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# Equilibrium fluctuations for the disordered harmonic chain perturbed by an energy conserving noise

Marielle Simon\*

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## Abstract

We investigate the macroscopic behaviour of the disordered harmonic chain of oscillators, through energy diffusion. The Hamiltonian dynamics of the system is perturbed by a degenerate conservative noise. After rescaling space and time diffusively, we prove that energy fluctuations in equilibrium evolve according to a linear heat equation. The diffusion coefficient is obtained from the non-gradient Varadhan's approach, and is equivalently defined through the Green-Kubo formula. Since the perturbation is very degenerate and the symmetric part of the generator does not have a spectral gap, the standard non-gradient method is reviewed under new perspectives.

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## 1 Introduction

In this paper we investigate diffusion behaviours in non-homogeneous media for interacting particle systems. More precisely, we address the problem of energy fluctuations for chains of oscillators with random defects. In the last fifty years, it has been recognized that introducing randomness in interacting particle systems has a drastic effect on the conduction properties of the material [8]. As far as we know, the only mathematically tractable model of oscillators is the one dimensional system with harmonic interactions [1]. The anharmonic case is poorly understood from a rigorous point of view, but since the works of Peierls [22, 23], it is now well understood that non-linear interactions between atoms should play a crucial role in the derivation of the Fourier law. In [2, 5, 7] (among others) it is proposed to model the anharmonicity by stochastic perturbations, in order to recover the expected diffusivity. Being inspired by all these previous works, the aim of this paper is to prove the diffusivity of disordered harmonic chains perturbed by an energy conserving noise. In some sense, the noise simulates the effect of non-linearities, and the conductivity of the one-dimensional chain becomes finite and positive. Moreover

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we prove that all the effects of the disorder are, on a sufficiently large scale, contained in a diffusion coefficient, which depends on the statistics of the field, but not on the randomness itself.

The disorder effect has already been investigated for lattice gas dynamics: the first article dealing with scaling limits of particle systems in random environment is the remarkable work of Fritz [12], and since then the subject has attracted a lot of interest, for example in [11, 14, 20, 24]. These papers share one main feature: the models are non-gradient due to the presence of the disorder. Except for [12], the non-gradient systems are often solved by establishing a microscopic Fourier's law up to a small fluctuating term, following the sophisticated method initially developed by Varadhan in [28], and generalized to non-reversible dynamics [16].

The previous works mainly deal with symmetric systems of particles that evolve according to an exclusion process in random environment: the particles are attempting jumps to nearest neighbour sites at rates which depend on both their position and the objective site, and the rates themselves come from a quenched random field. Different approaches are adopted to tackle the non-gradient feature: whereas the standard Varadhan's method is helpful in dimension  $d \geq 3$  only [11], the "long jump" variation developed by Quastel in [24] is valid in any dimension. The study of disordered chains of oscillators perturbed by a conservative noise has appeared more recently, see for instance [3, 4, 9]. In these papers, the thermal conductivity is defined by the Green-Kubo formula only. Here, the diffusion coefficient is also defined through hydrodynamics.

In [27], we have obtained the diffusive scaling limit for a homogeneous chain of coupled harmonic oscillators perturbed by a noise, which randomly flips the sign of the velocities, so that the energy is conserved but not the momentum. Our first motivation was to investigate the same chain of harmonic oscillators, still perturbed by the velocity-flip noise, but now provided with i.i.d. random masses. In [27], a system of non-linear homogeneous hydrodynamic equations has been derived thanks to the relative entropy method: this system involves the only two conserved quantities (energy and total length of the chain). One of the major ingredient for the proof was an exact *fluctuation-dissipation equation* (see for example [18]), which reproduces the Fourier law at the microscopic level.

The disorder assumption makes all previous computations pointless: in particular, the fluctuation-dissipation equations are not directly solvable any more. To overcome this difficulty, one replaces these exact equations by approximations: more precisely, there exists a sequence of local functions for which an approximate fluctuation-dissipation decomposition holds, in the sense that the difference has a small space-time variance with respect to the dynamics in equilibrium. The main ingredients of the usual non-reversible non-gradient method are: first, a *spectral gap* for the symmetric part of the dynamics, and second, a *sector condition* for the total generator.

Our model has special features that enforce Varadhan's method to be considered with new perspectives. In particular, the symmetric part of the generator (which comes from the stochastic noise) is poorly ergodic, and does not have a spectral gap when restricted to micro-canonical manifolds. Moreover, due to the degeneracy of the perturbation, the asymmetric part of the generator is difficult to control by its symmetric part (in technical terms, the sector condition does not hold), with the only velocity-flip noise. Finally, the energy current depends on the disorder, and has to be approximated by a fluctuation-dissipation equation which takes into account the fluctuations of the disorder itself.

Because of the high degeneracy of the velocity-flip noise, we add a second stochastic perturbation, that exchanges velocities (divided by the square root of mass) and positions at random independent Poissonian times, so that a kind of sector condition can be proved (see Proposition 5.7: we call it the *weak sector condition*). However, the spectral gap estimate and the usual sector condition still do not hold

when adding the exchange noise, meaning that the stochastic perturbation remains very degenerate; the noises are still far from ergodic.

Due to the harmonicity of the chain, the generator of the dynamics preserves the degree of polynomials, and even a degenerate noise is sufficient to apply Varadhan's approach. The sector condition and the non-gradient decomposition are only needed for a specific class of functions. Even if the stochastic noise still does not have a spectral gap, it does make no harm. Contrary to the standard Varadhan's approach, we do not need an in-depth study of the so-called *closed forms*: indeed, in the general theory (we invite the reader to see [26, 15] for more details), one usually proves some strong result concerning the decomposition of *every* closed form, which then is applied only to a specific family of functions. Here, it turns out that proving the decomposition only for these functions is successful and enough to achieving our goals. This is a clear advantage of our model: since some difficult technical parts are in some sense simplified, the usual approach to non-gradient problems becomes slightly neater. The main difficulties appear in the foundations that are necessary for using it. We have made the decision to adopt a detailed level in the redaction of Varadhan's method, even if some proofs may look rather standard for expert readers. We hope that this choice will be beneficial for the reader not familiar with it.

For the non-linear ordered chain, S. Olla and M. Sasada [21] do need a less degenerate noise than ours, in particular both the spectral gap and the sector condition hold. They show that ideas from Varadhan's method can be used to prove a diffusive behaviour of the energy: its fluctuations in equilibrium evolve following an infinite Ornstein-Uhlenbeck process. The covariances characterizing this linearised heat equation are given in terms of the diffusion coefficient, which is defined through a variational formula. We use their ideas to prove energy fluctuations at equilibrium as a consequence of the non-gradient approach.

Furthermore, we show that the diffusion coefficient can be equivalently defined by the Green-Kubo formula. The latter is the space-time variance of the current at equilibrium, which is only formal in the sense that a double limit (in space and time) has to be taken. As in [3], we prove here that the limit exists, and that the homogenization effect occurs for the Green-Kubo formula: for almost every realization of the disorder, the thermal conductivity exists, is independent of the disorder, is positive and finite. Finally, let us introduce  $\gamma > 0$  the intensity of the flip noise, and  $\lambda > 0$  the intensity of the exchange noise. We denote the diffusion coefficient by  $D(\lambda, \gamma)$  when obtained through the variational formula in the Varadhan's method, and by  $\bar{D}(\lambda, \gamma)$  when defined through the Green-Kubo formula. Then, we rigorously prove that the two conductivities are equal:  $D(\lambda, \gamma) = \bar{D}(\lambda, \gamma)$ , when the two intensities  $\lambda, \gamma$  are positive. In addition, we prove in Theorem 7.1 that the Green-Kubo formula remains well defined when  $\lambda = 0$ , that is  $\bar{D}(0, \gamma)$  exists, is finite and positive. Finally, Theorem 7.4 states that  $D(\lambda, \gamma)$  tends to  $\bar{D}(0, \gamma)$  as  $\lambda$  goes to 0. The existence question for  $D(0, \gamma)$  (the hydrodynamics diffusion coefficient) remains open.

Before ending the introduction, let us be more precise on the model and the fluctuation result. We introduce the harmonic Hamiltonian system described by the sequence  $\{p_x, r_x\}_{x \in \mathbb{Z}}$ , where  $p_x$  stands for the momentum of the oscillator at site  $x$ , and  $r_x$  represents the distance between oscillator  $x$  and oscillator  $x + 1$ . Each atom  $x \in \mathbb{Z}$  has a mass  $M_x > 0$ , the velocity of atom  $x$  is given by  $p_x/M_x$ . We assume the disorder  $\mathbf{M} := \{M_x\}_{x \in \mathbb{Z}}$  to be a collection of real i.i.d. positive random variables such that

$$\forall x \in \mathbb{Z}, \quad \frac{1}{C} \leq M_x \leq C,$$

for some finite constant  $C > 0$ . The equations of motions are given by

$$\begin{cases} \frac{dp_x}{dt} = r_x - r_{x-1}, \\ \frac{dr_x}{dt} = \frac{p_{x+1}}{M_{x+1}} - \frac{p_x}{M_x}. \end{cases}$$

The dynamics conserves the total energy

$$\mathcal{E} := \sum_{x \in \mathbb{Z}} \left\{ \frac{p_x^2}{2M_x} + \frac{r_x^2}{2} \right\}.$$

To overcome the lack of ergodicity of deterministic chains<sup>1</sup>, we add a stochastic perturbation to this new dynamics, and prove that the convergence of the energy fluctuations distribution holds (Theorem 3.1). The noise can be easily described: at independently distributed random Poissonian times, the quantity  $p_x/\sqrt{M_x}$  and the interdistance  $r_x$  are exchanged, or the momentum  $p_x$  is flipped into  $-p_x$ . This noise still conserves the total energy  $\mathcal{E}$ , and is very degenerate.

Even if Theorem 3.1 could be proved *mutatis mutandis* for this harmonic chain, for pedagogical reasons we now focus on a simplified model, which has the same features and involves simplified computations<sup>2</sup>. From now on, we study the dynamics on the new configurations  $\{\eta_x\}_{x \in \mathbb{Z}}$  written as

$$m_x d\eta_x = (\eta_{x+1} - \eta_{x-1})dt, \quad (1)$$

where  $\mathbf{m} := \{m_x\}_{x \in \mathbb{Z}}$  is the new disorder with the same characteristics as before. It is notationally convenient to change the variable  $\eta_x$  into  $\omega_x := \sqrt{m_x}\eta_x$ , and the total energy reads

$$\mathcal{E} = \sum_{x \in \mathbb{Z}} \omega_x^2.$$

Let us now introduce the corresponding stochastic energy conserving dynamics: the evolution is described by (1) between random exponential times, and at each ring one of the following interactions can happen:

- a. *Exchange noise* – two nearest neighbour variables  $\omega_x$  and  $\omega_{x+1}$  are exchanged;
- b. *Flip noise* – the variable  $\omega_x$  at site  $x$  is flipped into  $-\omega_x$ .

As a consequence of these two perturbations, the dynamics conserves the total energy only, the other important conservation laws of the Hamiltonian part being destroyed by the stochastic noises<sup>3</sup>. The following family  $\{\mu_\beta\}_{\beta > 0}$  of grand-canonical Gibbs measures is invariant for the process:

$$\mu_\beta(d\omega) := \prod_{x \in \mathbb{Z}} \sqrt{\frac{2\pi}{\beta}} \exp\left(-\frac{\beta}{2}\omega_x^2\right) d\omega_x. \quad (2)$$

The index  $\beta$  stands for the inverse of the temperature. Notice that with our notational convenience,  $\mu_\beta$  does not depend on the disorder. Observe also that the dynamics is not reversible with respect to the

<sup>1</sup>In the deterministic system of harmonic oscillators, it is well known that the energy is ballistic, destroying the validity of the Fourier law. The remarkable work of Liebowitz, Lieb and Rieder [19] is the standard reference for more details.

<sup>2</sup>We invite the reader to see [6] for the origin of this new particle system.

<sup>3</sup>It is now well understood that the ballisticity of the harmonic chain is due to the infinite number of conserved quantities. In 1994, Fritz, Funaki and Lebowitz [13] propose different stochastic noises that are sufficient to destroy the ballisticity of the chain: the *velocity-flip* noise is one of them.

measure  $\mu_\beta$ . We define  $\mathbf{e}_\beta$  as the thermodynamical energy associated to  $\beta$ , namely the expectation of  $\omega_0^2$  with respect to  $\mu_\beta$ , and  $\chi(\beta) = 2\beta^{-2}$  as the variance of  $\omega_0^2$  with respect to  $\mu_\beta$ .

We consider the system starting with  $\mu_\beta$  and we denote by  $\mathbb{E}_{\mu_\beta}$  the expectation for the stochastic dynamics starting with this invariant distribution. We prove a diffusive behaviour for the energy: first, define the distribution-valued energy fluctuation field

$$\mathcal{Y}^N := \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} \delta_{x/N} \{ \omega_x^2(0) - \mathbf{e}_\beta \}.$$

It is well known that  $\mathcal{Y}^N$  converges in distribution as  $N \rightarrow \infty$  towards a centred Gaussian field  $\mathcal{Y}$ , which satisfies

$$\mathbb{E}_{\mu_\beta} [\mathcal{Y}(F)\mathcal{Y}(G)] = \chi(\beta) \int_{\mathbb{R}} F(y)G(y)dy,$$

for smooth test functions  $F, G$ . In this paper we prove that these energy fluctuations evolve diffusively in time (Theorem 3.1). More precisely, the following distribution

$$\mathcal{Y}_t^N = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} \delta_{x/N} \{ \omega_x^2(tN^2) - \mathbf{e}_\beta \}$$

converges in law as  $N \rightarrow \infty$  to the solution of the linear Stochastic Partial Differential Equation (SPDE)

$$\partial_t \mathcal{Y} = D \partial_y^2 \mathcal{Y} + \sqrt{2D\chi(\beta)} \partial_y B(y, t), \quad t > 0, y \in \mathbb{R}.$$

where  $D$  is the diffusion coefficient, and  $B$  is the standard normalized space-time white noise.

Finally, we could think of using the entropy method to derive the hydrodynamic equation. For that purpose, the initial law is not assumed to be the equilibrium measure  $\mu_\beta$ , but a *local equilibrium measure* (see (60) below). We conjecture that this property of local equilibrium propagates in time. In other words, let  $\mathbf{e}_0 : \mathbb{T} \rightarrow \mathbb{R}$  be a bounded function, where  $\mathbb{T}$  denotes the torus  $[0, 1)$ . We would like to show that the empirical energy profile converges in the thermodynamic limit to the macroscopic profile  $\mathbf{e}(t, \cdot) : \mathbb{T} \rightarrow \mathbb{R}$  solution of

$$\begin{cases} \frac{\partial \mathbf{e}}{\partial t}(t, u) = D \frac{\partial^2 \mathbf{e}}{\partial u^2}(t, u), & t > 0, u \in \mathbb{T}, \\ \mathbf{e}(0, u) = \mathbf{e}_0(u). \end{cases}$$

Unfortunately, even if the diffusion coefficient is well defined through the non-gradient approach, this does not straightforwardly provide a method to derive the hydrodynamic limits.

Let us now give the plan of the paper. Section 2 is devoted to introduce the model and all notations and definitions that are needed. The convergence of the energy fluctuations field (in the sense of finite dimensional distributions) is proved in Section 3. The main point is to identify the diffusion coefficient  $D$  (Section 5), by adapting the non-gradient method introduced in [28]. In Section 4, we derive the Boltzmann-Gibbs principle. Finally, Section 6 gives a precise description of the diffusion coefficient through several variational formulas. In Section 7 we prove the convergence of the Green-Kubo formula, and demonstrate rigorously that both definitions of the diffusion coefficient are equivalent. In Section 8, we present a second disordered model, where the interaction is described by a potential  $V$ . For this anharmonic chain, we need a very strong stochastic perturbation, which has a spectral gap, and satisfies the sector condition. We conclude in Section 9 by highlighting the step where the usual techniques for proving hydrodynamic limits fail. In Appendices, technical points are detailed: in Appendix A, the space of square-integrable functions w.r.t. the standard Gaussian law is studied through its orthonormal basis

of Hermite polynomials. In Appendix B we replace the usual result concerning the decomposition of closed forms by some weaker version. The sector condition is proved for a specific class of functions in Appendix C. In Appendix D, the tightness for the energy fluctuation field is recalled for the sake of completeness.

## 2 The harmonic chain perturbed by stochastic jump noises

### 2.1 Generator of the Markov process

We first describe the dynamics on the finite torus  $\mathbb{T}_N := \{1, \dots, N\}$ , meaning that boundary conditions are periodic. The configuration  $\{\omega_x\}_{x \in \mathbb{T}_N}$  evolves according to a dynamics which can be divided into two parts, a deterministic one and a stochastic one. The space of configurations of our system is given by  $\Omega_N = \mathbb{R}^N$ . We recall that the disorder is an i.i.d. sequence  $\mathbf{m} = \{m_x\}_{x \in \mathbb{Z}}$  which satisfies:

$$\forall x \in \mathbb{Z}, \quad \frac{1}{C} \leq m_x \leq C,$$

for some finite constant  $C > 0$ . The corresponding product and translation invariant measure on the space  $\Omega_{\mathcal{D}} = [C^{-1}, C]^{\mathbb{Z}}$  is denoted by  $\mathbb{P}$  and its expectation is denoted by  $\mathbb{E}$ . For a fixed disorder field  $\mathbf{m} = \{m_x\}_{x \in \mathbb{Z}}$ , we consider the system of ODE's

$$\sqrt{m_x} d\omega_x = \left( \frac{\omega_{x+1}}{\sqrt{m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1}}} \right) dt, \quad t \geq 0, x \in \mathbb{T}_N$$

and we superpose to this deterministic dynamics a stochastic perturbation described as follows: to each atom  $x \in \mathbb{T}_N$ , and each bond  $\{x, x+1\}, x \in \mathbb{T}_N$  is associated an exponential clock of rate  $\lambda > 0$  and  $\gamma > 0$  respectively, such that each clock is independent of each other. When the clock attached to  $x$  rings,  $\omega_x$  is flipped into  $-\omega_x$ , and when the clock attached to the bond  $\{x, x+1\}$  rings, the values  $\omega_x$  and  $\omega_{x+1}$  are exchanged. This dynamics can be entirely defined by the generator of the Markov process  $\{\omega_x(t); x \in \mathbb{T}_N\}_{t \geq 0}$ , that is

$$\mathcal{L}_N^{\mathbf{m}} = \mathcal{A}_N^{\mathbf{m}} + \gamma \mathcal{S}_N^{\text{flip}} + \lambda \mathcal{S}_N^{\text{exch}}$$

where,

$$\mathcal{A}_N^{\mathbf{m}} = \sum_{x \in \mathbb{T}_N} \left\{ \left( \frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}} \right) \frac{\partial}{\partial \omega_x} \right\},$$

and, for all functions  $f : \Omega_{\mathcal{D}} \times \Omega_N \rightarrow \mathbb{R}$ ,

$$\mathcal{S}_N^{\text{flip}} f(\mathbf{m}, \omega) = \sum_{x \in \mathbb{T}_N} f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega),$$

$$\mathcal{S}_N^{\text{exch}} f(\mathbf{m}, \omega) = \sum_{x \in \mathbb{T}_N} f(\mathbf{m}, \omega^{x, x+1}) - f(\mathbf{m}, \omega).$$

Here, the configuration  $\omega^x$  is the configuration obtained from  $\omega$  by flipping the momentum of atom  $x$ :

$$(\omega^x)_z = \begin{cases} \omega_z & \text{if } z \neq x, \\ -\omega_x & \text{if } z = x. \end{cases}$$

The configuration  $\omega^{x,x+1}$  is obtained from  $\omega$  by exchanging the momenta of atoms  $x$  and  $x + 1$ :

$$(\omega^{x,x+1})_z = \begin{cases} \omega_z & \text{if } z \neq x, x + 1, \\ \omega_{x+1} & \text{if } z = x, \\ \omega_x & \text{if } z = x + 1. \end{cases}$$

We denote the total generator of the noise by  $\mathcal{S}_N := \gamma \mathcal{S}_N^{\text{flip}} + \lambda \mathcal{S}_N^{\text{exch}}$ , where  $\gamma, \lambda > 0$  are two positive parameters which regulate the respective strengths of noises.

One quantity is conserved: the total energy  $\sum \omega_x^2$ . The following translation invariant product Gibbs measures  $\mu_\beta^N$  on  $\Omega_N$  are invariant for the process:

$$d\mu_\beta^N(\omega) := \prod_{x \in \mathbb{T}_N} \sqrt{\frac{2\pi}{\beta}} \exp\left(-\frac{\beta}{2} \omega_x^2\right) d\omega_x.$$

In the following, the expectation of  $f$  with respect to  $\mu_\beta^N$  is denoted by  $\langle f \rangle_\beta$ . The index  $\beta$  stands for the inverse temperature, namely  $\langle \omega_0^2 \rangle_\beta = 1/\beta$ . Let us notice that the Gibbs measures do not depend on the disorder  $\mathbf{m}$ . From the definition, our model is not reversible with respect to the measure  $\mu_\beta^N$ . More precisely,  $\mathcal{A}_N^{\mathbf{m}}$  is an antisymmetric operator in  $\mathbf{L}^2(\mu_\beta^N)$ , whereas  $\mathcal{S}_N$  is symmetric.

We denote by  $\Omega$  the space of configurations in the infinite line, that is  $\Omega := \mathbb{R}^{\mathbb{Z}}$ , and by  $\mu_\beta$  the product Gibbs measure on  $\mathbb{R}^{\mathbb{Z}}$ . Hereafter, for every  $\beta > 0$ , we denote by  $\mathbb{P}_\beta^*$  the probability measure on  $\Omega_{\mathcal{D}} \times \Omega$  defined by

$$\mathbb{P}_\beta^* := \mathbb{P} \otimes \mu_\beta.$$

Along the article we will widely use the fact that  $\mathbb{P}_\beta^*$  is translation invariant. We write  $\mathbb{E}_\beta^*$  for the corresponding expectation.

## 2.2 Energy current

Since the dynamics conserves the total energy, there exist instantaneous currents of energy  $j_{x,x+1}$  such that  $\mathcal{L}_N^{\mathbf{m}}(\omega_x^2) = j_{x,x+1}(\mathbf{m}, \omega) - j_{x-1,x}(\mathbf{m}, \omega)$ . The quantity  $j_{x,x+1}$  is the amount of energy flowing between the particles  $x$  and  $x + 1$ , and is equal to

$$j_{x,x+1}(\mathbf{m}, \omega) = \frac{2\omega_x \omega_{x+1}}{\sqrt{m_x m_{x+1}}} + \lambda(\omega_{x+1}^2 - \omega_x^2).$$

The energy conservation law can be read locally as

$$\omega_x^2(t) - \omega_x^2(0) = J_{x,x+1}(t) - J_{x-1,x}(t),$$

where  $J_{x,x+1}(t)$  is the total energy current between  $x$  and  $x + 1$  up to time  $t$ . This can be written as

$$J_{x,x+1}(t) = \int_0^t j_{x,x+1}(s) ds + M_{x,x+1}(t),$$

where  $M_{x,x+1}(t)$  is a martingale which can be explicitly computed as an Itô stochastic integral:

$$M_{x,x+1}(t) = \int_0^t \left\{ \omega_{x+1}^2 - \omega_x^2 \right\} (s^-) d[N_{x,x+1}(s) - \lambda s] + \int_0^t \left\{ \omega_{x-1}^2 - \omega_x^2 \right\} (s^-) d[N_{x-1,x}(s) - \lambda s],$$



where  $(N_{x,x+1})_{x \in \mathbb{Z}}$  are independent Poisson processes of intensity  $\lambda$ . We write  $j_{x,x+1} = j_{x,x+1}^A + j_{x,x+1}^S$  where  $j_{x,x+1}^A$  (resp.  $j_{x,x+1}^S$ ) is the current associated to the antisymmetric (resp. symmetric) part of the generator:

$$j_{x,x+1}^A(\mathbf{m}, \omega) = \frac{2\omega_x \omega_{x+1}}{\sqrt{m_x m_{x+1}}}$$

$$j_{x,x+1}^S(\mathbf{m}, \omega) = j_{x,x+1}^S(\omega) = \lambda(\omega_{x+1}^2 - \omega_x^2).$$

Unfortunately, the current cannot be directly written as the gradient of a local function, neither by an exact fluctuation-dissipation equation (in other words, the current is not the sum of a gradient and a dissipative term of the form  $\mathcal{L}_N^{\mathbf{m}}(\tau_x h)$ , where  $h$  is a local function of the system configuration). This means that we are in the non-gradient case. We also define the *static compressibility* that is equal to the variance of the one-site energy  $\omega_0^2$  with respect to  $\mu_\beta$ , namely

$$\chi(\beta) := \langle \omega_0^4 \rangle_\beta - \langle \omega_0^2 \rangle_\beta^2 = \frac{2}{\beta^2}.$$

### 2.3 Cylinder functions

For every  $x \in \mathbb{Z}$  and every measurable function  $f$  on  $\Omega_{\mathcal{D}} \times \Omega$ , we consider the translated function  $\tau_x f$ , which is defined on  $\Omega_{\mathcal{D}} \times \Omega$  by:  $\tau_x f(\mathbf{m}, \omega) := f(\tau_x \mathbf{m}, \tau_x \omega)$ , where  $\tau_x \mathbf{m}$  and  $\tau_x \omega$  are the disorder and particle configurations translated by  $x \in \mathbb{Z}$ , respectively:

$$(\tau_x \mathbf{m})_z := m_{x+z}, \quad (\tau_x \omega)_z = \omega_{x+z}.$$

Let  $\Lambda$  be a finite subset of  $\mathbb{Z}$ , and denote by  $\mathcal{F}_\Lambda$  the  $\sigma$ -algebra generated by  $\{m_x, \omega_x ; x \in \Lambda\}$ . For a fixed positive integer  $\ell$ , we define  $\Lambda_\ell := \{-\ell, \dots, \ell\}$ . If the box is centred at site  $x \in \mathbb{Z}$ , we denote it by  $\Lambda_\ell(x) := \{-\ell + x, \dots, \ell + x\}$ . If  $f$  is a measurable function on  $\Omega_{\mathcal{D}} \times \Omega$ , the *support* of  $f$ , denoted by  $\Lambda_f$ , is the smallest subset of  $\mathbb{Z}$  such that  $f(\mathbf{m}, \omega)$  only depends on  $\{m_x, \omega_x ; x \in \Lambda_f\}$  and  $f$  is called a *cylinder* (or *local*) *function* if  $\Lambda_f$  is finite. In that case, we denote by  $s_f$  the smallest positive integer  $s$  such that  $\Lambda_s$  contains the support of  $f$  and then  $\Lambda_f = \Lambda_{s_f}$ . For every cylinder function  $f : \Omega_{\mathcal{D}} \times \Omega \rightarrow \mathbb{R}$ , consider the formal sum

$$\Gamma_f := \sum_{x \in \mathbb{Z}} \tau_x f$$

which does not make sense but for which

$$\nabla_0(\Gamma_f) := \Gamma_f(\mathbf{m}, \omega^0) - \Gamma_f(\mathbf{m}, \omega),$$

$$\nabla_{0,1}(\Gamma_f) := \Gamma_f(\mathbf{m}, \omega^{0,1}) - \Gamma_f(\mathbf{m}, \omega).$$

are well defined. Similarly, we define

$$(\nabla_x f)(\mathbf{m}, \omega) := f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega),$$

$$(\nabla_{x,x+1} f)(\mathbf{m}, \omega) := f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega).$$

**DEFINITION 2.1.** We denote by  $\mathcal{C}$  the set of measurable cylinder functions  $\varphi$  on  $\Omega_{\mathcal{D}} \times \Omega$ , such that

- (i) for all  $\omega \in \Omega$ , the random variable  $\mathbf{m} \mapsto \varphi(\mathbf{m}, \omega)$  is continuous on  $\Omega_{\mathcal{D}}$ ;
- (ii) for all  $\mathbf{m} \in \Omega_{\mathcal{D}}$ , the function  $\omega \mapsto \varphi(\mathbf{m}, \omega)$  belongs to  $\mathbf{L}^2(\mu_\beta)$  and has null average with respect to  $\mu_\beta$ .

Notice that (ii) + (iii) implies that the function  $\omega \mapsto \sup_{\mathbf{m} \in \Omega_{\mathcal{D}}} \varphi(\mathbf{m}, \omega)$  belongs to  $\mathbf{L}^2(\mu_{\beta})$  and has null average with respect to  $\mu_{\beta}$ .

**DEFINITION 2.2.** We introduce the set of quadratic cylinder functions on  $\Omega_{\mathcal{D}} \times \Omega$ , denoted by  $\mathcal{Q} \subset \mathcal{C}$ , and defined as follows:  $f \in \mathcal{Q}$  if there exists a sequence  $\{\psi_{i,j}(\mathbf{m})\}_{i,j \in \mathbb{Z}}$  of real cylinder measurable functions on  $\Omega_{\mathcal{D}}$ , with finite support in  $\Omega_{\mathcal{D}}$ , such that

- (i) for all  $i, j \in \mathbb{Z}$  and all  $\omega \in \Omega$ , the random variable  $\mathbf{m} \mapsto \psi_{i,j}(\mathbf{m}, \omega)$  is continuous on  $\Omega_{\mathcal{D}}$ ;
- (ii)  $\psi_{i,j}$  vanishes for all but a finite number of pairs  $(i, j)$ ,
- (iii)  $f$  is written as

$$f(\mathbf{m}, \omega) = \sum_{i \in \mathbb{Z}} \psi_{i,i}(\mathbf{m})(\omega_{i+1}^2 - \omega_i^2) + \sum_{\substack{i,j \in \mathbb{Z} \\ i \neq j}} \psi_{i,j}(\mathbf{m})\omega_i\omega_j, \quad (3)$$

In other words, quadratic functions are homogeneous polynomials of degree two in the variable  $\omega$ , that have null average with respect to  $\mu_{\beta}$  for every  $\mathbf{m} \in \Omega_{\mathcal{D}}$ . An other definition through *Hermite polynomials* is given in Appendix A. We are now ready to define two sets of functions that will play further a crucial role.

**DEFINITION 2.3.** Let  $\mathcal{C}_0$  be the set of cylinder functions  $\varphi$  on  $\Omega_{\mathcal{D}} \times \Omega$  such that there exists a finite subset  $\Lambda$  of  $\mathbb{Z}$ , and cylinder, measurable functions  $\{F_x, G_x\}_{x \in \Lambda}$  defined on  $\Omega_{\mathcal{D}} \times \Omega$ , that verify

$$\varphi = \sum_{x \in \Lambda} \left\{ \nabla_x(F_x) + \nabla_{x,x+1}(G_x) \right\},$$

and such that, for all  $x \in \Lambda$ ,

- (i) for all  $\omega \in \Omega$ , the functions  $\mathbf{m} \mapsto F_x(\mathbf{m}, \omega)$  and  $\mathbf{m} \mapsto G_x(\mathbf{m}, \omega)$  are continuous on  $\Omega_{\mathcal{D}}$ ;
- (ii) for all  $\mathbf{m} \in \Omega_{\mathcal{D}}$ , the functions  $\omega \mapsto F_x(\mathbf{m}, \omega)$  and  $\omega \mapsto G_x(\mathbf{m}, \omega)$  belong to  $\mathbf{L}^2(\mu_{\beta})$ .

Let  $\mathcal{Q}_0 \subset \mathcal{C}_0$  be the set of such functions  $\varphi$ , with the additional assumption that the cylinder functions  $F_x, G_x$  are homogeneous polynomials of degree two in the variable  $\omega$ .

Before giving a few properties of these two spaces, we redefine the generators of the Markov process on the whole discrete line: the operators  $\mathcal{L}^{\mathbf{m}}$ ,  $\mathcal{A}^{\mathbf{m}}$  and  $\mathcal{S}$  are acting on cylinder functions  $f$  defined on  $\Omega_{\mathcal{D}} \times \Omega$  as

$$\mathcal{L}^{\mathbf{m}}f = \mathcal{A}^{\mathbf{m}}f + \mathcal{S}f, \quad (4)$$

with

$$\mathcal{A}^{\mathbf{m}} = \sum_{x \in \mathbb{Z}} \left\{ \left( \frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}} \right) \frac{\partial}{\partial \omega_x} \right\},$$

and

$$\mathcal{S}f = \gamma \mathcal{S}^{\text{flip}}f + \lambda \mathcal{S}^{\text{exch}}f,$$

$$\begin{aligned} \mathcal{S}^{\text{flip}}f(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} (\nabla_x f)(\mathbf{m}, \omega) = \sum_{x \in \mathbb{Z}} \left\{ f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega) \right\}, \\ \mathcal{S}^{\text{exch}}f(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} (\nabla_{x,x+1} f)(\mathbf{m}, \omega) = \sum_{x \in \mathbb{Z}} \left\{ f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega) \right\}. \end{aligned}$$

We denote  $\mathcal{S}_x = \gamma \nabla_x + \lambda \nabla_{x,x+1}$  for  $x \in \mathbb{Z}$ . For  $\Lambda_{\ell}$  defined as above,  $\mathcal{L}_{\Lambda_{\ell}}^{\mathbf{m}}$ , resp.  $\mathcal{S}_{\Lambda_{\ell}}$ , is the restriction of the generator  $\mathcal{L}^{\mathbf{m}}$ , resp.  $\mathcal{S}$ , to the finite box  $\Lambda_{\ell}$ , assuming periodic boundary conditions.

## 2.4 Properties of $\mathcal{C}_0$ and $\mathcal{Q}_0$

Before giving the main properties of the sets introduced above, we define the quadratic form associated to the generator: for any  $x \in \mathbb{Z}$  and cylinder functions  $f, g \in \mathcal{C}$ , let us define

$$\mathcal{D}_\ell(\mu_\beta; f) := \langle (-\mathcal{L}_{\Lambda_\ell}^{\mathbf{m}})f, f \rangle_\beta = \langle (-\mathcal{S}_{\Lambda_\ell})f, f \rangle_\beta = \sum_{x \in \Lambda_\ell} \langle (-\mathcal{S}_x)f, f \rangle_\beta,$$

and notice that

$$\begin{aligned} \langle (-\mathcal{S}_x)f, g \rangle_\beta &= \frac{\gamma}{2} \langle (f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega))(g(\mathbf{m}, \omega^x) - g(\mathbf{m}, \omega)) \rangle_\beta \\ &\quad + \frac{\lambda}{2} \langle (f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega))(g(\mathbf{m}, \omega^{x,x+1}) - g(\mathbf{m}, \omega)) \rangle_\beta, \end{aligned}$$

The symmetric form  $\mathcal{D}_\ell$  is called the *Dirichlet form*, and is well defined on  $\mathcal{C}$ . This is a random variable with respect to the disorder  $\mathbf{m}$ .

**PROPOSITION 2.1.** *For every function  $\varphi \in \mathcal{C}_0$ , the quantity  $\langle \varphi, (-\mathcal{S})^{-1}\varphi \rangle_\beta$  is well defined. Moreover, the following elements belong to  $\mathcal{Q}_0$ :*

- (a)  $j_{0,1}^S, j_{0,1}^A$ .
- (b)  $\mathcal{L}^{\mathbf{m}}f, Sf$  and  $\mathcal{A}^{\mathbf{m}}f$ , for all  $f \in \mathcal{Q}$ .

*Proof.* The fact that the function  $\varphi \in \mathcal{C}_0$  can be written by definition as

$$\varphi = \sum_{x \in \Lambda_\varphi} \nabla_x(\mathbf{F}_x) + \nabla_{x,x+1}(\mathbf{G}_x)$$

implies that we can write

$$\langle \varphi, (-\mathcal{S})^{-1}\varphi \rangle_\beta = \frac{1}{\gamma} \sum_{x \in \Lambda_\varphi} \langle \nabla_x(\mathbf{F}_x), \mathbf{F}_x \rangle_\beta + \frac{1}{\lambda} \sum_{x \in \Lambda_\varphi} \langle \nabla_{x,x+1}(\mathbf{G}_x), \mathbf{G}_x \rangle_\beta,$$

and the right-hand side above is well defined. Besides, the first statement (a) is directly obtained from the following identities: for  $x \in \mathbb{Z}$ , and  $k \geq 1$ ,

$$\omega_{x+1}^2 - \omega_x^2 = \nabla_{x,x+1}(\omega_x^2) \tag{5}$$

$$\omega_x \omega_{x+k} = \mathcal{S}_x \left( \frac{-\omega_x \omega_{x+1}}{\gamma} \right) + \sum_{\ell=1}^{k-1} \mathcal{S}_{x+\ell} \left( \frac{-\omega_x \omega_{x+\ell+1}}{\lambda} \right). \tag{6}$$

Then, if  $f \in \mathcal{Q}$ , it is easy to see that (5) and (6) are sufficient to prove (b). For instance,

$$\begin{aligned} \mathcal{L}^{\mathbf{m}}(\omega_x \omega_{x+1}) &= \frac{\omega_x \omega_{x+2}}{\sqrt{m_{x+1} m_{x+2}}} - \frac{\omega_{x+1} \omega_{x-1}}{\sqrt{m_x m_{x-1}}} + \frac{\omega_{x+1}^2 - \omega_x^2}{\sqrt{m_x m_{x+1}}} \\ &\quad - 4\gamma \omega_x \omega_{x+1} + \lambda(\omega_{x+2} - \omega_{x+1})\omega_x + \lambda(\omega_{x-1} - \omega_x)\omega_{x+1}. \end{aligned}$$

The integrability and regularity conditions are easy to check.  $\square$

**PROPOSITION 2.2** (Dirichlet bound). *Let  $\varphi$  be a cylinder function in  $\mathcal{C}_0$ , written by definition as*

$$\varphi = \sum_{x \in \Lambda_\varphi} \nabla_x(F_x) + \nabla_{x,x+1}(G_x),$$

for some functions  $F_x$  and  $G_x$  satisfying the conditions of Definition 2.3. Let us consider  $h \in \mathcal{C}$  with support in  $\Lambda_\ell$ . Denote by  $\ell_\varphi$  the integer  $\ell_\varphi := \ell - s_\varphi - 1$  so that the support of  $\tau_x \varphi$  is included in  $\Lambda_\ell$  for every  $x \in \Lambda_\varphi$ . Then, the following constant

$$C_\varphi := (2s_\varphi + 1) \left\{ \left( \sup_{x \in \Lambda_\varphi} \mathbb{E}_\beta^* [F_x^2] \right)^{1/2} + \left( \sup_{x \in \Lambda_\varphi} \mathbb{E}_\beta^* [G_x^2] \right)^{1/2} \right\}$$

that only depends on  $\varphi$  satisfies

$$\left| \mathbb{E}_\beta^* \left[ \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, h \right] \right| \leq C_\varphi |2\ell_\varphi + 1|^{1/2} \left( \mathbb{E}[\mathcal{D}_\ell(\mu_\beta; h)] \right)^{1/2}. \quad (7)$$

*Proof.* First, let us assume first that  $\varphi = \nabla_0(F_0)$ , so that  $s_\varphi = 1$ . Then we have

$$\begin{aligned} \left| \mathbb{E}_\beta^* \left[ \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, h \right] \right| &= \left| \sum_{|x| \leq \ell_\varphi} \mathbb{E}_\beta^* [\tau_x F_0, \nabla_x h] \right| \leq \sum_{|x| \leq \ell_\varphi} \left\{ \mathbb{E}_\beta^* [(\tau_x F_0)^2]^{1/2} \mathbb{E}_\beta^* [(\nabla_x h)^2]^{1/2} \right\} \\ &\leq \left( \sum_{|x| \leq \ell_\varphi} \mathbb{E}_\beta^* [(\tau_x F_0)^2] \right)^{1/2} \left( \mathbb{E}[\mathcal{D}_\ell(\mu_\beta; h)] \right)^{1/2} \\ &\leq |2\ell_\varphi + 1|^{1/2} \mathbb{E}_\beta^* [F_0^2]^{1/2} \left( \mathbb{E}[\mathcal{D}_\ell(\mu_\beta; h)] \right)^{1/2}. \end{aligned}$$

In the same way, if  $\varphi = \sum_{y \in \Lambda} \nabla_y(F_y)$ , we have

$$\begin{aligned} \left| \mathbb{E}_\beta^* \left[ \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, h \right] \right| &= \left| \sum_{|x| \leq \ell_\varphi} \sum_{y \in \Lambda} \mathbb{E}_\beta^* [\tau_x F_y, \nabla_{y+x} h] \right| \\ &\leq \sum_{|x| \leq \ell_\varphi} \sum_{y \in \Lambda} \left\{ \mathbb{E}_\beta^* [(\tau_x F_y)^2]^{1/2} \mathbb{E}_\beta^* [(\nabla_{y+x} h)^2]^{1/2} \right\} \\ &\leq (2s_\varphi + 1) |2\ell_\varphi + 1|^{1/2} \left( \sup_{y \in \Lambda} \mathbb{E}_\beta^* [F_y^2] \right)^{1/2} \left( \mathbb{E}[\mathcal{D}_\ell(\mu_\beta; h)] \right)^{1/2}. \end{aligned}$$

The general case easily follows.  $\square$

Finally, if we use the decomposition of every function in  $\mathbf{L}^2(\mu_\beta)$  over the basis of Hermite polynomials, we can prove the following result (the details for the proof are given in Proposition A.3 in Section A):

**PROPOSITION 2.3** (Variance of quadratic functions). *If  $\varphi \in \mathcal{Q}_0$ , then*

$$\mathbb{E}_\beta^* [\varphi, (-\mathcal{S}_{\Lambda_\varphi})^{-1} \varphi] = \sup_{g \in \mathcal{Q}} \left\{ 2 \mathbb{E}_\beta^* [\varphi, g] - \mathbb{E}[\mathcal{D}_{s_\varphi}(\mu_\beta; g)] \right\}.$$

## 2.5 Semi-inner products and diffusion coefficient

For cylinder functions  $g, h \in \mathcal{C}$ , let us define:

$$\langle\langle g, h \rangle\rangle_{\beta, \star} := \sum_{x \in \mathbb{Z}} \mathbb{E}_{\beta}^{\star} [g \tau_x h], \quad \text{and} \quad \langle\langle g \rangle\rangle_{\beta, \star \star} := \sum_{x \in \mathbb{Z}} x \mathbb{E}_{\beta}^{\star} [g \omega_x^2]. \quad (8)$$

Both quantities are well defined because  $g$  and  $h$  belong to  $\mathcal{C}$  and therefore all but a finite number of terms on each sum vanish. Notice that  $\langle\langle \cdot, \cdot \rangle\rangle_{\beta, \star}$  is a semi inner product, since the following equality holds:

$$\langle\langle f, g \rangle\rangle_{\beta, \star} = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} \mathbb{E}_{\beta}^{\star} \left[ \sum_{x \in \Lambda} \tau_x f, \sum_{x \in \Lambda} \tau_x g \right].$$

Since  $\langle\langle f - \tau_x f, g \rangle\rangle_{\beta, \star} = 0$  for all  $x \in \mathbb{Z}$ , this scalar product is only semi-definite. In the next proposition we give explicit formulas for elements of  $\mathcal{C}_0$ .

**PROPOSITION 2.4.** *If  $\varphi \in \mathcal{C}_0$  with*

$$\varphi = \sum_{y \in \Lambda} \left\{ \nabla_y (F_y) + \nabla_{y, y+1} (G_y) \right\},$$

then

$$\begin{aligned} \langle\langle \varphi \rangle\rangle_{\beta, \star \star} &= \mathbb{E}_{\beta}^{\star} \left[ (\omega_0^2 - \omega_1^2) \sum_{y \in \Lambda} \tau_{-y} G_y \right], \\ \langle\langle \varphi, g \rangle\rangle_{\beta, \star} &= \mathbb{E}_{\beta}^{\star} \left[ \nabla_0 (\Gamma_g) \sum_{y \in \Lambda} \tau_{-y} F_y + \nabla_{0,1} (\Gamma_g) \sum_{y \in \Lambda} \tau_{-y} G_y \right] \quad \text{for all } g \in \mathcal{C}. \end{aligned}$$

*Proof.* The proof is straightforward. □

**DEFINITION 2.4.** *We define the diffusion coefficient  $D(\beta)$  for  $\beta > 0$  as*

$$D(\beta) := \lambda + \frac{1}{\chi(\beta)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \left\{ \langle\langle f, -Sf \rangle\rangle_{\beta, \star} + 2 \langle\langle j_{0,1}^A - \mathcal{A}^m f, g \rangle\rangle_{\beta, \star} - \langle\langle g, -Sg \rangle\rangle_{\beta, \star} \right\}.$$

The first term in the sum ( $\lambda$ ) is only due to the exchange noise, whereas the second one comes from the Hamiltonian part of the dynamics. Formally, this formula could be read as

$$D(\beta) = \lambda + \frac{1}{\chi(\beta)} \langle\langle j_{0,1}^A, (-\mathcal{L}^m)^{-1} j_{0,1}^A \rangle\rangle_{\beta, \star}, \quad (9)$$

but the last term is ill-defined because  $j_{0,1}^A$  is not in the range of  $\mathcal{L}^m$ . More rigorously, we should define  $\langle\langle j_{0,1}^A, (-\mathcal{L}^m)^{-1} j_{0,1}^A \rangle\rangle_{\beta, \star}$  as

$$\limsup_{z \rightarrow 0} \langle\langle j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \rangle\rangle_{\beta, \star}.$$

The last expression is now well defined, and the problem is reduced to prove convergence as  $z \rightarrow 0$ . Hille-Yosida Theorem (see Proposition 2.1 in [10] for instance) suggests that (9) is equal to the infinite volume Green-Kubo formula:

$$\bar{D}(\beta) = \lambda + \frac{1}{\chi(\beta)} \lim_{\substack{z \rightarrow 0 \\ z > 0}} \int_0^{+\infty} e^{-zt} \mathbb{E}_{\beta}^{\star} \left[ \sum_{x \in \mathbb{Z}} j_{x, x+1}^A(t), j_{0,1}^A(0) \right] dt, \quad (10)$$

In Section 7, we prove that (10) is well defined and converges, inspired by [3], and we also show that the diffusion coefficient can be equivalently defined in the two ways. Assuming the convergence in the Green-Kubo formula, one can easily see that  $\bar{D}(\beta)$  does not depend on  $\beta$ . We denote by  $L(z)$  the second term of the right-hand side of (10), that is

$$L(z) := \frac{1}{\chi(\beta)} \int_0^{+\infty} e^{-zt} \ll j_{0,1}^A(t), j_{0,1}^A(0) \gg_{\beta, \star} dt.$$

Let  $\mathbf{L}_{\beta, \star}^2$  be the Hilbert space generated by the set of square integrable local functions and the inner product  $\ll \cdot, \cdot \gg_{\beta, \star}$ . Consider  $h_z := h_z(\mathbf{m}, \omega; \beta)$  the solution of the resolvent equation in  $\mathbf{L}_{\beta, \star}^2$  i.e.

$$(z - \mathcal{L}^{\mathbf{m}})h_z = j_{0,1}^A.$$

Then we have

$$L(z) = \frac{1}{\chi(\beta)} \ll h_z, j_{0,1}^A \gg_{\beta, \star} = \frac{\beta^2}{2} \ll h_z, j_{0,1}^A \gg_{\beta, \star}.$$

Observe that if  $\omega$  is distributed according to  $\mu_\beta$  then  $\beta^{1/2}\omega$  is distributed according to  $\mu_1$ . Besides,  $j_{0,1}^A$  is a homogeneous function of degree two in  $\omega$  and  $\mathcal{L}^{\mathbf{m}}$  preserves the degree of polynomials, which implies that  $h_z$  is also homogeneous of degree two. It follows that  $h_z(\mathbf{m}, \omega; 1) = h_z(\mathbf{m}, \omega; \beta)$  and then the diffusion coefficient does not depend on  $\beta$ .

### 3 Macroscopic fluctuations of energy

In this section we are interested in the fluctuations of the empirical energy, when the system is at equilibrium. We prove that the limit fluctuation process is governed by a generalized Ornstein-Uhlenbeck process, whose covariances are given in terms of the diffusion coefficient. We adapt the non-gradient method introduced by Varadhan. In particular, we establish rigorously the variational formula that appears in the definition of the diffusion coefficient (Definition 2.4). Varadhan's approach is investigated in Sections 4, 5 and 6.

#### 3.1 Energy fluctuation field

Recall that we denote by  $\mathbf{e}_\beta$  the thermodynamical energy associated to the inverse of temperature  $\beta > 0$ , namely  $\mathbf{e}_\beta = \beta^{-1}$ . We define the energy empirical distribution  $\pi_{t, \mathbf{m}}^N$  on the torus  $\mathbb{T} = [0, 1)$  as

$$\pi_{t, \mathbf{m}}^N(du) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^2(t) \delta_{x/N}(du), \quad t \in [0, T], \quad u \in \mathbb{T},$$

where  $\delta_u$  stands for the Dirac measure. We denote by  $\{\omega(t)\}_{t \geq 0}$  the Markov process generated by  $N^2 \mathcal{L}_N^{\mathbf{m}}$  and by  $\mathcal{M}_1$  the set of positive, Radon measures on  $\mathbb{T}$ , endowed with the weak topology. The space of trajectories in  $\mathcal{M}_1$ , which are right-continuous and left-limited (i.e. the Skorokhod space) is denoted by  $\mathcal{D}([0, T], \mathcal{M}_1)$ . If the initial state of the dynamics is given by the equilibrium Gibbs measure  $\mu_\beta^N$ , then, for any fixed  $t \geq 0$ , the measure  $\pi_{t, \mathbf{m}}^N$  weakly converges towards the deterministic measure on  $\mathbb{T}$ , equal to  $\{\mathbf{e}_\beta du\}$ . Our goal is to investigate the fluctuations of the empirical measure  $\pi_{t, \mathbf{m}}^N$  with respect to this limit. Let us fix the disorder  $\mathbf{m}$ , and the inverse of temperature  $\beta > 0$ . Consider the system under the equilibrium measure  $\mu_\beta^N$ .

**DEFINITION 3.1** (Empirical energy fluctuations). We denote by  $\mathcal{Y}_{t,\mathbf{m}}^N$  the empirical energy fluctuation field defined as

$$\mathcal{Y}_{t,\mathbf{m}}^N(\mathbf{H}) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} \mathbf{H}\left(\frac{x}{N}\right) \{\omega_x^2(t) - \mathbf{e}_\beta\},$$

where  $\mathbf{H} : \mathbb{T} \rightarrow \mathbb{R}$  is a smooth function.

We are going to prove that the distribution  $\mathcal{Y}_{t,\mathbf{m}}^N$  converges in law towards the solution of the linear SPDE:

$$\partial_t \mathcal{Y} = \mathbf{D} \partial_y^2 \mathcal{Y} + \sqrt{2\mathbf{D}\chi(\beta)} \partial_y \mathbf{B}(y, t),$$

where  $\mathbf{B}$  is a standard normalized space-time white noise, and  $\mathbf{D}$  is the diffusion coefficient defined in Theorem 5.9. Observe that there is no dependence on the disorder  $\mathbf{m}$  in the limit process. In other words, the latter is described by the stationary generalized Ornstein-Uhlenbeck process with zero mean and covariances given by

$$\langle \mathcal{Y}_t(\mathbf{H}) \mathcal{Y}_0(\mathbf{G}) \rangle_\beta = \frac{\chi(\beta)}{\sqrt{4\pi t \mathbf{D}}} \int_{\mathbb{R}^2} du dv \bar{\mathbf{H}}(u) \bar{\mathbf{G}}(v) \exp\left(-\frac{(u-v)^2}{4t\mathbf{D}}\right),$$

for all  $t \geq 0$  and smooth functions  $\mathbf{H}, \mathbf{G} : \mathbb{T} \rightarrow \mathbb{R}$ . Here,  $\bar{\mathbf{H}}$  (resp.  $\bar{\mathbf{G}}$ ) is the periodic extension to the real line of  $\mathbf{H}$  (resp.  $\mathbf{G}$ ).

Consider for  $k > 5/2$  the Sobolev space  $\mathfrak{H}_{-k}$  of distributions  $\mathcal{Y}$  on  $\mathbb{T}$  such that they have finite norm

$$\|\mathcal{Y}\|_{-k}^2 = \sum_{n \geq 1} (\pi n)^{-2k} |\mathcal{Y}(\mathbf{e}_n)|^2,$$

where  $\mathbf{e}_n$  is the function  $x \mapsto \sqrt{2} \sin(\pi n x)$ . We denote by  $\mathfrak{Y}_{\mathbf{m}}^N$  the probability measure on the space  $\mathcal{C}([0, T], \mathfrak{H}_{-k})$  of continuous trajectories on the Sobolev space, induced by the energy fluctuation field  $\mathcal{Y}_{t,\mathbf{m}}^N$  and the Markov process  $\{\omega(t)\}_{t \geq 0}$  generated by  $N^2 \mathcal{L}_N^m$ , starting from the equilibrium probability measure  $\mu_\beta^N$ . Let  $\mathfrak{Y}$  be the probability measure on the space  $\mathcal{C}([0, T], \mathfrak{H}_{-k})$  corresponding to the generalized Ornstein-Uhlenbeck process  $\mathcal{Y}_t$  defined above. The main result of this section is the following.

**THEOREM 3.1.** For almost all realization of the disorder  $\mathbf{m} \in \Omega_{\mathcal{D}}$ , the sequence  $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$  weakly converges in  $\mathcal{C}([0, T], \mathfrak{H}_{-k})$  to the probability measure  $\mathfrak{Y}$ .

## 3.2 Strategy of the proof

We follow the lines of [21, Section 3]. The proof of Theorem 3.1 is divided into three steps. First, we need to show that the sequence  $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$  is tight. This point follows a standard argument, given for instance in [15, Section 11], and recalled in Appendix D for the sake of completeness. Then, we prove that the one-time marginal of any limit point  $\mathfrak{Y}^*$  of a convergent subsequence of  $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$  is the law of a centered Gaussian field  $\mathcal{Y}$  with covariances given by

$$\langle \mathcal{Y}(\mathbf{H}) \mathcal{Y}(\mathbf{G}) \rangle = \chi(\beta) \int_{\mathbb{T}} du \mathbf{H}(u) \mathbf{G}(u),$$

where  $\mathbf{H}, \mathbf{G} : \mathbb{T} \rightarrow \mathbb{R}$  are smooth functions. This statement comes from the central limit theorem for independent variables. Finally, we prove the main point in the next subsections: all limit points  $\mathfrak{Y}^*$  of convergent subsequences of  $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$  solve the martingale problems below.

**Martingale problems** – For each smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}$ ,

$$\mathfrak{M}_t(H) := \mathcal{Y}_t(H) - \mathcal{Y}_0(H) - \int_0^t D\mathcal{Y}_s(H'') ds, \quad (11)$$

and

$$\mathfrak{N}_t(H) := (\mathfrak{M}_t(H))^2 - 2t\chi(\beta)D \int_{\mathbb{T}} H'(u)^2 du \quad (12)$$

are  $L^1(\mathfrak{Y})$ -martingales.

### 3.3 Martingale decompositions

Let us fix a smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}$ . We rewrite  $\mathcal{Y}_{t,\mathbf{m}}^N(H)$  as

$$\mathcal{Y}_{t,\mathbf{m}}^N(H) = \mathcal{Y}_{0,\mathbf{m}}^N(H) + \int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) j_{x,x+1}(\mathbf{m}, s) ds + \mathcal{M}_{t,\mathbf{m}}^N(H)$$

where  $\mathcal{M}_{t,\mathbf{m}}^N$  is the martingale defined as

$$\mathcal{M}_{t,\mathbf{m}}^N(H) = \int_0^t \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) (\omega_{x+1}^2 - \omega_x^2)(s) d[N_{x,x+1}(s) - \lambda s].$$

Hereafter,  $(N_{x,x+1})_{x \in \mathbb{Z}}$  and  $(N_x)_{x \in \mathbb{Z}}$  are independent Poisson processes of intensity (respectively)  $\lambda$  and  $\gamma$ , and  $\nabla_N$  stands for the discrete gradient:

$$\nabla_N H\left(\frac{x}{N}\right) = N \left[ H\left(\frac{x+1}{N}\right) - H\left(\frac{x}{N}\right) \right].$$

The discrete Laplacian  $\Delta_N$  is defined in a similar way:

$$\Delta_N H\left(\frac{x}{N}\right) = N^2 \left[ H\left(\frac{x+1}{N}\right) + H\left(\frac{x-1}{N}\right) - 2H\left(\frac{x}{N}\right) \right].$$

To close the equation, we are going to replace the term involving the microscopic currents with a term involving  $\mathcal{Y}_{t,\mathbf{m}}^N$ . In other words, the most important part in the fluctuation field represented by

$$\int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) j_{x,x+1}(\mathbf{m}, s) ds$$

is its projection over the conservation field  $\mathcal{Y}_{t,\mathbf{m}}^N$  (recall that the total energy is the unique conserved quantity of the system). The non-gradient approach consists in using the fluctuation-dissipation approximation of the current  $j_{x,x+1}$  given by Theorem 5.9 below as  $D(\omega_{x+1}^2 - \omega_x^2) + \mathcal{L}^{\mathbf{m}}(\tau_x f)$ . For that purpose, we rewrite, for any  $f \in \mathcal{Q}$ ,

$$\mathcal{Y}_{t,\mathbf{m}}^N(H) = \mathcal{Y}_{0,\mathbf{m}}^N(H) + \int_0^t D\mathcal{Y}_{s,\mathbf{m}}^N(\Delta_N H) ds + \mathfrak{J}_{t,\mathbf{m},f}^{1,N}(H) + \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{1,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H), \quad (13)$$



where

$$\begin{aligned}\mathcal{J}_{t,\mathbf{m},f}^{1,N}(\mathbb{H}) &= \int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbb{H} \left( \frac{x}{N} \right) \left[ j_{x,x+1}(\mathbf{m},s) - \mathbb{D}(\omega_{x+1}^2 - \omega_x^2)(s) - \mathcal{L}^{\mathbf{m}}(\tau_x f)(\mathbf{m},s) \right] ds, \\ \mathcal{J}_{t,\mathbf{m},f}^{2,N}(\mathbb{H}) &= \int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbb{H} \left( \frac{x}{N} \right) \mathcal{L}^{\mathbf{m}}(\tau_x f)(\mathbf{m},s) ds, \\ \mathfrak{M}_{t,\mathbf{m},f}^{1,N}(\mathbb{H}) &= \int_0^t \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbb{H} \left( \frac{x}{N} \right) \left\{ \left[ \nabla_{x,x+1}(\omega_x^2 - \Gamma_f) \right](s) d[N_{x,x+1}(s) - \lambda s] \right. \\ &\quad \left. - \nabla_x(\Gamma_f)(s) d[N_x(s) - \gamma s] \right\}, \\ \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(\mathbb{H}) &= \int_0^t \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbb{H} \left( \frac{x}{N} \right) \left\{ \nabla_{x,x+1}(\Gamma_f)(s) d[N_{x,x+1}(s) - \lambda s] \right. \\ &\quad \left. + \nabla_x(\Gamma_f)(s) d[N_x(s) - \gamma s] \right\}.\end{aligned}$$

The strategy of the proof is based on the two following results.

**LEMMA 3.2.** *For every smooth function  $\mathbb{H} : \mathbb{T} \rightarrow \mathbb{R}$ , and every function  $f \in \mathcal{Q}$ ,*

$$\lim_{N \rightarrow \infty} \mathbb{E}_\beta^* \left[ \sup_{0 \leq t \leq T} \left( \mathcal{J}_{t,\mathbf{m},f}^{2,N}(\mathbb{H}) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(\mathbb{H}) \right)^2 \right] = 0.$$

**THEOREM 3.3** (Boltzmann-Gibbs principle). *There exists a sequence of functions  $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{Q}$  such that*

(i) *for every smooth function  $\mathbb{H} : \mathbb{T} \rightarrow \mathbb{R}$ ,*

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E}_\beta^* \left[ \sup_{0 \leq t \leq T} \left( \mathcal{J}_{t,\mathbf{m},f_k}^{1,N}(\mathbb{H}) \right)^2 \right] = 0, \quad (14)$$

(ii) *and moreover*

$$\lim_{k \rightarrow \infty} \mathbb{E}_\beta^* \left[ \lambda \left( \nabla_{0,1}(\omega_0^2 - \Gamma_{f_k}) \right)^2 + \gamma \left( \nabla_0(\Gamma_{f_k}) \right)^2 \right] = 2\mathbb{D}\chi(\beta). \quad (15)$$

As a result, the martingale  $\mathfrak{M}_{t,\mathbf{m},f_k}^{1,N}$  converges in  $\mathbf{L}^2(\mathbb{P}_\beta^*)$ , as  $N \rightarrow \infty$  and then  $k \rightarrow \infty$ , to a martingale  $\mathfrak{M}_t(\mathbb{H})$  of quadratic variation

$$2t\mathbb{D}\chi(\beta) \int_{\mathbb{T}} \mathbb{H}'(u)^2 du,$$

and the limit  $\mathcal{Y}_t(\mathbb{H})$  of  $\mathcal{Y}_{t,\mathbf{m}}^N(\mathbb{H})$  satisfies the equation

$$\mathcal{Y}_t(\mathbb{H}) = \mathcal{Y}_0(\mathbb{H}) + \int_0^t \mathcal{Y}_s(\mathbb{D}\mathbb{H}'' ) ds + \mathfrak{M}_t(\mathbb{H}).$$

We have proved that the limit solves the martingale problems (11) and (12), which uniquely characterized the generalized Ornstein-Uhlenbeck process  $\mathcal{Y}_t$ . The proof of Lemma 3.2 is the content of the next subsection. The proof of Theorem 3.3 is more challenging, and Sections 4, 5 and 6 are devoted to it.

### 3.4 Proof of Lemma 3.2

In this paragraph we give a proof of Lemma 3.2. We define

$$X_{\mathbf{m},f}^N(t) = \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \tau_x f(\mathbf{m}, t)$$

As before, we can rewrite

$$\begin{aligned} \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H) &= X_{\mathbf{m},f}^N(t) - X_{\mathbf{m},f}^N(0) \\ &+ \frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_{x,x+1} \left( \left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_z f \right\} - \nabla_N\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_{x,x+1}(s) - \lambda s] \\ &+ \frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_x \left( \left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_z f \right\} - \nabla_N H\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_x(s) - \lambda s]. \end{aligned}$$

Therefore,

$$\begin{aligned} \left( \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H) \right)^2 &\leq 3 \left( X_{\mathbf{m},f}^N(t) - X_{\mathbf{m},f}^N(0) \right)^2 \\ &+ 3 \left( \frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_{x,x+1} \left( \left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_z f \right\} - \nabla_N H\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_{x,x+1}(s) - \lambda s] \right)^2 \\ &+ 3 \left( \frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_x \left( \left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_z f \right\} - \nabla_N H\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_x(s) - \lambda s] \right)^2 \end{aligned} \quad (16)$$

On the one hand,

$$\mathbb{E}_\beta^* \left[ (X_{\mathbf{m},f}^N)^2 \right] = \frac{1}{N^3} \sum_{x,y \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \nabla_N H\left(\frac{y}{N}\right) \mathbb{E}_\beta^* [\tau_x f, \tau_y f].$$

This last quantity is of order  $1/N^2$ , because  $f$  is a local function of zero average, and  $H$  is smooth. On the other hand, let us define, at least formally,

$$Y_x(\mathbf{m}, \omega) := \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_z f - \nabla_N H\left(\frac{x}{N}\right) \sum_{z \in \mathbb{Z}} \tau_z f.$$

Then, the expectation of the second term of (16) is equal to

$$\frac{3\lambda^2 t N^2}{N^3} \sum_{x \in \mathbb{T}_N} \mathbb{E}_\beta^* \left[ \left\{ \nabla_{x,x+1}(Y_x) \right\}^2 \right].$$

Again, since  $f$  is local and  $H$  is smooth, this quantity is of order  $1/N^2$ . Indeed, in the expression  $\nabla_{x,x+1}(Y_x)$ , there is a sum over  $z \in \mathbb{Z}$ , but in which only terms with  $|z - x| \leq 2$  remain. The same holds for the third term of (16).

## 4 CLT variances at equilibrium

In this section we are going to identify the diffusion coefficient  $D$  that appears in (13). Roughly speaking,  $D$  can be viewed as the asymptotic component of the energy current  $j_{x,x+1}$  in the direction of the gradient  $\omega_{x+1}^2 - \omega_x^2$ , and makes the expression below vanish:

$$\inf_{f \in \mathcal{Q}} \limsup_{N \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{tN} \mathbb{E}_\beta^* \left[ \left( \int_0^t \sum_{x \in \mathbb{T}_N} [j_{x,x+1} - D(\omega_{x+1}^2 - \omega_x^2) - \mathcal{L}^{\mathbf{m}}(\tau_x f)] ds \right)^2 \right], \quad \text{for any } \beta > 0.$$

Let us start by giving already-known tools that will help understand the forthcoming results, staying at an informal level.

## 4.1 An insight through additive functionals of Markov processes

Consider a continuous time Markov process  $\{Y_s\}_{s \geq 0}$  on a complete and separable metric space  $E$ , which has an invariant measure  $\pi$ . We denote by  $\langle \cdot \rangle_\pi$  the inner product in  $L^2(\pi)$  and by  $\mathcal{L}$  the infinitesimal generator of the process. The adjoint of  $\mathcal{L}$  in  $L^2(\pi)$  is denoted by  $\mathcal{L}^*$ . Fix a function  $V : E \rightarrow \mathbb{R}$  in  $L^2(\pi)$  such that  $\langle V \rangle_\pi = 0$ . Theorem 2.7 in [17] gives conditions on  $V$  which guarantee a central limit theorem for

$$\frac{1}{\sqrt{t}} \int_0^t V(Y_s) ds$$

and shows that the limiting variance equals

$$\sigma^2(V, \pi) = 2 \lim_{\substack{z \rightarrow 0 \\ z > 0}} \langle V, (z - \mathcal{L})^{-1} V \rangle_\pi.$$

Let the generator  $\mathcal{L}$  be decomposed as  $\mathcal{L} = \mathcal{S} + \mathcal{A}$ , where  $\mathcal{S} = (\mathcal{L} + \mathcal{L}^*)/2$  and  $\mathcal{A} = (\mathcal{L} - \mathcal{L}^*)/2$  denote, respectively, the symmetric and antisymmetric parts of  $\mathcal{L}$ . Let  $\mathcal{H}_1$  be the completion of the quotient of  $L^2(\pi)$  with respect to constant functions, for the semi-norm  $\|\cdot\|_1$  defined as:

$$\|f\|_1^2 := \langle f, (-\mathcal{L})f \rangle_\pi = \langle f, (-\mathcal{S})f \rangle_\pi.$$

Let  $\mathcal{H}_{-1}$  be the dual space of  $\mathcal{H}_1$  with respect to  $L^2(\pi)$ , in other words, the Hilbert space generated by suitably regular functions and the norm  $\|\cdot\|_{-1}$  defined by

$$\|f\|_{-1}^2 := \sup_g \{2\langle f, g \rangle_\pi - \|g\|_1^2\},$$

where the supremum is carried over some good functions  $g$ . Formally,  $\|f\|_{-1}$  can also be thought as

$$\langle f, (-\mathcal{S})^{-1} f \rangle_\pi.$$

Notice the difference with the variance  $\sigma^2(V, \pi)$  which formally reads

$$2\langle V, (-\mathcal{L})^{-1} V \rangle_\pi = 2\langle V, [(-\mathcal{L})^{-1}]_s V \rangle_\pi.$$

Hereafter,  $B_s$  represents the symmetric part of the operator  $B$ . We can write, at least formally, that

$$\{[(-\mathcal{L})^{-1}]_s\}^{-1} = -\mathcal{S} + \mathcal{A}^* (-\mathcal{S})^{-1} \mathcal{A} \geq -\mathcal{S},$$

where  $\mathcal{A}^*$  stands for the adjoint of  $\mathcal{A}$ . We have therefore that  $[(-\mathcal{L})^{-1}]_s \leq (-\mathcal{S})^{-1}$ . The following result is a rigorous estimate of the time variance in terms of the  $\mathcal{H}_{-1}$  norm, which is proved in [17, Lemma 2.4].

**LEMMA 4.1.** *Given  $T > 0$  and a mean zero function  $V$  in  $L^2(\pi) \cap \mathcal{H}_{-1}$ ,*

$$\mathbb{E}_\pi \left[ \sup_{0 \leq t \leq T} \left( \int_0^t V(s) ds \right)^2 \right] \leq 24T \|V\|_{-1}^2. \quad (17)$$

If we compare the previous left-hand side to the Boltzmann-Gibbs principle (15), the next step should be to take  $V$  proportional to

$$\sum_{x \in \mathbb{T}_N} [j_{x,x+1} - D(\omega_{x+1}^2 - \omega_x^2) - \mathcal{L}^m(\tau_x f)]$$

and then take the limit as  $N$  goes to  $\infty$ . In the right-hand side of (17) we will obtain a variance that depends on  $N$ , and the main task will be to show that this variance converges: this is studied in more details in what follows. Precisely, we prove that the limit of the variance results in a semi-norm, which is denoted by  $||| \cdot |||_\beta$  and defined in (18). We are going to see that (18) involves a variational formula, which formally reads

$$|||f|||_\beta^2 = \ll f, (-S)^{-1}f \gg_{\beta,*} + \frac{2}{\lambda \chi(\beta)} \ll f \gg_{\beta,**}^2.$$

The final step consists in minimizing this semi-norm on a well-chosen subspace in order to get the Boltzmann-Gibbs principle, through orthogonal projections in Hilbert spaces. The hard point is that  $||| \cdot |||_\beta$  only depends on the symmetric part of the generator  $S$ , and the latter is really degenerate, since it does not have a spectral gap.

In Subsection 4.2, we relate the previous limiting variance (taking the limit as  $N$  goes to infinity) to the suitable semi-norm. Subsection 4.3 is devoted to prove the Boltzmann-Gibbs principle inspired by Lemma 4.1. Then, in Section 5 we investigate the Hilbert space generated by the semi-norm, and prove some decompositions into direct sums. Finally, Section 6 focuses on the diffusion coefficient and its different expressions.

## 4.2 Limiting variance and semi-norm

We now assume  $\beta = 1$ . All statements are valid for any  $\beta > 0$ , and the same argument can be easily written. In the following, we deliberately keep the notation  $\chi(1)$ , even if the latter could be replaced with its exact value. We are going to obtain a variational formula for the variance

$$\frac{1}{2\ell} \mathbb{E}_1^* \left[ (-S_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right]$$

where  $\varphi \in \mathcal{Q}_0$  and  $\ell_\varphi = \ell - s_\varphi - 1$ . We first introduce a semi-norm on  $\mathcal{Q}_0$ . For any cylinder function  $\varphi$  in  $\mathcal{Q}_0$ , let us define

$$|||\varphi|||_1^2 := \sup_{g \in \mathcal{Q}} \left\{ 2 \ll \varphi, g \gg_{1,*} + \frac{2}{\lambda} \frac{\ll \varphi \gg_{1,**}^2}{\chi(1)} - \frac{\lambda}{2} \mathbb{E}_1^* \left[ (\nabla_{0,1} \Gamma_g)^2 \right] - \frac{\gamma}{2} \mathbb{E}_1^* \left[ (\nabla_0 \Gamma_g)^2 \right] \right\} \quad (18)$$

$$= \sup_{\substack{g \in \mathcal{Q} \\ a \in \mathbb{R}}} \left\{ 2 \ll \varphi, g \gg_{1,*} + 2a \ll \varphi \gg_{1,**} - \mathbb{E} \left[ \mathcal{D}_0(\mu_1; a\omega_0^2 + \Gamma_g) \right] \right\}. \quad (19)$$

Notice that, if  $\varphi \in \mathcal{Q}_0$  with

$$\varphi = \sum_{y \in \Lambda} \left\{ \nabla_y(F_y) + \nabla_{y,y+1}(G_y) \right\},$$

then

$$|||\varphi|||_1^2 \leq \frac{2}{\gamma} \mathbb{E}_1^* \left[ \left( \sum_{y \in \Lambda} \tau_{-y} F_y \right)^2 \right] + \frac{2}{\lambda} \mathbb{E}_1^* \left[ \left( \sum_{y \in \Lambda} \tau_{-y} G_y \right)^2 \right]$$

We are now in position to state the main result of this subsection.

**THEOREM 4.2.** Consider a quadratic cylinder function  $\varphi \in \mathcal{Q}_0$ . Then

$$\lim_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] = \|\varphi\|_1^2.$$

Here,  $\ell_\varphi$  stands for  $\ell - s_\varphi - 1$  so that the support of  $\tau_x \varphi$  is included in  $\Lambda_\ell$  for every  $x \in \Lambda_{\ell_\varphi}$ .

This theorem is the key of the standard non-gradient Varadhan's method. As usual, the proof is done in two steps that we separate as two different lemmas for the sake of clarity. First, we bound the variance of a cylinder function  $\varphi \in \mathcal{Q}_0$ , with respect to the canonical measure  $\mu_1$ , by the semi-norm  $\|\varphi\|_1^2$  (Lemma 4.3). In the second step, a lower bound for the variance can be easily deduced from the variational formula which expresses the variance as a supremum (Lemma 4.4).

**LEMMA 4.3.** Under the assumptions of Theorem 4.2,

$$\limsup_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] \leq \|\varphi\|_1^2.$$

In this step, one need to know the weak limits of some particular sequences in  $\mathcal{Q}_0$ . In the typical approach, these weak limits are viewed as *germs of closed forms*, but for the harmonic chain, this way of thinking is not necessary.

*Proof.* We follow the proof given in [21, Lemma 4.3] and we assume for the sake of clarity that  $\varphi = \nabla_0(F) + \nabla_{0,1}(G)$ , for two quadratic cylinder functions  $F, G$  (the general case can then be deduced quite easily). We write the variational formula

$$\begin{aligned} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] &= \sup_{h \in \mathcal{C}} \left\{ 2\mathbb{E}_1^* \left[ \varphi, \frac{1}{2\ell} \sum_{|x| \leq \ell_\varphi} \tau_x h \right] - \frac{1}{2\ell} \mathbb{E}[\mathcal{D}_\ell(\mu_1; h)] \right\} \\ &= \sup_{h \in \mathcal{C}} \left\{ 2\mathbb{E}_1^* \left[ F\nabla_0 \left( \frac{1}{2\ell} \sum_{|x| \leq \ell_\varphi} \tau_x h \right) + G\nabla_{0,1} \left( \frac{1}{2\ell} \sum_{|x| \leq \ell_\varphi} \tau_x h \right) \right] - \frac{1}{2\ell} \mathbb{E}[\mathcal{D}_\ell(\mu_1; h)] \right\}. \end{aligned}$$

Since  $\varphi$  is quadratic, we can restrict the supremum in the class of quadratic functions  $h$  with support contained in  $\Lambda_\ell$  (the proof of that statement is detailed in Proposition A.3). We can also restrict the supremum to functions  $h$  such that  $\mathbb{E}[\mathcal{D}_\ell(\mu_1; h)] \leq C\ell$ , as a standard consequence of Proposition 2.2. Next, we want to replace the sums over  $\Lambda_{\ell_\varphi}$  with the same sums over  $\Lambda_\ell$  (recall that  $\ell_\varphi = \ell - s_\varphi - 1 \leq \ell$ ). For that purpose, we denote

$$\zeta_0^\ell(h) = \nabla_0 \left( \frac{1}{2\ell} \sum_{|x| \leq \ell} \tau_x h \right), \quad \zeta_1^\ell(h) = \nabla_{0,1} \left( \frac{1}{2\ell} \sum_{|x| \leq \ell} \tau_x h \right). \quad (20)$$

First of all, from Cauchy-Schwarz inequality, we have

$$\mathbb{E}_1^* \left[ \frac{\gamma}{2} \left( \zeta_0^\ell(h) \right)^2 + \frac{\lambda}{2} \left( \zeta_1^\ell(h) \right)^2 \right] \leq \frac{1}{2\ell} \mathbb{E}[\mathcal{D}_\ell(\mu_1; h)].$$

Then, from Proposition 2.2 we also can write

$$\left| \mathbb{E}_1^* \left[ \varphi, \frac{1}{2\ell} \sum_{\ell_\varphi \leq x \leq \ell} \tau_x h \right] \right| \leq \frac{1}{2\ell} C_\varphi \left( \mathbb{E}[\mathcal{D}_\ell(\mu_1; h)] \right)^{1/2},$$

where  $C_\varphi$  is a constant that depends only on  $\varphi$ . These last two inequalities give the upper bound

$$(2\ell)^{-1}\mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] \\ \leq \sup_h \left\{ 2\mathbb{E}_1^* \left[ \mathbb{F} \zeta_0^\ell(h) + \mathbb{G} \zeta_1^\ell(h) \right] - \mathbb{E}_1^* \left[ \frac{\gamma}{2} \left( \zeta_0^\ell(h) \right)^2 + \frac{\lambda}{2} \left( \zeta_1^\ell(h) \right)^2 \right] \right\} + \frac{C}{\sqrt{\ell}}.$$

Let us choose a sequence  $\{h_\ell\}$  satisfying  $\mathbb{E}[\mathcal{D}_\ell(\mu_1; h_\ell)] \leq C\ell$ . Then, the sequence  $\{\zeta_0^\ell(h_\ell), \zeta_1^\ell(h_\ell)\}$  is uniformly bounded in  $L^2(\mathbb{P}_1^*)$ , and this implies the existence of a weakly convergent subsequence. We denote by  $(\zeta_0, \zeta_1)$  a weak limit and assume that the sequence  $\{\zeta_0^\ell(h_\ell), \zeta_1^\ell(h_\ell)\}$  weakly converges to  $(\zeta_0, \zeta_1)$ . The conclusion is now based on the decomposition of a certain class of quadratic functions that we prove in Appendix B, Theorem B.2: the pair  $(\zeta_0, \zeta_1)$  can be written as the limit in  $L^2(\mathbb{P}_1^*)$  of functions of the form

$$(\nabla_0 \Gamma_g, a(\omega_0^2 - \omega_1^2) + \nabla_{0,1} \Gamma_g),$$

with  $g \in \mathcal{Q}$  and  $a \in \mathbb{R}$ . We have obtained that

$$(2\ell)^{-1}\mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] \leq \sup_{\zeta_0, \zeta_1} \left\{ 2\mathbb{E}_1^* [\mathbb{F} \zeta_0 + \mathbb{G} \zeta_1] - \frac{\gamma}{2} \mathbb{E}_1^* [\zeta_0^2] - \frac{\lambda}{2} \mathbb{E}_1^* [\zeta_1^2] \right\} \\ = \sup_{\substack{g \in \mathcal{Q} \\ a \in \mathbb{R}}} \left\{ 2\mathbb{E}_1^* \left[ \mathbb{F} \nabla_0 \Gamma_g + \mathbb{G} \left( a(\omega_0^2 - \omega_1^2) + \nabla_{0,1} \Gamma_g \right) \right] - \frac{\gamma}{2} \mathbb{E}_1^* \left[ (\nabla_0 \Gamma_g)^2 \right] - \frac{\lambda}{2} \mathbb{E}_1^* \left[ \left( a(\omega_0^2 - \omega_1^2) + \nabla_{0,1} \Gamma_g \right)^2 \right] \right\}.$$

The inequality above is a consequence of the following fact: the  $L^2$ -norm may only decrease along weakly convergent subsequences. The result follows, after recalling (19).  $\square$

We now turn to the upper bound.

**LEMMA 4.4.** *Under the assumptions of Theorem 4.2,*

$$\limsup_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] \geq \|\varphi\|_1^2.$$

*Proof.* We define, for  $f \in \mathcal{Q}$ ,

$$\mathbb{J}_\ell := \sum_{y, y+1 \in \Lambda_\ell} \tau_y \mathbb{J}_{0,1}^S, \quad \mathbb{H}_\ell^f = \sum_{|y| \leq \ell - s_f - 1} \mathcal{S}(\tau_y f).$$

The following limits hold:

$$\lim_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \mathbb{J}_\ell \right] = \ll \varphi \gg_{1, \star \star}, \quad (21) \\ \lim_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \mathbb{H}_\ell^f \right] = \ll \varphi, f \gg_{1, \star}, \\ \lim_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \left( a\mathbb{J}_\ell + \mathbb{H}_\ell^f \right), \left( a\mathbb{J}_\ell + \mathbb{H}_\ell^f \right) \right] = \\ \frac{\lambda}{2} \mathbb{E}_1^* \left[ \left( a(\omega_0^2 - \omega_1^2) + \nabla_{0,1} \Gamma_g \right)^2 \right] + \frac{\gamma}{2} \mathbb{E}_1^* \left[ (\nabla_0 \Gamma_g)^2 \right].$$

We only prove (21), the other relations can be obtained in a similar way. As previously, we assume for the sake of simplicity that  $\varphi = \nabla_0(\mathbb{F}) + \nabla_{0,1}(\mathbb{G})$ . We recall the elementary identity

$$\mathcal{S}_{\Lambda_\ell} \left( \sum_{x \in \Lambda_\ell} x \omega_x^2 \right) = \mathbb{J}_\ell(\omega). \quad (22)$$

Therefore,

$$\begin{aligned} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \mathbb{J}_\ell \right] &= -(2\ell)^{-1} \sum_{y, y+1 \in \Lambda_\ell} \sum_{|x| \leq \ell_\varphi} y \mathbb{E}_1^* [\varphi \omega_{y-x}^2] \\ &= -(2\ell)^{-1} \sum_{y, y+1 \in \Lambda_\ell} \sum_{|x| \leq \ell_\varphi} y \mathbb{E}_1^* [\mathbb{G} \nabla_{0,1}(\omega_{y-x}^2)] \\ &= -(2\ell)^{-1} \sum_{|x| \leq \ell_\varphi} x \mathbb{E}_1^* [\mathbb{G} \nabla_{0,1}(\omega_0^2)] + (x+1) \mathbb{E}_1^* [\mathbb{G} \nabla_{0,1}(\omega_1^2)] \\ &= (2\ell)^{-1} (2\ell_\varphi + 1) \mathbb{E}_1^* [\mathbb{G}, \omega_0^2 - \omega_1^2] \xrightarrow{\ell \rightarrow \infty} \ll \varphi \gg_{1, **}. \end{aligned}$$

The last limit comes from Proposition 2.4 and the fact that  $\ell_\varphi = \ell - s_\varphi - 1$ . We also have used the translation invariance of  $\mathbb{P}_1^*$ . Then, we write the variational formula with  $h = (-\mathcal{S}_{\Lambda_\ell})^{-1}(a\mathbb{J}_\ell + \mathbb{H}_\ell^f)$  and we obtain:

$$\begin{aligned} &\liminf_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right] \\ &\geq \liminf_{\ell \rightarrow \infty} (2\ell)^{-1} \left\{ 2\mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, a\mathbb{J}_\ell + \mathbb{H}_\ell^f \right] - \mathbb{E}_1^* \left[ \left( -\mathcal{S}_{\Lambda_\ell} \right)^{-1} (a\mathbb{J}_\ell + \mathbb{H}_\ell^f), a\mathbb{J}_\ell + \mathbb{H}_\ell^f \right] \right\} \\ &= 2 \ll \varphi, f \gg_{1, * } + 2a \ll \varphi \gg_{1, **} - \mathbb{E} [\mathcal{D}_0(\mu_1; a\omega_0^2 + \Gamma_f)]. \end{aligned}$$

The result follows after taking the supremum on  $f \in \mathcal{Q}$ , and recalling (19). □

### 4.3 Proof of Theorem 3.3

In this paragraph, we start the proof of Theorem 3.3 by using the result given in Theorem 4.2. First, we show how to relate (14) to such variances. Recall that we have assumed  $\beta = 1$ , but the same argument remains in force for any  $\beta > 0$ .

**PROPOSITION 4.5.** *Let  $\psi \in \mathcal{C}_0$ , with  $s_\psi \leq N$ . Then*

$$\left\langle \sup_{0 \leq t \leq T} \left[ \int_0^t \psi(s) ds \right]^2 \right\rangle_1 \leq \frac{24T}{N^2} \langle \psi, (-\mathcal{S}_N)^{-1} \psi \rangle_1. \quad (23)$$

This result is proved for example in [17, Section 2, Lemma 2.4]. We are going to use this bound for functions of type  $\sum_x \mathbb{G}(x/N) \tau_x \varphi$ , where  $\varphi$  belongs to  $\mathcal{Q}_0$ . The main result of this subsection is the following.

**THEOREM 4.6.** *Let  $\varphi \in \mathcal{Q}_0$ , and  $\mathbb{G}$  a smooth function on  $\mathbb{T}$ . Then,*

$$\limsup_{N \rightarrow \infty} \mathbb{E}_1^* \left[ \sup_{0 \leq t \leq T} \left\{ \sqrt{N} \int_0^t \sum_{x \in \mathbb{T}_N} \mathbb{G} \left( \frac{x}{N} \right) \tau_x \varphi(\mathbf{m}, s) ds \right\}^2 \right] \leq CT \|\varphi\|_1^2 \int_{\mathbb{T}} \mathbb{G}^2(u) du. \quad (24)$$

*Proof.* From Proposition 4.5, the left-hand side of (24) is bounded by

$$24T \mathbb{E}_1^* \left[ \sqrt{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \tau_x \varphi(\mathbf{m}), (-N^2 \mathcal{S}_N)^{-1} \left( \sqrt{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \tau_x \varphi(\mathbf{m}) \right) \right],$$

that can be written with the variational formula as

$$24T \sup_{f \in \mathcal{C}} \left\{ \sqrt{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \mathbb{E}_1^* [f \tau_x \varphi] - N^2 \mathbb{E} [\mathcal{D}_N(\mu_1; f)] \right\}.$$

Since  $\varphi \in \mathcal{Q}_0$ , from Proposition A.3 we can restrict the supremum over  $f \in \mathcal{Q}$ . Proposition 2.2 gives

$$\langle f \tau_x \varphi \rangle_1 \leq C_\varphi \left\langle \tau_{-x} f, (-\mathcal{S}_{\Lambda_\varphi})(\tau_{-x} f) \right\rangle_1^{1/2}$$

and by Cauchy-Schwarz inequality,

$$\sqrt{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \langle f \tau_x \varphi \rangle_1 \leq \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right)^2 \right)^{1/2} N C_\varphi \langle f, (-\mathcal{S}_N) f \rangle_1^{1/2}.$$

The supremum on  $f$  can be explicitly computed, and gives the final bound

$$\mathbb{E}_1^* \left[ \sup_{0 \leq t \leq T} \left\{ \sqrt{N} \int_0^t \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \tau_x \varphi(\mathbf{m}, s) ds \right\}^2 \right] \leq C'_\varphi T \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right)^2 \right). \quad (25)$$

We are now going to show that, after sending  $N$  to infinity, the constant on the right-hand side is proportional to  $\|\varphi\|_1^2$ . For that purpose, we average on microscopic boxes: for  $k \ll N$ , we denote

$$\bar{\varphi}_k = \sum_{y \in \Lambda_k} \tau_y \varphi,$$

and we want to substitute

$$\sqrt{N} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \tau_x \varphi$$

with

$$\frac{\sqrt{N}}{2k+1} \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \tau_x \bar{\varphi}_k.$$

The error term that appears is estimated by

$$\mathbb{E}_1^* \left[ \sup_{0 \leq t \leq T} \left\{ \sqrt{N} \int_0^t \sum_{\substack{x, y \in \mathbb{T}_N \\ |x-y| \leq k}} \frac{1}{2k+1} \left( G \left( \frac{x}{N} \right) - G \left( \frac{y}{N} \right) \right) \tau_x \varphi(\mathbf{m}, s) ds \right\}^2 \right].$$

From (25), the expression above is bounded by  $Ck/N^2$ , and then vanishes as  $N \rightarrow \infty$ . We are reduced to estimate

$$\mathbb{E}_1^* \left[ \sup_{0 \leq t \leq T} \left\{ \frac{\sqrt{N}}{2k+1} \int_0^t \sum_{x \in \mathbb{T}_N} G \left( \frac{x}{N} \right) \tau_x \bar{\varphi}_k(\mathbf{m}, s) ds \right\}^2 \right].$$

By the same argument, this is bounded by

$$\begin{aligned} & \frac{CT}{2k+1} \sum_{x \in \mathbb{T}_N} \sup_{f \in \mathcal{Q}} \left\{ \sqrt{N} G \left( \frac{x}{N} \right) \mathbb{E}_1^* [f \tau_x \bar{\varphi}_k] - \frac{N^2}{2k+1} \mathbb{E}_1^* [\tau_{-x} f, (-\mathcal{S}_{\Lambda_k}) \tau_{-x} f] \right\} \\ & \leq \frac{CT}{2k+1} \sum_{x \in \mathbb{T}_N} \sup_{f \in \mathcal{Q}} \left\{ C(\varphi) \sqrt{N} G \left( \frac{x}{N} \right) \mathbb{E}_1^* [\tau_{-x} f, (-\mathcal{S}_{\Lambda_k})(\tau_{-x} f)]^{1/2} - \frac{N^2}{2k+1} \mathbb{E}_1^* [\tau_{-x} f, (-\mathcal{S}_{\Lambda_k}) \tau_{-x} f] \right\} \end{aligned}$$



The supremum on  $f$  can be explicitly computed, and gives the final bound

$$C(\varphi)T \left( \frac{1}{N} \sum_{x \in \mathbb{T}_N} G^2 \left( \frac{x}{N} \right) \right) \frac{1}{2k+1} \mathbb{E}_1^* \left[ \overline{\varphi}_k, (-S_{\Lambda_k})^{-1} \overline{\varphi}_k \right].$$

Taking the limit as  $N \rightarrow \infty$  and then  $k \rightarrow \infty$ , we obtain (24) from the central limit theorem for variances at equilibrium (Theorem 4.2). □

We apply Theorem 4.6 to  $\mathcal{J}_{t,m,f}^{1,N}(\mathbb{H})$ , and we get

$$\limsup_{N \rightarrow \infty} \mathbb{E}_1^* \left[ \sup_{0 \leq t \leq T} \left( \mathcal{J}_{t,m,f}^{1,N}(\mathbb{H}) \right)^2 \right] \leq CT \|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f\|_1^2 \int_{\mathbb{T}} H'(u)^2 du.$$

To conclude the proof of Theorem 3.3, we show in Section 5 that there exists a sequence of local functions  $\{f_k\} \in \mathcal{Q}$  such that

$$\|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f_k\|_1 \xrightarrow{k \rightarrow \infty} 0,$$

and Section 6 is devoted to prove the second statement of Theorem 3.3.

## 5 Hilbert space and projections

We now focus on the semi-norm  $\|\cdot\|_1$  that was introduced in the previous section by (18). We can easily define from  $\|\cdot\|_1$  a semi-inner product on  $\mathcal{C}_0$  through polarization. Denote by  $\mathcal{N}$  the kernel of the semi-norm  $\|\cdot\|_1$  on  $\mathcal{C}_0$ . Then, the completion of  $\mathcal{Q}_0|_{\mathcal{N}}$  denoted by  $\mathcal{H}_1$  is a Hilbert space. Let us explain how Varadhan's non-gradient approach is modified. Usually, the Hilbert space on which orthogonal projections are performed is the completion of  $\mathcal{C}_0|_{\mathcal{N}}$ , in other words it involves all local functions. Then, the standard procedure aims at proving that each element of that Hilbert space can be approximated by a sequence of functions in the range of the generator plus an additional term which is proportional to the current. The crucial steps for obtaining this decomposition consist in: first, controlling the antisymmetric part of the generator by the symmetric one for every cylinder function, and second, proving a strong result on germs of closed forms (see Appendix B). These two key points are not satisfied in our model, but they can be proved when restricted to quadratic functions. It turns out that these weak versions are sufficient, since we are looking for a fluctuation-dissipation approximation that involves quadratic functions only.

In Subsection 5.1, we show that  $\mathcal{H}_1$  is the completion of  $\mathcal{S}\mathcal{Q}|_{\mathcal{N}} + \{j_{0,1}^S\}$ . In other words, all elements of  $\mathcal{H}_1$  can be approximated by  $aj_{0,1}^S + \mathcal{S}g$  for some  $a \in \mathbb{R}$  and  $g \in \mathcal{Q}$ . This is not irrelevant since the symmetric part of the generator preserves the degree of polynomial functions. Moreover, the two subspaces  $\{j_{0,1}^S\}$  and  $\overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}}$  are orthogonal, and we denote their sum by

$$\overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}} \oplus^\perp \{j_{0,1}^S\}.$$

Nevertheless, this decomposition is not satisfactory, because we want the fluctuating term to be on the form  $\mathcal{L}^m(f_k)$ , and not  $\mathcal{S}(f_k)$ . In order to make this replacement, we need to prove the weak sector condition, that gives a control of  $\|\mathcal{A}^m g\|_1$  by  $\|\mathcal{S}g\|_1$ , when  $g$  is a quadratic function. The argument is explained in Subsection 5.2 and 5.3, and the weak sector condition is proved in Appendix C. The only trouble is that this new decomposition is not orthogonal any more, so that we can not express the diffusion coefficient as a variational formula, like (31). This problem is solved in Section 6.

## 5.1 Decomposition according to the symmetric part

We begin this subsection with a table of calculus, very useful in the sequel.

**PROPOSITION 5.1.** For all  $g \in \mathcal{Q}_0, h \in \mathcal{Q}$ ,

$$\begin{aligned} \ll h, \mathcal{S}g \gg_1 &= - \ll h, g \gg_{1,\star} \\ \ll h, j_{0,1}^{\mathcal{S}} \gg_1 &= - \ll h \gg_{1,\star\star} \\ \ll j_{0,1}^{\mathcal{S}}, \mathcal{S}h \gg_1 &= 0 \\ \|\| j_{0,1}^{\mathcal{S}} \|\|_1^2 &= \lambda^2 \chi(1). \end{aligned}$$

*Proof.* The first two identities are direct consequences of Theorem 4.2 and of Equality (22). The last two ones follow directly.  $\square$

**COROLLARY 5.2.** For all  $a \in \mathbb{R}$  and  $g \in \mathcal{Q}$ ,

$$\|\| a j_{0,1}^{\mathcal{S}} + \mathcal{S}g \|\|_1^2 = a^2 \lambda \chi(1) + \frac{\lambda}{2} \mathbb{E}_1^* [(\nabla_{0,1} \Gamma_g)^2] + \frac{\gamma}{2} \mathbb{E}_1^* [(\nabla_0 \Gamma_g)^2].$$

In particular, the variational formula for  $\|\|h\|\|_1$ ,  $h \in \mathcal{Q}_0$ , writes

$$\|\|h\|\|_1^2 = \frac{1}{\lambda \chi(1)} \ll h, j_{0,1}^{\mathcal{S}} \gg_1^2 + \sup_{g \in \mathcal{Q}} \{-2 \ll h, \mathcal{S}g \gg_1 - \|\| \mathcal{S}g \|\|_1^2\}. \quad (26)$$

**PROPOSITION 5.3.** We denote by  $\mathcal{S}\mathcal{Q}$  the space  $\{\mathcal{S}h ; h \in \mathcal{Q}\}$ . Then,

$$\mathcal{H}_1 = \overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}} \oplus^\perp \{j_{0,1}^{\mathcal{S}}\}$$

*Proof.* We divide the proof into two steps.

**(a) The space is well generated** – The inclusion  $\overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}} + \{j_{0,1}^{\mathcal{S}}\} \subset \mathcal{H}_1$  is obvious. Moreover, from the variational formula (26) we know that: if  $h \in \mathcal{H}_1$  satisfies  $\ll h, j_{0,1}^{\mathcal{S}} \gg_1 = 0$  and  $\ll h, \mathcal{S}g \gg_1 = 0$  for all  $g \in \mathcal{Q}$ , then  $\|\|h\|\|_1 = 0$ .

**(b) The sum is orthogonal** – This follows directly from the previous proposition and from the fact that:  $\ll j_{0,1}^{\mathcal{S}}, \mathcal{S}h \gg_1 = 0$  for all  $h \in \mathcal{Q}$ .  $\square$

## 5.2 Replacement of $\mathcal{S}$ with $\mathcal{L}$

In this subsection, we prove identities which mix the antisymmetric and the symmetric part of the generator, which will be used to get the *weak sector condition* (Proposition 5.7).

**LEMMA 5.4.** For all  $g, h \in \mathcal{Q}$ ,

$$\ll \mathcal{S}g, \mathcal{A}^m h \gg_1 = - \ll \mathcal{A}^m g, \mathcal{S}h \gg_1.$$

*Proof.* This easily follows from the first identity of Proposition 5.1 and from the invariance by translations of the measure  $\mathbb{P}_1^*$ :

$$\begin{aligned} \ll \mathcal{S}g, \mathcal{A}^m h \gg_1 &= - \ll g, \mathcal{A}^m h \gg_{1,\star} = - \sum_{x \in \mathbb{Z}} \mathbb{E}_1^* [\tau_x g, \mathcal{A}^m h] = \sum_{x \in \mathbb{Z}} \mathbb{E}_1^* [\mathcal{A}^m(\tau_x g), h] \\ &= \sum_{x \in \mathbb{Z}} \mathbb{E}_1^* [\tau_x(\mathcal{A}^m g), h] = \sum_{x \in \mathbb{Z}} \mathbb{E}_1^* [\mathcal{A}^m g, \tau_{-x} h] = \sum_{x \in \mathbb{Z}} \mathbb{E}_1^* [\mathcal{A}^m g, \tau_x h] = - \ll \mathcal{A}^m g, \mathcal{S}h \gg_1. \end{aligned}$$

$\square$

**LEMMA 5.5.** For all  $g \in \mathcal{Q}$ ,

$$\ll \mathcal{S}g, j_{0,1}^A \gg_1 = - \ll \mathcal{A}^m g, j_{0,1}^S \gg_1.$$

*Proof.* By the first identity of Proposition 5.1,

$$\begin{aligned} \ll \mathcal{S}g, j_{0,1}^A \gg_1 &= - \ll g, j_{0,1}^A \gg_{1,*} = - \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\tau_x g, j_{0,1}^A] = - \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[g, j_{x,x+1}^A] \\ &= - \sum_{x \in \mathbb{Z}} x \mathbb{E}_1^*[g, j_{x-1,x}^A - j_{x,x+1}^A] = - \sum_{x \in \mathbb{Z}} x \mathbb{E}_1^*[g, \mathcal{A}^m(\omega_x^2)] \\ &= \sum_{x \in \mathbb{Z}} x \mathbb{E}_1^*[\mathcal{A}^m g, \omega_x^2] = - \ll \mathcal{A}^m g, j_{0,1}^S \gg_1. \end{aligned}$$

□

Then, these two lemmas together with the second identity of Proposition 5.1 imply the following:

**COROLLARY 5.6.** For all  $a \in \mathbb{R}$ ,  $g \in \mathcal{Q}$ ,

$$\ll a j_{0,1}^S + \mathcal{S}g, a j_{0,1}^A + \mathcal{A}^m g \gg_1 = 0.$$

We are now in position to state the main result of this subsection.

**PROPOSITION 5.7** (Weak sector condition). (i) There exists two constants  $C_0 := C(\gamma, \lambda)$  and  $C_1 := C(\gamma, \lambda)$  such that the following inequalities hold for all  $f, g \in \mathcal{Q}$ :

$$|\ll \mathcal{A}^m g, \mathcal{S}f \gg_1| \leq C_0 \| \mathcal{S}f \|_1 \| \mathcal{S}g \|_1. \quad (27)$$

$$|\ll \mathcal{A}^m g, \mathcal{S}f \gg_1| \leq C_1 \| \mathcal{S}g \|_1 + \frac{1}{2} \| \mathcal{S}f \|_1. \quad (28)$$

(ii) There exists a positive constant  $C$  such that, for all  $g \in \mathcal{Q}$ ,

$$\| \mathcal{A}^m g \|_1 \leq C \| \mathcal{S}g \|_1.$$

*Proof.* The proof is technical because made of explicit computations for quadratic functions. For that reason, we report it to Appendix C. □

### 5.3 Decomposition of the Hilbert space

We now deduce from the previous two subsections the expected decomposition of  $\mathcal{H}_1$ .

**PROPOSITION 5.8.** We denote by  $\mathcal{L}^m \mathcal{Q}$  the space  $\{\mathcal{L}^m g ; g \in \mathcal{Q}\}$ . Then,

$$\mathcal{H}_1 = \overline{\mathcal{L}^m \mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}^S\}.$$

*Proof.* We first prove that  $\mathcal{H}_1$  can be written as the sum of the two subspaces. Then, we show that the sum is direct.

**(a) The space is well generated** – The inclusion  $\overline{\mathcal{L}^m \mathcal{Q}}|_{\mathcal{N}} + \{j_{0,1}^S\} \subset \mathcal{H}_1$  follows from Proposition 2.1. To prove the converse inclusion, let  $h \in \mathcal{H}_1$  so that  $\ll h, j_{0,1}^S \gg_1 = 0$  and  $\ll h, \mathcal{L}^m g \gg_1 = 0$  for all  $g \in \mathcal{Q}$ . From Corollary 5.3,  $h$  can be written as

$$h = \lim_{k \rightarrow \infty} \mathcal{S}g_k$$

for some sequence  $\{g_k\} \in \mathcal{Q}$ . More precisely, since  $\ll \mathcal{S}g_k, \mathcal{A}^m g_k \gg_1 = 0$  by Lemma 5.4,

$$\| \|h\|_1^2 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{S}g_k \gg_1 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{L}^m g_k \gg_1.$$

Moreover, we also have by assumption that  $\ll h, \mathcal{S}g_k \gg_1 = 0$  for all  $k$ , and from Proposition 5.7,

$$\sup_{k \in \mathbb{N}} \| \mathcal{L}^m g_k \|_1 \leq (C + 1) \sup_{k \in \mathbb{N}} \| \mathcal{S}g_k \|_1 =: C_h$$

is finite. Therefore,

$$\| \|h\|_1^2 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{L}^m g_k \gg_1 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k - h, \mathcal{L}^m g_k \gg_1 \leq \lim_{k \rightarrow \infty} C_h \| \mathcal{S}g_k - h \|_1 = 0.$$

**(b) The sum is direct** – Let  $\{g_k\} \in \mathcal{Q}$  be a sequence such that, for some  $a \in \mathbb{R}$ ,

$$\lim_{k \rightarrow \infty} \mathcal{L}^m g_k = a j_{0,1}^S \quad \text{in } \mathcal{H}_1,$$

By a similar argument,

$$\limsup_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{S}g_k \gg_1 = \limsup_{k \rightarrow \infty} \ll \mathcal{L}^m g_k, \mathcal{S}g_k \gg_1 = \limsup_{k \rightarrow \infty} \ll \mathcal{L}^m g_k - a j_{0,1}^S, \mathcal{S}g_k \gg_1 = 0,$$

where the last equality comes from the fact that  $\ll j_{0,1}^S, \mathcal{S}g_k \gg_1 = 0$  for all  $k$ . On the other hand, by Proposition 5.7,  $\| \mathcal{L}^m g_k \|_1 \leq (C + 1) \| \mathcal{S}g_k \|_1^2$ . Then,  $a = 0$ . This concludes the proof.  $\square$

Recall that  $j_{0,1}^S(\mathbf{m}, \omega) = \lambda(\omega_1^2 - \omega_0^2)$ . We have obtained the following result.

**THEOREM 5.9.** *For every  $g \in \mathcal{Q}_0$ , there exists a unique constant  $a \in \mathbb{R}$ , such that*

$$g + a(\omega_1^2 - \omega_0^2) \in \overline{\mathcal{L}^m \mathcal{Q}} \quad \text{in } \mathcal{H}_1. \quad (29)$$

In particular, this theorem states that there exists a unique number  $D$ , and a sequence of cylinder functions  $\{f_k\} \in \mathcal{Q}$  such that

$$\| \|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f_k \|_1 \| \xrightarrow{k \rightarrow \infty} 0. \quad (30)$$

Let us notice that this convergence also holds with the same constant  $D$  and the same sequence  $f_k$  if we replace the semi-norm  $\| \cdot \|_1$  with  $\| \cdot \|_\beta$  for any  $\beta > 0$  (as a consequence of a standard change of variables argument). This concludes the first statement of Theorem 3.3. We prove the second statement (15) in Proposition 6.5 in Section 6.

## 6 On the diffusion coefficient

The main goal of this section is to express the diffusion coefficient in several variational formulas. We also prove the second statement of Theorem 3.3. First, recall Definition 2.4, which can be written as

$$D = \lambda^2 + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \mathbb{E} \left[ \mathcal{D}_0(\mu_1; \Gamma_f) + 2 \langle j_{0,1}^A - \mathcal{A}^m f, \Gamma_g \rangle_1 - \mathcal{D}_0(\mu_1; \Gamma_g) \right]. \quad (31)$$

From Theorem 5.9, there exists a unique number  $D$  such that

$$j_{0,1} - D(\omega_1^2 - \omega_0^2) \in \overline{\mathcal{L}^m \mathcal{Q}} \quad \text{in } \mathcal{H}_1.$$

We are going to obtain a more explicit formula for that  $D$ , and relate it to (31), by following the argument in [21]. We first rewrite the decomposition of the Hilbert space given in Proposition 5.8, by replacing  $j_{0,1}^S$  with  $j_{0,1}$ . This new statement is based on Corollary 5.6, which gives an orthogonality relation. The second step is to find an other orthogonal decomposition (see (32) below), which will enable us to prove the variational formula (31) for  $D$ .

Hereafter, we denote by  $\mathcal{L}^{m,*} := \mathcal{S} - \mathcal{A}^m$  the adjoint of the generator in  $L^2(\mu_1)$ , and

$$j_{0,1}^* := j_{0,1}^S - j_{0,1}^A.$$

**LEMMA 6.1.** *The following decompositions hold*

$$\mathcal{H}_1 = \overline{\mathcal{L}^m \mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}\} = \overline{\mathcal{L}^{m,*} \mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}^*\}.$$

*Proof.* We only sketch the proof of the first decomposition, since it is done in [21]. Let us recall from Proposition 5.8 that  $\overline{\mathcal{L}^m \mathcal{Q}}$  has a complementary subspace in  $\mathcal{H}_1$  which is one-dimensional. Therefore, it is sufficient to prove that  $\mathcal{H}_1$  is generated by  $\overline{\mathcal{L}^m \mathcal{Q}}$  and the total current. Let  $h \in \mathcal{H}_1$  such that  $\langle h, j_{0,1} \rangle_1 = 0$  and  $\langle h, \mathcal{L}^m g \rangle_1 = 0$  for all  $g \in \mathcal{Q}$ . By Corollary 5.3,  $h$  can be written as

$$h = \lim_{k \rightarrow \infty} \mathcal{S} g_k + a j_{0,1}^S$$

for some sequence  $\{g_k\} \in \mathcal{Q}$ , and  $a \in \mathbb{R}$ , and from Corollary 5.6,

$$\|h\|_1^2 = \lim_{k \rightarrow \infty} \langle a j_{0,1}^S + \mathcal{S} g_k, a j_{0,1} + \mathcal{L}^m g_k \rangle_1.$$

Moreover, from Proposition 5.7,

$$\sup_{k \in \mathbb{N}} \|a j_{0,1} + \mathcal{L}^m g_k\|_1^2 \leq 2a^2 \|j_{0,1}\|_1^2 + 2(C+1) \sup_{k \in \mathbb{N}} \|\mathcal{S} g_k\|_1^2 =: C_h$$

is finite. Therefore,

$$\begin{aligned} \|h\|_1^2 &= \lim_{k \rightarrow \infty} \langle a j_{0,1}^S + \mathcal{S} g_k - h, a j_{0,1} + \mathcal{L}^m g_k \rangle_1 \\ &\leq \limsup_{k \rightarrow \infty} C_h \|a j_{0,1}^S + \mathcal{S} g_k - h\|_1^2 = 0. \end{aligned}$$

The same arguments apply to the second decomposition. □

We define bounded linear operators  $T, T^* : \mathcal{H}_1 \rightarrow \mathcal{H}_1$  as

$$\begin{aligned} T(a j_{0,1} + \mathcal{L}^m f) &:= a j_{0,1}^S + \mathcal{S} f, \\ T^*(a j_{0,1}^* + \mathcal{L}^{m,*} f) &:= a j_{0,1}^S + \mathcal{S} f. \end{aligned}$$

From the following identity

$$\|aj_{0,1} + \mathcal{L}^m f\|_1^2 = \|aj_{0,1}^* + \mathcal{L}^{m,*} f\|_1^2 = \|aj_{0,1}^S + \mathcal{S}f\|_1^2 + \|aj_{0,1}^A + \mathcal{A}^m f\|_1^2,$$

we can easily see that  $T^*$  is the adjoint operator of  $T$  and we also have the relations

$$\begin{aligned} \langle Tj_{0,1}^S, j_{0,1}^* \rangle_1 &= \langle T^* j_{0,1}^S, j_{0,1} \rangle_1 = \lambda \chi(1) \\ \langle Tj_{0,1}^S, \mathcal{L}^{m,*} f \rangle_1 &= \langle T^* j_{0,1}^S, \mathcal{L}^m f \rangle_1 = 0, \text{ for all } f \in \mathcal{Q}. \end{aligned}$$

In particular,

$$\mathcal{H}_1 = \overline{\mathcal{L}^{m,*} \mathcal{Q}}|_{\mathcal{N}} \oplus^\perp \{Tj_{0,1}^S\} \quad (32)$$

and there exists a unique number  $Q$  such that

$$j_{0,1}^* - QTj_{0,1}^S \in \overline{\mathcal{L}^{m,*} \mathcal{Q}} \quad \text{in } \mathcal{H}_1.$$

We are going to show that  $D = \lambda Q$ .

**LEMMA 6.2.**

$$Q = \frac{\lambda \chi(1)}{\|Tj_{0,1}^S\|_1^2} = \frac{1}{\lambda \chi(1)} \inf_{f \in \mathcal{Q}} \|j_{0,1}^* - \mathcal{L}^{m,*} f\|_1^2. \quad (33)$$

*Proof.* The first identity follows from the fact that

$$\langle Tj_{0,1}^S, j_{0,1}^* - QTj_{0,1}^S \rangle_1 = \lambda \chi(1) - Q \|Tj_{0,1}^S\|_1^2 = 0.$$

The second identity is obtained from the following statement

$$\inf_{f \in \mathcal{Q}} \|j_{0,1}^* - QTj_{0,1}^S - \mathcal{L}^{m,*} f\|_1 = 0. \quad (34)$$

□

After an easy computation, we can also prove that  $\langle Tg, g \rangle_1 = \langle Tg, Tg \rangle_1$  for all  $g \in \mathcal{H}_1$ . Since  $j_{0,1}^S - Tj_{0,1}^S$  is orthogonal to  $Tj_{0,1}^S$ , we have:

$$j_{0,1}^S - Tj_{0,1}^S \in \overline{\mathcal{L}^{m,*} \mathcal{Q}}.$$

By the fact we obtain the variational formula for  $\|Tj_{0,1}^S\|_1$ :

**PROPOSITION 6.3.**

$$\|Tj_{0,1}^S\|_1^2 = \inf_{f \in \mathcal{Q}} \|j_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2. \quad (35)$$

*Proof.* With a similar argument (in the proof of the previous proposition), we have

$$\inf_{f \in \mathcal{Q}} \|j_{0,1}^S - Tj_{0,1}^S - \mathcal{L}^{m,*} f\|_1 = 0,$$

and

$$\inf_{f \in \mathcal{Q}} \|j_{0,1}^S - Tj_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2 = \inf_{f \in \mathcal{Q}} \|j_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2 - \inf_{f \in \mathcal{Q}} \|Tj_{0,1}^S\|_1^2,$$

which concludes the proof. □

**THEOREM 6.4.**

$$D = \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \|j_{0,1}^* - \mathcal{L}^{\mathbf{m},*} f\|_1^2 = \frac{\chi(1)}{4 \inf_{f \in \mathcal{Q}} \|j_{0,1}^S - \mathcal{L}^{\mathbf{m},*} f\|_1^2}. \quad (36)$$

*Proof.* By the definition,  $j_{0,1} - D j_{0,1}^S / \lambda \in \overline{\mathcal{L}^{\mathbf{m}} \mathcal{Q}}$  and therefore

$$\langle\langle j_{0,1} - j_{0,1}^S \frac{D}{\lambda}, T^* j_{0,1}^S \rangle\rangle_1 = \lambda \chi(1) - \frac{D}{\lambda} \|T j_{0,1}^S\|_1^2 = 0. \quad (37)$$

So,  $D = \lambda Q$ , and the variational formula for  $D$  can be deduced from the one for  $Q$ .  $\square$

**REMARK 6.1.** We can rewrite the variational formula for  $D$  as:

$$\begin{aligned} D &= \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \left\{ \|j_{0,1}^S\|_1^2 + \|Sf\|_1^2 + \|j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f\|_1^2 \right\} \\ &= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \left\{ \|Sf\|_1^2 + \|j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f\|_1^2 \right\} \end{aligned} \quad (38)$$

$$\begin{aligned} &= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \left\{ \|Sf\|_1^2 - 2 \langle\langle j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f, Sg \rangle\rangle_1 - \|Sg\|_1^2 \right\} \\ &= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \left\{ \langle\langle f, -Sf \rangle\rangle_{1,*} + 2 \langle\langle j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f, g \rangle\rangle_{1,*} - \langle\langle g, -Sg \rangle\rangle_{1,*} \right\} \end{aligned} \quad (39)$$

$$= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \mathbb{E} \left[ \mathcal{D}_0(\mu_1; \Gamma_f) + 2 \langle\langle j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f, \Gamma_g \rangle\rangle_1 - \mathcal{D}_0(\mu_1; \Gamma_g) \right]. \quad (40)$$

We use the fact that in (38), we can restrict the infimum on functions  $f$  satisfying  $\langle\langle j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f, j_{0,1}^S \rangle\rangle_1 = 0$ . Let us notice that (39) and (40) recover the variational formula (31).

We are now in position to prove the remaining statement of Theorem 3.3:

**PROPOSITION 6.5.** *For any sequence  $\{f_k\} \in \mathcal{Q}$  such that*

$$\lim_{k \rightarrow \infty} \|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^{\mathbf{m}} f_k\|_1 = 0$$

*we have*

$$\lim_{k \rightarrow \infty} \mathbb{E}_1^* \left[ \lambda \left( \nabla_{0,1}(\omega_0^2 - \Gamma_{f_k}) \right)^2 + \gamma \left( \nabla_0(\Gamma_{f_k}) \right)^2 \right] = 2D\chi(1).$$

*Proof.* By assumption,

$$\lim_{k \rightarrow \infty} \|T(j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^{\mathbf{m}} f_k)\|_1 = 0$$

and therefore

$$\lim_{k \rightarrow \infty} \|j_{0,1}^S - Sf_k\|_1^2 = D^2 \|T(\omega_1^2 - \omega_0^2)\|_1^2.$$

Then, the result follows from

$$D = \lambda Q = \frac{\chi(1)}{\|T(\omega_1^2 - \omega_0^2)\|_1^2}$$

and

$$\|j_{0,1}^S - Sf_k\|_1^2 = \frac{\lambda}{2} \mathbb{E}_1^* \left[ \left( \omega_0^2 - \omega_1^2 - \nabla_{0,1}(\Gamma_{f_k}) \right)^2 \right] + \frac{\gamma}{2} \mathbb{E}_1^* \left[ \left( \nabla_0(\Gamma_{f_k}) \right)^2 \right]. \quad (41)$$

$\square$

## 7 Green-Kubo formulas

In this section, we study the convergence of the Green-Kubo formula given in (10), and then, we investigate its behaviour when the intensity of the exchange noise vanishes.

### 7.1 Convergence of Green-Kubo formula

Linear response theory predicts that the diffusion coefficient is given by the homogenized *Green-Kubo formula*. In [3, Section 3] it is proved that this formula can be well defined in infinite volume as the following:

$$\bar{\kappa}(z) = \lambda + \frac{1}{2} \int_0^{+\infty} dt e^{-zt} \ll j_{0,1}^A(\mathbf{m}, t), j_{0,1}^A(\mathbf{m}, 0) \gg_{1,\star} \quad (42)$$

where  $\ll \cdot \gg_{1,\star}$  is the inner product defined by (8). Even if the measure  $\mathbb{P}_1^\star$  is not on the path space, the translation invariance of  $\mathbb{P}_1^\star$  permits to give a rigorous sense to (42) (see [3] for details).

Hereafter, we extend the inner-product  $\ll \cdot \gg_{1,\star}$  (originally defined on  $\mathcal{C}$ ) to the Hilbert space generated by the set of square integrable functions and denoted by  $\mathbf{L}_\star^2$ . We define  $h_z := h_z(\mathbf{m}, \omega; 1)$  as the solution of the resolvent equation in  $\mathbf{L}_\star^2$

$$(z - \mathcal{L}^{\mathbf{m}})h_z = j_{0,1}^A. \quad (43)$$

The Laplace transform is defined and is smooth on  $(0, +\infty)$ , and can be rewritten:

$$\bar{\kappa}(z) = \lambda + \frac{1}{2} \ll j_{0,1}^A, (z - \mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{1,\star}. \quad (44)$$

**THEOREM 7.1.** *The following limit*

$$\bar{\mathbf{D}} := \lim_{\substack{z \rightarrow 0 \\ z > 0}} \bar{\kappa}(z) \quad (45)$$

*exists, and is finite.*

*Proof.* We investigate the existence of the limit

$$\lim_{\substack{z \rightarrow 0 \\ z > 0}} \ll j_{0,1}^A, (z - \mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{1,\star}. \quad (46)$$

With the notations above, we have to prove that

$$\mathbf{L}(z) = \frac{1}{2} \ll h_z, j_{0,1}^A \gg_{1,\star}$$

converges as  $z$  goes to 0, and that the limit is finite and non-negative. Then, from (44) it will follow that  $\bar{\mathbf{D}} \geq \lambda > 0$  and  $\bar{\mathbf{D}}$  is positive. We denote by  $\|\cdot\|_1$  the semi-norm corresponding to the symmetric part of the generator due to the flip noise

$$\|f\|_1^2 = \ll f, (-\gamma \mathcal{S}^{\text{flip}})f \gg_{1,\star}$$

and  $\mathcal{H}_\star$  is the Hilbert space obtained by the completion of  $\mathbf{L}_\star^2$  w.r.t. that semi-norm. We multiply (43) by  $h_z$  and integrate with respect to  $\ll \cdot \gg_{1,\star}$  and we get:

$$z \ll h_z, h_z \gg_{1,\star} + \|h_z\|_1^2 + \ll h_z, (-\lambda \mathcal{S}^{\text{exch}})h_z \gg_{1,\star} = \ll h_z, j_{0,1}^A \gg_{1,\star}.$$



Let us notice that  $(-\gamma\mathcal{S}^{\text{flip}})(j_{0,1}^A) = 2\gamma j_{0,1}^A$ . As a consequence, the Cauchy-Schwarz inequality for the scalar product  $\ll \cdot, (-\gamma\mathcal{S}^{\text{flip}}) \cdot \gg_{1,\star}$  on the right-hand side gives

$$\|h_z\|_1^2 \leq C$$

for some positive constant  $C$ . Since  $\{h_z\}_z$  is bounded in  $\mathcal{H}_\star$ , we can extract a weakly converging subsequence in  $\mathcal{H}_\star$ . We continue to denote this subsequence by  $\{h_z\}_z$  and we denote by  $h_0$  the limit.

Now we are going to show that the convergence is stronger (see (iv) in Lemma 7.2 below) and that the limit is independent of the subsequence. Since the generator  $\mathcal{L}^m$  conserves the degree of homogeneous polynomial functions, we know that the solution of the resolvent equation is expected to be on the form

$$h_z(\omega) = \sum_{x,y \in \mathbb{Z}^2} \varphi_z(x,y) \omega_x \omega_y,$$

where  $\varphi_z : \mathbb{Z}^2 \rightarrow \mathbb{R}$  is a square-summable symmetric function. Let  $h_z = h_z^- + h_z^\neq$  be the decomposition of  $h_z$  according to the two subspaces  $\mathcal{Q}^-$  and  $\mathcal{Q}^\neq$ , where  $\mathcal{Q}^-$  is generated by  $\{\omega_x^2, x \in \mathbb{Z}\}$  and  $\mathcal{Q}^\neq$  is generated by  $\{\omega_x \omega_y, x \neq y\}$ . The main point in the following argument is that all gradient terms vanish in  $\mathbf{L}_\star^2$ .

First of all, one can easily see how the spaces  $\mathcal{Q}^-$  and  $\mathcal{Q}^\neq$  are mapped by the generators:

$$\begin{aligned} \mathcal{A}^m : \mathcal{Q}^- &\rightarrow \mathcal{Q}^\neq & \mathcal{A}^m : \mathcal{Q}^\neq &\rightarrow \mathcal{Q} \\ \mathcal{S}^{\text{flip}} : \mathcal{Q}^- &\rightarrow \{0\} & \mathcal{S}^{\text{flip}} : \mathcal{Q}^\neq &\rightarrow \mathcal{Q}^\neq \\ \mathcal{S}^{\text{exch}} : \mathcal{Q}^- &\rightarrow \mathcal{Q}^- & \mathcal{S}^{\text{exch}} : \mathcal{Q}^\neq &\rightarrow \mathcal{Q}^\neq \end{aligned}$$

Moreover, if  $f \in \mathcal{Q}^-$ , then  $\mathcal{A}^m(f)$  is a gradient in  $\mathcal{Q}^\neq$ , and  $\mathcal{S}^{\text{exch}}(f)$  is a gradient in  $\mathcal{Q}^-$ . With all these considerations, (43) rewrites in  $\mathbf{L}_\star^2$  as

$$\begin{cases} z h_z^- - \lambda \mathcal{S}^{\text{exch}}(h_z^-) = 0 \\ z h_z^\neq - \lambda \mathcal{S}^{\text{exch}}(h_z^\neq) - \gamma \mathcal{S}^{\text{flip}}(h_z^\neq) - \mathcal{A}^m(h_z^\neq) = j_{0,1}^A. \end{cases}$$

The first equation means that  $h_z^- = 0$  in  $\mathbf{L}_\star^2$  and therefore the solution  $h_z$  of the resolvent equation is an element of  $\mathcal{Q}^\neq$ . As a consequence, we can write  $(-\gamma\mathcal{S}^{\text{flip}})(h_z) = 2\gamma h_z$ , and this remark is one of the key points in the following argument.

**LEMMA 7.2.** *All the properties below are satisfied:*

- (i)  $\lim_{z \rightarrow 0} z \ll h_z, h_z \gg_{1,\star} = 0$
- (ii)  $\{h_z\}$  weakly converges as  $z$  goes to 0 towards  $h_0$  in  $\mathbf{L}_\star^2$
- (iii)  $\ll j_{0,1}^A, h_0 \gg_{1,\star} = \ll h_0, (-\mathcal{S})h_0 \gg_{1,\star}$
- (iv)  $\ll (h_z - h_0), (-\mathcal{S})(h_z - h_0) \gg_{1,\star}$  vanishes as  $z$  goes to 0
- (v) the weak limit of  $\{h_z\}$  does not depend on the subsequence.

We briefly prove the five points: (i) and (ii) come from the fact that  $(-\gamma\mathcal{S}^{\text{flip}})(h_z) = 2\gamma h_z$ . To get (iii), we multiply (43) by  $h_{z'}$  and integrate:

$$z \ll h_{z'}, h_z \gg_{1,\star} + \ll h_{z'}, (-\mathcal{S})h_z \gg_{1,\star} + \ll h_{z'}, (-\mathcal{A}^m)h_z \gg_{1,\star} = \ll h_{z'}, j_{0,1}^A \gg_{1,\star}. \quad (47)$$

We first take the limit as  $z' \rightarrow 0$  and then as  $z \rightarrow 0$ , and we use (i) and (ii) to obtain (iii). In the same way, multiplying (43) by  $h_z$  gives

$$z \ll h_z, h_z \gg_{1,\star} + \ll h_z, (-S)h_z \gg_{1,\star} = \ll h_z, j_{0,1}^A \gg_{1,\star}.$$

The first term of the left-hand side vanishes as  $z$  goes to 0, and the right-hand side converges to  $\ll h_0, (-S)h_0 \gg_{1,\star}$ . This implies (iv), that is

$$\ll (h_z - h_0), (-S)(h_z - h_0) \gg_{1,\star} \xrightarrow{z \rightarrow 0} 0.$$

The uniqueness of the limit follows by a standard argument with the same idea of (47). We have proved the first part: the limit (46) exists. To obtain its finiteness, we are going to give an upper bound, using the following variational formula:

$$\ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star} = \sup_f \left\{ 2 \ll f, j_{0,1}^A \gg_{1,\star} - \|f\|_{\pm 1, z}^2 - \|\mathcal{A}^m f\|_{-1, z}^2 \right\},$$

where the supremum is carried over local functions and the two norms  $\|\cdot\|_{\pm 1, z}$  are defined by

$$\|f\|_{\pm 1, z}^2 = \ll f, (z - S)^{\pm 1} f \gg_{1,\star}.$$

For the upper bound, we neglect the term coming from the antisymmetric part  $\mathcal{A}^m f$ , that gives

$$\ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star} \leq \ll j_{0,1}^A, (z - S)^{-1} j_{0,1}^A \gg_{1,\star}.$$

In the right-hand side we can also neglect the part coming from the exchange symmetric part  $\mathcal{S}^{\text{exch}}$ , and remind that  $\mathcal{S}^{\text{flip}}(j_{0,1}^A) = -2j_{0,1}^A$ . This gives an explicit finite upper bound. Then, we have from Lemma 7.2, Property (iii) that the limit

$$\lim_{z \rightarrow 0} \ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star} = \ll j_{0,1}^A, h_0 \gg_{1,\star} = \ll h_0, (-S)h_0 \gg_{1,\star} \geq 0,$$

and the positiveness is proved.  $\square$

## 7.2 Equivalence of the definitions

In this subsection we rigorously prove the equality between the variational formula for the diffusion coefficient and the Green-Kubo formula (see the end of Subsection 2.5).

**THEOREM 7.3.** *For every  $\lambda > 0$  and  $\gamma > 0$ ,*

$$\bar{D} := \lambda + \frac{1}{2} \lim_{\substack{z \rightarrow 0 \\ z > 0}} \ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star}$$

*coincides with the diffusion coefficient  $D$  defined in Theorem 6.4.*

*Proof.* From Subsection 6, we know that the diffusion coefficient can be written different ways. For instance, one can easily check that

$$D = \frac{2}{\|T(\omega_1^2 - \omega_0^2)\|_1^2}.$$

By definition of  $D$ , there exists a sequence  $\{f_\varepsilon\}_{\varepsilon > 0}$  of functions in  $\mathcal{Q}$  such that

$$g_\varepsilon := j_{0,1}^* - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^{m,\star}(f_\varepsilon)$$

satisfies  $\|g_\varepsilon\|_1 \rightarrow 0$  as  $\varepsilon$  goes to 0. By substitution in the equality above, we get

$$D^{-1} = \frac{1}{2D^2} \ll j_{0,1}^* - \mathcal{L}^{m,*} f_\varepsilon - g_\varepsilon, T^*(j_{0,1}^* - \mathcal{L}^{m,*} f_\varepsilon - g_\varepsilon) \gg_1$$

recalling that  $\ll Tg, Tg \gg_1 = \ll g, T^*g \gg_1$  for all  $g \in \mathcal{H}_1$ . Therefore,

$$\begin{aligned} D &= \frac{1}{2} \ll j_{0,1}^* - \mathcal{L}^{m,*} f_\varepsilon - g_\varepsilon, j_{0,1}^S - S f_\varepsilon - g_\varepsilon \gg_1 \\ &= \frac{1}{2} \ll j_{0,1}^* - \mathcal{L}^{m,*} f_\varepsilon, j_{0,1}^S - S f_\varepsilon \gg_1 + R_\varepsilon \end{aligned}$$

where  $R_\varepsilon$  is bounded by  $C\|g_\varepsilon\|_1^2$ , and then vanishes as  $\varepsilon$  goes to 0. Finally, from Proposition 5.1, we can write

$$D = \lambda + \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}.$$

The problem is now reduced to prove that

$$\lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*} = \lim_{\substack{z \rightarrow 0 \\ z > 0}} \ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,*}. \quad (48)$$

For every  $z > 0$  and  $\varepsilon > 0$ , we have by definition above and (43),

$$j_{0,1}^A = z h_z - \mathcal{L}^m h_z \quad (49)$$

$$j_{0,1}^* = D(\omega_1^2 - \omega_0^2) + g_\varepsilon + \mathcal{L}^{m,*} f_\varepsilon. \quad (50)$$

First, multiply (50) by  $f_\varepsilon$  and integrate with respect to  $\ll \cdot \gg_{1,*}$ , keeping in mind that all gradients give no contribution. We get

$$- \ll j_{0,1}^A, f_\varepsilon \gg_{1,*} = \ll f_\varepsilon, g_\varepsilon \gg_{1,*} - \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}$$

and using (49),

$$\ll \mathcal{L}^m h_z, f_\varepsilon \gg_{1,*} - z \ll h_z, f_\varepsilon \gg_{1,*} = \ll f_\varepsilon, g_\varepsilon \gg_{1,*} - \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}.$$

First, let  $z$  go to 0, and observe that  $z \ll h_z, f_\varepsilon \gg_{1,*}$  vanishes, from the Cauchy-Schwarz inequality together with Statement (i) of Lemma 7.2. The limit of  $\ll \mathcal{L}^m h_z, f_\varepsilon \gg_{1,*}$  exists from the weak convergence of  $\{h_z\}_z$ . Then, take the limit as  $\varepsilon$  goes to 0, and observe that

$$\begin{aligned} \ll f_\varepsilon, g_\varepsilon \gg_{1,*} &= \ll f_\varepsilon, (-S)(-S)^{-1} g_\varepsilon \gg_{1,*} \leq \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}^{1/2} \ll g_\varepsilon, (-S)^{-1} f_\varepsilon \gg_{1,*}^{1/2} \\ &\leq C \|g_\varepsilon\|_1 \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

The first equality is justified by the fact that  $g_\varepsilon$  belongs to  $\mathcal{Q}_0$ , and the last inequality comes from the definition of the semi-norm  $\|\cdot\|_1$  given in (18). As a consequence, we have obtained

$$\lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*} = \lim_{\varepsilon \rightarrow 0} \lim_{z \rightarrow 0} \ll -\mathcal{L}^m h_z, f_\varepsilon \gg_{1,*}.$$

In the same way, multiply (50) by  $h_z$  and integrate with respect to  $\ll \cdot \gg_{\beta,*}$  so that

$$- \ll j_{0,1}^A, h_z \gg_{1,*} = \ll g_\varepsilon, h_z \gg_{1,*} + \ll \mathcal{L}^{m,*} f_\varepsilon, h_z \gg_{1,*}.$$

If we send first  $z$  to 0, then  $\ll g_\varepsilon, h_z \gg_{1,*}$  converges to  $\ll g_\varepsilon, h_0 \gg_{1,*}$  from the weak convergence of  $\{h_z\}_z$ . With the same argument as before, we write

$$\ll g_\varepsilon, h_0 \gg_{1,*} \leq C \|g_\varepsilon\|_1 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore,

$$\begin{aligned} \lim_{z \rightarrow 0} \ll j_{0,1}^A, h_z \gg_{1,*} &= \lim_{\varepsilon \rightarrow 0} \lim_{z \rightarrow 0} \ll -\mathcal{L}^{\mathbf{m},*} f_\varepsilon, h_z \gg_{1,*} \\ &= \lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-\mathcal{S})f_\varepsilon \gg_{1,*} \end{aligned}$$

and the claim is proved.  $\square$

### 7.3 Vanishing exchange noise

With the same ideas of the previous subsection, it can be easily shown that the homogenized Green-Kubo formula also converges if the strength  $\lambda$  of the exchange noise vanishes. The aim of this paragraph is to study the limit of (45) as  $\lambda$  goes to 0. First, we turn (44) into a new definition that highlights the dependence on  $\lambda > 0$ . For that purpose we introduce new notations: we denote  $\mathcal{S}_0 = \gamma \mathcal{S}^{\text{flip}}$ ,  $\mathcal{S}_\lambda = \mathcal{S}_0 + \lambda \mathcal{S}^{\text{exch}}$ , and then

$$\begin{cases} \mathcal{L}_0^{\mathbf{m}} = \mathcal{A}^{\mathbf{m}} + \mathcal{S}_0 \\ \mathcal{L}_\lambda^{\mathbf{m}} = \mathcal{A}^{\mathbf{m}} + \mathcal{S}_\lambda = \mathcal{L}_0^{\mathbf{m}} + \lambda \mathcal{S}^{\text{exch}} \end{cases} \quad \text{and} \quad J_0(\mathbf{m})(\omega) = \frac{\omega_0 \omega_1}{\sqrt{m_0 m_1}} = j_{0,1}^A(\mathbf{m}, \omega).$$

Let us introduce the homogenized Green-Kubo formula for both noises:

$$\bar{\kappa}(\lambda, z) := \ll J_0(\mathbf{m}), (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} J_0(\mathbf{m}) \gg_{1,*} \quad (51)$$

and the homogenized Green-Kubo formula for flip noise only:

$$\bar{\kappa}_0(z) := \ll J_0(\mathbf{m}), (z - \mathcal{L}_0^{\mathbf{m}})^{-1} J_0(\mathbf{m}) \gg_{1,*}. \quad (52)$$

According to the previous paragraph, we already know that the Green-Kubo formulas (51) and (52) converge as  $z$  goes to 0. Then, the following diffusion coefficients are well defined, for all  $\lambda > 0$ ,

$$\begin{cases} \bar{D}(\lambda) := \lambda + \lim_{z \rightarrow 0} \bar{\kappa}(\lambda, z), \\ \bar{D}_0 := \lim_{z \rightarrow 0} \bar{\kappa}_0(z). \end{cases}$$

The main result of this subsection is stated in the following theorem.

**THEOREM 7.4.** *The function  $\lambda \mapsto \bar{D}(\lambda)$  is continuous at 0. More precisely,*

$$\lim_{\lambda \rightarrow 0} \bar{D}(\lambda) = \bar{D}_0.$$

Let us remark that the theorem above does not imply the existence of the hydrodynamics diffusion coefficient  $D(0, \gamma)$ . This question remains open.

*Proof.* The proof is divided into two steps. For the sake of readability, we erase the notation  $\mathbf{m}$  in  $J_0(\mathbf{m})$ , and keep in mind its dependence on the disorder.

**Step 1 - Convergence of the diffusion coefficient.** Let us denote by  $h_{z,0}$  and  $\bar{h}_{z,\lambda}$  the two solutions of the following resolvent equations in  $\mathbf{L}_*^2$ :

$$\begin{aligned}(z - \mathcal{L}_0^{\mathbf{m}})h_{z,0} &= J_0, \\ (z - \mathcal{L}_\lambda^{\mathbf{m},*})\bar{h}_{z,\lambda} &= J_0.\end{aligned}$$

We look at the following difference, for  $\lambda, z > 0$  fixed,

$$\begin{aligned}& \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} J_0 \gg_{1,*} - \ll J_0, (z - \mathcal{L}_0^{\mathbf{m}})^{-1} J_0 \gg_{1,*} \right| \\ &= \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} J_0 \gg_{1,*} - \ll J_0, h_{z,0} \gg_{1,*} \right| \\ &= \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} [(z - \mathcal{L}_0^{\mathbf{m}})h_{z,0} - (z - \mathcal{L}_\lambda^{\mathbf{m}})h_{z,0}] \gg_{1,*} \right| \\ &= \lambda \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,*} \right| \\ &= \lambda \left| \ll (z - \mathcal{L}_\lambda^{\mathbf{m},*})^{-1} J_0, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,*} \right|.\end{aligned}$$

To complete the proof, we are reduced to show that  $\lambda \left| \ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,*} \right|$  vanishes when we first let  $z \rightarrow 0$  and then  $\lambda \rightarrow 0$ . For that purpose, we need more precise information on the two solutions  $\bar{h}_{z,\lambda}$  and  $h_{z,0}$ . Since the generator  $\mathcal{L}_\lambda^{\mathbf{m}}$  (resp.  $\mathcal{L}_0^{\mathbf{m}}$ ) conserves the degree of homogeneous polynomial functions, we know that the solution of the resolvent equation  $\bar{h}_{z,\lambda}$  (resp.  $h_{z,0}$ ) has to be homogeneous polynomial of degree two, precisely:

$$\bar{h}_{z,\lambda}(\omega) = \sum_{x,y \in \mathbb{Z}} \varphi_{z,\lambda}(\mathbf{m}, x, y) \omega_x \omega_y,$$

where  $\varphi_{z,\lambda}(\mathbf{m}, \cdot, \cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$  is a square integrable symmetric function. As before, we decompose every degree two function  $h$  as  $h = h^- + h^\neq$ , where  $h^-$  belongs to  $\mathcal{Q}^-$  and  $h^\neq$  belongs to  $\mathcal{Q}^\neq$ . We have seen in the proof of Theorem 7.1 that the part belonging to  $\mathcal{Q}^-$  vanishes for the two solutions, in other words,  $\bar{h}_{z,\lambda}$  and  $h_{z,0}$  are elements of  $\mathcal{Q}^\neq$ . As a consequence,

$$\ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,*} = \ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,*}$$

which is bounded by

$$\sqrt{\ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,*}} \sqrt{\ll h_{z,0}, (-\mathcal{S}^{\text{exch}})(h_{z,0}) \gg_{1,*}}$$

according to the Cauchy-Schwarz inequality for the scalar product  $\ll \cdot, (-\mathcal{S}^{\text{exch}}) \cdot \gg_{1,*}$ . We treat separately the two terms into the two lemmas below. We prove that the first term is bounded by  $C/\sqrt{\lambda}$ , and the second one is uniformly bounded for  $\lambda, z > 0$ . Here we state the two lemmas:

**LEMMA 7.5.** *There exists a constant  $C > 0$  such that, for all  $z, \lambda > 0$ ,*

$$\ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,*} \leq \frac{C}{\lambda}.$$

**LEMMA 7.6.** *There exists a constant  $C > 0$  such that, for all  $z > 0$ ,*

$$\ll h_{z,0}, (-\mathcal{S}^{\text{exch}})(h_{z,0}) \gg_{1,*} \leq C.$$

From these statements we deduce

$$\lambda \left| \ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,*} \right| \leq C_0 \sqrt{\lambda}$$

where  $C_0$  does not depend on  $\lambda, z > 0$ , and Theorem 7.4 follows.

**Step 2 - Proofs of the two lemmas.** We begin with the proof of Lemma 7.5. We recall the resolvent equation in  $\mathbf{L}_*^2$ :

$$z\bar{h}_{z,\lambda} - (\lambda\mathcal{S}^{\text{exch}} + \mathcal{S}_0 - \mathcal{A}^{\mathbf{m}})\bar{h}_{z,\lambda} = J_0. \quad (53)$$

We multiply (53) by  $h_{z,\lambda}$  and integrate with respect to  $\ll \cdot \gg_{1,*}$ , in order to get

$$z \ll \bar{h}_{z,\lambda}, \bar{h}_{z,\lambda} \gg_{1,*} + \ll \bar{h}_{z,\lambda}, (-\mathcal{S}_0)(\bar{h}_{z,\lambda}) \gg_{1,*} + \lambda \ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,*} = \ll J_0, \bar{h}_{z,\lambda} \gg_{1,*}.$$

The right-hand side rewrites as

$$(2\gamma)^{-1} \ll (-\mathcal{S}_0)(J_0), h_{z,0} \gg_{1,*}.$$

Cauchy-Schwarz inequality for the scalar product  $\ll \cdot, (-\mathcal{S}_0) \cdot \gg_{1,*}$  on the right-hand side gives

$$z \ll \bar{h}_{z,\lambda}, \bar{h}_{z,\lambda} \gg_{1,*} \leq C$$

with  $C := (2\gamma)^{-1/2} \ll J_0, J_0 \gg_{1,*}^{1/2}$  and then

$$\lambda \ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,*} \leq C.$$

We now turn to Lemma 7.6. We prove a general result, precisely: there exists a constant  $C > 0$  such that, for all  $g \in \mathcal{Q}^\neq$ ,

$$\ll g, (-\mathcal{S}^{\text{exch}})g \gg_{1,*} \leq C \ll g, g \gg_{1,*}. \quad (54)$$

This fact is proved through explicit computations. Let us write  $g \in \mathcal{Q}^\neq$  in the form

$$g(\omega) = \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \phi_{x,k}(\mathbf{m}) \omega_x \omega_{x+k}.$$

A straightforward computation gives that

$$\begin{aligned} \ll g, (-\mathcal{S}^{\text{exch}})g \gg_{1,*} &= \frac{1}{2} \mathbb{E}_1^* \left[ (\nabla_{0,1} \Gamma_g)^2 \right] = \sum_{k \geq 2} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \phi_{z,k}(\tau_{-z} \mathbf{m}) - \phi_{z,k}(\tau_{1-z} \mathbf{m}) \right)^2 \right] \\ &\leq 4 \sum_{k \geq 2} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \phi_{z,k}(\mathbf{m}) \right)^2 \right]. \end{aligned}$$

In the last inequality, we use the fact that the measure  $\mathbb{P}$  on the disorder is translation invariant and that  $(a - b)^2 \leq 2(a^2 + b^2)$  for all  $a, b \in \mathbb{R}$ . Besides, one can also check that

$$\ll g, g \gg_{1,*} = \sum_{k \geq 1} \mathbb{E} \left[ \sum_{x, z \in \mathbb{Z}} \phi_{z,k}(\tau_{-z} \mathbf{m}) \phi_{x,k}(\tau_{-x} \mathbf{m}) \right] = \sum_{k \geq 1} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \phi_{z,k}(\mathbf{m}) \right)^2 \right],$$

thanks to the translation invariance of  $\mathbb{P}$ . The bound (54) follows directly, with  $C = 4$ . To prove Lemma 7.6, it remains to show that  $\ll h_{z,0}, h_{z,0} \gg_{1,*}$  is uniformly bounded in  $z$ . We recall the resolvent equation in  $\mathbf{L}_*^2$ :

$$zh_{z,0} - (\mathcal{S}_0 + \mathcal{A}^{\mathbf{m}})h_{z,0} = J_0. \quad (55)$$

Notice that we can write  $\mathcal{S}_0(h_{z,0}) = -2\gamma h_{z,0}$ . We multiply (55) by  $h_{z,0}$  and integrate with respect to  $\ll \cdot \gg_{1,*}$  in order to get

$$z \ll h_{z,0}, h_{z,0} \gg_{1,*} + 2\gamma \ll h_{z,0}, h_{z,0} \gg_{1,*} = \ll J_0, h_{z,0} \gg_{1,*}.$$

As previously, Cauchy-Schwarz inequality for the scalar product  $\ll \cdot, (-\mathcal{S}_0) \cdot \gg_{1,*}$  on the right-hand side gives

$$\ll h_{z,0}, h_{z,0} \gg_{1,*} \leq C,$$

with  $C := (2\gamma)^{-1} \ll J_0, J_0 \gg_{1,*}^{1/2}$ . □

## 8 The anharmonic chain perturbed by a diffusive noise

In this last main section we say a few words about the anharmonic chain, meaning that the interactions between atoms are non-linear, and given by a potential  $V$ . As in [21], we assume that the function  $V : \mathbb{R} \rightarrow \mathbb{R}_+$  satisfies the following properties:

- (i)  $V(\cdot)$  is a smooth symmetric function,
- (ii) there exist  $\delta_-$  and  $\delta_+$  such that  $0 < \delta_- \leq V''(\cdot) \leq \delta_+ < +\infty$ ,
- (iii)  $\delta_-/\delta_+ > (3/4)^{1/16}$ .

Using the same notations as in the introduction, the configuration  $\{p_x, r_x\}$  now evolves according to

$$\begin{cases} \frac{dp_x}{dt} = V'(r_{x+1}) - V'(r_x), \\ \frac{dr_x}{dt} = \frac{p_x}{M_x} - \frac{p_{x-1}}{M_{x-1}}. \end{cases} \quad (56)$$

We define  $\pi_x := p_x/\sqrt{M_x}$ , and the dynamics on  $\{\pi_x, r_x\}$  is rewritten as:

$$\begin{cases} \frac{d\pi_x}{dt} = \frac{1}{\sqrt{M_x}} [V'(r_{x+1}) - V'(r_x)], \\ \frac{dr_x}{dt} = \frac{\pi_x}{\sqrt{M_x}} - \frac{\pi_{x-1}}{\sqrt{M_{x-1}}}. \end{cases} \quad (57)$$

The total energy

$$\mathcal{E} := \sum_{x \in \mathbb{Z}} \left\{ \frac{\pi_x^2}{2} + V(r_x) \right\}$$

is conserved. The flip and exchange noises have poor ergodic properties, and can be used for harmonic chains only. For the anharmonic case, we introduce a stronger stochastic perturbation. Now, the total generator of the dynamics writes  $\mathcal{L}^m = \mathcal{A}^m + \gamma \mathcal{S}$ , where

$$\mathcal{A}^m := \sum_x \frac{1}{\sqrt{m_x}} (X_x - Y_{x,x+1}), \quad \mathcal{S} := \frac{1}{2} \sum_x X_x^2 + Y_{x,x+1}^2, \quad (58)$$

where  $Y_{x,y} = \pi_x \partial_{r_y} - V'(r_y) \partial_{\pi_x}$ , and  $X_x = Y_{x,x}$ . For this anharmonic case, the two needed ingredients can be proved directly from [21]. First, notice that the symmetric part of the generator does not depend on the disorder and is exactly the same as in [21]: the proof of the spectral gap is done in Section 12 of that paper. The sector condition can also be proved, inspired by [21]. After taking into account the disorder and its fluctuation, the same argument of Lemma 8.2, Section 8 can be applied: it is mainly based on the fact that both antisymmetric and symmetric parts involve the same operators  $Y_{x,y}$ .

## 9 Hydrodynamic limits

We briefly enlighten the failure in the derivation of the hydrodynamic limits. Let us assume that the initial law for the Markov process  $\{\omega(t)\}_{t \geq 0}$  (still generated by  $N^2 \mathcal{L}_N^m$ ), is not the equilibrium measure  $\mu_\beta^N$ , but a *local equilibrium measure* (see (60) below). The main goal would be to prove that this property of local equilibrium propagates in time: in other words hydrodynamics limits hold, with an energy profile solution of the diffusion equation with constant coefficient  $D$ .

## 9.1 Statement of the hydrodynamic limits conjecture

The distribution at time  $t$  of the Markov chain on  $\mathbb{T}_N$  with the generator  $N^2 \mathcal{L}_N^{\mathbf{m}}$  and the initial probability measure  $\mu_N$  is denoted by  $\mathbb{P}_{\mu_N, t}^{\mathbf{m}}$ . The measure induced by  $\mathbb{P}_{\mu_N, t}^{\mathbf{m}}$  on  $\mathcal{D}([0, T], \Omega_N)$  is denoted by  $\mathbb{P}_N^{\mathbf{m}}$ .

Recall that we denote by  $\mathcal{M}_1$  the set of probability measures on  $\mathbb{T}$ , endowed with the weak topology and by  $\mathcal{D}([0, T], \mathcal{M}_1)$  the Skorokhod space of trajectories in  $\mathcal{M}_1$ . The measure induced by  $\mathbb{P}_N^{\mathbf{m}}$  on  $\mathcal{D}([0, T], \mathcal{M}_1)$  is denoted by  $\mathcal{Q}_N^{\mathbf{m}} := \mathbb{P}_N^{\mathbf{m}} \circ (\pi^N)^{-1}$ , where

$$\pi^N := \frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^2 \delta_{\frac{x}{N}}.$$

**CONJECTURE 9.1.** *Let  $T > 0$  be a time-horizon. Let  $\{\mu^N\}_N$  be a sequence of probability measures on  $\Omega_N$ . Under suitable conditions on the initial law  $\mu^N$ , for almost every realization of the disorder  $\mathbf{m}$ , the measure  $\mathcal{Q}_N^{\mathbf{m}}$  weakly converges in  $\mathcal{D}([0, T], \mathcal{M}_1)$  to the probability measure concentrated on the path  $\{\mathbf{e}(t, u) du\}_{t \in [0, T]}$ , where  $\mathbf{e}$  is the unique weak solution of the system*

$$\begin{cases} \frac{\partial \mathbf{e}}{\partial t}(t, u) = \mathbf{D} \frac{\partial^2 \mathbf{e}}{\partial u^2}(t, u), & t > 0, u \in \mathbb{T} \\ \mathbf{e}(0, u) = \mathbf{e}_0(u). \end{cases}$$

What we expect as for “suitable assumptions” on the initial law are the common ones in the literature of hydrodynamic limits, when dealing with non compact spaces. The first one is natural and related on the relative entropy:

**ASSUMPTION 9.2.** *We suppose that there exists a positive constant  $K_0$  such that the relative entropy  $H(\mu^N | \mu_\star^N)$  of  $\mu^N$  with respect to a reference measure  $\mu_\star^N$  is bounded by  $K_0 N$ :*

$$H(\mu^N | \mu_\star^N) \leq K_0 N. \quad (59)$$

For instance, if  $\mu^N$  is defined as a Gibbs local equilibrium state:

$$\prod_{x \in \mathbb{T}_N} \sqrt{\frac{2\pi}{\beta_0(x/N)}} \exp\left(-\frac{\beta_0(x/N)}{2} \omega_x^2\right) d\omega_x \quad (60)$$

for some continuous function  $\beta_0 : \mathbb{T} \rightarrow \mathbb{R}_+$ , then (59) is satisfied. The second one is related to energy boundness, that has already been a major concern in [27]. More precisely,

**ASSUMPTION 9.3.** *We assume there exists a positive constant  $E_0$  such that*

$$\limsup_{N \rightarrow \infty} \mu^N \left[ \frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^4 \right] \leq E_0. \quad (61)$$

In the derivation of hydrodynamic limits with the usual *entropy method*, we need the following two estimates: first, there exists a positive constant  $C$  such that, for any  $t > 0$

$$\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[ \frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^2(t) \right] \leq C. \quad (62)$$

This can be easily established using (61) and the Cauchy-Schwarz inequality. The second control that we need is

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[ \int_0^t \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \omega_x^4(s) ds \right] = 0. \quad (63)$$



If  $\mu^N$  is a convex combination of Gibbs local equilibrium states, then the same argument of [27] shows that the law of the process remains a convex combination of Gaussian measures, and that (63) holds.

Contrary to the velocity-flip model, we do not need to assume a good control of every energy moment if we expect the usual entropy method to work. This technical need was only due to the relative entropy method.

With Assumptions 9.2 and 9.3 we could try to prove Theorem 9.1 by using the entropy method, which permits to consider more general initial profiles (for example, the profile  $\beta_0$  can be assumed only bounded, not smooth). The usual technical points of this well-known procedure are the one and two-blocks estimates, as well as tightness. In this model, they are somehow easy to prove because the diffusion coefficient is constant, and there is no need to prove its regularity.

## 9.2 Replacement of the current by a gradient

In this subsection we recall the main steps of the usual entropy method, and explain which ones can be proved for our system. We fix the disorder  $\mathbf{m} = \{m_x\}_{x \in \mathbb{T}_N}$  and  $T > 0$ . For  $t \in [0, T]$ , we denote by  $\mathcal{Z}_{t,\mathbf{m}}^N$  the empirical energy field defined as

$$\mathcal{Z}_{t,\mathbf{m}}^N(H) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \omega_x^2(t),$$

where  $H : \mathbb{T} \rightarrow \mathbb{R}$  is a smooth function. We rewrite  $\mathcal{Z}_{t,\mathbf{m}}^N(H)$  as

$$\mathcal{Z}_{t,\mathbf{m}}^N(H) = \mathcal{Z}_{0,\mathbf{m}}^N(H) + \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) j_{x,x+1}(\mathbf{m}, \omega)(s) ds + \mathfrak{M}_{t,\mathbf{m}}^N(H),$$

where  $\mathfrak{M}_{t,\mathbf{m}}^N(H)$  is a martingale. The strategy consists in replacing the current  $j_{x,x+1}$  by the linear combination given in Theorem 5.9. For that purpose, for any  $f \in \mathcal{Q}$  we rewrite

$$\mathcal{Z}_{t,\mathbf{m}}^N(H) = \mathcal{Z}_{0,\mathbf{m}}^N(H) + \int_0^t D \mathcal{Z}_{s,\mathbf{m}}^N(\Delta_N H) ds + \mathfrak{J}_{t,\mathbf{m},f}^{1,N}(H) + \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m}}^N(H),$$

where

$$\begin{aligned} \mathfrak{J}_{t,\mathbf{m},f}^{1,N}(H) &= \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \left[ j_{x,x+1}(\mathbf{m}, \omega)(s) - D(\omega_{x+1}^2 - \omega_x^2)(s) - \mathcal{L}_N^{\mathbf{m}}(\tau_x f)(\mathbf{m}, \omega)(s) \right] ds, \\ \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) &= \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \mathcal{L}_N^{\mathbf{m}}(\tau_x f)(\mathbf{m}, \omega)(s) ds. \end{aligned}$$

Theorem 9.1 would follow from the three lemmas below.

**LEMMA 9.4.** *For every smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}$  and every  $\delta > 0$ ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_N^{\mathbf{m}} \left[ \sup_{[0,T]} \left| \mathfrak{M}_{t,\mathbf{m}}^N(H) \right| > \delta \right] = 0.$$

**LEMMA 9.5.** *For every  $f \in \mathcal{Q}$  and every smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}$ ,*

$$\limsup_{N \rightarrow \infty} \mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[ \left| \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) \right| \right] = 0.$$

**LEMMA 9.6.** *There exists a sequence of functions  $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{Q}$  such that, for every smooth function  $H : \mathbb{T} \rightarrow \mathbb{R}$ ,*

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E} \left[ \mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[ \left| \mathfrak{J}_{t, \mathbf{m}, f_k}^{1, N}(H) \right| \right] \right] = 0.$$

Lemma 9.4 and Lemma 9.5 can be proved, following the same standard arguments given for example in [15, Section 7]. We need the energy moment estimate (63) in Lemma 9.4, in the computation of the quadratic variation of the martingale. The next subsection is devoted to highlight what fails in Lemma 9.6, which should be related to the results of Sections 4, 6 and 5.

**REMARK 9.1.** Conditioned to proving Lemma 9.6, Theorem 9.1 would follow: recall that  $\mathcal{Q}_N^{\mathbf{m}}$  is the distribution on the path space  $\mathcal{D}([0, T], \mathcal{M}_1)$  of the process  $\pi_t^N$ . Following the same argument as for the generalized exclusion process in [15, Section 7.6], we can show that the sequence  $\{\mathcal{Q}_N^{\mathbf{m}}, N \geq 1\}$  is weakly relatively compact. It remains to prove that every limit point  $\mathcal{Q}_*^{\mathbf{m}}$  is concentrated on absolutely continuous paths  $\mathbf{e}(t, du) = \mathbf{e}(t, u)du$  whose densities are solutions of the hydrodynamic equations given in Theorem 9.1. It could be seen from Lemma 9.6 by following the proof of [15, Theorem 7.0.1].

### 9.3 Failed variance estimate

In this paragraph we fix the disorder  $\mathbf{m}$ , and we erase it whenever no confusion arises. We are going to recall here the usual main steps of the entropy method. We rewrite  $\mathfrak{J}_{t, \mathbf{m}, f}^{1, N}(H)$  as

$$\mathfrak{J}_{t, \mathbf{m}, f}^{1, N}(H) = \int_0^t \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi(\mathbf{m}, \omega)(s) ds,$$

where

$$\begin{cases} \varphi(\mathbf{m}, \omega) := j_{0,1}(\mathbf{m}, \omega) - D(\omega_1^2 - \omega_0^2) - \mathcal{L}_N^{\mathbf{m}}(f)(\mathbf{m}, \omega) \\ G\left(\frac{x}{N}\right) := \nabla_N H\left(\frac{x}{N}\right). \end{cases}$$

**Entropy inequality –** In Lemma 9.6, note that the expectation with respect to the law of the process  $\mathbb{P}_N^{\mathbf{m}}$  is taken. There is *a priori* no hope to get any estimate of this expectation, apart from the well-known entropy inequality. More precisely, let us denote by  $X_N^f(\omega)$  the following quantity:

$$X_N^f(\omega) := \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi(\omega).$$

From the entropy inequality, we obtain

$$\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[ \left| \int_0^T X_N^f(\omega)(s) ds \right| \right] \leq \frac{1}{\alpha N} H(\mathbb{P}_N^{\mathbf{m}} | \mu_\beta^N) + \frac{1}{\alpha N} \log \mathbb{E}_{\mu_\beta^N} \left[ \exp \left( \alpha N \left| \int_0^T X_N^f(\omega)(s) ds \right| \right) \right],$$

for all  $\alpha > 0$ . Since the entropy is decreasing in time, we know that, for all disorder field  $\mathbf{m}$ ,  $H(\mathbb{P}_N^{\mathbf{m}} | \mu_\beta^N)$  is bounded. From the arbitrariness of  $\alpha$ , we are reduced to investigate the convergence of the second term in the previous right-hand side.

**Feynman-Kac formula –** Usually, the purpose is to reduce the dynamics problem to the study of the largest eigenvalue for a small perturbation of the generator  $N^2 S_N$ . This reduction relies on Feynman-Kac formula and on a variational formula for the largest eigenvalue of a symmetric operator. By Feynman-Kac formula,

$$\mathbb{E}_{\mu_\beta^N} \left[ \exp \left\{ N \int_0^T X_N^f(\omega)(s) ds \right\} \right] \leq \exp \left\{ \int_0^T \lambda_N(s) ds \right\}$$

where  $\lambda_N(s)$  is the largest eigenvalue of the symmetric operator  $N^2 \mathcal{S}_N(\cdot) + NX_N^f(\omega)$ . From the variational formula for the largest eigenvalue of an operator in a Hilbert space, we also know that

$$\lambda_N(s) \leq \sup_g \left\{ \langle NX_N^f(\cdot) g(\cdot) \rangle_\beta - N^2 \mathcal{D}_N(\mu_\beta; \sqrt{g}) \right\}$$

where the supremum is taken over all measurable functions  $g$  which are densities with respect to  $\mu_\beta^N$ . In particular,

$$\frac{1}{N} \log \mathbb{E}_{\mu_\beta^N} \left[ \exp \left\{ \int_0^T NX_N^f(\omega)(s) ds \right\} \right] \leq \int_0^T \sup_g \left\{ \langle X_N^f(\omega) g(\omega) \rangle_\beta - N \mathcal{D}_N(\mu_\beta; \sqrt{g}) \right\} ds.$$

**Reduction to microscopic blocks** – With the same spirit of the one-block estimate presented in [27], it is then crucial to replace microscopic quantities with their spatial averages. Here, with the same ideas of [15], we can replace

$$\begin{aligned} j_{0,1} & \text{ with } \frac{1}{2\ell+1} \sum_{x \in \Lambda_\ell} j_{x,x+1} \\ \omega_0^2 & \text{ with } \frac{1}{2\ell+1} \sum_{x \in \Lambda_\ell} \omega_x^2 \\ \mathcal{L}_N^m(f)(\omega) & \text{ with } \frac{1}{2\ell_f+1} \sum_{x \in \Lambda_{\ell_f}} \mathcal{L}_{s_f+1}^m(\tau_x f) \end{aligned}$$

where  $\ell_f = \ell - s_f - 1$  so that  $\mathcal{L}_{s_f+1}(\tau_y f)$  is  $\mathcal{F}_{\Lambda_{\ell_f}}$ -measurable for every  $y \in \Lambda_{\ell_f}$ . Let us introduce the following notation

$$W^{f,\ell} := \frac{1}{2\ell'+1} \sum_{y \in \Lambda_{\ell'}} j_{y,y+1} + D \left[ \frac{1}{2\ell+1} \sum_{|x| \leq \ell} \omega_x^2 - \frac{1}{2\ell+1} \sum_{|x-1| \leq \ell} \omega_x^2 \right] - \frac{1}{2\ell_f+1} \sum_{y \in \Lambda_{\ell_f}} \mathcal{L}_{s_f+1}(\tau_y f) \quad (64)$$

with  $\ell' = \ell - 1$ . Finally, thanks to the regularity of the function  $G$  and the fact that  $D$  is constant, we are able to reduce Lemma 9.6 to Lemma 9.7 below. We also need to perform a cut-off in order to control high energy values, and this is valid thanks to (63).

**LEMMA 9.7.** *For all  $\delta > 0$ ,*

$$\inf_{f \in \mathcal{Q}} \limsup_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_g \left\{ \langle Y_{N,\ell}^f(\omega) g(\omega) \rangle_\beta - \delta N \mathcal{D}_N(\mu_\beta; \sqrt{g}) \right\} \leq 0, \quad (65)$$

where

$$Y_{N,\ell}^f(\omega) := \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x W^{f,\ell}(\omega).$$

**Reduction to a variance estimate** – Then, the challenge is to reduce the proof of Lemma 9.7 to the following result:

$$\inf_{f \in \mathcal{Q}} \lim_{\ell \rightarrow \infty} 2\ell \times \mathbb{E} \left[ \langle (-\mathcal{S}_{\Lambda_\ell})^{-1} W^{f,\ell}, W^{f,\ell} \rangle_\beta \right] = 0 \quad (66)$$

This convergence holds, since it is equivalent to the conclusion of Theorem 5.9, where the diffusion coefficient  $D$  is defined through the non-gradient approach. Here is the main obstacle. If we follow the strategy given in [15, Section 7.3], we can bound the supremum in (65) by the largest eigenvalue of

$\mathcal{S}_{\Lambda_\ell} + \mathbf{b}W^{f,\ell}$  where  $\mathbf{b}$  is a small constant. In order to estimate this largest eigenvalue, we usually use a perturbation method which provides a bound on the largest eigenvalue in terms of the variance of  $W^{f,\ell}$ . This can not be proved, and suggests that the entropy inequality together with the Feynman-Kac formula are not the good tools to prove the hydrodynamic limits for systems which do not have a spectral gap (see the last concluded section).

We conclude this section by explaining why the perturbation theory does not work. Let us try to prove Lemma 9.7. Since  $\mu_\beta$  is translation invariant, we may rewrite  $\langle Y_N^{f,\ell}(\cdot) g(\cdot) \rangle_\beta$  as

$$\sum_{x \in \mathbb{T}_N} \left\langle G\left(\frac{x}{N}\right) W^{f,\ell}(\omega) \tau_{-x} g(\omega) \right\rangle_\beta.$$

Since the Dirichlet form is convex, the supremum in (65) is bounded from above by

$$\frac{\delta N}{2\ell} \sum_{x \in \mathbb{T}_N} \sup_g \left\{ \mathbf{b} \langle W^{f,\ell} g \rangle_\beta - \mathcal{D}_\ell(\mu_\beta; \sqrt{g}) \right\}, \quad (67)$$

where the constant  $\mathbf{b} = \mathbf{b}(x, \ell, \delta, N)$  satisfies

$$|\mathbf{b}| := \left| G\left(\frac{x}{N}\right) \frac{2\ell}{\delta N} \right| \leq \|G\|_\infty \frac{2\ell}{\delta N}.$$

Let us denote by  $\lambda_{N,\ell,f}$  this last supremum inside the sum (67), which does not depend on  $x$ . We consider a sequence  $\{g_k\}_{k \in \mathbb{N}}$  that approaches this supremum, such that

$$\lim_{k \rightarrow \infty} \langle \sqrt{g_k}, (\mathcal{S}_{\Lambda_\ell} + \mathbf{b}W^{f,\ell}) \sqrt{g_k} \rangle_\beta = \lambda_{N,\ell,f}.$$

The idea of the perturbation theory is to expand  $\sqrt{g_k}$  around the constant value 1. We write

$$\langle \sqrt{g_k}, (\mathcal{S}_{\Lambda_\ell} + \mathbf{b}W^{f,\ell}) \sqrt{g_k} \rangle_\beta = \mathbf{b} \left( \langle W^{f,\ell} \rangle_\beta + 2 \langle W^{f,\ell}(\sqrt{g_k} - 1) \rangle_\beta + \langle W^{f,\ell}(\sqrt{g_k} - 1)^2 \rangle_\beta \right) - \mathcal{D}_\ell(\mu_\beta; \sqrt{g_k}). \quad (68)$$

We know that  $\langle W^{f,\ell} \rangle_\beta = 0$ , and we use the Cauchy-Schwarz inequality for the scalar product  $\langle \cdot, (-\mathcal{S}_{\Lambda_\ell}) \cdot \rangle_\beta$  in the second term. We obtain that (68) is bounded, for every  $A > 0$ , by

$$\mathbf{b} \left( \frac{\mathbf{b}}{A} \langle W^{f,\ell}, (-\mathcal{S}_{\Lambda_\ell})^{-1} W^{f,\ell} \rangle_\beta + \frac{A}{\mathbf{b}} \mathcal{D}_\ell(\mu_\beta; \sqrt{g_k}) \right) + \mathbf{b} \langle W^{f,\ell}(\sqrt{g_k} - 1)^2 \rangle_\beta - \mathcal{D}_\ell(\mu_\beta; \sqrt{g_k}).$$

It remains to bound the third term in the expression above. This could be done if we had the following lemma.

**LEMMA 9.8.** *There exists a constant  $C := C(\ell, f, \beta, \gamma, \lambda)$  such that, for every  $g \geq 0$ ,*

$$\langle W^{f,\ell}(\sqrt{g} - 1)^2 \rangle_\beta \leq C \mathcal{D}_\ell(\mu_\beta; \sqrt{g}). \quad (69)$$

As before, we could try to use the fact that  $W^{f,\ell}$  is a quadratic function. Even this fact is not helpful, and we give now a counter-example to this last lemma. We denote by  $H_n$  the normalized one-variable Hermite polynomial of degree  $n \geq 3$  (see Appendix A). Let us consider

$$\begin{cases} \sqrt{g}(\omega) = |H_n(\omega_0)| \\ W^{f,\ell}(\omega) = H_2(\omega_0) = \omega_0^2 - 1. \end{cases}$$

Let us notice that  $\langle H_n^2 \rangle_\beta = 1$ , and  $\langle H_2 \rangle_\beta = 0$ , so that the two test functions  $g$  and  $W^{f,\ell}$  satisfy all expected conditions. By using the recursive relation

$$H_{n+1}(\omega_0) = \omega_0 H_n(\omega_0) - n H_{n-1}(\omega_0),$$

we get for the left-hand side of (69),

$$\begin{aligned} \langle H_2(|H_n| - 1)^2 \rangle_\beta &= \langle \omega_0^2 H_n^2(\omega_0) \rangle_\beta - \langle H_n^2 \rangle_\beta - 2\langle H_2 |H_n| \rangle_\beta + \langle H_2 \rangle_\beta \\ &= \langle H_{n+1}^2 + 2n H_{n+1} H_{n-1} + n^2 H_{n-1}^2 \rangle_\beta - 1 - 2\langle H_2 |H_n| \rangle_\beta \\ &= 1 + n^2 - 1 - 2\langle H_2 |H_n| \rangle_\beta \geq n^2 - 2. \end{aligned}$$

Above the last equality comes from the orthonormality of the polynomial basis, and the last inequality is a consequence of the Cauchy-Schwarz inequality  $\langle H_2 |H_n| \rangle_\beta^2 \leq \langle H_2^2 \rangle_\beta \langle H_n^2 \rangle_\beta = 1$ . Let us assume that there exists a constant  $C > 0$  which does not depend on  $n$  such that

$$n^2 - 2 \leq \langle H_2(|H_n| - 1)^2 \rangle_\beta \leq C \mathcal{D}_\ell(\mu_\beta; |H_n|).$$

From the convexity of the Dirichlet form, we have

$$\mathcal{D}_\ell(\mu_\beta; |H_n|) \leq \mathcal{D}_\ell(\mu_\beta; H_n).$$

In the case where  $n$  is an even positive integer, the flip noise gives a zero contribution to the Dirichlet form, and then, for all  $n$  even, we have

$$\mathcal{D}_\ell(\mu_\beta; H_n) = \frac{\lambda}{2} \langle (H_n(\omega_1) - H_n(\omega_0))^2 \rangle_\beta = \lambda \langle H_n^2 \rangle_\beta - \lambda \langle H_n(\omega_0) H_n(\omega_1) \rangle_\beta = \lambda.$$

In the last equality, we use the fact that  $H_n$  is unitary, and that  $H_n(\omega_0)H_n(\omega_1)$  constitutes another element of the Hermite polynomial basis, then is orthogonal to the constant polynomial 1. Letting  $n$  go to infinity, we obtain a contradiction to (69).

**Ergodic decomposition** – Another idea would be to use the ergodic decomposition. The generator  $\mathcal{S}_\ell$  restricted to finite boxes does not have a spectral gap, but it becomes ergodic when restricted to some finite orbits. However, this approach fails, because the space is not compact, and we need to disintegrate the measure  $\mu_\beta$  with respect to all energy levels in  $(0, +\infty)$ . This enforces us to introduce a cut-off in the variational formula giving the largest eigenvalue. In other words, an indicator function  $\mathbf{1}_{\{|\omega_x| \leq E_0\}}$  will appear in front of  $W^{f,\ell}$ . Finally, we will have to deal with functions of the configurations that are not quadratic any more, and we do not know how to prove the convergence result (66) for general functions.

## 9.4 Conclusion

Even if the non-gradient method can be applied in some cases when the spectral gap does not hold, (and then the diffusion coefficient is well defined), this does not straightforwardly imply the hydrodynamic limits.

In order to derive the hydrodynamic theorem, we would need to bypass the entropy inequality together with the Feynman-Kac formula. The entropy inequality is however a convenient mean to transform the averages w.r.t. the unknown law  $\mu_t^N$  into equilibrium averages w.r.t.  $\mu_\beta^N$ , which are more easily tractable. The same problem would arise in the relative entropy method, because of the entropy inequality.

## A Hermite polynomials and quadratic functions

In the whole section we assume  $\beta = 1$ . Every result can be restated for the general case after multiplying the process by  $\beta^{-1/2}$ .

### A.1 Hermite polynomials on $\mathbb{R}^{\mathbb{Z}}$

Let  $\chi$  be the set of positive integer-valued functions  $\xi : \mathbb{Z} \rightarrow \mathbb{N}$ , such that  $\xi_x$  vanish for all but a finite number of  $x \in \mathbb{Z}$ . The *length* of  $\xi$ , denoted by  $|\xi|$ , is defined as

$$|\xi| := \sum_{x \in \mathbb{Z}} \xi_x.$$

For  $\xi \in \chi$ , we define the polynomial function on  $\Omega$

$$H_\xi(\omega) = \prod_{x \in \mathbb{Z}} h_{\xi_x}(\omega_x),$$

where  $\{h_n\}_{n \in \mathbb{N}}$  are the normalized Hermite polynomials w.r.t. the centered one-dimensional Gaussian law with variance 1. The sequence  $\{H_\xi\}_{\xi \in \chi}$  forms an orthonormal basis of the Hilbert space  $\mathbf{L}^2(\mu_1)$ , where  $\mu_1$  is the infinite product Gibbs measure defined by (2). As a result, every function  $f \in \mathbf{L}^2(\mu_1)$  can be decomposed in the form

$$f(\omega) = \sum_{\xi \in \chi} F(\xi) H_\xi(\omega).$$

Moreover, we can compute the scalar product  $\langle f, g \rangle_1$  for  $f = \sum_{\xi} F(\xi) H_\xi$  and  $g = \sum_{\xi} G(\xi) H_\xi$  as

$$\langle f, g \rangle_1 = \sum_{\xi \in \chi} F(\xi) G(\xi).$$

**DEFINITION A.1.** We denote by  $\chi_n \subset \chi$  the subset sequences of length  $n$ , i.e.  $\chi_n := \{\xi \in \chi ; |\xi| = n\}$ . A function  $f \in \mathbf{L}^2(\mu_1)$  is of degree  $n$  if its decomposition

$$f = \sum_{\xi \in \chi} F(\xi) H_\xi$$

satisfies:  $F(\xi) = 0$  for all  $\xi \notin \chi_n$ .

In the next paragraph we focus on degree 2 functions, which are by definition on the form

$$\sum_{x \in \mathbb{Z}} \varphi(x, x)(\omega_x^2 - 1) + \sum_{x \neq y} \varphi(x, y) \omega_x \omega_y \quad (70)$$

where  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$  is a square summable symmetric function.

**Local functions** – On the set of  $n$ -tuples  $\mathbf{x} := (x_1, \dots, x_n)$  of  $\mathbb{Z}^n$ , we introduce the equivalence relation  $\mathbf{x} \sim \mathbf{y}$  if there exists a permutation  $\sigma$  on  $\{1, \dots, n\}$  such that  $x_{\sigma(i)} = y_i$  for all  $i \in \{1, \dots, n\}$ . The class of  $\mathbf{x}$  for the relation  $\sim$  is denoted by  $[\mathbf{x}]$  and its cardinal by  $c(\mathbf{x})$ . Then the set of configurations of  $\chi_n$  can be identified with the set of  $n$ -tuples classes for  $\sim$  by the one-to-one application:

$$\begin{aligned} \mathbb{Z}^n / \sim &\rightarrow \chi_n \\ [\mathbf{x}] = [(x_1, \dots, x_n)] &\mapsto \xi^{[\mathbf{x}]} \end{aligned}$$

where for any  $y \in \mathbb{Z}$ ,  $(\xi^{[x]})_y = \sum_{i=1}^n \mathbf{1}_{y=x_i}$ . We identify  $\xi \in \chi_n$  with the occupation numbers of a configuration with  $n$  particles, and  $[x]$  corresponds to the positions of those  $n$  particles. A function  $F : \chi_n \rightarrow \mathbb{R}$  is nothing but a symmetric function  $F : \mathbb{Z}^n \rightarrow \mathbb{R}$  through the identification of  $\xi$  with  $[x]$ . We denote (with some abuse of notations) by  $\langle \cdot, \cdot \rangle$  the scalar product on  $\oplus \mathbf{L}^2(\chi_n)$ , each  $\chi_n$  being equipped with the counting measure. Hence, for two functions  $F, G : \chi \rightarrow \mathbb{R}$ , we have

$$\langle F, G \rangle = \sum_{n \geq 0} \sum_{\xi \in \chi_n} F_n(\xi) G_n(\xi) = \sum_{n \geq 0} \sum_{\mathbf{x} \in \mathbb{Z}^n} \frac{1}{c(\mathbf{x})} F_n(\mathbf{x}) G_n(\mathbf{x}),$$

with  $F_n, G_n$  the restrictions of  $F, G$  to  $\chi_n$ .

**Dirichlet form** – It is not hard to check the following proposition, which is a direct consequence of the fact that  $h_n$  has the same parity of the integer  $n$ .

**PROPOSITION A.1.** *If a local function  $f \in \mathbf{L}^2(\mu_1)$  is written in the form  $f = \sum_{\xi \in \chi} F(\xi) H_\xi$ , then*

$$Sf(\omega) = \sum_{\xi \in \chi} (\mathfrak{S}F)(\xi) H_\xi(\omega),$$

where  $\mathfrak{S}$  is the operator acting on functions  $F : \chi \rightarrow \mathbb{R}$  as

$$\mathfrak{S}F(\xi) = \lambda \sum_{x \in \mathbb{Z}} [F(\xi^{x, x+1}) - F(\xi)] + \gamma \sum_{x \in \mathbb{Z}} ((-1)^{\xi_x} - 1) F(\xi).$$

Here,  $\xi^{x, y}$  is obtained from  $\xi$  by exchanging  $\xi_x$  and  $\xi_y$ .

From this result we deduce:

**COROLLARY A.2.** *For any  $f = \sum_{\xi \in \chi} F(\xi) H_\xi \in \mathbf{L}^2(\mu_1)$ , we have*

$$\mathcal{D}(f; \mu_1) = \langle f, -Sf \rangle_1 = \sum_{\xi \in \chi} \left\{ \frac{\lambda}{2} \sum_{x \in \mathbb{Z}} (F(\xi^{x, x+1}) - F(\xi))^2 + \gamma \sum_{x \in \mathbb{Z}} ((-1)^{\xi_x} - 1) F^2(\xi) \right\}$$

**Quadratic functions** – In this paper, we deal with the set of *quadratic* functions  $f$  in  $\mathbf{L}^2(\mu_1)$ , namely degree two functions that are *homogeneous*, i.e. satisfies the algebraic relation

$$\forall \lambda \in \mathbb{R}, \quad f(\lambda \omega) = \lambda^2 f(\omega), \quad \mu_1 - \text{a.s.} \quad (71)$$

We also assume that  $f$  has zero average with respect to  $\mu_1$ . Therefore, we could also rewrite every  $f$  as

$$f(\omega) = \sum_{x \in \mathbb{Z}} \psi(x, x) (\omega_x^2 - \omega_{x+1}^2) + \sum_{x \neq y} \psi(x, y) \omega_x \omega_y,$$

for a suitable function  $\psi : \mathbb{Z}^2 \rightarrow \mathbb{R}$  square summable and symmetric, and we recover the form given in (3). We first restrict some variational formula to this class of functions, and then we study sequences of functions that weakly converge in  $\mathbf{L}^2(\mu_1)$ .

**PROPOSITION A.3.** *If  $f \in \mathbf{L}^2(\mu_1)$  is quadratic in the sense above with zero average w.r.t  $\mu_1$ , then the following variational formula*

$$\sup_{g \in \mathbf{L}^2(\mu_1)} \{2\langle f, g \rangle_1 - \mathcal{D}(g; \mu_1)\}$$

can be restricted over quadratic functions  $g$  of zero mean w.r.t  $\mu_1$ .

*Proof.* This fact follows after decomposing  $g$  as  $\sum_{\xi \in \chi} G(\xi)H_\xi$ . Corollary A.2 and the orthogonality of Hermite polynomials imply that we can restrict the supremum over functions  $g$  of degree two (70). As a result,  $g$  writes as the sum of a quadratic function plus an additional constant term.

Then, notice that the constant term gives a zero contribution in the quantity to maximise: indeed, the Dirichlet form does not change if we add a constant, and the function  $f$  is supposed to be centered, so that  $\langle f \rangle_1 = 0$ . Therefore, we can assume that the supremum is taken over homogeneous functions of degree two, and the same argument shows that  $g$  can also be taken with zero average.  $\square$

**PROPOSITION A.4.** *Let  $\{f_n\}_n$  be a sequence of quadratic functions in  $\mathbf{L}^2(\mu_1)$ . Suppose that  $\{f_n\}$  weakly converges to  $f$  in  $\mathbf{L}^2(\mu_1)$ . Then,  $f$  is quadratic.*

*Proof.* For all  $n \in \mathbb{N}$ , and  $\xi \notin \chi_2$ , the scalar product  $\langle f_n, H_\xi \rangle_1$  vanishes (by definition). From the weak convergence, we know that

$$\langle f_n, H_\xi \rangle_1 \rightarrow \langle f, H_\xi \rangle_1,$$

as  $n$  goes to infinity, for all  $\xi \in \chi$ . This implies:  $\langle f, H_\xi \rangle_1 = 0$  for all  $\xi \notin \chi_2$ .

Besides, the algebraic relation (71) is still valid after taking the weak limit in  $\mathbf{L}^2(\mu_1)$ , as well as the zero-average property (with respect to  $\mu_1$ ). This implies that the weak limit  $f \in \mathbf{L}^2(\mu_1)$  is quadratic, and of zero mean if every  $f_n$  is centered.  $\square$

Notice that the set denoted by  $\mathcal{Q}$  and defined in Definition 2.2 is restricted to *cylinder* quadratic functions. The conclusions of Propositions A.3 and A.4 can be restated as:

**COROLLARY A.5.** *If  $f \in \mathcal{Q}$ , then the following variational formula*

$$\sup_{g \in \mathbf{L}^2(\mathbb{P}_1^*)} \{2\mathbb{E}_1^*[f, g] - \mathbb{E}[\mathcal{D}(g; \mu_1)]\}$$

*can be restricted over functions  $g$  in  $\mathcal{Q}$ . Moreover, if  $\{f_n\}_n$  is a sequence of functions in  $\mathcal{Q}$  such that  $\{f_n\}$  weakly converges to  $f$  in  $\mathbf{L}^2(\mathbb{P}_1^*)$ , then  $f$  belongs to  $\mathcal{Q}$ .*

## B A weak version of closed forms results

In that section we prove a theorem that should be thought as a kind of closed forms results, as they are stated in [26] or in [15] (Section A.3.4). We give the link between Theorem B.2 below and closed forms at the end of this paragraph.

### B.1 Decomposition of quadratic functions

First, we erase the dependence on the disorder  $\mathbf{m}$ , and consider the functions to be defined on  $\Omega$ , and square integrable w.r.t. the Gibbs measure  $\mu_1$ . Then, we restate the same result for functions defined on  $\Omega_{\mathcal{D}} \times \Omega$ .

**THEOREM B.1** (Decomposition in  $\mathbf{L}^2(\mu_1)$ ). *Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of quadratic functions in  $\mathbf{L}^2(\mu_1)$  and let us define*

$$g_n := \nabla_0(\Gamma_{f_n}) \quad \text{and} \quad h_n := \nabla_{0,1}(\Gamma_{f_n}).$$



If  $\{g_n\}$ , respectively  $\{h_n\}$ , weakly converges in  $\mathbf{L}^2(\mu_1)$  towards  $g$ , respectively  $h$ , then there exist  $a \in \mathbb{R}$  and  $f$  quadratic function in  $\mathbf{L}^2(\mu_1)$  such that

$$g(\omega) = \nabla_0(\Gamma_f)(\omega), \quad (72)$$

$$h(\omega) = a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}(\Gamma_f)(\omega). \quad (73)$$

In view of (72) and (73), and in order to generalize that result, we introduce the set:

$$\Theta := \left\{ (g, h) \in \mathbf{L}^2(\mathbb{P}_1^*) \times \mathbf{L}^2(\mathbb{P}_1^*); \text{ there exist } a \in \mathbb{R}, f \in \mathcal{Q}, \text{ such that } \begin{cases} g &= \nabla_0(\Gamma_f), \\ h &= a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}(\Gamma_f). \end{cases} \right\}$$

Before proving Theorem B.1, we restate it in  $\mathbf{L}^2(\mathbb{P}_1^*)$  as:

**THEOREM B.2** (Decomposition in  $\mathbf{L}^2(\mathbb{P}_1^*)$ ). *Let  $\{f_n\}_{n \in \mathbb{N}}$  be a sequence of quadratic functions in  $\mathbf{L}^2(\mathbb{P}_1^*)$ . Let us define*

$$g_n := \nabla_0(\Gamma_{f_n}) \quad \text{and} \quad h_n := \nabla_{0,1}(\Gamma_{f_n}).$$

*If  $\{g_n\}$ , respectively  $\{h_n\}$ , weakly converges in  $\mathbf{L}^2(\mathbb{P}_1^*)$  towards  $g$ , respectively  $h$ , then  $(g, h)$  belongs to  $\overline{\Theta}$ .*

*Proof of Theorem B.1.* From Proposition A.4, we already know that  $g$  and  $h$  are quadratic functions in  $\mathbf{L}^2(\mu_1)$ . Hence, we look for  $g$  and  $h$  of the form

$$g(\omega) = \sum_{x,y \in \mathbb{Z}} \psi_1(x,y) \omega_x \omega_y \quad (74)$$

$$h(\omega) = \sum_{x,y \in \mathbb{Z}} \psi_2(x,y) \omega_x \omega_y \quad (75)$$

where  $\psi_1, \psi_2 : \mathbb{Z}^2 \rightarrow \mathbb{R}$  are square integrable symmetric functions. We are now going to give a list or equalities, being satisfied by the pair of sequences. Let us be more precise. We define, for a pair  $(\mathbf{f}^1, \mathbf{f}^2)$  of two  $\mathbf{L}^2(\mu_1)$  functions, the following identities, stated in  $\mathbf{L}^2(\mu_1)$  sense:

**(R1)**  $(\tau_x \mathbf{f}^1)(\omega) + (\tau_x \mathbf{f}^1)(\omega^x) = 0$ , for all  $x \in \mathbb{Z}$ .

**(R2)**  $(\tau_x \mathbf{f}^2)(\omega) + (\tau_x \mathbf{f}^2)(\omega^{x,x+1}) = 0$ , for all  $x \in \mathbb{Z}$ .

**(R3)**  $(\tau_x \mathbf{f}^1)(\omega) + (\tau_x \mathbf{f}^2)(\omega^x) = (\tau_x \mathbf{f}^2)(\omega) + (\tau_{x+1} \mathbf{f}^1)(\omega^{x,x+1})$ , for all  $x \in \mathbb{Z}$ ,

It is straightforward to check that, for all  $n \in \mathbb{N}$ , the pair  $(g_n, h_n)$  satisfies identities **(R1–R3)**. Easily, one can show that the latter always take place after passing to the weak limit in  $\mathbf{L}^2(\mu_1)$ . Precisely, the weak limit  $(g, h)$  of  $\{g_n, h_n\}$  also satisfy **(R1–R3)**. This follows from the following easy lemma (which is a consequence of the translation invariance of  $\mu_1$ ):

**LEMMA B.3.** *If  $\{g_n\}_n$  weakly converges in  $\mathbf{L}^2(\mu_1)$  towards  $g$ , then, for all  $x \in \mathbb{Z}$ ,*

$$\begin{aligned} \{g_n(\omega^x)\}_n &\text{ weakly converges towards } g(\omega^x), \\ \{g_n(\omega^{x,x+1})\}_n &\text{ weakly converges towards } g(\omega^{x,x+1}). \end{aligned}$$

Notice that all equalities **(R1–R3)** – now stated for  $(g, h)$  – turn into identities for  $\psi_1$  and  $\psi_2$ , defined in (74) and (75). Namely,  $\psi_1$  and  $\psi_2$  have to satisfy

**(R1)**  $\begin{cases} \psi_1(x,y) = 0 & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \psi_1(0,0) = 0. \end{cases}$

$$(R2) \begin{cases} \psi_2(x, y) = 0 & \text{if } x \notin \{0, 1\} \text{ and } y \notin \{0, 1\}, \\ \psi_2(x, x) + \psi_2(x+1, x+1) = 0 & \text{for all } x \in \mathbb{Z}. \end{cases}$$

$$(R3) \begin{cases} 2\psi_2(x, 0) = \psi_1(x-1, 0) - \psi_1(x, 0) & \text{if } x \notin \{0, 1\}, \\ \psi_1(-1, 0) = \psi_1(1, 0). \end{cases}$$

The first two identities imply that  $g$  writes on the form

$$g(\omega) = \sum_{x \neq 0} \psi_1(x, 0) \omega_x \omega_0, \quad (76)$$

and  $h$  rewrites as

$$h(\omega) = \sum_{x \neq 0, 1} \psi_2(x, 0) [\omega_1 \omega_x - \omega_0 \omega_x] + \psi_2(0, 0) (\omega_0^2 - \omega_1^2), \quad (77)$$

whereas the final equality makes a connection between  $g$  and  $h$ . In view of (72) and (73), we are going to need the following straightforward lemma:

**LEMMA B.4.** *Let  $f \in \mathbf{L}^2(\mu_1)$  be of the form*

$$f(\omega) = \sum_{x, y \in \mathbb{Z}} \varphi(x, y) \omega_x \omega_y,$$

where  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$  is a symmetric function. Then,

$$\nabla_0(\Gamma_f) = -4 \sum_{\substack{z \in \mathbb{Z} \\ x \neq 0}} \varphi(x+z, z) \omega_0 \omega_x, \quad (78)$$

$$\nabla_{0,1}(\Gamma_f) = 2 \sum_{\substack{z \in \mathbb{Z} \\ x \neq 0, 1}} [\varphi(x+z, z) - \varphi(x+z, z+1)] (\omega_1 \omega_x - \omega_0 \omega_x). \quad (79)$$

Confronting (78)–(79) with (76)–(77), and keeping in mind the expected result of Theorem B.2, we are now looking for a symmetric function  $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$  which is square-summable on  $\mathbb{Z}^2$  and satisfies

$$\begin{cases} \sum_{z \in \mathbb{Z}} \varphi(x+z, z) = -\frac{1}{4} \psi_1(x, 0) & \text{for all } x \neq 0, \\ \sum_{z \in \mathbb{Z}} [\varphi(x+z, z) - \varphi(x+z, z+1)] = \frac{1}{2} \psi_2(x, 0) & \text{for all } x \notin \{0, 1\}. \end{cases}$$

Such a function  $\varphi$  exists if and only if, for all  $x \notin \{0, 1\}$ ,

$$2\psi_2(x, 0) = \psi_1(x-1, 0) - \psi_1(x, 0).$$

This last equality is true according to (R3), and the result is proved, with  $a = \psi_2(0, 0)$  and  $f \in \mathcal{Q}$  defined as

$$f(\omega) = \sum_{x, y \in \mathbb{Z}} \varphi(x, y) \omega_x \omega_y.$$

□

*Proof of Theorem B.2.* Every result that involves the Gibbs measure  $\mu_1$  can be translated into the same result involving the product measure  $\mathbb{P}_1^*$ . If  $f \in \mathbf{L}^2(\mathbb{P}_1^*)$  is homogeneous of degree two, it rewrites

$$f(\mathbf{m}, \omega) = \sum_{x, y \in \mathbb{Z}} \varphi(\mathbf{m}, x, y) \omega_x \omega_y,$$

where, for all  $\mathbf{m} \in \Omega_{\mathcal{D}}$ ,  $\varphi(\mathbf{m}, \cdot, \cdot)$  is a symmetric function on  $\mathbb{Z}^2$ . Moreover, the translation operator  $\tau_x$  that is involved in identities **(R1-R3)** also translates the disorder, as it is defined at the beginning of Subsection 2.3. The crucial point here it that  $\mathbb{P}_1^*$  is space translation invariant.

Following line to line the previous proof, one can show that there exists  $a(\mathbf{m}) \in \mathbf{L}^2(\mathbb{P})$  and  $f \in \mathcal{Q}$  such that the following decompositions hold in  $\mathbf{L}^2(\mathbb{P}_1^*)$ :

$$\begin{aligned} g(\mathbf{m}, \omega) &= \nabla_0(\Gamma_f)(\mathbf{m}, \omega), \\ h(\mathbf{m}, \omega) &= a(\mathbf{m})(\omega_0^2 - \omega_1^2) + \nabla_{0,1}(\Gamma_f)(\mathbf{m}, \omega). \end{aligned}$$

The conclusion of Theorem B.2 follows from the following lemma:

**LEMMA B.5.** *Let  $\xi \in \mathbf{L}^2(\mathbb{P}_1^*)$  defined on  $\Omega_{\mathcal{D}} \times \Omega$  as  $\xi(\mathbf{m}, \omega) := a(\mathbf{m})(\omega_1^2 - \omega_0^2)$ , for some function  $a \in \mathbf{L}^2(\mathbb{P})$ . Then, for any  $f \in \mathcal{Q}$ , the pair*

$$(\nabla_0(\Gamma_f), \xi + \nabla_{0,1}(\Gamma_f))$$

belongs to  $\overline{\Theta}$ .

Let us prove this lemma, which has been inspired by [11, Lemma 7.7]. By subtracting the function  $\mathbb{E}[a](\omega_1^2 - \omega_0^2)$ , we can assume  $\mathbb{E}[a] = 0$ . We first define for any  $x \in \mathbb{Z}$  the function  $g_x \in \mathbf{L}^2(\mathbb{P})$  as

$$g_x(\mathbf{m}) = \sum_{k=0}^{x-1} \tau_k a(\mathbf{m}),$$

with the convention  $g_0(\mathbf{m}) = 0$ . One can check that, for any  $n \in \mathbb{N}$ , for any  $z \in \mathbb{Z}$  such that  $\{z; z+1\} \subset \Lambda_n$ , we have

$$\tau_z \xi(\mathbf{m}, \omega) = \nabla_{z,z+1}(h_n), \quad \text{with } h_n(\mathbf{m}, \omega) = - \sum_{x \in \Lambda_n} g_x(\mathbf{m}) \omega_x^2.$$

Let us define

$$\xi_n = \frac{1}{2n} \nabla_{0,1}(\Gamma_{h_n}).$$

We are going to show that  $\xi_n$  converges to  $\xi$  in  $\mathbf{L}^2(\mathbb{P}_1^*)$ . Since  $\nabla_0(h_n) = 0$  for every  $n \in \mathbb{N}$ , this will prove the convergence

$$(\nabla_0(\Gamma_{f+h_n}), \nabla_{0,1}(\Gamma_{f+h_n})) \xrightarrow[n \rightarrow \infty]{\mathbf{L}^2(\mathbb{P}_1^*) \times \mathbf{L}^2(\mathbb{P}_1^*)} (\nabla_0(\Gamma_f), \xi + \nabla_{0,1}(\Gamma_f)),$$

and since  $(\nabla_0(\Gamma_{f+h_n}), \nabla_{0,1}(\Gamma_{f+h_n}))$  belongs to  $\Theta$ , we got exactly the conclusion of Lemma B.5. After basic computations, one can check that

$$\xi_n = \frac{2n+1}{2n} \xi + \frac{1}{2n} \tau_{-n} \left\{ g_n(\mathbf{m})(\omega_n^2 - \omega_{n+1}^2) \right\} + \frac{1}{2n} \tau_{n+1} \left\{ g_{-n-1}(\mathbf{m})(\omega_{-n-1}^2 - \omega_{-n}^2) \right\}.$$

Therefore, from Cauchy-Schwarz inequality and translation invariance of  $\mathbb{P}$ , it is enough to show that

$$\frac{\mathbb{E}[g_n^2]}{n^2} = \frac{1}{n^2} \mathbb{E} \left[ \left( \sum_{k=0}^{n-1} \tau_k a(\mathbf{m}) \right)^2 \right] \xrightarrow[n \rightarrow \infty]{} 0,$$

and that the same holds for  $\mathbb{E}[g_{-n-1}^2]/n^2$ . This convergence is a standard consequence of the translation invariance of  $\mathbb{P}$ . More precisely, let us fix a positive integer  $\ell$  and introduce for any  $x \in \mathbb{Z}$  the conditional expectation

$$\tilde{a}_x^{(\ell)} = \mathbb{E} \left[ \tau_x a(\mathbf{m}) \mid m_y ; y \in \Lambda_\ell(x) \right].$$

From our assumptions, notice that  $\tilde{a}_x^{(\ell)} = \tau_x \tilde{a}_0^{(\ell)}$  and  $\mathbb{E}[\tilde{a}_x^{(\ell)}] = 0$ . As a result,

$$\begin{aligned} \frac{1}{n^2} \mathbb{E} \left[ \left( \sum_{k=0}^{n-1} \tau_k a(\mathbf{m}) \right)^2 \right] &\leq \frac{2}{n^2} \mathbb{E} \left[ \left( \sum_{k=0}^{n-1} \left\{ \tau_k a(\mathbf{m}) - \tilde{a}_k^{(\ell)} \right\} \right)^2 \right] + \frac{2}{n^2} \mathbb{E} \left[ \left( \sum_{k=0}^{n-1} \tilde{a}_k^{(\ell)} \right)^2 \right] \\ &\leq 2 \mathbb{E} \left[ \left\{ a(\mathbf{m}) - \tilde{a}_0^{(\ell)} \right\}^2 \right] + \frac{C(\ell)}{n}. \end{aligned}$$

The last inequality comes from the fact that  $\sum \tilde{a}_k^{(\ell)}$  is a sum of identically distributed variables (because of the translation invariance of  $\mathbb{P}$ ), for which we have a good control of its variance. Lemma B.5 is proved by letting  $n$  go to infinity and then  $\ell$  go to infinity.  $\square$

## B.2 Connection with closed forms results

Let us briefly explain the connection between Theorem B.1 and the closed forms as they are defined for example in [26]. For that purpose, we are going to reformulate identities (R1–R3). First, interpret  $\mathbf{f}_x^1(\omega)$ , respectively  $\mathbf{f}_x^2(\omega)$ , as the price to change the configuration  $\omega \in \Omega$  into  $\omega^x$ , respectively to change  $\omega$  into  $\omega^{x,x+1}$ . In particular,

- the price to flip  $\omega_x$  when the configuration is  $\omega$  should be equal to  $-\mathbf{f}_x^1(\omega^x)$  : this is (R1),
- the price to exchange  $\omega_x$  and  $\omega_{x+1}$  when the configuration is  $\omega$  should also be equal to  $-\mathbf{f}_x^2(\omega^{x,x+1})$  : this is (R2).

In the context of interacting particle systems, *closed forms* are expected to give the same price for any 2-step path with equal end points. In our setting, the last equality (R3) can be translated into: “The quantity at site  $x$  is flipped, and then exchanged with the quantity at site  $x + 1$ . Equally, the quantities at site  $x$  and  $x + 1$  are exchanged first, and then the quantity at site  $x + 1$  is flipped.” There are three other such paths, that we do not need to prove our statement:

- two quantities are exchanged at sites  $x, x + 1$ , and also independently at sites  $y, y + 1$ , with  $\{x, x + 1\} \cap \{y, y + 1\} = \emptyset$ ,
- two quantities are flipped independently at sites  $x$  and  $y$ , with  $x \neq y$ ,
- the quantity at site  $x$  is flipped, and then the quantities at sites  $y$  and  $y + 1$  are exchanged, for  $y \notin \{x, x + 1\}$ , and the converse is also possible.

Recall that we have defined  $\Omega := \mathbb{R}^{\mathbb{Z}}$ . We denote by  $\mathcal{B}$  the space of real-valued functions

$$\mathcal{B} := \{f : \Omega \rightarrow \mathbb{R}\}. \quad (80)$$

We are now interested in the space of *forms*, which are defined as  $(\mathbf{f}_x^1, \mathbf{f}_x^2)_{x \in \mathbb{Z}}$  where  $\mathbf{f}_x^1 \in \mathcal{B}$ , and  $\mathbf{f}_x^2 \in \mathcal{B}$ , for every  $x \in \mathbb{Z}$ . To each function  $F : \Omega \rightarrow \mathbb{R}$  is associated a form:

**DEFINITION B.1.** A form  $\mathbf{f} = (\mathbf{f}_x^1, \mathbf{f}_x^2)_{x \in \mathbb{Z}}$  is an exact form if there exists a continuous function  $F : \Omega \rightarrow \mathbb{R}$  such that

$$\forall x \in \mathbb{Z}, \forall \omega \in \Omega, \begin{cases} \mathbf{f}_x^1(\omega) = F(\omega^x) - F(\omega), \\ \mathbf{f}_x^2(\omega) = F(\omega^{x,x+1}) - F(\omega). \end{cases} \quad (81)$$

Easily, one can prove that all exact forms are closed forms. We now present two examples of closed forms that play a central role.

EXAMPLE B.1. We denote by  $\mathfrak{a} = (\mathfrak{a}^1, \mathfrak{a}^2)$  the closed form defined by

$$\begin{cases} \mathfrak{a}_x^1(\omega) = 0, \\ \mathfrak{a}_x^2(\omega) = \omega_x^2 - \omega_{x+1}^2, \end{cases}$$

for all  $x \in \mathbb{Z}$  and configurations  $\omega \in \Omega$ . This closed form corresponds to the formal function  $F(\omega) = \sum_x x \omega_x^2$ , but this is not an exact form.

EXAMPLE B.2. Let  $h$  be a cylinder function. Let us recall that we denote by  $\Gamma_h$  the formal sum  $\sum_x \tau_x h$ , and define  $\mathfrak{u}_h = (\mathfrak{u}_h^1, \mathfrak{u}_h^2)$  as

$$\begin{cases} (\mathfrak{u}_h^1)_x(\omega) = \Gamma_h(\omega^x) - \Gamma_h(\omega), \\ (\mathfrak{u}_h^2)_x(\omega) = \Gamma_h(\omega^{x,x+1}) - \Gamma_h(\omega), \end{cases}$$

for all  $x \in \mathbb{Z}$ , and configurations  $\omega \in \Omega$ . Though  $\sum_x \tau_x h$  is a formal sum, these two equalities are well defined. Let us notice that  $\mathfrak{u}_h$  is a closed form that is not exact, unless  $h$  is constant.

These two examples show that closed forms on  $\Omega$  are not always exact forms. Let us introduce the notion of a germ of a closed form.

**DEFINITION B.2.** A pair of continuous functions  $f = (f^1, f^2)$ , where  $f^i : \Omega \rightarrow \mathbb{R}$ , is a germ of closed form if  $\mathfrak{f} = (\tau_x f)_{x \in \mathbb{Z}}$  is a closed form.

Examples B.1 and B.2 provide two types of germs of closed forms. Consider the cylinder function  $\mathfrak{Q}(\omega) = (0, \omega_0^2 - \omega_1^2)$ . The collection  $(\tau_x \mathfrak{Q})_{x \in \mathbb{Z}}$  is the closed form  $\mathfrak{a}$  of Example B.1. For a cylinder function  $h$ , the collection  $(\nabla_x \Gamma_h, \nabla_{x,x+1} \Gamma_h)_{x \in \mathbb{Z}}$  obtained through translations of the cylinder function  $(\nabla_0 \Gamma_h, \nabla_{0,1} \Gamma_h)$  is the closed form of Example B.2. For a pair of  $\mathbf{L}^2(\mathbb{P}_1^*)$ -functions  $f = (f^1, f^2)$ , we called it a *germ of closed form* if  $\mathfrak{f} = (\tau_x f)_{x \in \mathbb{Z}}$  satisfies all of conditions as a closed form in  $\mathbf{L}^2(\mathbb{P}_1^*)$ -sense. Usually, Theorem B.2 is replaced with a similar result that concerns every germ of closed form in  $\mathbf{L}^2(\mathbb{P}_1^*)$ : see [26, Theorem 5.1] or [15, Theorem A.3.4.14].

## C Proof of the weak sector condition

In this section we prove Proposition 5.7 that we recall here for the sake of clarity.

**PROPOSITION C.1** (Weak Sector condition). (i) There exists two constants  $C_0(\gamma, \lambda)$  and  $C_1(\gamma, \lambda)$  such that the following inequality hold for all  $f, g \in \mathcal{Q}$ :

$$\begin{aligned} || \ll \mathcal{A}^m g, \mathcal{S} f \gg_\beta || &\leq C_0 || \mathcal{S} f ||_\beta || \mathcal{S} g ||_\beta. \\ || \ll \mathcal{A}^m g, \mathcal{S} f \gg_\beta || &\leq C_1 || \mathcal{S} g ||_\beta + \frac{1}{2} || \mathcal{S} f ||_\beta. \end{aligned}$$

(ii) There exists a positive constant  $C(\beta)$  such that, for all  $g \in \mathcal{Q}$ ,

$$|| \mathcal{A}^m g ||_\beta \leq C(\beta) || \mathcal{S} g ||_\beta.$$

*Proof.* We prove (i). We assume that

$$\begin{aligned} g(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} \psi_{x,0}(\mathbf{m})(\omega_{x+1}^2 - \omega_x^2) + \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \psi_{x,k}(\mathbf{m})\omega_x\omega_{x+k} \\ f(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} \varphi_{x,0}(\mathbf{m})(\omega_{x+1}^2 - \omega_x^2) + \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \varphi_{x,k}(\mathbf{m})\omega_x\omega_{x+k}. \end{aligned}$$

We denote by  $\nabla^{\mathbf{m}}\psi$  the discrete Laplacian in the variable  $\mathbf{m}$ , that is

$$\nabla^{\mathbf{m}}\psi(\mathbf{m}) = 2\psi(\mathbf{m}) - \psi(\tau_1\mathbf{m}) - \psi(\tau_{-1}\mathbf{m}),$$

and  $\tau_x\nabla^{\mathbf{m}}$  is the operator defined as

$$(\tau_x\nabla^{\mathbf{m}})\psi(\mathbf{m}) := \nabla^{\mathbf{m}}\psi(\tau_x\mathbf{m}).$$

Straightforward computations show that

$$\begin{aligned} \|\mathcal{S}g\|_{\beta}^2 &= \frac{\gamma}{2}\mathbb{E}_{\beta}^* \left[ (\nabla_0\Gamma_g)^2 \right] + \frac{\lambda}{2}\mathbb{E}_{\beta}^* \left[ (\nabla_{0,1}\Gamma_g)^2 \right] \\ &= \frac{4\gamma}{\beta^2} \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \mathbb{E}[\psi_{x,k}^2] + \frac{2\lambda}{\beta^2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_x(\nabla^{\mathbf{m}}\psi_{x,0}) \right)^2 \right] \\ &\quad + \frac{\lambda}{\beta^2} \sum_{k \geq 2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} [\tau_{-x}(\psi_{x,k}) - \tau_{1-x}(\psi_{x,k})] \right)^2 \right], \\ \|\mathcal{S}f\|_{\beta}^2 &\geq \|\mathcal{S}^{\text{flip}}f\|_{\beta}^2 = \frac{\gamma}{2}\mathbb{E}_{\beta}^* \left[ \left( 2 \sum_{\substack{z \in \mathbb{Z} \\ k \geq 1}} \varphi_{z,k}(\mathbf{m})\omega_0\omega_k \right)^2 \right] = \frac{2\gamma}{\beta^2} \sum_{k \geq 1} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \varphi_{z,k}(\mathbf{m}) \right)^2 \right]. \end{aligned} \quad (82)$$

Now we deal with  $\langle\langle \mathcal{A}^{\mathbf{m}}g, \mathcal{S}f \rangle\rangle_{\beta}$ . From Proposition 5.1, and by definition,

$$\begin{aligned} \langle\langle \mathcal{A}^{\mathbf{m}}g, \mathcal{S}f \rangle\rangle_{\beta} &= - \sum_{z \in \mathbb{Z}} \mathbb{E}_{\beta}^* [f, \tau_z(\mathcal{A}^{\mathbf{m}}g)] \\ &= - \sum_{x,z \in \mathbb{Z}} \mathbb{E} \left[ \varphi_{x,0}(\mathbf{m}) \langle \omega_{x+1}^2 - \omega_x^2, \tau_z(\mathcal{A}^{\mathbf{m}}g) \rangle_{\beta} \right] - \sum_{\substack{x,z \in \mathbb{Z} \\ k \geq 1}} \mathbb{E} \left[ \varphi_{x,k}(\mathbf{m}) \langle \omega_x\omega_{x+k}, \tau_z(\mathcal{A}^{\mathbf{m}}g) \rangle_{\beta} \right] \\ &= \frac{2}{\beta^2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[ \frac{\tau_x(\nabla^{\mathbf{m}}\psi_{x,0})}{\sqrt{m_x m_{x+1}}} \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,1}) \right] \\ &\quad + \frac{1}{\beta^2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[ \left( \frac{\tau_1\psi_{x,1}}{\sqrt{m_x m_{x+1}}} - \frac{\psi_{x,1}}{\sqrt{m_{x+1} m_{x+2}}} \right) \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,2}) \right] \\ &\quad + \frac{1}{\beta^2} \sum_{k \geq 2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[ \left( \frac{\tau_1\psi_{x,k}}{\sqrt{m_x m_{x+1}}} - \frac{\psi_{x,k}}{\sqrt{m_{x+k} m_{x+k+1}}} \right) \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,k+1}) \right] \\ &\quad + \frac{1}{\beta^2} \sum_{k \geq 2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[ \left( \frac{\tau_{-1}\psi_{x,k}}{\sqrt{m_x m_{x+1}}} - \frac{\psi_{x,k}}{\sqrt{m_{x+k} m_{x+k-1}}} \right) \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,k-1}) \right]. \end{aligned}$$

From Cauchy-Schwarz inequality, and recalling  $1/\sqrt{m_0 m_1} \leq C$  ( $\mathbb{P}$ -a.s.), we obtain the following bound:

$$\| \ll \mathcal{A}^m g, S f \gg_\beta \| \leq \frac{2C}{\beta^2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_x (\nabla^m \psi_{x,0}) \right)^2 \right]^{1/2} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,1} \right)^2 \right]^{1/2} \quad (83)$$

$$+ \frac{3C}{\beta^2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_1 \psi_{x,1} - \psi_{x,1} \right)^2 \right]^{1/2} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,2} \right)^2 \right]^{1/2} \quad (84)$$

$$+ \frac{3C}{\beta^2} \sum_{k \geq 2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_1 \psi_{x,k} - \psi_{x,k} \right)^2 \right]^{1/2} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,k+1} \right)^2 \right]^{1/2} \quad (85)$$

$$+ \frac{3C}{\beta^2} \sum_{k \geq 2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_{-1} \psi_{x,k} - \psi_{x,k} \right)^2 \right]^{1/2} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,k-1} \right)^2 \right]^{1/2}. \quad (86)$$

Now we are going to use two times the trivial inequality  $\sqrt{ab} \leq a/\varepsilon + \varepsilon b/2$  for a particular choice of  $\varepsilon > 0$ : in (83) we take  $\varepsilon = \gamma/C$  and in (84) we take  $\varepsilon = 2\gamma/(3C)$ . This trick gives the final bound

$$\| \ll \mathcal{A}^m g, S f \gg_\beta \| \leq \frac{2C^2}{\gamma\beta^2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_x (\nabla^m \psi_{x,0}) \right)^2 \right] + \frac{2\gamma}{\beta^2} \sum_{k \geq 1} \mathbb{E} \left[ \left( \sum_{z \in \mathbb{Z}} \varphi_{z,k} (\tau_{-z} \mathbf{m}) \right)^2 \right] \\ + \frac{9C^2}{\gamma\beta^2} \sum_{k \geq 2} \mathbb{E} \left[ \left( \sum_{x \in \mathbb{Z}} \tau_1 \psi_{x,k} - \psi_{x,k} \right)^2 \right].$$

Recalling (82), we obtain

$$\| \ll \mathcal{A}^m g, S f \gg_\beta \| \leq \frac{9C^2}{\gamma\lambda} \| \| S g \| \|_\beta^2 + \frac{1}{2} \| \| S f \| \|_\beta^2.$$

If we use the Cauchy-Schwarz inequality, we get:

$$\ll \mathcal{A}^m g, S f \gg_\beta^2 \leq \frac{18C^2}{\gamma\lambda} \| \| S g \| \|_\beta^2 \| \| S f \| \|_\beta^2.$$

We have proved (i) with  $C_0 = \sqrt{18C^2/(\gamma\lambda)}$  and  $C_1 = 9C^2/(\gamma\lambda)$ . Now we turn to (ii). From Lemma 5.5 and Statement (i),

$$\ll \mathcal{A}^m g, j_{0,1}^S \gg_\beta = \ll S g, j_{0,1}^A \gg_\beta \leq \| \| S g \| \|_\beta \| \| j_{0,1}^A \| \|_\beta.$$

Moreover, from Statement (i), we also get, for all  $f \in \mathcal{Q}_0$ ,

$$-2 \ll \mathcal{A}^m g, S f \gg_\beta \leq \| \| S f \| \|_\beta^2 + \frac{2C}{\gamma\lambda} \| \| S g \| \|_\beta^2.$$

As a result, the variational formula (26) for  $\| \| \mathcal{A}^m g \| \|_\beta^2$  gives:

$$\| \| \mathcal{A}^m g \| \|_\beta^2 \leq \frac{1}{\lambda\chi(\beta)} \ll \mathcal{A}^m g, j_{0,1}^S \gg_\beta^2 + \frac{9C^2}{\gamma\lambda} \| \| S g \| \|_\beta^2 \leq \left( \frac{\| \| j_{0,1}^A \| \|_\beta^2}{\lambda\chi(\beta)} + \frac{9C^2}{\gamma\lambda} \right) \| \| S g \| \|_\beta^2.$$

The result is proved.  $\square$

## D Tightness

In this section we prove the tightness of the sequence  $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$ , by using standard arguments. First, let us recall that the space  $\mathcal{M}_1$  is equipped with the norm defined as

$$\|\mathfrak{Y}\|_{-k}^2 = \sum_{n \geq 1} (\pi n)^{-2k} |\mathfrak{Y}(\mathbf{e}_n)|^2.$$

**THEOREM D.1.** *For almost all realization of the disorder  $\mathbf{m} \in \Omega_{\mathcal{D}}$ , the sequence  $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$  is tight in  $\mathcal{C}([0, T], \mathfrak{H}_{-k})$ .*

*Proof.* The tightness of the sequence  $\{\mathfrak{Y}_{\mathbf{m}}^N\}$  follows from two conditions (see [15], page 299):

$$\lim_{A \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_{\beta}^N} \left[ \sup_{0 \leq t \leq T} \|\mathfrak{Y}_{t, \mathbf{m}}^N\|_{-k} > A \right] = 0 \quad (87)$$

$$(88)$$

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_{\beta}^N} \left[ w(\mathfrak{Y}_{\mathbf{m}}^N, \delta) > \varepsilon \right] = 0, \quad \text{for all } \varepsilon > 0, \quad (89)$$

where the modulus of continuity  $w(\mathfrak{Y}, \delta)$  is defined by

$$w(\mathfrak{Y}, \delta) = \sup_{\substack{\|t-s\| < \delta \\ 0 \leq s \leq t \leq T}} \|\mathfrak{Y}_t - \mathfrak{Y}_s\|_{-k}.$$

Let us remind the decomposition of  $\mathfrak{Y}_{t, \mathbf{m}}^N$  given in (13):

$$\mathfrak{Y}_{t, \mathbf{m}}^N(\mathbf{H}) = \mathfrak{Y}_{0, \mathbf{m}}^N(\mathbf{H}) + \int_0^t \mathcal{D}\mathfrak{Y}_{s, \mathbf{m}}(\Delta_N \mathbf{H}) ds + \mathfrak{M}_{t, \mathbf{m}, f_k}^{1, N}(\mathbf{H}) + Z_{t, \mathbf{m}, f_k}^N(\mathbf{H}),$$

where  $\mathfrak{M}_{t, \mathbf{m}, f_k}^{1, N}(\mathbf{H})$  is the martingale defined in Subsection 3.3, and  $Z_{t, \mathbf{m}, f_k}^N(\mathbf{H})$  is defined as the sum of the remaining terms in the decomposition. On the first hand,

$$\mathbb{E}_{\mu_{\beta}^N} \left[ \sup_{0 \leq t \leq T} \left( Z_{t, \mathbf{m}, f_k}^N(\mathbf{H}) \right)^2 \right]$$

can be estimated by the proof of Lemma 3.2 and Theorem 3.3. On the other hand,

$$\mathbb{E}_{\mu_{\beta}^N} \left[ \left( \mathfrak{M}_{t, \mathbf{m}, f_k}^{1, N}(\mathbf{H}) \right)^2 \right]$$

can be computed explicitly. □

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