogeneous manifolds such that D is not Stein is given in [KLY11]. The incorporation of methods of Hirschowitz [Hir75] allows us to

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simplify their proof and to show that the assumptions on the smoothness of the boundary and relative compactness of D are not needed. One of the essential tools that Hirschowitz uses is the concept of an *inner integral curve*, i.e., a non-constant holomorphic image of \mathbb{C} in D that is relatively compact and is the integral curve of a vector field. Indeed, Hirschowitz proves that if a non-compact pseudoconvex domain D is spread over an infinitesimally homogeneous complex manifold and D has no inner integral curves, then D is Stein.

Our generalization of the main result of [KLY11] reads then as follows.

Theorem 3.1. *Let $p: D \rightarrow X$ be a pseudoconvex domain spread over the complex homogeneous manifold $X = G/H$ such that $p(D)$ contains the base point $eH \in X$. If D is not Stein, then there exist a connected complex Lie subgroup \widehat{H} of G with $H^0 \subset \widehat{H}$ and $\dim H < \dim \widehat{H}$ and a foliation $\mathcal{F} = \{F_x\}_{x \in D}$ of D such that*

- (1) *every leaf of \mathcal{F} is a relatively compact immersed complex submanifold of D ,*
- (2) *every inner integral curve in D passing through $x \in D$ lies in the leaf F_x containing x , and*
- (3) *the leaves of \mathcal{F} are homogeneous under a covering group of \widehat{H} .*

In Proposition 3.6 we show furthermore the existence of an open subgroup H^* of H such that D can be realized as a domain spread over G/H^* and that H^* normalizes \widehat{H} . This will be important in the proof of our Main Theorem.

Moreover, we have the following strengthening of this theorem in the projective setting.

Theorem 5.1. *Suppose G is a connected complex Lie group acting holomorphically on $\mathbb{P}_n(\mathbb{C})$ and $X = G/H$ is an orbit. Then, every pseudoconvex domain spread over X is holomorphically convex and the fibers of its Remmert reduction are rational homogeneous manifolds.*

Let us briefly outline the organization of this paper. In Section 2 we summarize Hirschowitz' results in a form suitable for our needs. In Section 3 we prove Theorem 3.1 and discuss several applications of it. In the fourth section we investigate when the foliation of a pseudoconvex domain has compact leaves, which leads in Section 5 to the proof of Theorem 5.1. The sixth section contains a generalization of Kiselman's minimum principle that is used in the last two sections in order to prove our Main Theorem in the reductive and solvable case, respectively.

Throughout this paper we will denote Lie groups by upper case letters and their Lie algebras by the corresponding fracture letters.

2. A RESULT OF HIRSCHOWITZ

Following Hirschowitz [Hir74] and [Hir75] we call a complex manifold X *infinitesimally homogeneous* if every tangent space of X is generated by global holomorphic vector fields. Every complex manifold homogeneous under the action of a Lie group of holomorphic transformations is infinitesimally homogeneous, and more generally every domain spread over such a manifold has this property. Here we understand by a *domain spread over X* a pair (D, p) where D is a connected complex manifold and $p: D \rightarrow X$ is locally biholomorphic.

Throughout the rest of the paper we fix a complex homogeneous space $X = G/H$ where G is a connected complex Lie group and H is a (not necessarily connected) closed complex subgroup of G . We view an element $\xi \in \mathfrak{g}$ as a right invariant vector field on G which we push down to a holomorphic vector field ξ_X on $X = G/H$. If $p: D \rightarrow X$ is a domain spread over X , we denote by $\tilde{\xi}_X$ the lift of ξ_X to D . An *inner integral curve in D* is a non-constant holomorphic map $\tilde{\mathbb{C}} \rightarrow D$ with relatively compact image in D which is the integral curve of some vector field $\tilde{\xi}_X$ with $\xi \in \mathfrak{g}$.

In this paper a complex manifold is called *pseudoconvex* if it admits a continuous plurisubharmonic exhaustion function. Note that every holomorphically convex complex manifold is pseudoconvex. We will use the following special case of [Hir75, Theorem 4.1].

Theorem 2.1. *Let $p: D \rightarrow X$ be a non-compact pseudoconvex domain spread over a complex homogeneous manifold $X = G/H$.*

- (1) *If D is not Stein, then D contains an inner integral curve.*
- (2) *If X is a compact rational variety, then D is holomorphically convex.*
- (3) *If X is an irreducible compact rational variety, then D is Stein.*

Remark 2.2. Let X be infinitesimally homogeneous. If X is compact rational, then $X = G/P$ where G is complex semisimple and P is a parabolic subgroup of G . Such X is irreducible if and only if G is simple and P is maximal parabolic.

The following example shows that Hirschowitz' theorem does not hold for locally Stein domains in complex homogeneous manifolds.

Example 2.3. Let D be the punctured unit ball in $\mathbb{C}^2 \setminus \{0\} \simeq G/H$ where $G = \mathrm{SL}(2, \mathbb{C})$ and $H = \begin{pmatrix} 1 & \mathbb{C} \\ 0 & 1 \end{pmatrix}$. Since ∂D is strictly Levi-convex, D is locally Stein in G/H , but not pseudoconvex (in which case D would be Stein). Note that D does not contain an inner integral curve.

The following result (see [Hir74, Theorem 2.1]) implies that such an example does not exist when the domain $p(D)$ is relatively compact in X and p has finite fibers.

Theorem 2.4. *Let $p: D \rightarrow X$ be a locally Stein domain spread over $X = G/H$. If $p(D)$ is relatively compact and p has finite fibers, then D is pseudoconvex.*

Remark 2.5. As a direct application of Theorem 2.1 we obtain the following: *If each holomorphic map $\mathbb{C} \rightarrow X$ with relatively compact image is constant, then every pseudoconvex domain spread over X is Stein.* This observation applies e.g. when X is Brody-hyperbolic or if there exists a holomorphic map $f: X \rightarrow Y$ such that Y and all fibers of f are holomorphically separable.

The following example comes from a construction in [CL85] and illustrates the second case in Remark 2.5.

Example 2.6. Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and take $D := \log(A)$ the unique real logarithm of A . For $K = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ we define G_K to be the semi-direct product $K \ltimes K^2$ with group law

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1 + x_2, \exp(x_1 D)y_2 + y_1).$$

Then $G_{\mathbb{R}}/G_{\mathbb{Z}}$ is a compact totally real submanifold of $G_{\mathbb{C}}/G_{\mathbb{Z}}$ and the map $p: G_{\mathbb{C}}/G_{\mathbb{Z}} \rightarrow G_{\mathbb{C}}/(G'_{\mathbb{C}}G_{\mathbb{Z}}) \simeq \mathbb{C}^*$ is a holomorphic fiber bundle with fiber $\mathbb{C}^* \times \mathbb{C}^*$. It is known that $\mathcal{O}(G_{\mathbb{C}}/G_{\mathbb{Z}}) = p^* \mathcal{O}(\mathbb{C}^*)$, in particular $G_{\mathbb{C}}/G_{\mathbb{Z}}$ is not Stein. However, since base and fiber are Stein, $G_{\mathbb{C}}/G_{\mathbb{Z}}$ cannot contain an inner integral curve. Therefore every pseudoconvex domain spread over $G_{\mathbb{C}}/G_{\mathbb{Z}}$ is Stein by Theorem 2.1.

We would like to point out that $G_{\mathbb{C}}/G_{\mathbb{Z}}$ contains pseudoconvex domains of the form $D = p^{-1}(U)$ for a domain $U \subset \mathbb{C}^*$ which are non-trivial in the sense that the restricted fiber bundle $p|_D$ is non-trivial. In fact, Dan Zaffran proved in [Zaf08] that for an annulus $U \subset \mathbb{C}^*$ of modulus smaller than a certain constant, the inverse image $p^{-1}(U) \subset G_{\mathbb{C}}/G_{\mathbb{Z}}$ is Stein, so in particular it is pseudoconvex. Of course every domain of the form $p^{-1}(U)$ is locally Stein in $G_{\mathbb{C}}/G_{\mathbb{Z}}$, but in general not pseudoconvex.

3. EXISTENCE OF THE FOLIATION

In this section we consider a non-Stein pseudoconvex domain $p: D \rightarrow X$ spread over the complex homogeneous manifold $X = G/H$. Generalizing [KLY11, Theorem 6.4] we will show that there exists a connected complex Lie subgroup \widehat{H} of G that induces a holomorphic foliation of D having relatively compact leaves such that every leaf of this foliation contains all of the inner integral curves of D passing through any of its points. Note that it is not assumed that $p(D)$ is relatively compact or has smooth boundary. What happens when these leaves are closed, and thus compact, is addressed in Section 4.

Theorem 3.1. *Let $p: D \rightarrow X$ be a pseudoconvex domain spread over $X = G/H$ such that $p(D)$ contains the base point $eH \in X$. If D is not Stein, then there exist a (not necessarily closed) connected complex Lie subgroup \widehat{H} of G with $H^0 \subset \widehat{H}$ and $\dim H < \dim \widehat{H}$ as well as a foliation $\mathcal{F} = \{F_x\}_{x \in D}$ of D such that*

- (1) every leaf of \mathcal{F} is a relatively compact immersed complex submanifold of D ,
- (2) every inner integral curve in D passing through $x \in D$ lies in the leaf F_x containing x , and
- (3) the leaves of \mathcal{F} are homogeneous under a covering group of \widehat{H} and the restriction of p to a leaf in D is a finite covering map onto its image in X .

We will prove Theorem 3.1 in several steps and start by defining the group \widehat{H} . For this let $p: D \rightarrow X$ be a domain over $X = G/H$ and choose $x_0 \in D$ such that $p(x_0) = eH$. We will denote the local flow of the holomorphic vector field $\widetilde{\xi}_X$ by $(t, x) \mapsto e^{t\xi} \cdot x$ and its maximal complex integral manifold through x by F_x^ξ . Note that t is a complex parameter and that F_x^ξ is an immersed complex curve. Then we set

$$(3.1) \quad \widehat{\mathfrak{h}} := \{\xi \in \mathfrak{g}; \widetilde{\xi}_X \varphi(x_0) = 0 \text{ for every continuous plurisubharmonic function } \varphi \text{ on } D\}$$

where the derivative $\widetilde{\xi}_X \varphi(x_0) = \frac{d}{dt} \Big|_0 \varphi(e^{t\xi} \cdot x_0)$ of a continuous function has to be understood in the distributional sense and is taken with respect to the complex parameter t . If $X = G/H$ is unimodular, i.e., if X has a G -invariant Borel measure, and if $D \subset X$, then we can use the usual convolution technique in order to approximate continuous plurisubharmonic functions on D by smooth ones. In this case we may replace “continuous” by “smooth” in the definition of $\widehat{\mathfrak{h}}$. We shall see later that $\widehat{\mathfrak{h}} = \mathfrak{h}$ if and only if a pseudoconvex D is Stein.

Example 3.2. We present here an example which turns out to be important in the rest of the paper. Let G be a *Cousin group*, i.e., a connected complex Lie group without non-constant holomorphic functions. It is well-known that G is a quotient of $(\mathbb{C}^n, +)$ by a discrete subgroup Γ of rank $n + m$, $1 \leq m \leq n$, generating \mathbb{C}^n over \mathbb{C} . With $V := \langle \Gamma \rangle_{\mathbb{R}}$ and $W := V \cap iV$, one has furthermore that V/Γ is the maximal compact subgroup of G and that $W + \Gamma$ is dense in V . Hence we have $\widehat{\mathfrak{h}} = W$ under the identification $\mathfrak{g} \simeq \mathbb{C}^n$.

Since we can lift the vector field ξ_X from X to D for every $\xi \in \mathfrak{g}$, we obtain a *local* holomorphic action of G on D such that $p: D \rightarrow X$ is equivariant. This means that there exist an open neighborhood $\Omega \subset G \times D$ of $\{e\} \times D$ such that $\{g \in G; (g, x) \in \Omega\}$ is connected for every $x \in D$, as well as a holomorphic map $\Phi: \Omega \rightarrow D$, $\Phi(g, x) =: g \cdot x$, fulfilling the usual axioms of a group action. For more details we refer the reader to [HI97]. Note that the local G -action on D is in general not globalizable unless $p: D \rightarrow X$ is schlicht. For this reason there will in general be no maximal domain of definition Ω of the local G -action.

For the readers' convenience we repeat some arguments from [Hir75] and [KLY11] in order to give the proof of the following.

Lemma 3.3. *Let $p: D \rightarrow X$ be pseudoconvex. Then the set $\widehat{\mathfrak{h}}$ defined in equation (3.1) is a complex Lie subalgebra of \mathfrak{g} .*

Proof. The key point is to observe that if $\widetilde{\xi}_X \varphi(x_0) = 0$ for every continuous plurisubharmonic function on D , then every such φ must be constant on $F_{x_0}^\xi$. To see this, let ρ be a continuous plurisubharmonic exhaustion function of D and set $D_\alpha := \{x \in D; \rho(x) < \alpha\}$. Choose $\alpha \in \mathbb{R}$ such that $x_0 \in D_\alpha$. For $|t|$ sufficiently small we have

$$e^{t\xi} \cdot D_{\alpha+1} \supset D_\alpha \ni x_t := e^{t\xi} \cdot x_0$$

Hence, we have the holomorphic map $e^{t\xi}: D_\alpha \rightarrow D_{\alpha+1}$ and therefore $\varphi_{-t} := \varphi \circ e^{t\xi}$ is continuous plurisubharmonic on D_α whenever φ is continuous plurisubharmonic on $D_{\alpha+1}$. Following [Hir75, Proposition 1.6] we construct a continuous plurisubharmonic function ψ_{-t} on D which coincides with φ_{-t} in a neighborhood of x_0 . Choose $\beta \in \mathbb{R}$ such that $\varphi_{-t}(x_0) < \beta < \alpha$ and note that $K := \rho^{-1}(\beta) \subset D_\alpha$ is compact. Then choose a convex increasing function χ on \mathbb{R} fulfilling

$$\begin{aligned} \chi(\rho(x_0)) &< \varphi_{-t}(x_0) \quad \text{and} \\ \chi(\beta) &> \|\varphi_{-t}\|_K. \end{aligned}$$

Finally, define $\psi_{-t}: D \rightarrow \mathbb{R}$ by

$$\psi_{-t}(x) := \begin{cases} \max(\varphi_{-t}(x), \chi \circ \rho(x)) & : \rho(x) \leq \beta \\ \chi \circ \rho(x) & : \rho(x) \geq \beta. \end{cases}$$

One checks directly that ψ_{-t} is continuous plurisubharmonic and coincides with φ_{-t} in some neighborhood of x_0 . Consequently, we may calculate

$$\widetilde{\xi}_X \varphi(x_{-t}) = \frac{d}{ds} \Big|_t \varphi(e^{-s\xi} \cdot x_0) = \widetilde{\xi}_X \varphi_{-t}(x_0) = \widetilde{\xi}_X \psi_{-t}(x_0) = 0.$$

Hence, the set of x_t such that $\widetilde{\xi}_X \varphi(x_t) = 0$ holds for every continuous plurisubharmonic φ is open in $F_{x_0}^\xi$, and since it is also closed, we see that $\widetilde{\xi}_X \varphi$ vanishes on $F_{x_0}^\xi$ as a distribution for every continuous plurisubharmonic φ . But this implies in turn that φ is constant on $F_{x_0}^\xi$. Thus the proof of our first claim is finished.

Applying this fact to the continuous exhaustion function ρ of D , we conclude that for every $\xi \in \widehat{\mathfrak{h}}$ the maximal integral manifold $F_{x_0}^\xi$ is relatively compact and contained in D . In particular, we have $F_{x_0}^\xi = \mathbb{C}^\xi \cdot x_0$ where \mathbb{C}^ξ is the universal covering of $\exp(\mathbb{C}\xi)$. In order to finish the proof that $\widehat{\mathfrak{h}}$ is a complex Lie subalgebra of \mathfrak{g} , we will show that for every finite collection $\xi_1, \dots, \xi_k \in \widehat{\mathfrak{h}}$ the map

$$(t_1, \dots, t_k) \mapsto \rho(e^{t_1 \xi_1} \dots e^{t_k \xi_k} \cdot x_0)$$

defined on a neighborhood of $0 \in \mathbb{C}^k$ has vanishing differential wherever it is defined. To see this choose $\alpha \in \mathbb{R}$ such that $\mathbb{C}^{\xi_k} \cdot x_0$ is relatively compact in $D_\alpha \Subset D$. If $|t_j|$ is sufficiently small for all $j = 1, \dots, k-1$, the map

$$\rho \circ e^{t_1 \xi_1} \circ \dots \circ e^{t_{k-1} \xi_{k-1}}$$

is defined and continuous plurisubharmonic on D_α . Consequently, this map is constant on $\mathbb{C}^{\xi_k} \cdot x_0$, and the claim follows by induction over k .

Having established this, we use the following argument based on the Campbell-Baker-Hausdorff formula (cf. [KLY11, pp.31–32]). Let $\xi_1, \xi_2 \in \widehat{\mathfrak{h}}$ and $\lambda_1, \lambda_2 \in \mathbb{C}$. Then we conclude

from $\rho(e^{t\lambda_1\xi_1}e^{t\lambda_2\xi_2} \cdot x_0) = \rho(x_0)$ and $e^{t\lambda_1\xi_1}e^{t\lambda_2\xi_2} = e^{t\lambda_1\xi_1+t\lambda_2\xi_2+O(t^2)}$ that $\lambda_1\xi_1 + \lambda_2\xi_2 \in \widehat{\mathfrak{h}}$ holds. Hence $\widehat{\mathfrak{h}}$ is a complex subspace of \mathfrak{g} . To see that $[\xi_1, \xi_2]$ lies in $\widehat{\mathfrak{h}}$ we use

$$\rho(e^{\sqrt{t}\xi_1}e^{\sqrt{t}\xi_2}e^{-\sqrt{t}\xi_1}e^{-\sqrt{t}\xi_2} \cdot x_0) = \rho(x_0)$$

together with

$$e^{\sqrt{t}\xi_1}e^{\sqrt{t}\xi_2}e^{-\sqrt{t}\xi_1}e^{-\sqrt{t}\xi_2} = e^{t[\xi_1, \xi_2]+O(t^{3/2})}$$

hence completing the proof that $\widehat{\mathfrak{h}}$ is a complex Lie subalgebra of \mathfrak{g} . \square

The following example shows that even if $\widehat{\mathfrak{h}}$ is a subalgebra, without pseudoconvexity of D we do not obtain a foliation with relatively compact leaves.

Example 3.4. Consider the homogeneous space $X = \mathbb{P}_2 \setminus \{[e_1]\} \simeq P/H$ for $P = \left\{ \begin{pmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\} \simeq \mathrm{GL}(2, \mathbb{C}) \ltimes \mathbb{C}^2$ and $H = \left\{ \begin{pmatrix} * & 0 & * \\ 0 & * & * \\ 0 & 0 & * \end{pmatrix} \right\} \simeq (\mathbb{C}^*)^2 \ltimes \mathbb{C}^2$. It follows from Theorem 2.1 that every pseudoconvex domain spread over X is Stein. However, X itself is not pseudoconvex and there does not exist a maximal complex subgroup \widehat{H} of P such that $\widehat{H} \cdot eH$ is relatively compact in X . Therefore the existence of such an \widehat{H} is not a purely Lie theoretic property.

After these preparations we are in the position to prove the theorem.

Proof of Theorem 3.1. We define \widehat{H} to be the analytic subgroup of G with Lie algebra $\widehat{\mathfrak{h}}$. Since \mathfrak{h} is contained in $\widehat{\mathfrak{h}}$, we have $H^0 \subset \widehat{H}$. In order to show that $\dim H < \dim \widehat{H}$ we suppose from now on that the pseudoconvex domain D is not Stein. It follows from Theorem 2.1 that there exists $\xi \in \mathfrak{g}$ such that $\widetilde{\xi}_X(x_0) \neq 0$ and such that $\mathbb{C}^\xi \cdot x_0$ is relatively compact in D . Since subharmonic functions on \mathbb{C} which are bounded from above must be constant, we see that $\xi \in \widehat{\mathfrak{h}} \setminus \mathfrak{h}$. In particular, we have that D is Stein if and only if $\widehat{\mathfrak{h}} = \mathfrak{h}$.

In order to define the foliation \mathcal{F} of D we first construct a relatively compact immersed complex submanifold of D which contains $x_0 \in D$ and which will become the leaf F_{x_0} . For this let \widetilde{H} be the simply-connected complex Lie group with Lie algebra $\widehat{\mathfrak{h}}$. As we have seen, for every $\xi \in \widehat{\mathfrak{h}}$ the integral manifold $F_{x_0}^\xi$ is relatively compact in D . This implies that we can define a map Φ on $\exp(\widehat{\mathfrak{h}}) \subset \widetilde{H}$ with values in D such that $F_{x_0}^\xi$ is the image of $\exp(\mathbb{C}\xi)$ under Φ . Since $\exp(\widehat{\mathfrak{h}})$ is dense in \widetilde{H} (see [HM78]), we can extend Φ to \widetilde{H} and obtain an equivariant holomorphic map $\widetilde{H} \rightarrow D$ whose image is an immersed complex submanifold F_{x_0} . Note that ρ is constant on F_{x_0} , so that F_{x_0} is relatively compact in D . Moreover, it follows from the definition of F_{x_0} that every inner integral curve of D passing through x_0 is contained in F_{x_0} .

We define the foliation \mathcal{F} of D by moving around F_{x_0} with the local G -action on D . To make this precise, note that for every $x \in D$ there exists elements $g_1, \dots, g_k \in G$ such that $g_1 \cdot (\dots (g_k \cdot x_0))$ is defined and equals x . Then we set

$$F_x := g_1 \cdot (\dots (g_k \cdot F_{x_0})).$$

The product $g_k \cdot F_{x_0}$ on the right hand side is defined since F_{x_0} is relatively compact. The foliation is well-defined because of the following observation. Suppose we have $x_0 = g_1 \cdot (\dots (g_k \cdot x_0))$. Then F_{x_0} and $g_1 \cdot (\dots (g_k \cdot F_{x_0}))$ must coincide since F_{x_0} contains every inner integral curve of D passing through x_0 as noted above. One checks locally that \mathcal{F} indeed defines a foliation of D fulfilling the first three statements of Theorem 3.1. For the last statement note the following: Let $p: D \rightarrow X$ be a locally biholomorphic map between complex manifolds, $A \subset D$ be a relatively compact set and $B := \pi(A)$. Then for every $b \in B$ the set $p^{-1}(b) \cap A$ is finite. It follows that the restriction of p to a leaf in D is a finite covering map onto its image in X . \square

Example 3.5. A simple method to produce examples for Theorem 3.1 is the following. Let $Y = G/L$ be a complex homogeneous space and let H be a closed complex subgroup of L such that L/H is compact. Then $X = G/H$ contains many pseudoconvex non-Stein domains, e.g., pre-images of balls in Y under the fibration $G/H \rightarrow G/L$. For these domains we have $\widehat{H} = L$. Concretely, consider an embedding of $L = \mathrm{SL}(2, \mathbb{C})$ into $G = \mathrm{SL}(3, \mathbb{C})$ and let H be a discrete cocompact subgroup of L . In this case, the homogeneous space G/H is even holomorphically convex.

Since H^0 normalizes the group \widehat{H} it is clear that there exists a maximal open subgroup of H with this property. Defining

$$\widetilde{D} := D \times_X G := \{(x, g) \in D \times G; p(x) = gH\}$$

we have the commutative diagram

$$\begin{array}{ccc} \widetilde{D} & \xrightarrow{\widetilde{p}} & G \\ \widetilde{\pi} \downarrow & & \downarrow \pi \\ D & \xrightarrow{p} & X, \end{array}$$

where $\widetilde{p}: \widetilde{D} \rightarrow G$ is locally biholomorphic and $\widetilde{\pi}: \widetilde{D} \rightarrow D$ is a principal H -bundle over D . Note however that \widetilde{D} is not a domain spread over G since it is in general not connected. Choose the reference point $\widetilde{x}_0 := (x_0, e) \in \widetilde{D}$ and let \widetilde{D}^0 be the connected component of \widetilde{D} containing \widetilde{x}_0 . Then we define the open subgroup

$$(3.2) \quad H^* = \{h \in H; h \cdot \widetilde{x}_0 \in \widetilde{D}^0\}$$

of H . In particular, if $D = X = G/H$ is pseudoconvex, then $\widetilde{D} \simeq G$ and $H^* = H$. Hence, the group \widehat{H} is normalized by H as we shall see in

Proposition 3.6. *The group H^* defined in equation (3.2) normalizes \widehat{H} , and D is biholomorphic to \widetilde{D}^0/H^* which can be realized as a domain spread over $X^* := G/H^*$. Moreover, we have $\widehat{H} = \widehat{H}^*$. In other words, after replacing $X = G/H$ by a covering we may suppose that H normalizes \widehat{H} .*

Proof. In a first step one checks that $h^{-1} \cdot \widetilde{x}_0 \in \widetilde{D}^0$ for all $h \in H^*$. This implies then that H^* is indeed a subgroup of H and that $H^* = \{h \in H; h \cdot \widetilde{D}^0 = \widetilde{D}^0\}$. Moreover, H^0 is contained in H^* . Hence H^* is open in H , closed in G , and acts properly and freely on \widetilde{D}^0 . By definition, we have $\widetilde{\pi}^{-1}(\widetilde{\pi}(x)) \cap \widetilde{D}^0 = H^* \cdot x$ for all $x \in \widetilde{D}^0$, and thus $\widetilde{D}^0/H^* \simeq D$. Finally, one sees directly that the map $\widetilde{D}^0 \rightarrow D^* := D \times_X (G/H^*)$, $(x, g) \mapsto (x, gH^*)$, induces an isomorphism between \widetilde{D}^0/H^* and the connected component of D^* containing (x_0, eH^*) , so that D can be realized as a domain spread over G/H^* .

To finish the proof we must show that H^* normalizes $\widehat{H} = \widehat{H}^*$. For this we define the set

$$\Omega_D := \{(x, g) \in \widetilde{D}^0; (x, g\widehat{H}) \subset \widetilde{D}^0 \text{ and } \widetilde{\pi}(x, g\widehat{H}) \text{ is relatively compact in } D\}.$$

Since $\widetilde{\pi}(x_0, \widehat{H}) = F_{x_0}$ is relatively compact in D , we see that Ω_D is a non-empty open subset of \widetilde{D}^0 . Moreover, since the leaves of \mathcal{F} are relatively compact in D , the set Ω_D is also closed in \widetilde{D}^0 . Hence $\Omega_D = \widetilde{D}^0$. This implies that the groups H^* and \widehat{H} act on \widetilde{D}^0 by

$$h \cdot (x, g) := (x, gh^{-1}).$$

Consequently, for each $h \in H^*$ we have $(x_0, h\widehat{H}h^{-1}) \subset \widetilde{D}^0$ and can conclude that

$$\widetilde{\pi}(x_0, h\widehat{H}) = \widetilde{\pi}(x_0, h\widehat{H}h^{-1})$$

is relatively compact in D . Since $x_0 \in \tilde{\pi}(x_0, h\widehat{H})$ holds, we conclude from the maximality of F_{x_0} that $F_{x_0} = \tilde{\pi}(x_0, h\widehat{H})$. Hence $h\widehat{H}h^{-1} = \widehat{H}$. \square

The following example shows that in general \widehat{H} is not normalized by the whole group H .

Example 3.7. Let $G := \mathrm{SL}(2, \mathbb{C})$ and $H = \mathrm{SL}(2, \mathbb{Z}) \subset G$. Since H is Zariski dense in G there is no proper connected complex subgroup of G normalized by H . Let $X := G/H$,

$$A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \in G,$$

and let $\widehat{H} \simeq \mathbb{C}^*$ denote the Zariski closure of $H^* := \{A^k \mid k \in \mathbb{Z}\} \subset H$ in G . Then the orbit of \widehat{H} through the point eH in X is an elliptic curve, say, E . Let $B \subset G$ be a Borel subgroup transversal to \widehat{H} . Then there is a small open neighborhood $U \subset B$ biholomorphic to the unit ball \mathbb{B}_2 in \mathbb{C}^2 , such that $D := U \cdot E \subset X$ is isomorphic to $\mathbb{B}_2 \times E$. Therefore D is a pseudoconvex domain in X and we see that $H\widehat{H}$ is not a subgroup of G . Taking the covering $\pi: Y := G/H^* \rightarrow G/H = X$, one sees that the map

$$\pi|_{\pi^{-1}(D)}: \pi^{-1}(D) \rightarrow D$$

is biholomorphic. So by taking the open subgroup H^* of H and the associated covering, we do not change the pseudoconvex domain D but we now have that $H^*\widehat{H}$ is a group.

As an application of Theorem 3.1 we have the following

Corollary 3.8. *Let $p: D \rightarrow X$ be a pseudoconvex domain over $X = G/H$ such that $eH \in p(D)$. If the subgroup H is connected and maximal in G , then D is either compact or Stein.*

Proof. Since $H \subset \widehat{H} \subset G$ and H is maximal, either $\widehat{H} = G$ or $\widehat{H} = H$. In the first case D itself consists of exactly one leaf of the foliation and thus is compact. Otherwise $\widehat{H} = H$, i.e., D contains no inner integral curve, and so D is Stein. \square

Note that the isotropy subgroups of projective space and also of $Gr(k, n)$, the Grassmann manifold of k -dimensional subspaces of an n -dimensional vector space, are connected and maximal. Hence we have reproduced some classical results, e.g., see [Fuj63], [Hir75], [Nis62], [Tak64] and [Ue80].

Corollary 3.9. *Let G be a simple complex Lie group and Γ a discrete Zariski dense subgroup of G . If $X = G/\Gamma$ is pseudoconvex, then X is compact.*

Proof. Suppose that $X = G/\Gamma$ is pseudoconvex. Since X cannot be Stein, there is a connected Lie subgroup $\widehat{\Gamma} \subset G$ of positive dimension such that $\widehat{\Gamma} \cdot x_0$ is relatively compact in X . Since Γ normalizes $\widehat{\Gamma}$ and is Zariski dense in G , we have $\widehat{\Gamma} \triangleleft G$. Hence $\widehat{\Gamma} = G$ which proves the claim. \square

Let $p: D \rightarrow X$ be a pseudocovex domain spread over $X = G/H$ with $x_0 \in p(D)$. For later use we note the following technical

Lemma 3.10. *Let \widetilde{G} be a connected closed complex subgroup of G such that $\widetilde{G} \cdot x_0$ is closed in X . Then every connected component of $\widetilde{D} := p^{-1}(\widetilde{G} \cdot x_0)$ is a pseudoconvex domain spread over $\widetilde{G} \cdot x_0 \simeq \widetilde{G}/(\widetilde{G} \cap H)$ and we have*

$$\widehat{\widetilde{G} \cap H} = (\widetilde{G} \cap \widehat{H})^0$$

where the left hand side is the connected subgroup of \widetilde{G} corresponding to $\widetilde{D} \rightarrow \widetilde{G}/(\widetilde{G} \cap H)$.

Proof. Since \tilde{D} is a closed complex submanifold of D , all of its connected components are pseudoconvex domains spread over $\tilde{G} \cdot x_0$ by the map $\tilde{p} := p|_{\tilde{D}}$. By definition, the Lie algebra of $\widehat{\tilde{G} \cap H}$ is

$$\widehat{\tilde{\mathfrak{g}} \cap \tilde{\mathfrak{h}}} := \{\xi \in \tilde{\mathfrak{g}}; \tilde{\xi}_X \varphi(x_0) = 0 \text{ for every continuous plurisubharmonic function } \varphi \text{ on } \tilde{D}\},$$

hence contains $\tilde{\mathfrak{g}} \cap \tilde{\mathfrak{h}}$. Conversely, every element of $\widehat{\tilde{\mathfrak{g}} \cap \tilde{\mathfrak{h}}}$ induces an inner integral curve in \tilde{D} , thus also in D since \tilde{D} is closed in D . This implies $\widehat{\tilde{\mathfrak{g}} \cap \tilde{\mathfrak{h}}} = \tilde{\mathfrak{g}} \cap \tilde{\mathfrak{h}}$ as was to be shown. \square

4. A CHARACTERIZATION OF HOLOMORPHIC CONVEXITY

If a pseudoconvex non-Stein domain D is spread over X , in general, the leaf F_{x_0} is not closed in D . In this section we investigate exactly when this happens.

Theorem 4.1. *Suppose D is a pseudoconvex domain spread over the complex homogeneous manifold G/H . Then the complex group $H^* \hat{H}$ constructed in Proposition 3.6 is closed in G if and only if D is holomorphically convex.*

Moreover, when these conditions hold, the Remmert reduction of D is a holomorphic fiber bundle $\hat{\pi}: D \rightarrow D_0$ that is induced by the bundle $\pi: G/H \rightarrow G/H^ \hat{H}$. The fiber of $\hat{\pi}$ is compact and is biholomorphic to $H^* \hat{H}/H_1$, where H_1 is a subgroup of H^* having finite index. Its base D_0 is a Stein domain spread over the homogeneous manifold $G/H^* \hat{H}$.*

Proof. Since D is biholomorphic to a connected component of $D \times_X G/H^*$ we may assume $H = H^*$ throughout the proof in order to simplify the notation. Notice that $H \hat{H}$ is closed in G if and only if the leaves of \mathcal{F} are compact.

We first consider what happens when $D \subset X$ is a domain in the homogeneous manifold with the group $H \hat{H}$ closed. In this case the fibration $\pi: G/H \rightarrow G/H \hat{H}$ induces the foliation \mathcal{F} of D that we constructed in Theorem 3.1. Since the plurisubharmonic exhaustion function on D is constant on the compact \hat{H} -orbits in D , it descends to $\pi(D) \subset G/H \hat{H}$. Hence, $D_0 := \pi(D)$ is pseudoconvex and by the maximality of \hat{H} a Stein domain. Moreover, we have $H_1 = H$ in this case.

In order to be able to repeat this argument in the general case, we need to find a domain D_0 spread over $G/H \hat{H}$ such that the diagram

$$\begin{array}{ccc} D & \xrightarrow{p} & G/H \\ \downarrow & & \downarrow \\ D_0 & \xrightarrow{p_0} & G/H \hat{H} \end{array}$$

commutes. The idea is to define $D_0 := D/\mathcal{F}$. Since the leaves of the foliation \mathcal{F} are compact, by [Hol78, Proposition 6.2] the leaf space D/\mathcal{F} carries a canonical complex structure as soon as it is Hausdorff.

In order to see that D/\mathcal{F} is indeed Hausdorff let F_x be the leaf through $x \in D$ and let U be any open neighborhood of it. We must show that U contains a saturated open neighborhood of F_x . Since the domain $p(D)$ is foliated by the images $p(F_x)$, $F_x \in \mathcal{F}$, and since this foliation is induced by the fiber bundle $G/H \rightarrow G/H \hat{H}$ having compact fibers we find a saturated open neighborhood V of $p(F_x)$ inside $p(U)$, such that the connected component W of $p^{-1}(V)$ containing F_x lies in U and covers V . Now W is a saturated open neighborhood of F_x .

Having established the existence of the commutative diagram it follows that D_0 is Stein in exactly the same way as above. Consequently, the quotient map $D \rightarrow D/\mathcal{F} = D_0$ is

the Remmert reduction of D whose fibers are the leaves. Since F_{x_0} is connected, we have $F_{x_0} \simeq (H\widehat{H})/H_1$ where H_1 is the smallest open subgroup of H such that $H\widehat{H} = H_1\widehat{H}$ holds.

Conversely, if D is holomorphically convex, then it has a Remmert reduction, i.e., there exists a holomorphic map $\sigma: D \rightarrow D_1$ that has compact connected fibers and the target space D_1 is Stein. Since D is pseudoconvex, it admits a plurisubharmonic exhaustion. This function is clearly constant on the fibers of the map σ . It is clear from the construction given in Theorem 3.1 that the foliation given by the subgroup \widehat{H} gives the same partition of D as is given by the fibers of the map σ . The fact that D is the total space of a holomorphic fiber bundle follows from our considerations above and this observation. \square

5. DOMAINS SPREAD OVER PROJECTIVE ORBITS

In this section we again consider $p: D \rightarrow X$ which is a pseudoconvex domain spread over a homogeneous manifold $X = G/H$, as in Theorem 3.1, but now assume that X is an orbit of the connected complex linear group G in some projective space \mathbb{P}_N . One is in an algebraic setting, if X is compact and thus a flag manifold. In very stark contrast to this, if X is not compact, then X need not be closed or even locally closed in \mathbb{P}_N and the setting is not algebraic. Nonetheless, there are specific facts at hand concerning the *holomorphic actions* of complex groups in the projective case that allow us to prove the next result.

Theorem 5.1. *Let $X = G/H$ be an orbit of a connected complex Lie group G acting holomorphically on some projective space \mathbb{P}_N . Then any pseudoconvex domain D spread over X is holomorphically convex. Moreover, the fibers of the Remmert reduction of D given in Theorem 4.1 are homogeneous rational manifolds that are biholomorphic to $H\widehat{H}/H$.*

Proof. If D itself is Stein, then $\widehat{H} = H$ and there is nothing to prove in this case. So we assume throughout the rest of the proof that D is not Stein.

The first step of the proof consists in reducing the general situation to an algebraic one. Denoting \overline{G} the algebraic Zariski closure of the image of G in $\mathrm{PGL}(N+1, \mathbb{C})$ and G' its commutator subgroup, we have $G' = \overline{G}'$, and in particular G' is algebraic (for a proof of this result of Chevalley see [Bo91, Corollary II.7.9]). Consequently, the boundary of every G' -orbit in X consists of G' -orbits of strictly smaller dimension. Since G' is a normal subgroup of G , this implies that every G' -orbit is closed in X , and in particular, $G'H$ is a closed subgroup of G .

We claim that the relatively compact orbit $\widehat{H} \cdot y_0$ is contained in the neutral fiber of $G/H \rightarrow G/(G'H)$. To see this, we will repeat an argument from [HO81, p. 173]. Since $G \cap (\overline{G}_{y_0} \overline{G}') = G \cap (\overline{G}_{y_0} G') = HG'$, the Abelian algebraic group $\overline{G}/(\overline{G}_{y_0} \overline{G}') \simeq \mathbb{C}^k \times (\mathbb{C}^*)^l$ contains $G/(HG')$ as a G -orbit. As a consequence, the fiber bundle $G/H \rightarrow G/(HG')$ has holomorphically separable base which proves the claim. Moreover, since G' acts transitively on this fiber, we have $(HG')/H \simeq G'/(H \cap G')$ and $\dim \widehat{H} \cdot y_0 = \dim \widehat{H}_1 \cdot y_0$ where $\widehat{H}_1 := \widehat{H} \cap G'$ due to Lemma 3.10. Note that as a fiber $G'/(H \cap G')$ is closed in G/H . Thus $\widehat{H}_1 \cdot y_0$ is still relatively compact in the quasi-projective variety $G'/(H \cap G')$. Therefore we may replace G , H and \widehat{H} by $G_1 := G'$, $H_1 := G' \cap H = G'_{y_0}$ and \widehat{H}_1 respectively. Now we can iterate this procedure, thus replacing G_1 by $G_2 := G'_1$, and so on. As above we keep the orbit $\widehat{H} \cdot y_0$ as a relatively compact subset in G_2/H_2 . Therefore this iteration must terminate after finitely many steps and we end up with an algebraic group G_k such that $G_k = G'_k$. Consequently, we may assume without loss of generality that $X = G/H$ is a quasi-projective variety containing a relatively compact orbit $\widehat{H} \cdot y_0$ and that $G = G'$.

Since H is an algebraic subgroup of G , the map $G/H^0 \rightarrow G/H$ is a finite covering. Thus there exists a finite proper map between each connected component of $D \times_X (G/H^0)$

to D , which implies that D is pseudoconvex or holomorphically convex if and only if each component of $D \times_X (G/H^0)$ has this property. Therefore we may assume that H is connected. In particular this implies $H \subset \widehat{H}$. Hence it suffices to show that \widehat{H} is closed in G . If this is not the case, let L_1 be the topological closure of \widehat{H} and let G_1 be the connected Lie subgroup of G having Lie algebra $\mathfrak{l}_1 + i\mathfrak{l}_1$. It follows that $\widehat{\mathfrak{h}} = \mathfrak{l}_1 \cap i\mathfrak{l}_1$, therefore $\widehat{\mathfrak{h}} \triangleleft \mathfrak{g}_1$. We iterate this procedure until we arrive at a group G_n which is closed in G . Then we have $H \subset \widehat{H} \subset G_n$ and G_n/H closed in G/H , so that we can apply the sequence of commutator fibrations to G_n/H . Again these two reduction procedures must terminate after finitely many steps. Hence replacing G by the group G_n finally obtained we are in the situation that $H \subset \widehat{H} \subset G = G'$, that $X = G/H$ is quasi-projective and that there is a sequence of connected complex Lie subgroups

$$\widehat{H} =: G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G$$

such that $\mathfrak{g}_j = \mathfrak{l}_j + i\mathfrak{l}_j$ and $\mathfrak{g}_{j-1} = \mathfrak{l}_j \cap i\mathfrak{l}_j$ where L_j is the topological closure of G_{j-1} for all $j = 1, \dots, n$.

Now a purely algebraic argument gives

$$\mathfrak{l}_n \supset [\mathfrak{g}, \mathfrak{l}_n] = \mathfrak{l}'_n + i\mathfrak{l}'_n = \mathfrak{g}' = \mathfrak{g}.$$

Consequently we obtain $\mathfrak{g} = \mathfrak{l}_n$, hence $\mathfrak{g} = \mathfrak{g}_{n-1}$. Repeating this, we see $\widehat{\mathfrak{h}} = \mathfrak{g}$, so that \widehat{H} is an algebraic subgroup of G and in particular closed as was to be shown. \square

Remark 5.2. Let $H \subset G$ be linear algebraic groups and $X = G/H$. The proof of Theorem 5.1 shows that if there is a non-Stein pseudoconvex domain $p: D \rightarrow X$ spread over X with $eH \in p(D)$, then the group \widehat{H} constructed in Theorem 3.1 is likewise an algebraic subgroup of G .

In passing, we also note what happens in the case of complex orbits of real groups acting holomorphically on projective space.

Corollary 5.3. *Let $G_{\mathbb{R}}$ be a real subgroup of $\mathrm{PSL}(N+1, \mathbb{C})$ that is acting holomorphically and effectively on \mathbb{P}_N . Let $X := G_{\mathbb{R}} \cdot x$ be a complex orbit of some point $x \in \mathbb{P}_N$. Then any pseudoconvex domain spread over X is holomorphically convex.*

Proof. Let G denote the smallest connected complex subgroup of $\mathrm{PSL}(N+1, \mathbb{C})$ that contains $G_{\mathbb{R}}$. Then X is open in $G \cdot x$. Hence any domain spread over $G_{\mathbb{R}} \cdot x$ is also a domain spread over $G \cdot x$. The result now follows from Theorem 5.1. \square

For a general complex homogeneous manifold $X = G/H$ we have the normalizer fibration $X = G/H \rightarrow G/\mathcal{N}_G(H^0)$ whose base is an orbit of the linear Lie group $\mathrm{Ad}(G)$ in a projective space. However, even if G/H is pseudoconvex, in general $G/\mathcal{N}_G(H^0)$ does not have to be.

The following examples show that one cannot control how \widehat{H} is related to $\mathcal{N}_G(H^0)$.

Example 5.4. Let $G = \mathrm{SL}(3, \mathbb{C})$ with Borel B and a maximal parabolic subgroup P . Then $G/B \rightarrow G/P$ is a \mathbb{P}_1 -bundle over \mathbb{P}_2 . Taking the inverse image of e.g. the unit ball in \mathbb{P}_2 , we obtain a pseudoconvex non-Stein domain in G/B such that $\widehat{B} = P$ which is not contained in $\mathcal{N}_G(B) = B$.

Example 5.5. Let $S := \mathrm{SL}(3, \mathbb{C})$ and B be a standard Borel in S . We denote by T a maximal algebraic torus in S and take a holomorphic proper injection of \mathbb{C} into T as a closed subgroup A_1 such that the quotient $T/A_1 =: E$ is an elliptic curve. Set $H_1 = A_1 \times U$, where U denotes the unipotent radical of B . Then one has the homogeneous fibration $S/H_1 \rightarrow S/B$ with compact fiber E . Suppose $D_1 \subset S/H_1$ is a pseudoconvex domain that is not Stein. Then $\widehat{H}_1 = B = \mathcal{N}_S(H_1)$ and $\mathcal{O}(S/H_1) \simeq \mathbb{C}$.

Example 5.6. With the same set up as in the previous example we now take A_2 to be the closed image of a representation of \mathbb{Z} into T such that T/A_2 is a Cousin group C and set $H_2 := A_2 \times U$. Then $S/H_2 \rightarrow S/B$ is a homogeneous fibration with the Cousin group as fiber. Let $D_2 \subset S/H_2$ be a pseudoconvex domain that is not Stein. Then $\widehat{H}_2 = \mathbb{C} \times U$, where the image of \mathbb{C} is one of the complex leaves of the foliation of C , while $\mathcal{N}_S(H_2^0) = B$.

6. PUSHING DOWN PLURISUBHARMONIC FUNCTIONS

In this section we prove the following technical result.

Lemma 6.1. *Let $p: X \rightarrow Y$ be a holomorphic fiber bundle of complex manifolds, where X is pseudoconvex. Then Y is also pseudoconvex if either of the following holds:*

- (1) *the fiber is a Cousin group C*
- (2) *the bundle is a principal $(\mathbb{C}^*)^k$ -bundle.*

Proof. We consider case (1) first. Let $\rho: X \rightarrow \mathbb{R}$ be a continuous plurisubharmonic exhaustion of X . We define a function $\rho_Y: Y \rightarrow \mathbb{R}$ by

$$\rho_Y(y) := \inf\{\rho(x); x \in p^{-1}(y)\}.$$

As an exhaustion ρ attains a minimum on every closed complex submanifold of X . In particular, we see that ρ_Y is indeed real-valued.

Let us first show that ρ_Y is continuous. For this let (y_n) be a sequence which converges to $y_0 \in Y$. By the above remark there exist elements $x_n, x_0 \in X$ such that $\rho_Y(y_n) = \rho(x_n)$ and $\rho_Y(y_0) = \rho(x_0)$. For every $\varepsilon > 0$ the set $U_\varepsilon := \{\rho < \rho(x_0) + \varepsilon\}$ is open and relatively compact in X and contains x_0 and hence almost all x_n . Therefore, there is $z \in \overline{U_\varepsilon}$ such that $x_n \rightarrow z$ for a subsequence. We have $p(z) = \lim p(x_n) = y_0$ and consequently $\rho(z) \geq \rho_Y(y_0)$. On the other hand,

$$\rho_Y(y_n) = \rho(x_n) \rightarrow \rho(z) \leq \rho(x_0) + \varepsilon = \rho_Y(y_0) + \varepsilon$$

for all $\varepsilon > 0$ by continuity of ρ which implies $\lim \rho_Y(y_n) = \rho_Y(y_0)$ as was to be shown.

By continuity, $\{\rho_Y \leq c\}$ is closed in Y for every $c \in \mathbb{R}$. One checks directly that $\{\rho_Y \leq c\}$ is contained in $p\{\rho \leq c\}$ which implies that $\{\rho_Y \leq c\}$ is compact, hence that ρ_Y is exhaustive.

Finally we show that ρ_Y is plurisubharmonic. Since this can be checked locally, let $U \subset Y$ be open and isomorphic to the unit ball \mathbb{B}_n so that $p^{-1}(U) \simeq \mathbb{B}_n \times C$ where $C \simeq \mathbb{C}^k/\Gamma_{k+l}$ is a Cousin group. For fixed $z \in \mathbb{B}_n$ let ρ_z be the plurisubharmonic function $C \rightarrow \mathbb{R}$, $g \mapsto \rho(z, g)$. Its pull-back to \mathbb{C}^k is Γ_{k+l} -invariant and plurisubharmonic. Since the image of the complex vector subspace $V := \langle \Gamma_{k+l} \rangle_{\mathbb{R}} \cap i \langle \Gamma_{k+l} \rangle_{\mathbb{R}}$ of \mathbb{C}^k in C is an immersed complex submanifold which is dense in the compact torus $\langle \Gamma_{k+l} \rangle_{\mathbb{R}}/\Gamma_{k+l}$, the pull-back of ρ_z is invariant under $\langle \Gamma_{k+l} \rangle_{\mathbb{R}}$. Hence, it pushes down to a plurisubharmonic function $\overline{\rho}_z$ on \mathbb{C}^k/V which is still invariant under $\langle \Gamma_{k+l} \rangle_{\mathbb{R}}/V$. Since $\langle \Gamma_{k+l} \rangle_{\mathbb{R}}/V$ is a real form of \mathbb{C}^k/V , we may apply Kiselman's minimum principle (see [Kis78]) to the plurisubharmonic function $\overline{\rho}: \mathbb{B}_n \times (\mathbb{C}^k/V) \rightarrow \mathbb{R}$ and obtain plurisubharmonicity of

$$z \mapsto \rho_Y(z) = \inf\{\overline{\rho}(z, w); w \in \mathbb{C}^k/V\}$$

as was to be shown.

In the second case we may apply Kiselman's minimum principle to an $(S^1)^k$ -invariant plurisubharmonic exhaustion of X and essentially repeat the above argument. \square

Example 7.2 will show that pseudoconvexity of Y does not imply pseudoconvexity of X .

7. PSEUDOCONVEX REDUCTIVE HOMOGENEOUS SPACES

Recall that the holomorphic reduction of $X = G/H$ is given by $\pi: G/H \rightarrow G/J$ where J is a closed complex subgroup of G containing H such that G/J is holomorphically separable and $\mathcal{O}(G/H) \simeq \pi^* \mathcal{O}(G/J)$. More precisely, one has

$$J = \{g \in G; f(gH) = f(eH) \text{ for all } f \in \mathcal{O}(G)\}.$$

If $X = G/H$ is holomorphically convex, then the holomorphic reduction $X = G/H \rightarrow Y = G/J$ coincides with the Remmert reduction of X , i.e., Y is Stein and the fiber J/H is connected and compact. Conversely, if $X = G/H$ admits an equivariant map onto a Stein manifold with connected compact fibers, then X is holomorphically convex.

If D is a pseudoconvex non-Stein domain in $X = G/H$ containing $x_0 = eH$, then the complex subgroup \widehat{H} constructed in Theorem 3.1 must be contained in J . However, even if $D = X = G/H$ is pseudoconvex non-Stein, the group \widehat{H} does not necessarily coincide with J as the example of a non-compact Cousin group shows.

In this section we suppose that G is connected complex reductive. We shall see that the base of the holomorphic reduction of a pseudoconvex $X = G/H$ is Stein. We begin with the case that G is semisimple.

Theorem 7.1. *Let $X = G/H$ be a complex homogeneous manifold with G a complex semisimple Lie group. If X is pseudoconvex, then X is holomorphically convex.*

Proof. Due to [BO73] the base of the holomorphic reduction $\pi: X = G/H \rightarrow G/J$ is quasi-affine and J is an algebraic subgroup of G . Hence, π factorizes as

$$\begin{array}{ccc} X = G/H & \xrightarrow{\pi} & G/J \\ & \searrow & \nearrow \\ & G/\overline{H} & \end{array}$$

where \overline{H} is the Zariski-closure of H in G . Moreover, we have $\mathcal{O}(G)^H = \mathcal{O}(G)^{\overline{H}}$, see [BO73]. This shows that G/J is also the holomorphic reduction of G/\overline{H} .

Let $\varphi: X \rightarrow \mathbb{R}$ be a plurisubharmonic exhaustion function. By [Ber87], [BO88] we get that φ , considered as a function on G , is already invariant under the right \overline{H} -action on G . Therefore φ pushes down to a plurisubharmonic exhaustion function on the homogeneous quotient of algebraic groups G/\overline{H} and \overline{H}/H is compact. Then Theorem 5.1 gives the existence of an algebraic reductive subgroup $\widehat{H} \subset G$ containing \overline{H} with \widehat{H}/\overline{H} compact. Note that $\widehat{H} = J$. Considering now the fibration $G/H \rightarrow G/\widehat{H}$, the claim follows. Here we used implicitly the fact that quotients of reductive groups are Stein, if and only if the isotropy is also reductive (see [Mat60], [On60]). \square

Example 7.2. Let us give an example of a non-pseudoconvex semisimple manifold. Let $G = \mathrm{SL}(3, \mathbb{C})$ and take $\Gamma \simeq \mathbb{Z}$ as a discrete subgroup of its maximal torus $T \simeq \mathbb{C}^* \times \mathbb{C}^*$ such that T/Γ is a non-compact Cousin group. Then we have $\overline{\Gamma} = T$ and the holomorphic reduction of $X = G/\Gamma$ is the Stein manifold G/T . However, $X = G/\Gamma$ is not pseudoconvex since the fiber T/Γ is not compact.

The above theorem does not hold in the reductive case as the simple example of a Cousin groups shows. The following non-abelian and (in general) non-holomorphically convex examples of pseudoconvex reductive manifolds indicate their complexity.

Example 7.3. Let $G := \mathrm{SL}(2, \mathbb{C}) \times \mathbb{C}^*$ and $H \simeq \mathbb{C}^* \times \mathbb{C}^*$ the subgroup of G given by the product of the diagonal matrices D in $\mathrm{SL}(2, \mathbb{C})$ and the second factor in G . Let $\Gamma \simeq \mathbb{Z}$ be

a discrete subgroup of H and $J \subset H$ the smallest connected real subgroup containing Γ and the maximal compact subgroup of H . Then $J \simeq S^1 \times S^1 \times \mathbb{R}$. Consider the complex homogeneous space $X := G/\Gamma$. We shall see that X is pseudoconvex.

If $\dim_{\mathbb{R}} \mathrm{SL}(2, \mathbb{C}) \cap J = 1$, we get that the real fibration $X = G/\Gamma \rightarrow G/J$ has compact fibers and that $\mathrm{SL}(2, \mathbb{C})$ acts transitively on the base $G/J = \mathrm{SL}(2, \mathbb{C})/K$ where $K \subset D$ is the compact diagonal in $\mathrm{SL}(2, \mathbb{C})$. Therefore a left K -invariant plurisubharmonic exhaustion function on $\mathrm{SL}(2, \mathbb{C})$ induces a plurisubharmonic exhaustion function on X . Note that there are two possibilities for the Zariski-closure of Γ : First it might be isomorphic to \mathbb{C}^* in which case X is an elliptic curve bundle over $\mathrm{SL}(2, \mathbb{C})$, then holomorphically convex and Kähler, see [GMO11, Theorem 5.1]. On the other hand, for generic Γ , the manifold X is a holomorphic fiber bundle with non-compact Cousin fibers over the affine quadric, so it is not holomorphically convex. Depending on whether the projection of Γ to the central \mathbb{C}^* -factor is closed or not, X may or may not be Kähler.

If $\dim_{\mathbb{R}} \mathrm{SL}(2, \mathbb{C}) \cap J = 2$, then $\Gamma \subset D \subset \mathrm{SL}(2, \mathbb{C})$ and X is an elliptic curve bundle over the product of the affine quadric and \mathbb{C}^* . Hence X is pseudoconvex and holomorphically convex but, since the $\mathrm{SL}(2, \mathbb{C})$ -orbits are not closed in X , these examples are not Kähler.

Theorem 7.4. *Let G be connected complex reductive and $X = G/H$ be pseudoconvex. Then the holomorphic reduction G/J of X is Stein and we have $\mathcal{O}(J/H) = \mathbb{C}$.*

Proof. We define N to be the union of all connected components of the normalizer $\mathcal{N}_G(H)$ of H in G which meet H so that N/H is a connected complex Lie group. Note that N contains $Z := \mathcal{L}(G)^0 \simeq (\mathbb{C}^*)^k$.

Consider the holomorphic principal bundle $X = G/H \rightarrow G/N$ with structure group N/H and let $N/H \rightarrow N/I$ be the holomorphic reduction of its fiber. We obtain a new principal bundle $X = G/H \rightarrow G/I$ whose fiber I/H is now a Cousin group. Due to Lemma 6.1 the base G/I is again pseudoconvex. Moreover, G/H and G/I have the same holomorphic reduction. Suppose that $\dim G/I < \dim G/H$ holds. Arguing by induction over $\dim G/H$ we may assume that the holomorphic reduction of G/I is Stein, hence the same is true for $X = G/H$.

Therefore we must deal with the case $\dim G/I = \dim G/H$. This implies that $I = H$, i.e., that N/H is Stein. As noted above, the group $Z \simeq (\mathbb{C}^*)^k$ acts holomorphically on N/H . Since the latter is Stein, Z has a closed orbit in N/H and since Z is normal, all of its orbits are closed. In particular, we have $Z \cap H \simeq (\mathbb{C}^*)^{l_1}$. Thus we obtain a fibration $X = G/H \rightarrow G/(HZ)$ which is a principal bundle with structure group $(\mathbb{C}^*)^{k-l_1}$. Due to Lemma 6.1 its base $G/(HZ)$ is pseudoconvex. Since the derived group G' acts transitively on $G/(HZ)$, Theorem 7.1 yields that $G/(HZ)$ is holomorphically convex. Let us consider the commutative diagram

$$\begin{array}{ccc}
 G/H & \xrightarrow{\quad} & G/\overline{H} \\
 \downarrow & \searrow \pi & \downarrow \\
 & G/(\overline{H} \cap (HZ)) & \\
 \downarrow & \swarrow & \downarrow \\
 G/HZ & \xrightarrow{\quad} & G/\overline{HZ}
 \end{array}$$

where the bars denote the Zariski closure inside G . It is sufficient to show that G/\overline{H} is pseudoconvex since G/\overline{H} is then holomorphically convex by Theorem 5.1 and since the holomorphic reductions of G/H and G/\overline{H} coincide by [BO73].

The proof of Theorem 7.1 shows that \overline{HZ}/HZ is compact. Since Z is an algebraic subgroup of G , we have $\overline{HZ} = \overline{HZ}$. This implies that $\overline{HZ}/\overline{H} \simeq Z/(Z \cap \overline{H}) \simeq (\mathbb{C}^*)^{k-l_2}$. Consider the fiber bundle $\overline{H}/H \rightarrow \overline{H}/(\overline{H} \cap (HZ))$. The torus $\overline{H} \cap Z$ acts transitively on its fiber. Hence this fiber satisfies $(\overline{H} \cap (HZ))/H \simeq (\overline{H} \cap Z)/(H \cap Z) \simeq (\mathbb{C}^*)^{l_2-l_1}$. Consequently, the map π is a $(\mathbb{C}^*)^{l_2-l_1}$ -principal bundle and Lemma 6.1 shows that $G/(\overline{H} \cap (HZ))$ is pseudoconvex. Again from $\overline{HZ} = \overline{HZ}$ we conclude that \overline{H} acts transitively on \overline{HZ}/HZ , hence that $\overline{H}/(\overline{H} \cap (HZ)) \simeq \overline{HZ}/HZ$ is compact. Therefore G/\overline{H} is pseudoconvex, thus G/J is Stein.

We still must prove $\mathcal{O}(J/H) = \mathbb{C}$. Since G/J is Stein, J is reductive by [Mat60] or [On60]. Now consider the holomorphic reduction $J/H \rightarrow J/I$. Since J/H is a closed complex submanifold of $X = G/H$, it is pseudoconvex, hence by the above J/I is Stein. Therefore, I is reductive and G/I is Stein. From the factorization $G/H \rightarrow G/I \rightarrow G/J$ we obtain $I = J$. Hence $\mathcal{O}(J/H) = \mathbb{C}$, as was to be shown. \square

A compact Kähler G/H is a product of a compact complex torus and a homogeneous flag manifold; see Matsushima [Mat57] and Borel-Remmert [BR62]. For G reductive we have the following extension of this result for a Kähler pseudoconvex G/H under the assumption that $\mathcal{O}(G/H) \simeq \mathbb{C}$. In particular, Theorem 7.5 applies to the fiber of the holomorphic reduction of any Kähler pseudoconvex reductive homogeneous manifold because of Theorem 7.4.

Theorem 7.5. *Let G be connected complex reductive and $X = G/H$ be pseudoconvex and Kähler with $\mathcal{O}(X) \simeq \mathbb{C}$. Then G/\overline{H} is a homogeneous rational manifold, \overline{H}/H is a Cousin group, and the bundle $G/H \rightarrow G/\overline{H}$ is holomorphically trivial.*

Remark 7.6. Note that for any Kähler pseudoconvex reductive homogeneous manifold, since the base G/J of its holomorphic reduction $G/H \rightarrow G/J$ is Stein, J/\overline{H} is compact if and only if G/\overline{H} is holomorphically convex. Moreover, J is reductive and $\widehat{\mathfrak{h}} = \mathfrak{j}' \oplus V$ in this setting, where \mathfrak{j}' denotes the derived Lie algebra of \mathfrak{j} and V is the complex vector subspace of \mathbb{C}^k defined in the proof of Lemma 6.1 with the space \mathbb{C}^k considered as the Lie algebra of the Cousin group C . Note also that Example 7.3 shows that in general $G/H \rightarrow G/J$ is not holomorphically trivial.

Proof. In order to prove the first claim we will show that G/\overline{H} is pseudoconvex. Since X is Kähler, $G' \cap H$ is algebraic by [GMO11, Theorem 5.1]. Note that $G' \cap H$ is a closed normal subgroup of H and that $H/(G' \cap H)$ is an Abelian complex Lie group. Since $G' \cap H$ is algebraic, $\mathcal{N}_G(G' \cap H)$ is also an algebraic subgroup of G , hence must contain \overline{H} . Thus $\overline{H}/(G' \cap H)$ is an affine algebraic group containing $H/(G' \cap H)$ as a Zariski-dense closed complex subgroup. This implies that $\overline{H}/(G' \cap H)$ is Abelian as well. Hence

$$\overline{H}/H \cong (\overline{H}/(G' \cap H))/(H/(G' \cap H))$$

is an Abelian complex Lie group which cannot have a factor isomorphic to \mathbb{C} . Therefore we have shown that $G/H \rightarrow G/\overline{H}$ is a holomorphic principal bundle with fiber the product of a Cousin group and possibly $(\mathbb{C}^*)^k$. Applying Lemma 6.1 we see that G/\overline{H} is pseudoconvex, thus holomorphically convex by Theorem 5.1. Since $\mathcal{O}(G/H) = \mathbb{C}$, the space G/\overline{H} is compact and thus homogeneous rational.

In order to complete the proof we will show that the principal bundle $G/H \rightarrow G/\overline{H}$ is holomorphically trivial and that \overline{H}/H is a Cousin group and we show the latter first. Since $\overline{H}/H = C \times (\mathbb{C}^*)^k$ where C is a Cousin group, we have the factorization $G/H \rightarrow G/L \rightarrow G/\overline{H}$ where $L/H = C$ et $\overline{H}/L = (\mathbb{C}^*)^k$. Due to Lemma 7.7 below, the principal bundle $G/L \rightarrow G/\overline{H}$ is holomorphically trivial. Hence, $k \geq 1$ would contradict the fact that every holomorphic function on G/H is constant. This shows that \overline{H}/H is a Cousin group.

Finally, let us consider the two fibrations

$$\begin{array}{ccc} G/H & \xrightarrow{p_1} & G/\overline{H} \simeq G'/(G' \cap \overline{H}) \\ & & \downarrow p_2 \\ & & G/G'H, \end{array}$$

where $G'/(G' \cap \overline{H})$ is homogeneous rational and $G/G'H$ is a Cousin group. The restriction of p_1 to the p_2 -fiber $G'/(G' \cap H)$ is still surjective. Thus it is a holomorphic bundle with fiber the algebraic variety $(G' \cap \overline{H})/(G' \cap H)$ which is a closed subgroup of the Cousin group \overline{H}/H . This implies that $(G' \cap \overline{H})/(G' \cap H)$ is finite. Since the parabolic group $G' \cap \overline{H}$ is connected, we obtain $G' \cap \overline{H} = G' \cap H$, and hence G/H is the product of G/\overline{H} and \overline{H}/H . \square

Lemma 7.7. *Let S be a connected semisimple complex Lie group, let P be a parabolic subgroup of S , and let $p: X \rightarrow S/P$ be an equivariant holomorphic principal bundle with structure group $T = (\mathbb{C}^*)^k$. If X is pseudoconvex, then the bundle is trivial.*

Proof. An equivariant holomorphic T -principal bundle $p: X \rightarrow S/P$ is of the form $X \simeq S \times_P T$ where the twisted product is defined by a holomorphic group homomorphism $P \rightarrow T$. Since T is Abelian, this homomorphism factorizes over $P/P' \simeq (\mathbb{C}^*)^l$. Since P/P' is reductive, the factorized homomorphism is algebraic and in particular its image is an algebraic subtorus \tilde{T} of T . Hence, we obtain a second fiber bundle

$$\begin{array}{ccc} X \simeq S \times_P T & \longrightarrow & T/\tilde{T} \\ & & \downarrow p \\ & & S/P. \end{array}$$

Let us assume in a first step that $S = \mathrm{SL}(2, \mathbb{C})$ and thus that P is a Borel subgroup of S . Then $P/P' \simeq \mathbb{C}^*$ and the bundle $X \simeq S \times_P T \rightarrow S/P \simeq \mathbb{P}_1$ is non-trivial if and only if $P/P' \rightarrow T$ is non-constant. If this is the case, the fiber of $S \times_P T \rightarrow T/\mathbb{C}^*$ is isomorphic to a finite quotient of $S/P' \simeq \mathbb{C}^2 \setminus \{0\}$, hence we find a closed embedding of such a finite quotient of $\mathbb{C}^2 \setminus \{0\}$ inside X . Since this implies that X is not pseudoconvex, we have proved the claim for $S = \mathrm{SL}(2, \mathbb{C})$.

For arbitrary S we find root subgroups $S_\alpha \simeq \mathrm{SL}(2, \mathbb{C})$ of S such that $S_\alpha \cap P$ is a Borel in S_α . If the homomorphism $P \rightarrow T$ is not trivial, then its restriction to $S_\alpha \cap P$ is not trivial for some root α . Since $S_\alpha \times_{(S_\alpha \cap P)} T$ is a closed complex submanifold of X , this is in contradiction with the previous case. \square

The following example shows that a general pseudoconvex reductive homogeneous manifold is *not* a Cousin bundle over a holomorphically convex manifold.

Example 7.8. Let $\Gamma \subset S = \mathrm{SL}(2, \mathbb{C})$ be a cocompact discrete subgroup such that Γ/Γ' contains an element of infinite order. Existence of such Γ is shown in e.g. [Mill76]. Then there is a homomorphism $\varphi: \Gamma \rightarrow \mathbb{C}^*$ with dense image in $S^1 \subset \mathbb{C}^*$. We define the reductive homogeneous manifold $X = G/\Gamma_G$ where $G := S \times \mathbb{C}^*$ and Γ_G is the graph of φ , hence a discrete subgroup of G . By construction, X is the total space of a holomorphic \mathbb{C}^* -principal bundle over the compact base S/Γ .

We claim that X is pseudoconvex. Let ρ be an S^1 -invariant strictly plurisubharmonic exhaustion of \mathbb{C}^* . Then the function $G \rightarrow \mathbb{R}^{\geq 0}$, $(s, z) \mapsto \rho(z)$, is a Γ_G -invariant plurisubharmonic function of G , hence descends to a plurisubharmonic function on $X = G/\Gamma_G$. The

fibers of this function are the closures of finitely many S -orbits in X , thus compact. Therefore X is indeed pseudoconvex.

One sees that $\widehat{\Gamma}_G = S$ has no locally closed orbit in X and that $\mathcal{O}(X) = \mathbb{C}$ so that X is neither Kähler nor holomorphically convex.

8. THE STRUCTURE OF PSEUDOCONVEX SOLVMANIFOLDS

In this section we prove a structure theorem for pseudoconvex solvmanifolds, i.e., homogeneous spaces $X = G/H$ where G is connected solvable. Replacing G by its universal covering we will assume from now on that G is simply connected. Note that then every connected Lie subgroup of G is automatically closed and simply connected.

We start with the following observation from [Huc10] which deals with the nilpotent case.

Theorem 8.1. *Assume that G is nilpotent and that H is a closed complex subgroup of G . Then the complex nilmanifold $X = G/H$ is pseudoconvex.*

Proof. The normalizer $\mathcal{N}_G(H^0)$ in G of the connected component H^0 of the identity of H is connected, hence also simply connected, e.g., see Lemma 2 in [Mat60]. So $G/\mathcal{N}_G(H^0)$ is biholomorphic to \mathbb{C}^k for some k . By the Oka Principle the bundle $G/H \rightarrow G/\mathcal{N}_G(H^0)$ is holomorphically trivial. Therefore we need only consider its fiber which has the form N/Γ , where $\Gamma := H/H^0$ is a discrete subgroup of the simply connected group $N := \mathcal{N}_G(H^0)/H^0$. In the case of a connected, simply connected nilpotent Lie group the exponential map $\exp: \mathfrak{n} \rightarrow N$ is biholomorphic. The pre-image of Γ in \mathfrak{n} spans a (real) Lie subalgebra \mathfrak{n}_Γ whose associated connected Lie group N_Γ contains Γ cocompactly. We now set $\widehat{\mathfrak{n}}_\Gamma := \mathfrak{n}_\Gamma + i\mathfrak{n}_\Gamma$ and let \widehat{N}_Γ denote the corresponding connected complex Lie group. Then N/\widehat{N}_Γ is biholomorphic to \mathbb{C}^l for some l and applying the Oka Principle again we see that N/Γ is biholomorphic to the product $\widehat{N}_\Gamma/\Gamma \times \mathbb{C}^l$. So finally we see that it suffices to consider the nilmanifold $\widehat{N}_\Gamma/\Gamma$ in order to ensure the existence of a pseudoconvex exhaustion on X .

Setting $\mathfrak{m} := \mathfrak{n}_\Gamma \cap i\mathfrak{n}_\Gamma$ and letting M denote the corresponding closed complex subgroup of \widehat{N}_Γ , we consider the pair $(\widehat{N}_\Gamma/M, N_\Gamma/M)$. Note that this pair is pseudoconvex in the sense of Loeb [Loe85], since nilpotent Lie algebras always have purely imaginary spectra. So there exists an (N_Γ/M) -right invariant strictly plurisubharmonic exhaustion function on \widehat{N}_Γ/M that pulls back to an N_Γ -right invariant plurisubharmonic exhaustion function on \widehat{N}_Γ . Thus we see that the nilmanifold $\widehat{N}_\Gamma/\Gamma$ is pseudoconvex. It then follows that the original nilmanifold G/H is also pseudoconvex. \square

Definition 8.2. A (principal) Cousin group tower of length one is a Cousin group. A (principal) Cousin group tower of length $n > 1$ is a (principal) holomorphic bundle with fiber a Cousin group and base a (principal) Cousin group tower of length $n - 1$.

The fiber of the holomorphic reduction of a nilmanifold carries no non-constant holomorphic functions, in fact it is a Cousin group tower. For a general solvmanifold $X = G/H$ this is no longer true as we have seen in Example 2.6 where the fiber of the holomorphic reduction is $\mathbb{C}^* \times \mathbb{C}^*$. However, in the rest of this section we will prove that pseudoconvexity of the solvmanifold $X = G/H$ is sufficient (though not necessary, see Example 8.6) to ensure that the fiber of its holomorphic reduction is a Cousin group tower and therefore has no non-constant holomorphic functions.

We will need the following observation.

Lemma 8.3. *Let G be a connected complex Lie group and $\Gamma \subset G$ a discrete subgroup. Furthermore let $H_1 \subset H_2 \subset G$ be two closed connected complex subgroups which are both normalized by Γ . Suppose that ΓH_2 is closed in G and that $(\Gamma \cap H_2)H_1$ is closed in H_2 . Then ΓH_1 is closed in G .*

Proof. Any subgroup of a not necessarily connected Lie group L is closed in L if and only if its intersection with L^0 is closed in L^0 . Therefore ΓH_1 is closed in ΓH_2 (and hence closed in G) if and only if $\Gamma H_1 \cap H_2$ is closed in H_2 . Since $\Gamma H_1 \cap H_2 = (\Gamma \cap H_2)H_1$, the lemma is proved. \square

We prove our result first for a special class of solvmanifolds.

Proposition 8.4. *Let $X = G/\Gamma$ be a solvmanifold with $\Gamma \subset G$ discrete such that $\mathcal{O}(X) \simeq \mathbb{C}$. If X is pseudoconvex, then there is a connected normal Abelian subgroup $C \subset G$ such that ΓC is closed in G and the fibration $G/\Gamma \rightarrow G/\Gamma C$ has the Cousin group $C\Gamma/\Gamma$ as fiber.*

Proof. Since G' is a connected nilpotent subgroup of G , the exponential map $\exp: \mathfrak{g}' \rightarrow G'$ is biholomorphic and we can construct the smallest connected complex subgroup of G' containing Γ' as in the proof of Theorem 8.1, i.e., by $S_1 := \widehat{G}'_{\Gamma'}$. Let $L \subset G$ be the identity component of the centralizer of S_1 in G . As remarked in the proof of [BO69, Abspaltungssatz 3.2], the group $L\Gamma$ is a closed subgroup of G . Therefore, it follows from [BO69, Satz 1.1] that L is normal in G . One gets the fibration

$$X = G/\Gamma \rightarrow G/L\Gamma.$$

By hypothesis, X is a pseudoconvex manifold and therefore there is a maximal connected complex subgroup $\widehat{H} \subset G$ such that \widehat{H} is normalized by Γ and $\widehat{H}\Gamma/\Gamma$ is relatively compact in X . Since $\mathcal{O}(X) \simeq \mathbb{C}$, we have $\widehat{H} \triangleleft G$. Consider the Lie algebra $[\mathfrak{g}, \widehat{\mathfrak{h}}] \subset \mathfrak{g}'$. If $[\mathfrak{g}, \widehat{\mathfrak{h}}] = 0$, then $\widehat{\mathfrak{h}} \subset \mathfrak{z}(\mathfrak{g})$ and hence $\widehat{\mathfrak{h}} \subset \mathfrak{l}$. If $[\mathfrak{g}, \widehat{\mathfrak{h}}] \neq 0$, then $0 \neq \widehat{\mathfrak{h}} \cap \mathfrak{g}' \triangleleft \mathfrak{g}$ and in particular $\widehat{\mathfrak{h}} \cap \mathfrak{g}' \triangleleft \mathfrak{g}'$. The Lie algebra $\widehat{\mathfrak{h}} \cap \mathfrak{g}'$ being normal in \mathfrak{g}' , it follows that $(\widehat{\mathfrak{h}} \cap \mathfrak{g}') \cap \mathfrak{z}(\mathfrak{g}') \neq 0$. Since $\mathfrak{z}(\mathfrak{g}') \subset \mathfrak{l}$, it follows that in both cases $\widehat{\mathfrak{h}}_{\mathfrak{l}} := \widehat{\mathfrak{h}} \cap \mathfrak{l} \neq 0$ is a non-trivial ideal in \mathfrak{g} .

Let $\tilde{L} \subset L$ be the smallest connected subgroup containing $\widehat{H}_L := \widehat{H} \cap L$ such that \tilde{L} is normalized by Γ , $\tilde{L}\Gamma$ is closed in G and $\mathcal{O}(\tilde{L}\Gamma/\Gamma) \simeq \mathbb{C}$. Then \tilde{L} again is a normal subgroup of G . In order to simplify the notation, we omit the tilde and now have the fibration

$$X = G/\Gamma \rightarrow G/L\Gamma$$

with $\mathcal{O}(L\Gamma/\Gamma) \simeq \mathbb{C}$. Note that the fiber is isomorphic to $L/(L \cap \Gamma)$ where L is a simply connected solvable Lie group. Now define $\Lambda := \Gamma \cap L$. Since $L \cap \widehat{H}$ is non-trivial, the discrete group Λ is also non-trivial and a normal subgroup of Γ . We will now apply the same reduction steps as before to the homogeneous manifold L/Λ which has the same properties as $X = G/\Gamma$. In order to be able to carry over the results of this iteration to $L\Gamma/\Gamma$ we must check carefully that all the groups constructed inside L are normalized by Γ .

If Λ' is trivial, then Λ and hence L are abelian and the lemma is proved. Therefore suppose that $S_2 := \widehat{L}'_{\Lambda'} \subset L'$ is non-trivial and let L_2 be the connected component of the centralizer of S_2 in L . Since $\Lambda \triangleleft \Gamma$, we have that Γ normalizes S_2 and hence L_2 . As a consequence we get that $L_2 \triangleleft G$. Furthermore $\widehat{\mathfrak{h}}_{\mathfrak{l}} \cap \mathfrak{l}_2 \neq 0$, for the same reasons as above. By Lemma 8.3 we get that ΓL_2 is a closed subgroup of G .

Now we can iterate the construction from the proof of [BO69, Abspaltungssatz 3.2] to produce a chain of subgroups $L \supset L_2 \supset \dots$ in G which are normalized by Γ and such that ΓL_j is closed in G . Since this process must terminate after finitely many steps, we finally get the Abelian group C such that $\mathcal{O}(C\Gamma/\Gamma) \simeq \mathbb{C}$ as claimed. \square

After these preparations we are able to prove the main result of this section.

Theorem 8.5. *Suppose $X = G/H$ is a pseudoconvex solvmanifold with holomorphic reduction $G/H \rightarrow G/J$. Then the base G/J is Stein and the fiber J/H is a Cousin group tower. In particular, $\mathcal{O}(J/H) \simeq \mathbb{C}$.*

Proof. The fact that G/J is Stein is proven in [HO86]. We have $\widehat{\mathfrak{h}} \subset \mathfrak{j}$ since the base is holomorphically separable by definition. We claim that $\mathcal{O}(J/H) \simeq \mathbb{C}$. If not, then there would exist a holomorphic reduction $J/H \rightarrow J/J_1$ of J/H with $\dim J > \dim J_1$. We would continue taking holomorphic reductions in this way: given J_{n-1} we define the closed complex subgroup J_n by means of the holomorphic reduction $J_{n-1}/H \rightarrow J_{n-1}/J_n$, where $\dim J_n < \dim J_{n-1}$. The process stops after a finite number of steps and we obtain a subgroup J_k with $\mathcal{O}(J_k/H) \simeq \mathbb{C}$, where we assume that k is the smallest positive integer such that J_k/H has this property. Note that $\dim J_k/H > 0$, since recursively we have $\widehat{\mathfrak{h}} \subset \mathfrak{j}_n$ for $1 \leq n \leq k$ and $\dim \widehat{\mathfrak{h}} > \dim \mathfrak{h}$.

We claim that J_k/H is a Cousin group tower. Since $J_k/\mathcal{N}_{J_k}(H^0)$ is Stein by Lie's Flag Theorem, it follows that $\mathcal{N}_{J_k}(H^0) = J_k$ and thus the isotropy is discrete. In this case we write Γ instead of H . We now apply Proposition 8.4 to J_k/Γ and get the requisite subgroup $C \subset J_k$ such that $J_k/\Gamma \rightarrow J_k/C\Gamma$ has a Cousin group as fiber. By Lemma 6.1 the base $J_k/C\Gamma$ is pseudoconvex and $\mathcal{O}(J_k/C\Gamma) \simeq \mathbb{C}$. By recursion, J_k/Γ is a Cousin group tower and so by repeated use of Lemma 6.1 we conclude that G/J_k is pseudoconvex.

Finally, we shall show that $k \geq 1$ yields a contradiction. Consider the subgroups $H \subset J_k \subset J_{k-1} \subset J_{k-2} \subset G$, where we set $J_{-1} := G$ and $J_0 := J$. Then J_{k-2}/J_k , as a closed submanifold of G/J_k , is pseudoconvex. Moreover, we have the fibration $J_{k-2}/J_k \rightarrow J_{k-2}/J_{k-1}$. Both J_{k-2}/J_{k-1} and J_{k-1}/J_k are holomorphically separable implying J_{k-2}/J_k is Stein by Remark 2.5. This implies J_{k-2}/J_k is the holomorphic reduction of J_{k-2}/H . But, by construction J_{k-2}/J_{k-1} is the holomorphic reduction of J_{k-2}/H . Since $\dim J_k < \dim J_{k-1}$, we obtain the desired contradiction. *A posteriori* we see that $J = J_k$, i.e., that $\mathcal{O}(J/H) \simeq \mathbb{C}$. The argument given in the previous paragraph then shows that J/H is indeed a Cousin group tower. \square

We finish with an example which shows that the converse of Theorem 8.5 does not hold.

Example 8.6. Let $n \geq 3$ and $A \in \mathrm{SL}(n, \mathbb{Z})$ such that A is diagonalizable over \mathbb{C} and admits $s > 0$ positive real eigenvalues $\alpha_1, \dots, \alpha_s$ and $t > 0$ pairs of complex conjugate eigenvalues $\beta_1, \bar{\beta}_1, \dots, \beta_t, \bar{\beta}_t$. Note that $n = s + 2t$. Assume furthermore that the characteristic polynomial of A is irreducible over \mathbb{Q} . In particular $\alpha_j \neq 1$ for all j . The existence of such an A is easily seen by using elementary number theory, see e.g. [OT05]. The fact that A is diagonalizable implies that there is a *real* logarithm $D \in \mathfrak{sl}(n, \mathbb{R})$ of A . This means that the one-parameter group $\{\exp(xD) \mid x \in \mathbb{R}\}$ lies in $\mathrm{SL}(n, \mathbb{R})$, which justifies the following construction. For $K = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ define a solvable group structure $G_K := K \times K^n$ on the cartesian product by

$$(x_1, v_1) \cdot (x_2, v_2) := (x_1 + x_2, \exp(x_1 D)v_2 + v_1).$$

Note that $G_{\mathbb{Z}}$ is discrete cocompact in $G_{\mathbb{R}}$ and that $G_{\mathbb{R}}$ is a real form of the simply-connected solvable complex Lie group $G_{\mathbb{C}}$. Now let $H \subset \{0\} \times \mathbb{C}^n \subset G_{\mathbb{C}}$ be the t -dimensional complex subgroup generated by A -eigenvectors corresponding to the eigenvalues $\beta_1, \beta_2, \dots, \beta_t$. It was shown in [OT05] that $C := \{0\} \times \mathbb{C}^n / (\{0\} \times \mathbb{Z}^n)H$ is a Cousin group. Note that H is a normal subgroup of $G_{\mathbb{C}}$. Now define $X := G_{\mathbb{C}}/G_{\mathbb{Z}}H$. We get a (non-principal) fibration $X := G_{\mathbb{C}}/G_{\mathbb{Z}}H \rightarrow G_{\mathbb{C}}/G_{\mathbb{Z}}(\{0\} \times \mathbb{C}^n)$ with the Cousin group C as fiber and \mathbb{C}^* as base.

Suppose that X is pseudoconvex and choose a plurisubharmonic exhaustion function. Since $G_{\mathbb{R}}H/G_{\mathbb{Z}}H$ is compact, an integration argument as in [Loe85] gives a plurisubharmonic function on $G_{\mathbb{C}}$ which is $G_{\mathbb{R}}H$ -invariant and is an exhaustion function on $G_{\mathbb{C}}/G_{\mathbb{R}}H$. The maximal connected complex subgroup of $G_{\mathbb{R}}H$ being $H \oplus \bar{H}$, we get the pseudoconvex couple $(G_{\mathbb{C}}/H \oplus \bar{H}, G_{\mathbb{R}}/H \oplus \bar{H})$ in the sense of Loeb. But this couple has eigenvalues $\ln(\alpha_j) \in \mathbb{R}^*$ of the adjoint representation of its real form on itself. This is a contradiction. Therefore X is a Cousin fiber bundle over a Stein manifold but is *not* pseudoconvex.

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