



HAL
open science

Equilibrium fluctuations for the disordered harmonic chain perturbed by an energy conserving noise

Marielle Simon

► **To cite this version:**

Marielle Simon. Equilibrium fluctuations for the disordered harmonic chain perturbed by an energy conserving noise. 2014. hal-01065976v1

HAL Id: hal-01065976

<https://hal.science/hal-01065976v1>

Preprint submitted on 18 Sep 2014 (v1), last revised 6 Feb 2019 (v3)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Equilibrium fluctuations for the disordered harmonic chain perturbed by an energy conserving noise

Marielle Simon*

September 18, 2014

Abstract

We investigate the macroscopic behavior of the disordered harmonic chain of oscillators, through energy diffusion. The Hamiltonian dynamics of the system is perturbed by a degenerate conservative noise. After rescaling space and time diffusively, we prove that energy fluctuations in equilibrium evolve according to a linear heat equation. The diffusion coefficient is obtained from the non-gradient Varadhan's approach, and is equivalently defined through the Green-Kubo formula. Since the perturbation is very degenerate and the symmetric part of the generator does not have a spectral gap, the standard non-gradient method is reviewed under new perspectives.

Acknowledgements. This problem was suggested by Cédric Bernardin, and I am grateful to him for helpful and valuable remarks. I warmly thank Makiko Sasada and Stefano Olla for their interest and constructive discussions on this work.

1 Introduction

In this paper we investigate diffusion problems in non homogeneous media for interacting particle systems. More precisely, we address the problem of energy fluctuations for chains of oscillators with random defects. In the last fifty years, it has been recognized that introducing randomness in interacting particle systems has a drastic effect on the conduction properties of the material. Mathematically the only tractable model is the one dimensional system with harmonic interactions [1]. The aim of this paper is to study the diffusive behavior of disordered harmonic chains perturbed by an energy conserving noise. In some sense, the noise should simulate the non linearities effect, and the conductivity of the one-dimensional chain should become finite and positive. We also expect that some homogenization effect occurs and that the conductivity does not depend on the statistics of the disorder in the thermodynamic limit.

The disorder effect has already been investigated for lattice gas dynamics, for example in [7, 8, 13, 15]. These papers share one main feature: the models are non-gradient due to the presence of the environment. Non-gradient systems are usually solved by establishing a microscopic Fourier's law

*UMPA, UMR-CNRS 5669, ENS de Lyon, 46 allée d'Italie, 69007 Lyon – marielle.simon@ens-lyon.fr

up to a small fluctuating term, following the sophisticated method initially developed by Varadhan in [19], and generalized to non-reversible dynamics [10]. The previous works mainly deal with symmetric systems of particles that evolve according to an exclusion process in random environment: the particles are attempting jumps to nearest neighbor sites at rates which depend on both their position and the objective site, and the rates themselves come from a quenched random field. Different approaches are adopted to tackle this non-gradient system: whereas the standard Varadhan's method is helpful only in dimension $d \geq 3$ [7], the "long jump" variation developed by Quastel in [15] is valid in all dimensions.

The study of disordered chains of oscillators perturbed by a conservative noise has appeared more recently, see by instance [2, 3, 5]. In these papers, only the behavior of the thermal conductivity defined by the Green-Kubo formula is investigated. Here, the diffusion coefficient is defined through hydrodynamics.

In [18], we have obtained the diffusive scaling limit for a homogeneous chain of coupled harmonic oscillators perturbed by a noise, which randomly flips the sign of the velocities, so that the energy is conserved but not the momentum. We have derived a system of nonlinear hydrodynamic equations on the only two conserved quantities: the energy and the total length of the chain, thanks to the relative entropy method. One of the major ingredient is an exact *fluctuation-dissipation equation* (see for example [12]), which reproduces at the microscopic level the Fourier law up to a small fluctuating term.

Our first motivation was to investigate the same chain of harmonic oscillators, still perturbed by the velocity-flip noise, but now provided with i.i.d. random masses. This makes all previous computations pointless: in particular, the fluctuation-dissipation equations are not directly computable any more. As a consequence, the fluctuation-dissipation decomposition can only be approximated by a sequence of local functions, in the sense that the difference has a small space-time variance with respect to the dynamics in equilibrium. The main ingredients of the usual non-gradient method are: first, a spectral gap for the symmetric part of the dynamics, and second, a *sector condition* for the total generator. The model has then special features that enforce the Varadhan's method to be considered with new perspectives. In particular, the symmetric part of the generator is poorly ergodic, and does not have a spectral gap when restricted to microcanonical manifolds. Moreover, due to the degeneracy of the noise, the asymmetric part of the generator is difficult to control by its symmetric part (in technical terms, the sector condition does not hold), with the only velocity-flip noise. Besides, let us remark that the energy current depends on the disorder, and has to be approximated by a fluctuation-dissipation equation which takes into account the fluctuations of the disorder itself.

Because of the high degeneracy of the velocity-flip noise, we add a second stochastic perturbation, that exchanges velocities (divided by the square root of mass) and positions at random independent Poissonian times, so that a kind of sector condition can be proved (see Proposition 5.7: we call it the *weak sector condition*). However, the spectral gap estimate and the usual sector condition still do not hold when adding the exchange noise, meaning that the stochastic perturbation is still very degenerate. The harmonic chain has helpful features, in particular the generator of the dynamics preserves the degree of polynomials, and even a degenerate noise is sufficient to apply Varadhan's approach. The sector condition and the non-gradient decomposition are only needed for a specific class of functions. The stochastic noise still does not have a spectral gap, but it does make no harm. Contrary to the standard Varadhan's approach, we do not need to prove any general result concerning the so-called *closed forms* (we refer to [17, 9] for the general theory). As far as we know, this is the first time that the non-gradient method is used successfully without the spectral gap estimate nor the usual sector condition.

For the non-linear ordered chain investigated in [14], one need a less degenerate noise than ours,

in particular both the spectral gap and the sector condition hold. The authors show that ideas from Varadhan's method can be used to prove a diffusive behavior of the energy: its fluctuations in equilibrium evolve following an infinite Ornstein-Uhlenbeck process. The covariances characterizing this linearised heat equation are given in terms of the diffusion coefficient, which is defined through a variational formula.

We show that the diffusion coefficient can be equivalently defined by the Green-Kubo formula. More precisely, the latter is the space-time variance of the current at equilibrium, which is only formal in the sense that a double limit (in space and time) has to be taken. As in [2], we prove here that the limit exists, and that the homogenization effect occurs for the Green-Kubo formula: for almost every realization of the disorder, the thermal conductivity exists, is independent of the disorder, is positive and finite. Finally, let us introduce $\gamma > 0$ the intensity of the flip noise, and $\lambda > 0$ the intensity of the exchange noise. We denote the diffusion coefficient by $D(\lambda, \gamma)$ when obtained through the variational formula in the Varadhan's method, and by $\bar{D}(\lambda, \gamma)$ when defined through the Green-Kubo formula. Then, we prove that the two conductivities are equal: $D(\lambda, \gamma) = \bar{D}(\lambda, \gamma)$, when the two intensities λ, γ are positive. In addition, we prove in Theorem 7.1 that the Green-Kubo formula remains well-defined when $\lambda = 0$, that is $\bar{D}(0, \gamma)$ exists, is finite and positive. Finally, Theorem 7.4 states that $D(\lambda, \gamma)$ tends to $\bar{D}(0, \gamma)$ as λ goes to 0. The existence question for $D(0, \gamma)$ (the hydrodynamics diffusion coefficient) remains open.

Before ending the introduction, let us be more precise on the model and the convergence result. We introduce the harmonic Hamiltonian system described by the sequence $\{p_x, r_x\}_{x \in \mathbb{Z}}$, where p_x stands for the momentum of the oscillator at site x , and r_x represents the distance between oscillator x and oscillator $x + 1$. Each atom $x \in \mathbb{Z}$ has a mass $M_x > 0$, the velocity of atom x is given by p_x/M_x . We assume the disorder $\mathbf{M} := \{M_x\}_{x \in \mathbb{Z}}$ to be a collection of real i.i.d. positive random variables such that

$$\forall x \in \mathbb{Z}, \quad \frac{1}{C} \leq M_x \leq C,$$

for some finite constant $C > 0$. The equations of motions are given by

$$\begin{cases} \frac{dp_x}{dt} = r_x - r_{x-1}, \\ \frac{dr_x}{dt} = \frac{p_{x+1}}{M_{x+1}} - \frac{p_x}{M_x}. \end{cases}$$

The dynamics conserves the total energy

$$\mathcal{E} := \sum_{x \in \mathbb{Z}} \left\{ \frac{p_x^2}{2M_x} + \frac{r_x^2}{2} \right\}.$$

To overcome the lack of ergodicity of deterministic chains, we add a stochastic perturbation to this new dynamics, so that the convergence of the energy fluctuations distribution holds (Theorem 3.1). The noise can be easily described: at independently distributed random Poissonian times, the quantity $p_x/\sqrt{M_x}$ and the interdistance r_x are exchanged, or the momentum p_x is flipped into $-p_x$. This noise still conserves the total energy \mathcal{E} , and is very degenerate.

Even if Theorem 3.1 could be proved *mutatis mutandis* for this harmonic chain, for pedagogical reasons we now focus on a simplified model (as in [4]), which has exactly the same features and involves less painful computations. From now on, we study the dynamics on the new configurations $\{\eta_x\}_{x \in \mathbb{Z}}$ written as

$$m_x d\eta_x = (\eta_{x+1} - \eta_{x-1})dt, \quad (1)$$

where $\mathbf{m} := \{m_x\}_{x \in \mathbb{Z}}$ is the new disorder with the same characteristics as before. It is notationally convenient to change the variable η_x into $\omega_x := \sqrt{m_x} \eta_x$, and the total energy reads

$$\mathcal{E} = \sum_{x \in \mathbb{Z}} \omega_x^2.$$

Let us now introduce the corresponding stochastic energy conserving dynamics: the evolution is described by (1) between random exponential times, and at each ring one of the following interactions can happen:

- a. *Exchange noise* – two nearest neighbour variables ω_x and ω_{x+1} are exchanged;
- b. *Flip noise* – the variable ω_x at site x is flipped into $-\omega_x$.

With these two perturbations, the dynamics conserves the total energy only, the other important conservation laws of the Hamiltonian part being destroyed by the stochastic noises. As a result, the following family $\{\mu_\beta\}_{\beta > 0}$ of grand-canonical Gibbs measures is invariant for the process:

$$\mu_\beta(d\omega) := \prod_{x \in \mathbb{Z}} \sqrt{\frac{2\pi}{\beta}} \exp\left(-\frac{\beta}{2} \omega_x^2\right) d\omega_x. \quad (2)$$

The index β stands for the inverse of the temperature. Notice that μ_β does not depend on the disorder, and that the dynamics is not reversible with respect to the measure μ_β . We define \mathbf{e}_β as the thermodynamical energy associated to β , namely the expectation of ω_0^2 with respect to μ_β , and $\chi(\beta) = 2\beta^{-2}$ as the variance of ω_0^2 with respect to μ_β .

We consider the system starting with μ_β and we denote by \mathbb{E}_β the expectation for the stochastic dynamics starting with this invariant distribution. We prove a diffusive behavior for the energy: first, define the distributions-valued energy fluctuation field

$$\mathcal{Y}^N := \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} \delta_{x/N} \{\omega_x^2(0) - \mathbf{e}_\beta\}.$$

It is well-known that \mathcal{Y}^N converges in distribution as $N \rightarrow \infty$ towards a centered Gaussian field \mathcal{Y} , which satisfies

$$\mathbb{E}_\beta[\mathcal{Y}(F)\mathcal{Y}(G)] = \chi(\beta) \int_0^1 F(y)G(y)dy,$$

for good test functions F, G . In this paper we prove that these energy fluctuations evolve diffusively in time (Theorem 3.1). More precisely, the following distribution

$$\mathcal{Y}_t^N = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{Z}} \delta_{x/N} \{\omega_x^2(tN^2) - \mathbf{e}_\beta\}$$

converges in law as $N \rightarrow \infty$ to the solution of the linear Stochastic Partial Differential Equation (SPDE)

$$\partial_t \mathcal{Y} = D \partial_y^2 \mathcal{Y} dt + \sqrt{2D\chi(\beta)} \partial_y B(y, t),$$

where D is the diffusion coefficient which has explicit expressions, and B is the standard normalized space-time white noise.

Finally, we could think of using the entropy method to derive the hydrodynamic equation. For that purpose, the initial law is not the equilibrium measure μ_β , but a *local equilibrium measure* (Definition ?? above). We conjecture that this property of local equilibrium propagates in time. In other words, let $\mathbf{e}_0 : \mathbb{T} \rightarrow \mathbb{R}$ be a bounded function, where \mathbb{T} denotes the torus $[0, 1)$. The goal would be to show that the

empirical energy profile converges in the thermodynamic limit to the macroscopic profile $\mathbf{e}(t, \cdot) : \mathbb{T} \rightarrow \mathbb{R}$ solution of

$$\begin{cases} \frac{\partial \mathbf{e}}{\partial t}(t, u) = D \frac{\partial^2 \mathbf{e}}{\partial u^2}(t, u), & t > 0, u \in \mathbb{T}, \\ \mathbf{e}(0, u) = \mathbf{e}_0(u). \end{cases}$$

Unfortunately, even if the diffusion coefficient is well-defined through the non-gradient approach, this does not straightforwardly provide a method to derive the hydrodynamic limits.

Let us now give the plan of the paper. We start with Section 2, which is devoted to introduce the model and all notations and definitions that are needed. The main point is to identify the diffusion term D (Section 5), by adapting the method introduced in [19]. In Section 4, we derive the Boltzmann-Gibbs principle. The convergence of the energy fluctuations field (in the sense of finite dimensional distributions) is proved in Section 3. Finally, Section 6 gives a precise description of the diffusion coefficient through several variational formulas. In Section 8, we present a second disordered model, where the interaction is described by a potential V . For this anharmonic chain, we need a very strong stochastic perturbation, which has a spectral gap, and satisfies the sector condition. We conclude in Section ?? by highlighting the step where the usual techniques for proving hydrodynamic limits fail. In Appendices, technical points are detailed: in Appendix A, the space of square integrable functions w.r.t. the standard Gaussian law is studied through its orthonormal basis of Hermite polynomials. In Appendix B, the weak version of closed forms usual result is investigated. The sector condition is proved for a specific class of functions in Appendix C. Appendix ?? is devoted to prove the convergence of the Green-Kubo formula. In Appendix D, the tightness for the energy fluctuation field is investigated.

2 The harmonic chain perturbed by stochastic jump noises

2.1 Generator of the Markov process

We first describe the dynamics on the finite torus $\mathbb{T}_N := \{0, \dots, N\}$, meaning that boundary conditions are periodic. The configuration $\{\omega_x\}_{x \in \mathbb{T}_N}$ evolves according to a dynamics which can be divided into two parts, a deterministic one and a stochastic one. The space of configurations of our system is given by $\Omega_N = \mathbb{R}^N$. We recall that the disorder is an i.i.d. sequence $\mathbf{m} = \{m_x\}_{x \in \mathbb{Z}}$ which satisfies:

$$\forall x \in \mathbb{Z}, \quad \frac{1}{C} \leq m_x \leq C,$$

for some finite constant $C > 0$. The corresponding product and translation invariant measure on the space $\Omega_{\mathcal{D}} = [C^{-1}, C]^{\mathbb{Z}}$ is denoted by \mathbb{P} and its expectation is denoted by \mathbb{E} . For a fixed disorder field $\mathbf{m} = \{m_x\}_{x \in \mathbb{T}_N}$, we consider the system of ODE's

$$\sqrt{m_x} d\omega_x = \left(\frac{\omega_{x+1}}{\sqrt{m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1}}} \right) dt, \quad t \geq 0, x \in \mathbb{T}_N$$

and we superpose to this deterministic dynamics a stochastic perturbation described as follows: to each atom $x \in \mathbb{T}_N$, and each bond $\{x, x+1\}, x \in \mathbb{T}_N$ is associated an exponential clock of rate one, such that each clock is independent of each other. When the clock attached to x rings, ω_x is flipped into $-\omega_x$, and when the clock attached to the bond $\{x, x+1\}$ rings, the values ω_x and ω_{x+1} are exchanged. This dynamics can be entirely defined by the generator of the Markov process $\{\omega_x(t); x \in \mathbb{T}_N\}_{t \geq 0}$, that is

$$\mathcal{L}_N^{\mathbf{m}} = \mathcal{A}_N^{\mathbf{m}} + \gamma \mathcal{S}_N^{\text{flip}} + \lambda \mathcal{S}_N^{\text{exch}}$$

where, for all functions $f : \Omega_{\mathcal{D}} \times \Omega_{\mathbb{N}} \rightarrow \mathbb{R}$,

$$\mathcal{A}_{\mathbb{N}}^{\mathbf{m}} f(\mathbf{m}, \omega) = \sum_{x \in \mathbb{T}_{\mathbb{N}}} \left(\frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}} \right) \frac{\partial f}{\partial \omega_x}(\mathbf{m}, \omega),$$

$$\mathcal{S}_{\mathbb{N}}^{\text{flip}} f(\mathbf{m}, \omega) = \sum_{x \in \mathbb{T}_{\mathbb{N}}} f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega),$$

$$\mathcal{S}_{\mathbb{N}}^{\text{exch}} f(\mathbf{m}, \omega) = \sum_{x \in \mathbb{T}_{\mathbb{N}}} f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega).$$

Here, the configuration ω^x is the configuration obtained from ω by flipping the momentum of particle x :

$$(\omega^x)_z = \begin{cases} \omega_z & \text{if } z \neq x, \\ -\omega_x & \text{if } z = x. \end{cases}$$

The configuration $\omega^{x,x+1}$ is obtained from ω by exchanging the momenta of particles x and $x+1$:

$$(\omega^{x,x+1})_z = \begin{cases} \omega_z & \text{if } z \neq x, x+1, \\ \omega_{x+1} & \text{if } z = x, \\ \omega_x & \text{if } z = x+1. \end{cases}$$

We denote the total generator of the noise by $\mathcal{S}_{\mathbb{N}} := \gamma \mathcal{S}_{\mathbb{N}}^{\text{flip}} + \lambda \mathcal{S}_{\mathbb{N}}^{\text{exch}}$, where $\gamma, \lambda > 0$ are two positive parameters which regulate the respective strengths of noises.

One quantity is conserved: the total energy $\sum \omega_x^2$. The following translation invariant product Gibbs measures $\mu_{\beta}^{\mathbb{N}}$ on $\Omega_{\mathbb{N}}$ are invariant for the process:

$$d\mu_{\beta}^{\mathbb{N}}(\omega) := \prod_{x \in \mathbb{T}_{\mathbb{N}}} \sqrt{\frac{2\pi}{\beta}} \exp\left(-\frac{\beta}{2} \omega_x^2\right) d\omega_x.$$

In the following, the expectation of f with respect to $\mu_{\beta}^{\mathbb{N}}$ is denoted by $\langle f \rangle_{\beta}$. The index β stands for the inverse temperature, namely $\langle \omega_0^2 \rangle_{\beta} = 1/\beta$. Let us highlight the fact that the Gibbs measures do not depend on the disorder \mathbf{m} . This obvious remark will play further a crucial role. From the definition, our model is not reversible with respect to the measure $\mu_{\beta}^{\mathbb{N}}$. Precisely, $\mathcal{A}_{\mathbb{N}}^{\mathbf{m}}$ is an antisymmetric operator in $L^2(\mu_{\beta}^{\mathbb{N}})$, whereas $\mathcal{S}_{\mathbb{N}}$ is symmetric.

We denote by Ω the space of configurations in the infinite line, that is $\Omega := \mathbb{R}^{\mathbb{Z}}$, and by μ_{β} the product Gibbs measure on $\mathbb{R}^{\mathbb{Z}}$. Hereafter, for every $\beta > 0$, we denote by \mathbb{P}_{β}^* the probability measure on $\Omega_{\mathcal{D}} \times \Omega$ defined by

$$\mathbb{P}_{\beta}^* := \mathbb{P} \otimes \mu_{\beta}.$$

We notice that \mathbb{P}_{β}^* is translation invariant and we write \mathbb{E}_{β}^* for the corresponding expectation.

2.2 Energy current

Since the dynamics conserves the total energy, there exist instantaneous currents of energy $j_{x,x+1}$ such that $\mathcal{L}_{\mathbb{N}}^{\mathbf{m}}(\omega_x^2) = j_{x,x+1}(\mathbf{m}, \omega) - j_{x-1,x}(\mathbf{m}, \omega)$. The quantity $j_{x,x+1}$ is the amount of energy between the particles x and $x+1$, and is equal to

$$j_{x,x+1}(\mathbf{m}, \omega) = \frac{2\omega_x \omega_{x+1}}{\sqrt{m_x m_{x+1}}} + \lambda(\omega_{x+1}^2 - \omega_x^2).$$

The energy conservation law can be read locally as

$$\omega_x^2(t) - \omega_x^2(0) = J_{x,x+1}(t) - J_{x-1,x}(t),$$

where $J_{x,x+1}(t)$ is the total energy current between x and $x+1$ up to time t . This can be written as

$$J_{x,x+1}(t) = \int_0^t j_{x,x+1}(s) ds + M_{x,x+1}(t),$$

where $M_{x,x+1}(t)$ is a martingale which can be explicitly computed as Itô stochastic integral:

$$M_{x,x+1}(t) = \int_0^t \left[\omega_{x+1}^2 - \omega_x^2 \right] (s^-) d[N_{x,x+1}(s) - \lambda s] + \int_0^t \left[\omega_{x-1}^2 - \omega_x^2 \right] (s^-) d[N_{x-1,x}(s) - \lambda s],$$

where $(N_{x,x+1})_{x \in \mathbb{Z}}$ are independent Poisson processes of intensity λ . We also write $j_{x,x+1} = j_{x,x+1}^A + j_{x,x+1}^S$ where $j_{x,x+1}^A$ (resp. $j_{x,x+1}^S$) is the current associated to the antisymmetric (resp. symmetric) part of the generator:

$$j_{x,x+1}^A(\mathbf{m}, \omega) = \frac{2\omega_x \omega_{x+1}}{\sqrt{m_x m_{x+1}}}$$

$$j_{x,x+1}^S(\mathbf{m}, \omega) = j_{x,x+1}^S(\omega) = \lambda(\omega_{x+1}^2 - \omega_x^2).$$

One can check that the current cannot be directly written as the gradient of a local function, neither by an exact fluctuation-dissipation equation (in other words, the current is not the sum of a gradient and a dissipative term of the form $\mathcal{L}_N^m(\tau_x h)$, where h is a local function of the system configuration). This means that we are in the non-gradient case. We also define the *static compressibility* that is equal to

$$\chi(\beta) := \langle \omega_0^4 \rangle_\beta - \langle \omega_0^2 \rangle_\beta^2 = \frac{2}{\beta^2}.$$

2.3 Cylinder functions

For every $x \in \mathbb{Z}$ and f a measurable function on $\Omega_{\mathcal{D}} \times \Omega$, we consider the translated function $\tau_x f$, which is the function on $\Omega_{\mathcal{D}} \times \Omega$ defined by: $\tau_x f(\mathbf{m}, \omega) := f(\tau_x \mathbf{m}, \tau_x \omega)$, where $\tau_x \mathbf{m}$ and $\tau_x \omega$ are the disorder and particle configurations translated by $x \in \mathbb{Z}$, respectively:

$$(\tau_x \mathbf{m})_z := m_{x+z}, \quad (\tau_x \omega)_z = \omega_{x+z}.$$

If f is a measurable function on $\Omega_{\mathcal{D}} \times \Omega$, the *support* of f , denoted by Λ_f , is the smallest subset of \mathbb{Z} such that $f(\mathbf{m}, \omega)$ only depends on $\{m_x, \omega_x; x \in \Lambda_f\}$ and f is called a *cylinder function* if Λ_f is finite. For every cylinder function $f : \Omega_{\mathcal{D}} \times \Omega \rightarrow \mathbb{R}$, consider the formal sum

$$\Gamma_f := \sum_{x \in \mathbb{Z}} \tau_x f$$

which does not make sense but for which

$$\nabla_0(\Gamma_f) := \Gamma_f(\mathbf{m}, \omega^0) - \Gamma_f(\mathbf{m}, \omega),$$

$$\nabla_{0,1}(\Gamma_f) := \Gamma_f(\mathbf{m}, \omega^{0,1}) - \Gamma_f(\mathbf{m}, \omega).$$

are well defined. Similarly, we define

$$\begin{aligned} (\nabla_x f)(\mathbf{m}, \omega) &:= f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega), \\ (\nabla_{x,x+1} f)(\mathbf{m}, \omega) &:= f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega). \end{aligned}$$

Let $\Lambda \subseteq \mathbb{Z}$ be a finite subset of \mathbb{Z} , and denote by \mathcal{F}_Λ the σ -algebra generated by $\{m_x, \omega_x ; x \in \Lambda\}$. For a fixed positive integer ℓ , we define $\Lambda_\ell := \{-\ell, \dots, \ell\}$. If the box is centered at site $x \in \mathbb{Z}$, we denote it by $\Lambda_\ell(x) := \{-\ell + x, \dots, \ell + x\}$.

We denote by \mathcal{C} the set of cylinder functions on $\Omega_{\mathcal{D}} \times \Omega$ with compact support and null average with respect to μ_β (for \mathbb{P} -almost every $m \in \Omega_{\mathcal{D}}$). We also introduce the set of *quadratic* cylinder functions on $\Omega_{\mathcal{D}} \times \Omega$, denoted by $\mathcal{Q} \subset \mathcal{C}$, and defined as follows: $f \in \mathcal{Q}$ if there exists a finite sequence $(\psi_{i,j}(\mathbf{m}))_{i,j \in \mathbb{Z}}$ of real cylinder functions on $\Omega_{\mathcal{D}}$ such that

$$f(\mathbf{m}, \omega) = \sum_{i \in \mathbb{Z}} \psi_{i,i}(\mathbf{m})(\omega_{i+1}^2 - \omega_i^2) + \sum_{\substack{i,j \in \mathbb{Z} \\ i \neq j}} \psi_{i,j}(\mathbf{m})\omega_i\omega_j. \quad (3)$$

In other words, quadratic functions are homogeneous polynomials of degree two in the variable ω , that have null average with respect to μ_β for every $\mathbf{m} \in \Omega_{\mathcal{D}}$. An other definition through *Hermite polynomials* is given in Appendix A.

For $f \in \mathcal{C}$, denote by s_f the smallest positive integer s such that Λ_s contains the support of f and then $\Lambda_f = \Lambda_{s_f}$. Hereafter, we consider operators $\mathcal{L}^{\mathbf{m}}$, $\mathcal{A}^{\mathbf{m}}$ and \mathcal{S} acting on functions $f \in \mathcal{C}$ as

$$\mathcal{L}^{\mathbf{m}} f = \mathcal{A}^{\mathbf{m}} f + \mathcal{S} f, \quad (4)$$

$$\mathcal{A}^{\mathbf{m}} f(\mathbf{m}, \omega) = \sum_{x \in \mathbb{Z}} \left(\frac{\omega_{x+1}}{\sqrt{m_x m_{x+1}}} - \frac{\omega_{x-1}}{\sqrt{m_{x-1} m_x}} \right) \frac{\partial f}{\partial \omega_x}(\mathbf{m}, \omega),$$

$$\mathcal{S} f = \gamma \mathcal{S}^{\text{flip}} f + \lambda \mathcal{S}^{\text{exch}} f,$$

$$\begin{aligned} \mathcal{S}^{\text{flip}} f(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} (\nabla_x f)(\mathbf{m}, \omega) = \sum_{x \in \mathbb{Z}} f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega), \\ \mathcal{S}^{\text{exch}} f(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} (\nabla_{x,x+1} f)(\mathbf{m}, \omega) = \sum_{x \in \mathbb{Z}} f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega). \end{aligned}$$

We also denote $\mathcal{S}_x = \gamma \nabla_x + \lambda \nabla_{x,x+1}$ for $x \in \mathbb{Z}$. For $\Lambda_\ell \subseteq \mathbb{Z}$ defined as above, we denote by $\mathcal{L}_{\Lambda_\ell}^{\mathbf{m}}$, resp. $\mathcal{S}_{\Lambda_\ell}$, the restriction of the generator $\mathcal{L}^{\mathbf{m}}$, resp. \mathcal{S} , to the finite box Λ_ℓ , assuming periodic boundary conditions.

We are now ready to define two sets of functions that will play further a crucial role.

DEFINITION 2.1. Let \mathcal{C}_0 (respectively \mathcal{Q}_0) be the set of cylinder (respectively quadratic cylinder) functions φ on $\Omega_{\mathcal{D}} \times \Omega$ such that there exists a finite subset $\Lambda \subseteq \mathbb{Z}$, and cylinder functions $\{F_x, G_x\}_{x \in \Lambda}$, satisfying

$$\varphi = \sum_{x \in \Lambda} \nabla_x(F_x) + \nabla_{x,x+1}(G_x).$$

If φ belongs to \mathcal{Q}_0 , we assume the cylinder functions F_x, G_x to be quadratic.

To conclude this section we introduce the quadratic form associated to the generator: for any $x \in \mathbb{Z}$ and cylinder functions $f, g \in \mathcal{C}$, let us define

$$\mathcal{D}_\ell(\mu_\beta; f) := \langle (-\mathcal{L}_{\Lambda_\ell}^{\mathbf{m}})f, f \rangle_\beta = \langle (-\mathcal{S}_{\Lambda_\ell})f, f \rangle_\beta = \sum_{x \in \Lambda_\ell} \langle (-\mathcal{S}_x)f, g \rangle_\beta,$$

and notice that

$$\begin{aligned} \langle (-\mathcal{S}_x)f, g \rangle_\beta &= \frac{\gamma}{2} \langle (f(\mathbf{m}, \omega^x) - f(\mathbf{m}, \omega)) (g(\mathbf{m}, \omega^x) - g(\mathbf{m}, \omega)) \rangle_\beta \\ &\quad + \frac{\lambda}{2} \langle (f(\mathbf{m}, \omega^{x,x+1}) - f(\mathbf{m}, \omega)) (g(\mathbf{m}, \omega^{x,x+1}) - g(\mathbf{m}, \omega)) \rangle_\beta, \end{aligned}$$

The symmetric form \mathcal{D}_ℓ is called the *Dirichlet form*, and is well-defined on \mathcal{C} . This is a random variable with respect to the disorder \mathbf{m} .

2.4 Semi-inner products and diffusion coefficient

For cylinder functions $g, h \in \mathcal{C}$, let

$$\ll g, h \gg_{\beta, \star} := \sum_{x \in \mathbb{Z}} \mathbb{E}_\beta^* [g \tau_x h] \quad \text{and} \quad \ll g \gg_{\beta, \star\star} := \sum_{x \in \mathbb{Z}} x \mathbb{E}_\beta^* [g \omega_x^2] \quad (5)$$

which are well defined because g and h belong to \mathcal{C} and therefore all but a finite number of terms vanish. Notice that $\ll \cdot, \cdot \gg_{\beta, \star}$ is an inner-product, since the following equality holds:

$$\ll f, g \gg_{\beta, \star} = \lim_{\Lambda \uparrow \mathbb{Z}} \frac{1}{|\Lambda|} \mathbb{E} \left[\left\langle \sum_{x \in \Lambda} \tau_x f, \sum_{x \in \Lambda} \tau_x g \right\rangle_\beta \right]$$

Since $\ll f - \tau_x f, g \gg_{\beta, \star} = 0$ for all $x \in \mathbb{Z}$, this scalar product is only semidefinite. In the next proposition we give explicit formulas for elements of \mathcal{C}_0 .

PROPOSITION 2.1. *If $\varphi \in \mathcal{C}_0$ with*

$$\varphi = \sum_{y \in \Lambda} \nabla_y (F_y) + \nabla_{y, y+1} (G_y),$$

then

$$\begin{aligned} \ll \varphi \gg_{\beta, \star\star} &= \mathbb{E} \left\langle (\omega_0^2 - \omega_1^2) \sum_{y \in \Lambda} \tau_{-y} G_y \right\rangle_\beta \\ \ll \varphi, g \gg_{\beta, \star} &= \mathbb{E} \left\langle \nabla_0(\Gamma_g) \sum_{y \in \Lambda} \tau_{-y} F_y + \nabla_{0,1}(\Gamma_g) \sum_{y \in \Lambda} \tau_{-y} G_y \right\rangle_\beta \quad \text{for all } g \in \mathcal{C}. \end{aligned}$$

Proof. The proof is straightforward. □

DEFINITION 2.2. *We define the diffusion coefficient $D(\beta)$ for $\beta > 0$ as*

$$D(\beta) := \lambda + \frac{1}{\chi(\beta)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \left\{ \ll f, -\mathcal{S}f \gg_{\beta, \star} + 2 \ll j_{0,1}^A - \mathcal{A}^{\mathbf{m}} f, g \gg_{\beta, \star} - \ll g, -\mathcal{S}g \gg_{\beta, \star} \right\}.$$

The first term in the sum is only due to the exchange noise, whereas the second one comes from the Hamiltonian part of the dynamics. Formally, this formula could be read as

$$D(\beta) = \lambda + \frac{1}{\chi(\beta)} \ll j_{0,1}^A, (-\mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{\beta, \star}, \quad (6)$$

but the last term is ill-defined because $j_{0,1}^A$ is not in the range of $\mathcal{L}^{\mathbf{m}}$. More rigorously, we should define $\ll j_{0,1}^A, (-\mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{\beta, \star}$ as

$$\limsup_{z \rightarrow 0} \ll j_{0,1}^A, (z - \mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{\beta, \star}.$$

The last expression is now well-defined, and the problem is reduced to prove convergence as $z \rightarrow 0$. From Hille-Yosida Theorem (see Proposition 2.1 in [6] by instance) (6) is equal to the infinite volume Green-Kubo formula:

$$D(\beta) = \lambda + \frac{1}{\chi(\beta)} \lim_{\substack{z \rightarrow 0 \\ z > 0}} \mathbb{E} \left[\int_0^{+\infty} e^{-zt} \left\langle \sum_{x \in \mathbb{Z}} j_{x,x+1}^A(t), j_{0,1}^A(0) \right\rangle_{\beta} dt \right], \quad (7)$$

In Section 7.1, we prove that (7) converges, by inspiring the argument from [2], and we show that the diffusion coefficient can be equivalently defined in the two ways. Assuming the convergence in the Green-Kubo formula, one can easily see that $D(\beta)$ does not depend on β . We denote by $L(z)$ the second term of the right-hand side of (7), that is

$$L(z) := \frac{1}{\chi(\beta)} \int_0^{+\infty} e^{-zt} \ll j_{0,1}^A(t), j_{0,1}^A(0) \gg_{\beta, \star} dt.$$

Let $\mathbf{L}^2(\ll \cdot, \cdot \gg_{\beta, \star})$ be the Hilbert space generated by the set of bounded local functions and the inner product $\ll \cdot, \cdot \gg_{\beta, \star}$, and $h_z := h_z(\mathbf{m}, \omega; \beta)$ be the solution of the resolvent equation in $\mathbf{L}^2(\ll \cdot, \cdot \gg_{\beta, \star})$ i.e.

$$(z - \mathcal{L}^{\mathbf{m}})h_z = j_{0,1}^A.$$

Then we have

$$L(z) = \frac{1}{\chi(\beta)} \ll h_z, j_{0,1}^A \gg_{\beta, \star} = \frac{\beta^2}{2} \ll h_z, j_{0,1}^A \gg_{\beta, \star}.$$

Observe that if ω is distributed according to μ_{β} then $\beta^{1/2}\omega$ is distributed according to μ_1 . Since $h_z(\mathbf{m}, \omega; 1) = h_z(\mathbf{m}, \omega; \beta)$ and $j_{x,x+1}^A$ is an homogeneous function of degree two in ω , it follows that the diffusion coefficient does not depend on β .

3 Macroscopic fluctuations of energy

In this section we are interested in the fluctuations of the empirical energy, when the system is at equilibrium. We prove that the limit fluctuation process is governed by a generalized Ornstein-Uhlenbeck process, whose covariances are given in terms of the diffusion coefficient. We adapt the non-gradient method introduced by Varadhan. In particular, we establish rigorously the variational formula that appears in the definition of the diffusion coefficient (Definition 2.2). Varadhan's approach is investigated in Sections 4, 5 and 6.

3.1 Energy fluctuation field

Recall that we denote by \mathbf{e}_β the thermodynamical energy associated to the inverse temperature $\beta > 0$, namely $\mathbf{e}_\beta = \beta^{-1}$. We define the energy empirical distribution $\pi_{t,\mathbf{m}}^N$ on the torus $\mathbb{T} = [0, 1)$ as

$$\pi_{t,\mathbf{m}}^N(\mathrm{d}u) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^2(t) \delta_{x/N}(\mathrm{d}u), \quad t \in [0, T], \quad u \in \mathbb{T},$$

where δ_u states for the Dirac measure. We denote by $\{\omega(t)\}_{t \geq 0}$ the Markov process generated by $N^2 \mathcal{L}_N^{\mathbf{m}}$ and by \mathcal{M}_1 the set of probability measures on \mathbb{T} , endowed with the weak topology. The space of trajectories in \mathcal{M}_1 , which are right-continuous and left-limited (i.e. the Skorokhod space) is denoted by $\mathcal{D}([0, T], \mathcal{M}_1)$. If the initial state of the dynamics is given by the equilibrium Gibbs measure μ_β^N , then $\pi_{t,\mathbf{m}}^N$ weakly converges towards the deterministic measure on \mathbb{T} , equal to $\{\mathbf{e}_\beta \mathrm{d}u\}$. Our goal is to investigate the fluctuations of the empirical measure π^N with respect to this limit. Let us fix the disorder \mathbf{m} , and the inverse of temperature $\beta > 0$. Consider the system under the equilibrium measure μ_β^N .

DEFINITION 3.1 (Empirical energy fluctuations). *We denote by $\mathcal{Y}_{t,\mathbf{m}}^N$ the empirical energy fluctuation field defined as*

$$\mathcal{Y}_{t,\mathbf{m}}^N(\mathrm{H}) = \frac{1}{\sqrt{N}} \sum_{x \in \mathbb{T}_N} \mathrm{H}\left(\frac{x}{N}\right) \{\omega_x^2(t) - \mathbf{e}_\beta\},$$

where $\mathrm{H} : \mathbb{T} \rightarrow \mathbb{R}$ is a smooth function.

We are going to prove that the distribution $\mathcal{Y}_{t,\mathbf{m}}^N$ converges in law towards the solution of the linear SPDE:

$$\partial_t \mathcal{Y} = \mathrm{D} \partial_y^2 \mathcal{Y} \mathrm{d}t + \sqrt{2\mathrm{D}\chi(\beta)} \partial_y \mathcal{B}(y, t)$$

where \mathcal{B} is a standard normalized space-time white noise, and D is the diffusion coefficient defined in Theorem 5.9. Observe that there is no dependence on the disorder \mathbf{m} in the limit process. In other words, the latter is described by the stationary generalized Ornstein-Uhlenbeck process with zero mean and covariances given by

$$\langle \mathcal{Y}_t(\mathrm{H}) \mathcal{Y}_0(\mathrm{G}) \rangle = \frac{\chi(\beta)}{\sqrt{4\pi t \mathrm{D}}} \int_{\mathbb{R}^2} \mathrm{d}u \mathrm{d}v \bar{\mathrm{H}}(u) \bar{\mathrm{G}}(v) \exp\left(-\frac{(u-v)^2}{4t\mathrm{D}}\right),$$

for all $t \geq 0$ and smooth functions $\mathrm{H}, \mathrm{G} : \mathbb{T} \rightarrow \mathbb{R}$. Here, $\bar{\mathrm{H}}$ (resp. $\bar{\mathrm{G}}$) is the periodic extension to the real line of H (resp. G).

We denote by $\mathfrak{Y}_{\mathbf{m}}^N$ the probability measure on $\mathcal{D}([0, T], \mathcal{M}_1)$ induced by the energy fluctuation field $\mathcal{Y}_{t,\mathbf{m}}^N$ and the Markov process $\{\omega(t)\}_{t \geq 0}$ generated by $N^2 \mathcal{L}_N^{\mathbf{m}}$, starting from the equilibrium probability measure μ_β^N . Let \mathfrak{Y} be the probability measure on the space $\mathcal{D}([0, T], \mathcal{M}_1)$ corresponding to the generalized Ornstein-Uhlenbeck process \mathcal{Y}_t . The main result of this section is the following.

THEOREM 3.1. *For almost all realization of the disorder $\mathbf{m} \in \Omega_{\mathcal{D}}$, the sequence $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$ weakly converges in $\mathcal{D}([0, T], \mathcal{M}_1)$ to the probability measure \mathfrak{Y} .*

3.2 Strategy of the proof

We follow the lines of [14, Section 3]. The proof of Theorem 3.1 is divided into three steps. First, we need to show that the sequence $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$ is tight. This point follows a standard argument, given for instance in [9, Section 11], and recalled in Appendix D for the sake of completeness. Then, we prove

that the one-time marginal of any limit point \mathfrak{Y}^* of a convergent subsequence of $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$ is the law of a centered Gaussian field \mathcal{Y} with covariances given by

$$\langle \mathcal{Y}(\mathbf{H})\mathcal{Y}(\mathbf{G}) \rangle = \chi(\beta) \int_{\mathbb{T}} du \mathbf{H}(u)\mathbf{G}(u),$$

where $\mathbf{H}, \mathbf{G} : \mathbb{T} \rightarrow \mathbb{R}$ are smooth functions. This statement comes from the central limit theorem for independent variables. Finally, we prove the main point in the next subsections: all limit points \mathfrak{Y}^* of convergent subsequences of $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$ solve the martingale problems below.

Martingale problems – For each smooth function $\mathbf{H} : \mathbb{T} \rightarrow \mathbb{R}$,

$$\mathfrak{M}_t(\mathbf{H}) := \mathcal{Y}_t(\mathbf{H}) - \mathcal{Y}_0(\mathbf{H}) - \int_0^t D\mathcal{Y}_s(\mathbf{H}'') ds, \quad (8)$$

and

$$\mathfrak{N}_t(\mathbf{H}) := (\mathfrak{M}_t(\mathbf{H}))^2 - 2t\chi(\beta)D \int_{\mathbb{T}} \mathbf{H}'(u)^2 du \quad (9)$$

are $L^1(\mathfrak{Y})$ -martingales.

3.3 Martingale decompositions

Let us fix a smooth function $\mathbf{H} : \mathbb{T} \rightarrow \mathbb{R}$. We rewrite $\mathcal{Y}_{t,\mathbf{m}}^N(\mathbf{H})$ as

$$\mathcal{Y}_{t,\mathbf{m}}^N(\mathbf{H}) = \mathcal{Y}_{0,\mathbf{m}}^N(\mathbf{H}) + \int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbf{H} \left(\frac{x}{N} \right) j_{x,x+1}(\mathbf{m}, s) ds + \mathcal{M}_{t,\mathbf{m}}^N(\mathbf{H})$$

where $\mathcal{M}_{t,\mathbf{m}}^N$ is the martingale defined as

$$\mathcal{M}_{t,\mathbf{m}}^N(\mathbf{H}) = \int_0^t \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbf{H} \left(\frac{x}{N} \right) (\omega_{x+1}^2 - \omega_x^2)(s) d[N_{x,x+1}(s) - \lambda s].$$

Hereafter, $(N_{x,x+1})_{x \in \mathbb{Z}}$, $(N_x)_{x \in \mathbb{Z}}$ are independent Poisson processes of intensity (respectively) λ and γ , and ∇_N stands for the discrete gradient:

$$\nabla_N \mathbf{H} \left(\frac{x}{N} \right) = N \left[\mathbf{H} \left(\frac{x+1}{N} \right) - \mathbf{H} \left(\frac{x}{N} \right) \right],$$

and the discrete Laplacian Δ_N is defined in a similar way:

$$\Delta_N \mathbf{H} \left(\frac{x}{N} \right) = N^2 \left[\mathbf{H} \left(\frac{x+1}{N} \right) + \mathbf{H} \left(\frac{x-1}{N} \right) - 2\mathbf{H} \left(\frac{x}{N} \right) \right].$$

To close the equation, we are going to replace the term involving the microscopic currents with a term involving $\mathcal{Y}_{t,\mathbf{m}}^N$. In other words, the most important part in the fluctuation field represented by

$$\int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N \mathbf{H} \left(\frac{x}{N} \right) j_{x,x+1}(\mathbf{m}, s) ds$$

is its projection over the conservation field $\mathcal{Y}_{t,\mathbf{m}}^N$ (recall that the total energy is the unique conserved quantity of the system). The non-gradient approach consists in using the fluctuation-dissipation approximation of the current $j_{x,x+1}$ given by Theorem 5.9 below as $D(\omega_{x+1}^2 - \omega_x^2) + \mathcal{L}^{\mathbf{m}}(\tau_x f)$. For that purpose, we rewrite, for any $f \in \mathcal{Q}$,

$$\mathcal{Y}_{t,\mathbf{m}}^N(H) = \mathcal{Y}_{0,\mathbf{m}}^N(H) + \int_0^t D\mathcal{Y}_{s,\mathbf{m}}^N(\Delta_N H) ds + \mathcal{J}_{t,\mathbf{m},f}^{1,N}(H) + \mathcal{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{1,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H), \quad (10)$$

where

$$\mathcal{J}_{t,\mathbf{m},f}^{1,N}(H) = \int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N H \left(\frac{x}{N} \right) \left[j_{x,x+1}(\mathbf{m}, s) - D(\omega_{x+1}^2 - \omega_x^2)(s) - \mathcal{L}^{\mathbf{m}}(\tau_x f)(\mathbf{m}, s) \right] ds,$$

$$\mathcal{J}_{t,\mathbf{m},f}^{2,N}(H) = \int_0^t \sqrt{N} \sum_{x \in \mathbb{T}_N} \nabla_N H \left(\frac{x}{N} \right) \mathcal{L}^{\mathbf{m}}(\tau_x f)(\mathbf{m}, s) ds,$$

$$\mathfrak{M}_{t,\mathbf{m},f}^{1,N}(H) = \int_0^t \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N H \left(\frac{x}{N} \right) \left\{ \left[\nabla_{x,x+1}(\omega_0^2 - \Gamma_f) \right](s) d[N_{x,x+1}(s) - \lambda s] \right. \\ \left. - \nabla_x(\Gamma_f)(s) d[N_x(s) - \gamma s] \right\},$$

$$\mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H) = \int_0^t \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N H \left(\frac{x}{N} \right) \left\{ \nabla_{x,x+1}(\Gamma_f)(s) d[N_{x,x+1}(s) - \lambda s] \right. \\ \left. + \nabla_x(\Gamma_f)(s) d[N_x(s) - \gamma s] \right\}.$$

The strategy of the proof is based on the two following results.

LEMMA 3.2. *For every smooth function $H : \mathbb{T} \rightarrow \mathbb{R}$, and every function $f \in \mathcal{Q}$,*

$$\lim_{N \rightarrow \infty} \mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left(\mathcal{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H) \right)^2 \right\rangle_{\beta} = 0.$$

THEOREM 3.3 (Boltzmann-Gibbs principle). *There exists a sequence of functions $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{Q}$ such that*

(i) *for every smooth function $H : \mathbb{T} \rightarrow \mathbb{R}$,*

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left(\mathcal{J}_{t,\mathbf{m},f_k}^{1,N}(H) \right)^2 \right\rangle_{\beta} = 0, \quad (11)$$

(ii) *and moreover*

$$\lim_{k \rightarrow \infty} \mathbb{E} \left\langle \lambda \left(\nabla_{0,1}(\omega_0^2 - \Gamma_{f_k}) \right)^2 + \gamma \left(\nabla_0(\Gamma_{f_k}) \right)^2 \right\rangle_{\beta} = 2D\mathcal{X}(\beta). \quad (12)$$

As a result, the martingale $\mathfrak{M}_{t,\mathbf{m},f_k}^{1,N}$ converges in $\mathbf{L}^2(\mathbb{P}_{\beta}^*)$, as $N \rightarrow \infty$ and $k \rightarrow \infty$, to a martingale $\mathfrak{M}_t(H)$ of quadratic variation

$$2tD\mathcal{X}(\beta) \int_{\mathbb{T}} H'(u)^2 du,$$

and the limit $\mathcal{Y}_t(H)$ of $\mathcal{Y}_{t,\mathbf{m}}^N(H)$ satisfies the equation

$$\mathcal{Y}_t(H) = \mathcal{Y}_0(H) + \int_0^t \mathcal{Y}_s(DH'') ds + \mathfrak{M}_t(H).$$

We have proved that the limit solves the martingale problems (8) and (9), which uniquely characterized the generalized Ornstein-Uhlenbeck process \mathcal{Y}_t . The proof of Lemma 3.2 is the content of the next subsection. The proof of Theorem 3.3 is more challenging, and Sections 4, 5 and 6 are devoted to it.

3.4 Proof of Lemma 3.2

In this paragraph we give a proof of Lemma 3.2. We define

$$X_{\mathbf{m},f}^N(t) = \frac{1}{N\sqrt{N}} \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \tau_{xf}(\mathbf{m}, t)$$

As before, we can rewrite

$$\begin{aligned} \mathcal{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H) &= X_{\mathbf{m},f}^N(t) - X_{\mathbf{m},f}^N(0) \\ &+ \frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_{x,x+1} \left(\left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_{zf} \right\} - \nabla_N \left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_{x,x+1}(s) - \lambda s] \\ &+ \frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_x \left(\left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_{zf} \right\} - \nabla_N H\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_x(s) - \lambda s]. \end{aligned}$$

Therefore,

$$\begin{aligned} (\mathcal{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m},f}^{2,N}(H))^2 &\leq 3 \left(X_{\mathbf{m},f}^N(t) - X_{\mathbf{m},f}^N(0) \right)^2 \\ &+ 3 \left(\frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_{x,x+1} \left(\left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_{zf} \right\} - \nabla_N H\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_{x,x+1}(s) - \lambda s] \right)^2 \\ &+ 3 \left(\frac{1}{N\sqrt{N}} \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_x \left(\left\{ \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_{zf} \right\} - \nabla_N H\left(\frac{x}{N}\right) \Gamma_f \right) (\mathbf{m}, s) d[N_x(s) - \lambda s] \right)^2 \quad (13) \end{aligned}$$

On the one hand,

$$\mathbb{E} \left\langle \left(X_{\mathbf{m},f}^N \right)^2 \right\rangle_{\beta} = \frac{1}{N^3} \sum_{x,y \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \nabla_N H\left(\frac{y}{N}\right) \mathbb{E} \langle \tau_{xf}, \tau_{yf} \rangle_{\beta}.$$

This last quantity is of order $1/N^2$, because f is a local function of zero average, and H is smooth. On the other hand, let us define

$$Y_x(\mathbf{m}, \omega) := \sum_{z \in \mathbb{T}_N} \nabla_N H\left(\frac{z}{N}\right) \tau_{zf} - \nabla_N H\left(\frac{x}{N}\right) \sum_{z \in \mathbb{Z}} \tau_{zf}.$$

Then, the expectation of the second term of (13) is equal to

$$\frac{3\lambda^2 t N^2}{N^3} \sum_{x \in \mathbb{T}_N} \mathbb{E} \left\langle [\nabla_{x,x+1}(Y_x)]^2 \right\rangle_{\beta}.$$

Again, since f is local and H is smooth, this quantity is of order $1/N^2$. Indeed, in the expression $\nabla_{x,x+1}(Y_x)$, there is a sum over $z \in \mathbb{Z}$, but in which only terms with $|z - x| \leq 2$ remain. The same holds for the third term of (13).

4 Central limit theorem variances at equilibrium

In this section we are going to identify the diffusion coefficient D that appears in (10). Roughly speaking, D can be viewed as the asymptotic component of the energy current $j_{x,x+1}$ in the direction of

the gradient $\omega_{x+1}^2 - \omega_x^2$, and makes the expression below vanish:

$$\inf_{f \in \mathcal{Q}} \limsup_{N \rightarrow \infty} \limsup_{t \rightarrow \infty} \frac{1}{tN} \mathbb{E}_\beta^* \left[\left(\int_0^t \sum_{x \in \mathbb{T}_N} [j_{x,x+1} - D(\omega_{x+1}^2 - \omega_x^2) - \mathcal{L}^m(\tau_x f)] ds \right)^2 \right], \quad \text{for any } \beta > 0.$$

Let us start by giving well-known tools that will help understand the forthcoming results.

4.1 An insight through additive functionals of Markov processes

Consider a continuous time Markov process $\{Y_s\}_{s \geq 0}$ on a complete and separable metric space E , which has an invariant and ergodic measure π . We denote by $\langle \cdot \rangle_\pi$ the inner product in $L^2(\pi)$ and by \mathcal{L} the infinitesimal generator of the process. The adjoint of \mathcal{L} in $L^2(\pi)$ is denoted by \mathcal{L}^* . Fix a function $V : E \rightarrow \mathbb{R}$ in $L^2(\pi)$ such that $\langle V \rangle_\pi = 0$. Theorem 2.7 in [11] gives conditions on V which guarantee a central limit theorem for

$$\frac{1}{\sqrt{t}} \int_0^t V(Y_s) ds$$

and shows that the limiting variance equals

$$\sigma^2(V, \pi) = 2 \lim_{\substack{z \rightarrow 0 \\ z > 0}} \langle V, (z - \mathcal{L})^{-1} V \rangle_\pi.$$

Let the generator \mathcal{L} be decomposed as $\mathcal{L} = \mathcal{S} + \mathcal{A}$, where $\mathcal{S} = (\mathcal{L} + \mathcal{L}^*)/2$ and $\mathcal{A} = (\mathcal{L} - \mathcal{L}^*)/2$ denote, respectively, the symmetric and antisymmetric parts of \mathcal{L} . Let \mathcal{H}_1 be the completion of $L^2(\pi)$ with respect to the semi-norm $\| \cdot \|_1$ defined as:

$$\|f\|_1^2 := \langle f, (-\mathcal{L})f \rangle_\pi = \langle f, (-\mathcal{S})f \rangle_\pi.$$

Let \mathcal{H}_{-1} be the dual space of \mathcal{H}_1 with respect to $L^2(\pi)$, in other words, the Hilbert space generated by local functions and the norm $\| \cdot \|_{-1}$ defined by

$$\|f\|_{-1}^2 := \sup_g \{2\langle f, g \rangle_\pi - \|g\|_1^2\},$$

where the supremum is carried over all local functions g . Formally, $\|f\|_{-1}$ can also be thought as

$$\langle f, (-\mathcal{S})^{-1} f \rangle_\pi.$$

Notice the difference with the variance $\sigma^2(V, \pi)$ which formally reads

$$2\langle V, (-\mathcal{L})^{-1} V \rangle_\pi = 2\langle V, [(-\mathcal{L})^{-1}]_s V \rangle_\pi.$$

Hereafter, B_s represents the symmetric part of the operator B . We can write, at least formally, that

$$\{[(-\mathcal{L})^{-1}]_s\}^{-1} = -\mathcal{S} + \mathcal{A}^*(-\mathcal{S})^{-1}\mathcal{A} \geq -\mathcal{S},$$

where \mathcal{A}^* stands for the adjoint of \mathcal{A} . We have therefore that $[(-\mathcal{L})^{-1}]_s \leq (-\mathcal{S})^{-1}$. The following result is a rigorous estimate of the time variance in terms of the \mathcal{H}_{-1} norm, which is proved in [11, Lemma 2.4].

LEMMA 4.1. *Given $T > 0$ and a mean zero function V in $L^2(\pi) \cap \mathcal{H}_{-1}$,*

$$\mathbb{E}_\pi \left[\sup_{0 \leq t \leq T} \left(\int_0^t V(s) ds \right)^2 \right] \leq 24T \|V\|_{-1}^2. \quad (14)$$

If we compare the previous left-hand side to the Boltzmann-Gibbs principle (12), the next step should be to take V proportional to

$$\sum_{x \in \mathbb{T}_N} [j_{x,x+1} - D(\omega_{x+1}^2 - \omega_x^2) - \mathcal{L}^m(\tau_x f)]$$

and then take the limit as N goes to ∞ . In the right-hand side of (14) we will obtain a variance that depends on N , and the main task will be to show that this variance converges: this is studied in more details in what follows. Precisely, we prove that the limit of the variance results in a semi-norm, which is denoted by $||| \cdot |||_\beta$ and defined in (21). We are going to see that (21) involves a variational formula, which formally reads

$$|||f|||_\beta^2 = \langle f, (-S)^{-1}f \rangle_{\beta, \star} + \frac{2}{\lambda \chi(\beta)} \langle f \rangle_{\beta, \star}^2.$$

The final step consists in minimizing this semi-norm on a well-chosen subspace in order to get the Boltzmann-Gibbs principle, through orthogonal projections in Hilbert spaces. The hard point is that $||| \cdot |||_\beta$ only depends on the symmetric part of the generator S , and the latter is really degenerate, since it does not have a spectral gap.

In Subsection 4.2, we investigate the variance $\langle f, (-S)^{-1}f \rangle_\beta$, and prove its well-posedness for every function f in \mathcal{C}_0 . In Subsection 4.3, we relate the previous limiting variance (taking the limit as N goes to infinity) to the suitable semi-norm. Subsection 4.4 is devoted to prove the Boltzmann-Gibbs principle inspired from Lemma 4.1. Then, in Section 5 we investigate the Hilbert space generated by the semi-norm, and prove some decompositions into direct sums. Finally, Section 6 focuses on the diffusion coefficient and its different expressions.

4.2 Microcanonical measures and integration by part

4.2.1 Decomposition on microcanonical measures

The thermodynamic ensemble which is naturally associated with a Hamiltonian dynamics is the *microcanonical ensemble*, which describes the system at fixed energy. It is possible to devise a probability measure on the configurations $\omega \in \Omega_N$ with constant energy $\beta^{-1} > 0$ such that the measure is stationary with respect to the Hamiltonian flow. The corresponding probability measure denoted by $\mu_{N,\beta}^{mc}$ is the normalized uniform probability measure on the sphere

$$\mathfrak{S}_{N,\beta} := \left\{ \omega \in \Omega_N ; \sum_{x \in \mathbb{T}_N} \omega_x^2 = \beta^{-1} \right\}.$$

Now, for $\beta^{-1} > 0$ fixed, we disintegrate the microcanonical measure $\mu_{N,\beta}^{mc}$ on $\mathfrak{S}_{N,\beta}$. Let \mathcal{G} be the group generated by the following matrices: the *permutation matrices* P_σ , defined for any permutation σ of $\{1, \dots, N\}$ by

$$(P_\sigma)_{i,j} = \begin{cases} 1 & \text{if } i = \sigma(j), \\ 0 & \text{otherwise,} \end{cases}$$

and the *sign matrices* S_k , defined for $k \in \{1, \dots, N\}$ by

$$(S_k)_{i,j} = \begin{cases} 1 & \text{if } i = j \text{ and } i \neq k, \\ -1 & \text{if } i = j = k, \\ 0 & \text{otherwise.} \end{cases}$$

The group \mathcal{G} acts on $\mathfrak{S}_{N,\beta}$. For $\mathbf{x} \in \mathfrak{S}_{N,\beta}$, we denote by $\mathcal{G}_{\mathbf{x}}^{N,\beta}$ the orbit of $\mathbf{x} = (\mathbf{x}_1, \dots, \mathbf{x}_N)$ under the action of \mathcal{G} . More precisely,

$$\mathcal{G}_{\mathbf{x}}^{N,\beta} = \left\{ \omega \in \mathfrak{S}_{N,\beta} ; \forall i \in \{1, \dots, N\}, \omega_i \in \{\pm \mathbf{x}_1; \dots; \pm \mathbf{x}_N\} \right\},$$

and each orbit is finite, with cardinality $2^{N(N+1)/2}$. This group action defines a projection

$$\pi : \mathfrak{S}_{N,\beta} \rightarrow \mathcal{Q}_{N,\beta}$$

onto the quotient space $\mathcal{Q}_{N,\beta} := \mathfrak{S}_{N,\beta}/\mathcal{G}$. We can define the pushforward measure $\nu_{N,\beta} := \pi \star \mu_{N,\beta}^{mc}$ on $\mathcal{Q}_{N,\beta}$. Then, the disintegration of $\mu_{N,\beta}^{mc}$ with respect to π writes as follows: for all test functions $f : \mathfrak{S}_{N,\beta} \rightarrow \mathbb{R}$, the measure $\mu_{N,\beta}^{mc}(\cdot|\mathbf{x})$ with support on $\mathcal{G}_{\mathbf{x}}^{N,\beta}$ satisfies

$$\int_{\mathfrak{S}_{N,\beta}} f(\omega) d\mu_{N,\beta}^{mc}(\omega) = \int_{\mathcal{Q}_{N,\beta}} \int_{\mathfrak{S}_{N,\beta}} f(\omega) d\mu_{N,\beta}^{mc}(\omega|\mathbf{x}) d\nu_{N,\beta}(\mathbf{x}).$$

It is not difficult to see that $\mu_{N,\beta,\mathbf{x}}(\cdot) := \mu_{N,\beta}^{mc}(\cdot|\mathbf{x})$ is the uniform measure on the orbit $\mathcal{G}_{\mathbf{x}}^{N,\beta}$, since its support is invariant under a subgroup of rotations (of the total sphere). Let us denote by $\langle \cdot \rangle_{N,\mathbf{x}}$ the corresponding expectation. We obtain that, for all test functions $f : \mathfrak{S}_{N,\beta} \rightarrow \mathbb{R}$,

$$\langle f \rangle_{N,\mathbf{x}} = \frac{1}{2^{N(N+1)/2}} \sum_{y \in \mathcal{G}_{\mathbf{x}}^{N,\beta}} f(y).$$

To conclude, let us fix the energy $\beta^{-1} > 0$, take \mathbf{x} in the microcanonical sphere $\mathfrak{S}_{N,\beta}$, and look at the dynamics generated by \mathcal{L}_N restricted on the orbit $\mathcal{G}_{\mathbf{x}}^{N,\beta}$. Then, observe that the kernel of \mathcal{S}_N in the Hilbert space $\mathbf{L}^2(\mu_{N,\beta,\mathbf{x}})$ has dimension 1. As a result, the range of \mathcal{S}_N in $\mathbf{L}^2(\mu_{N,\beta,\mathbf{x}})$ has codimension 1, and is equal to the mean-zero functions.

In the sequel, we give some properties of the two spaces \mathcal{C}_0 and \mathcal{Q}_0 : for instance, the energy current is among the elements of \mathcal{Q}_0 (Proposition 4.2). We also prove an *integration by parts* formula for the functions of \mathcal{C}_0 (Proposition 4.3).

4.2.2 Properties of \mathcal{C}_0 and \mathcal{Q}_0

PROPOSITION 4.2. *Every $\varphi \in \mathcal{C}_0$ has zero mean with respect to $\mu_{s_\varphi, E, \mathbf{x}}$:*

$$\langle \varphi \rangle_{s_\varphi, E, \mathbf{x}} = 0, \text{ for all disorder } \mathbf{m}, \text{ for all } E > 0, \text{ and } \mathbf{x} \in \mathfrak{S}_{s_\varphi, E}.$$

Moreover, the following elements belong to \mathcal{Q}_0 :

- (a) $j_{0,1}^S, j_{0,1}^A$.
- (b) $\mathcal{L}^{\mathbf{m}}f, Sf$ and $\mathcal{A}^{\mathbf{m}}f$, for all $f \in \mathcal{Q}$.

Proof. The first statement is straightforward as a consequence of the definition. Besides, (a) is directly obtain from the following identities: for $x \in \mathbb{Z}$, and $k \geq 1$,

$$\omega_{x+1}^2 - \omega_x^2 = \nabla_{x,x+1}(\omega_x^2) \tag{15}$$

$$\omega_x \omega_{x+k} = \mathcal{S}_x \left(\frac{-\omega_x \omega_{x+1}}{\gamma} \right) + \sum_{\ell=1}^{k-1} \mathcal{S}_{x+\ell} \left(\frac{-\omega_x \omega_{x+\ell+1}}{\lambda} \right). \tag{16}$$

Then, if $f \in \mathcal{Q}$, it is easy to see that (15) and (16) are sufficient to prove (b). For instance,

$$\begin{aligned} \mathcal{L}^{\mathbf{m}}(\omega_x \omega_{x+1}) &= \frac{\omega_x \omega_{x+2}}{\sqrt{m_{x+1} m_{x+2}}} - \frac{\omega_{x+1} \omega_{x-1}}{\sqrt{m_x m_{x-1}}} + \frac{\omega_{x+1}^2 - \omega_x^2}{\sqrt{m_x m_{x+1}}} \\ &\quad - 4\gamma \omega_x \omega_{x+1} + \lambda(\omega_{x+2} - \omega_{x+1})\omega_x + \lambda(\omega_{x-1} - \omega_x)\omega_{x+1}. \end{aligned}$$

□

Conversely, let us now consider a function $\varphi \in \mathcal{C}_0$. From the previous subsection together with Proposition 4.2, we can write the cylinder function φ as $\varphi = (-\mathcal{S}_{\Lambda_\varphi})(-\mathcal{S}_{\Lambda_\varphi})^{-1}\varphi$ for some mean-zero function $(-\mathcal{S}_{\Lambda_\varphi})^{-1}\varphi$, measurable with respect to the variables $\{m_x, \omega_x; x \in \Lambda_\varphi\}$. The reversibility of the measure $\mu_{\ell, \beta, \mathbf{x}}$ implies that the following decomposition holds in $\mathbf{L}^2(\mu_{\ell, \beta, \mathbf{x}})$:

$$\varphi = \sum_{x \in \Lambda_\varphi} \nabla_x(\mathbf{F}_x) + \nabla_{x, x+1}(\mathbf{G}_x),$$

with

$$\begin{cases} \mathbf{F}_x = \frac{\gamma}{2} \nabla_x \left[\left(-\mathcal{S}_{\Lambda_\varphi} \right)^{-1} \varphi \right] \\ \mathbf{G}_x = \frac{\lambda}{2} \nabla_{x, x+1} \left[\left(-\mathcal{S}_{\Lambda_\varphi} \right)^{-1} \varphi \right]. \end{cases}$$

The following proposition is a direct consequence of these comments.

PROPOSITION 4.3 (Integration by parts formula). *Let φ be a cylinder function in \mathcal{C}_0 . There exists a family of cylinder functions $\{\mathbf{F}_x^\varphi, \mathbf{G}_x^\varphi; x \in \Lambda_\varphi\}$ measurable with respect to $\mathcal{F}_{\Lambda_\varphi}$ such that*

$$\langle \varphi, g \rangle_{\ell, \beta, \mathbf{x}} = \sum_{x \in \Lambda_\varphi} \langle \mathbf{F}_x^\varphi, \nabla_x g \rangle_{\ell, \beta, \mathbf{x}} + \langle \mathbf{G}_x^\varphi, \nabla_{x, x+1} g \rangle_{\ell, \beta, \mathbf{x}} \quad (17)$$

for all rectangles Λ_ℓ that contain Λ_φ , for all $\beta > 0$, and for all functions $g \in \mathbf{L}^2(\mu_{\ell, \beta, \mathbf{x}})$. For all $y \in \mathbb{Z}$,

$$\begin{aligned} \tau_y \mathbf{F}_x^\varphi &= \mathbf{F}_{x+y}^{\tau_y \varphi} \\ \tau_y \mathbf{G}_x^\varphi &= \mathbf{G}_{x+y}^{\tau_y \varphi}. \end{aligned}$$

Moreover,

$$\langle \varphi, g \rangle_{\ell, \beta, \mathbf{x}}^2 \leq \mathbf{C}(\varphi, \beta, \mathbf{x}) \langle g, \left(-\mathcal{S}_{\Lambda_\varphi} \right) g \rangle_{\ell, \beta, \mathbf{x}} \quad (18)$$

where $\mathbf{C}(\varphi, \beta, \mathbf{x})$ is equal to

$$\mathbf{C}(\varphi, \beta, \mathbf{x}) := 2 \left\langle \varphi, \left(-\mathcal{S}_{\Lambda_\varphi} \right)^{-1} \varphi \right\rangle_{\ell, \beta, \mathbf{x}}.$$

By reintegration of the desintegrated measure $\mu_{\ell, \beta, \mathbf{x}}$ the same result (17) may be restated with microcanonical measures $\mu_{\ell, \beta, \mathbf{x}}$ replaced by μ_β and \mathbb{P}_β^* and (18) becomes:

$$\langle \varphi, g \rangle_\beta^2 \leq \mathbf{C}_1(\varphi) \langle g, \left(-\mathcal{S}_{\Lambda_\varphi} \right) g \rangle_\beta \quad (19)$$

$$\mathbb{E} \langle \varphi, g \rangle_\beta^2 \leq \mathbf{C}_2(\varphi) \mathbb{E} \langle g, \left(-\mathcal{S}_{\Lambda_\varphi} \right) g \rangle_\beta \quad (20)$$

for some constants $\mathbf{C}_1, \mathbf{C}_2$ which can be written in terms of variances and do not depend on β .

Proof. Inequalities (18) and (19) follow from Cauchy-Schwarz inequality applied to (17). Let us notice that the last inequality (20) uses the translation invariance of the measure \mathbb{P}_β^* . \square

PROPOSITION 4.4 (Variance of quadratic functions). *If $\varphi \in \mathcal{Q}_0$, then*

$$\langle \varphi, (-S_{\Lambda_\varphi})^{-1} \varphi \rangle_\beta = \sup_{g \in \mathcal{Q}} \left\{ 2 \langle \varphi, g \rangle_\beta - \mathcal{D}_{s_\varphi}(\mu_\beta; g) \right\}.$$

This result can be restated with μ_β replaced with \mathbb{P}_β^ as*

$$\mathbb{E} \langle \varphi, (-S_{\Lambda_\varphi})^{-1} \varphi \rangle_\beta = \sup_{g \in \mathcal{Q}} \left\{ 2 \mathbb{E} \langle \varphi, g \rangle_\beta - \mathbb{E} [\mathcal{D}_{s_\varphi}(\mu_\beta; g)] \right\}.$$

Proof. This follows from the decomposition of every function in $L^2(\mu_\beta)$ over the Hermite polynomials basis: see Proposition A.3 in Section A. \square

4.3 Limiting variance and semi-norm

We now assume $\beta = 1$. All statements are valid for any $\beta > 0$, and the same argument can be easily written. In the following, we deliberately keep the notation $\chi(1)$, even if the latter could be replaced with its exact value. We are going to obtain a variational formula for the variance

$$(2\ell)^{-1} \mathbb{E} \left\langle (-S_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right\rangle_1$$

where $\varphi \in \mathcal{Q}_0$ and $\ell_\varphi = \ell - s_\varphi - 1$. We first introduce a semi-norm on \mathcal{Q}_0 . For any cylinder function φ in \mathcal{Q}_0 , let us define

$$\|\varphi\|_1^2 := \sup_{g \in \mathcal{Q}} \left\{ 2 \ll \varphi, g \gg_{1,\star} + \frac{2}{\lambda} \frac{\ll \varphi \gg_{1,\star\star}^2}{\chi(1)} - \frac{\lambda}{2} \mathbb{E}_1^* \left[(\nabla_{0,1} \Gamma_g)^2 \right] - \frac{\gamma}{2} \mathbb{E}_1^* \left[(\nabla_0 \Gamma_g)^2 \right] \right\} \quad (21)$$

$$= \sup_{\substack{g \in \mathcal{Q} \\ a \in \mathbb{R}}} \left\{ 2 \ll \varphi, g \gg_{1,\star} + 2a \ll \varphi \gg_{1,\star\star} - \mathbb{E} [\mathcal{D}_0(\mu_1; a\omega_0^2 + \Gamma_g)] \right\}. \quad (22)$$

As we previously noticed, this formula can be restated as

$$\|\varphi\|_1^2 = \ll \varphi, (-S)^{-1} \varphi \gg_{1,\star} + \frac{2}{\lambda \chi(1)} \ll \varphi \gg_{1,\star\star}^2. \quad (23)$$

Since φ belongs to \mathcal{Q}_0 , the results of Subsection 4.2 are valid, namely: the first term in the right-hand side of (23) is well-defined (Proposition 4.3), and the supremum in the variational formula (21) can be restricted over functions in \mathcal{Q} (Proposition 4.4). We are now in position to state the main result of this subsection.

THEOREM 4.5. *Consider a quadratic cylinder function $\varphi \in \mathcal{Q}_0$. Then*

$$\lim_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E} \left\langle (-S_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right\rangle_1 = \|\varphi\|_1^2.$$

Here, ℓ_φ stands for $\ell - s_\varphi - 1$ so that the support of $\tau_x \varphi$ is included in Λ_ℓ for every $x \in \Lambda_{\ell_\varphi}$.

This theorem is the key of the standard non-gradient Varadhan's method. As usual, the proof is done in two steps that we separate as two different lemmas for the sake of clarity. First, we bound the variance of a cylinder function $\varphi \in \mathcal{Q}_0$, with respect to the canonical measure μ_1 , by the semi-norm $\|\varphi\|_1^2$ (Lemma 4.6). In the second step, a lower bound for the variance can be easily deduced from the variational formula which expresses the variance as a supremum (Lemma 4.7).

LEMMA 4.6. *Under the assumptions of Theorem 4.5,*

$$\limsup_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E} \left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right\rangle_1 \leq \|\varphi\|_1^2.$$

In this step, one need to know the weak limits of some particular sequences in \mathcal{Q}_0 . In the typical approach, these weak limits are viewed as *germs of closed forms*, but for the harmonic chain, this way of thinking is not necessary.

Proof. We follow the proof given in [14], Lemma 4.3 and we assume first that $\varphi = \nabla_0(F) + \nabla_{0,1}(G)$, for two quadratic cylinder functions F, G . Then, the general case follows by linearity. We write the variational formula

$$\begin{aligned} (2\ell)^{-1} \mathbb{E} \left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right\rangle_1 &= \sup_{h \in \mathcal{C}} \left\{ 2\mathbb{E} \left\langle \varphi, \frac{1}{2\ell} \sum_{|x| \leq \ell_\varphi} \tau_x h \right\rangle_1 - \frac{1}{2\ell} \mathbb{E} [\mathcal{D}_\ell(\mu_1; h)] \right\} \\ &= \sup_{h \in \mathcal{C}} \left\{ 2\mathbb{E} \left\langle F\nabla_0 \left(\frac{1}{2\ell} \sum_{|x| \leq \ell_\varphi} \tau_x h \right) + G\nabla_{0,1} \left(\frac{1}{2\ell} \sum_{|x| \leq \ell_\varphi} \tau_x h \right) \right\rangle_1 - \frac{1}{2\ell} \mathbb{E} [\mathcal{D}_\ell(\mu_1; h)] \right\}. \end{aligned}$$

Since φ is quadratic, we can restrict the supremum in the class of quadratic functions h that are localized in Λ_ℓ (the proof of that statement is detailed in Proposition A.3). It turns out that we can also restrict the supremum to functions h such that $\mathbb{E}[\mathcal{D}_\ell(\mu_1; h)] \leq C\ell$. This follows from the fact that the first term can be bounded as follows (according Proposition 4.3 in addition to the convexity of the Dirichlet form):

$$\mathbb{E} \left\langle \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x h \right\rangle_1 \leq C_\varphi^{1/2} \left(\mathbb{E} [\mathcal{D}_\ell(\mu_1; h)] \right)^{1/2}.$$

Recall that C_φ is a constant that depends on φ . Next, we want to replace the sums over Λ_{ℓ_φ} with the same sums over Λ_ℓ (recall that $\ell_\varphi = \ell - s_\varphi - 1 \leq \ell$). For that purpose, we denote

$$\zeta_0^\ell(h) = \nabla_0 \left(\frac{1}{2\ell} \sum_{|x| \leq \ell} \tau_x h \right), \quad \zeta_1^\ell(h) = \nabla_{0,1} \left(\frac{1}{2\ell} \sum_{|x| \leq \ell} \tau_x h \right). \quad (24)$$

First of all, from Cauchy-Schwarz inequality, we have

$$\mathbb{E} \left\langle \frac{\gamma}{2} [\zeta_0^\ell(h)]^2 + \frac{\lambda}{2} [\zeta_1^\ell(h)]^2 \right\rangle_1 \leq \frac{1}{2\ell} \mathbb{E} [\mathcal{D}_\ell(\mu_1; h)].$$

Then, we also can write as before

$$\left| \mathbb{E} \left\langle \varphi, \frac{1}{2\ell} \sum_{\ell_\varphi \leq x \leq \ell} \tau_x h \right\rangle_1 \right| \leq \frac{1}{2\ell} C_\varphi^{1/2} \left(\mathbb{E} [\mathcal{D}_\ell(\mu_1; h)] \right)^{1/2}.$$

These last two inequalities give the upper bound

$$(2\ell)^{-1}\mathbb{E}\left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x|\leq\ell_\varphi} \tau_x\varphi, \sum_{|x|\leq\ell_\varphi} \tau_x\varphi \right\rangle_1 \\ \leq \sup_h \left\{ 2\mathbb{E}\left\langle \mathbb{F}\zeta_0^\ell(h) + \mathbb{G}\zeta_1^\ell(h) \right\rangle_1 - \mathbb{E}\left\langle \frac{\gamma}{2}[\zeta_0^\ell(h)]^2 + \frac{\lambda}{2}[\zeta_1^\ell(h)]^2 \right\rangle_1 \right\} + \frac{\mathbb{C}}{\sqrt{\ell}}.$$

Let us choose a sequence $\{h_\ell\}$ satisfying $\mathbb{E}[\mathcal{D}_\ell(\mu_1; h_\ell)] \leq \mathbb{C}\ell$. Then, the sequence $\{\zeta_0^\ell(h_\ell), \zeta_1^\ell(h_\ell)\}$ is uniformly bounded in $\mathbf{L}^2(\mathbb{P}_1^*)$, and this implies the existence of a weakly convergent subsequence. We denote by (ζ_0, ζ_1) a weak limit and assume that the sequence $\{\zeta_0^\ell(h_\ell), \zeta_1^\ell(h_\ell)\}$ weakly converges to (ζ_0, ζ_1) . The conclusion is now based on the weak version of closed forms result that we prove in Appendix B, Theorem B.1: the pair (ζ_0, ζ_1) can be written in $\mathbf{L}^2(\mathbb{P}_1^*)$ as

$$(\nabla_0\Gamma_g, a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}\Gamma_g),$$

with $g \in \mathcal{Q}$ and $a \in \mathbb{R}$. We have obtained that

$$(2\ell)^{-1}\mathbb{E}\left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x|\leq\ell_\varphi} \tau_x\varphi, \sum_{|x|\leq\ell_\varphi} \tau_x\varphi \right\rangle_1 \leq \sup_{\zeta_0, \zeta_1} \left\{ 2\mathbb{E}\left\langle \mathbb{F}\zeta_0 + \mathbb{G}\zeta_1 \right\rangle_1 - \frac{\gamma}{2}\mathbb{E}_1^*[\zeta_0^2] - \frac{\lambda}{2}\mathbb{E}_1^*[\zeta_1^2] \right\} \\ = \sup_{\substack{g \in \mathcal{Q} \\ a \in \mathbb{R}}} \left\{ 2\mathbb{E}\left\langle \mathbb{F}\nabla_0\Gamma_g + \mathbb{G}(a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}\Gamma_g) \right\rangle_1 - \frac{\gamma}{2}\mathbb{E}_1^*[(\nabla_0\Gamma_g)^2] - \frac{\lambda}{2}\mathbb{E}_1^*[(a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}\Gamma_g)^2] \right\}$$

The inequality above is a consequence of the following fact: the \mathbf{L}^2 -norm may only decrease along weakly convergent subsequences. The result follows, after recalling (22). \square

We now turn to the upper bound.

LEMMA 4.7. *Under the assumptions of Theorem 4.5,*

$$\limsup_{\ell \rightarrow \infty} (2\ell)^{-1}\mathbb{E}\left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x|\leq\ell_\varphi} \tau_x\varphi, \sum_{|x|\leq\ell_\varphi} \tau_x\varphi \right\rangle_1 \geq \|\varphi\|_1^2.$$

Proof. We define, for $f \in \mathcal{Q}$,

$$\mathbb{J}_\ell := \sum_{y, y+1 \in \Lambda_\ell} \tau_y \mathbb{J}_{0,1}^S, \quad \mathbb{H}_\ell^f = \sum_{|y|\leq\ell-s_f-1} \mathcal{S}(\tau_y f).$$

The following limits hold:

$$\lim_{\ell \rightarrow \infty} (2\ell)^{-1}\mathbb{E}\left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x|\leq\ell_\varphi} \tau_x\varphi, \mathbb{J}_\ell \right\rangle_1 = \ll \varphi \gg_{1, \star\star}, \tag{25} \\ \lim_{\ell \rightarrow \infty} (2\ell)^{-1}\mathbb{E}\left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x|\leq\ell_\varphi} \tau_x\varphi, \mathbb{H}_\ell^f \right\rangle_1 = \ll \varphi, f \gg_{1, \star}, \\ \lim_{\ell \rightarrow \infty} (2\ell)^{-1}\mathbb{E}\left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} (a\mathbb{J}_\ell + \mathbb{H}_\ell^f), a\mathbb{J}_\ell + \mathbb{H}_\ell^f \right\rangle_1 = \\ \frac{\lambda}{2}\mathbb{E}_1^*[(a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}\Gamma_g)^2] + \frac{\gamma}{2}\mathbb{E}_1^*[(\nabla_0\Gamma_g)^2].$$

We only prove (25), the other relations can be obtained in a similar way. As previously, we assume for the sake of simplicity that $\varphi = \nabla_0(F) + \nabla_{0,1}(G)$. We recall the elementary identity

$$\mathcal{S}_{\Lambda_\ell} \left(\sum_{x \in \Lambda_\ell} x \omega_x^2 \right) = J_\ell(\omega). \quad (26)$$

Therefore,

$$\begin{aligned} (2\ell)^{-1} \mathbb{E} \left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, J_\ell \right\rangle_1 &= -(2\ell)^{-1} \sum_{y, y+1 \in \Lambda_\ell} \sum_{|x| \leq \ell_\varphi} y \mathbb{E}_1^* [\varphi \omega_{y-x}^2] \\ &= -(2\ell)^{-1} \sum_{y, y+1 \in \Lambda_\ell} \sum_{|x| \leq \ell_\varphi} y \mathbb{E}_1^* [G \nabla_{0,1}(\omega_{y-x}^2)] \\ &= -(2\ell)^{-1} \sum_{|x| \leq \ell_\varphi} x \mathbb{E}_1^* [G \nabla_{0,1}(\omega_0^2)] + (x+1) \mathbb{E}_1^* [G \nabla_{0,1}(\omega_1^2)] \\ &= (2\ell)^{-1} (2\ell_\varphi + 1) \mathbb{E}_1^* [G, \omega_0^2 - \omega_1^2] \xrightarrow{\ell \rightarrow \infty} \ll \varphi \gg_{1, \star\star}. \end{aligned}$$

The last limit comes from Proposition 2.1 and the fact that $\ell_\varphi = \ell - s_\varphi - 1$. Then, we obtain from the variational formula written with $h = (-\mathcal{S}_{\Lambda_\ell})^{-1}(aJ_\ell + H_\ell^f)$:

$$\begin{aligned} \liminf_{\ell \rightarrow \infty} (2\ell)^{-1} \mathbb{E} \left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, \sum_{|x| \leq \ell_\varphi} \tau_x \varphi \right\rangle_1 \\ \geq \liminf_{\ell \rightarrow \infty} (2\ell)^{-1} \left\{ 2 \mathbb{E} \left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} \sum_{|x| \leq \ell_\varphi} \tau_x \varphi, aJ_\ell + H_\ell^f \right\rangle_1 - \mathbb{E} \left\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} (aJ_\ell + H_\ell^f), aJ_\ell + H_\ell^f \right\rangle_1 \right\} \\ = 2 \ll \varphi, f \gg_{1, \star} + 2a \ll \varphi \gg_{1, \star\star} - \mathbb{E} [\mathcal{D}_0(\mu_1; a\omega_0^2 + \Gamma_f)]. \end{aligned}$$

The result follows after taking the supremum on $f \in \mathcal{Q}$, and recalling (22). \square

4.4 Proof of Theorem 3.3

In this paragraph, we start the proof of Theorem 3.3 by using the central limit theorem variances given in Theorem 4.5. First, we show how to relate (11) to such variances. Recall that we have assumed $\beta = 1$, but the same argument remains in force for any $\beta > 0$.

PROPOSITION 4.8. *Let $\psi \in \mathcal{C}_0$, with $s_\psi \leq N$. Then*

$$\left\langle \sup_{0 \leq t \leq T} \left[\int_0^t \psi(s) ds \right]^2 \right\rangle_1 \leq \frac{24T}{N^2} \langle \psi, (-\mathcal{S}_N)^{-1} \psi \rangle_1. \quad (27)$$

The previous result is proved for example in [11, Section 2, Lemma 2.4]. We are going to use this bound for functions of type $\sum_x G(x/N) \tau_x \varphi$, where φ belongs to \mathcal{Q}_0 . The main result of this subsection is the following.

THEOREM 4.9. *Let $\varphi \in \mathcal{Q}_0$, and G a smooth function on \mathbb{T} . Then,*

$$\limsup_{N \rightarrow \infty} \mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left[\sqrt{N} \int_0^t \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi(\mathbf{m}, s) ds \right]^2 \right\rangle_1 \leq CT \|\varphi\|_1^2 \int_{\mathbb{T}} G^2(u) du. \quad (28)$$

Proof. From Proposition 4.8, the left-hand side of (28) is bounded by

$$24T \mathbb{E} \left\langle \sqrt{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi(\mathbf{m}), (-N^2 \mathcal{S}_N)^{-1} \left(\sqrt{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi(\mathbf{m}) \right) \right\rangle_1,$$

that can be written with the variational formula as

$$24T \sup_{f \in \mathcal{C}} \left\{ \sqrt{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \mathbb{E}[\langle f \tau_x \varphi \rangle_1] - N^2 \mathbb{E}[\mathcal{D}_N(\mu_1; f)] \right\}.$$

Since $\varphi \in \mathcal{Q}_0$, from Proposition A.3 we can restrict the supremum over $f \in \mathcal{Q}$. Proposition 4.3 gives

$$\langle f \tau_x \varphi \rangle_1 \leq C_\varphi \left\langle \tau_{-x} f, (-\mathcal{S}_{\Lambda_\varphi})(\tau_{-x} f) \right\rangle_1^{1/2}$$

and by Cauchy-Schwarz inequality,

$$\sqrt{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \langle f \tau_x \varphi \rangle_1 \leq \left(\frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right)^2 \right)^{1/2} N C_\varphi \langle f, (-\mathcal{S}_N) f \rangle_1^{1/2}.$$

The supremum on f can be explicitly computed, and gives the final bound

$$\mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left[\sqrt{N} \int_0^t \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi(\mathbf{m}, s) ds \right]^2 \right\rangle_1 \leq C'_\varphi T \left(\frac{1}{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right)^2 \right). \quad (29)$$

We are now going to show that the constant on the right-hand side is proportional to $\|\varphi\|_1^2$. For that purpose, we average on microscopic boxes: for $k \ll N$, we denote

$$\bar{\varphi}_k = \sum_{y \in \Lambda_k} \tau_y \varphi,$$

and we want to substitute

$$\sqrt{N} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \varphi$$

with

$$\frac{\sqrt{N}}{2k+1} \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \bar{\varphi}_k.$$

The error term that appears is estimated by

$$\mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left[\sqrt{N} \int_0^t \sum_{\substack{x, y \in \mathbb{T}_N \\ |x-y| \leq k}} \frac{1}{2k+1} \left(G\left(\frac{x}{N}\right) - G\left(\frac{y}{N}\right) \right) \tau_x \varphi(\mathbf{m}, s) ds \right]^2 \right\rangle_1.$$

From (29), the expression above is bounded by Ck/N , and then vanishes as $N \rightarrow \infty$. We are reduced to estimate

$$\mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left[\frac{\sqrt{N}}{2k+1} \int_0^t \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x \bar{\varphi}_k(\mathbf{m}, s) ds \right]^2 \right\rangle_1.$$

By the same argument, this is bounded by

$$\begin{aligned} & \frac{\text{CT}}{2k+1} \sum_{x \in \mathbb{T}_N} \sup_{f \in \mathcal{Q}} \left\{ \sqrt{N} \mathbb{G} \left(\frac{x}{N} \right) \mathbb{E} [\langle f \tau_x \bar{\varphi}_k \rangle_1] - \frac{N^2}{2k+1} \mathbb{E} [\langle \tau_{-x} f, (-\mathcal{S}_{\Lambda_k}) \tau_{-x} f \rangle_1] \right\} \\ & \leq \frac{\text{CT}}{2k+1} \sum_{x \in \mathbb{T}_N} \sup_{f \in \mathcal{Q}} \left\{ C(\varphi) \sqrt{N} \mathbb{G} \left(\frac{x}{N} \right) \mathbb{E} [\langle \tau_{-x} f, (-\mathcal{S}_{\Lambda_k}) (\tau_{-x} f) \rangle_1]^{1/2} - \frac{N^2}{2k+1} \mathbb{E} [\langle \tau_{-x} f, (-\mathcal{S}_{\Lambda_k}) \tau_{-x} f \rangle_1] \right\} \end{aligned}$$

The supremum on f can be explicitly computed, and gives the final bound

$$C(\varphi) \mathbb{T} \left(\frac{1}{N} \sum_{x \in \mathbb{T}_N} \mathbb{G}^2 \left(\frac{x}{N} \right) \right) \frac{1}{2k+1} \mathbb{E} \langle \bar{\varphi}_k, (-\mathcal{S}_{\Lambda_k})^{-1} \bar{\varphi}_k \rangle_1.$$

Taking the limit as $N \rightarrow \infty$ and then $k \rightarrow \infty$, we obtain (28) from the central limit theorem for variances at equilibrium (Theorem 4.5). \square

We apply Theorem 4.9 to $\mathcal{J}_{t,m,f}^{1,N}(\mathbb{H})$, and we get

$$\limsup_{N \rightarrow \infty} \mathbb{E} \left\langle \sup_{0 \leq t \leq T} \left(\mathcal{J}_{t,m,f}^{1,N}(\mathbb{H}) \right)^2 \right\rangle_1 \leq \text{CT} \| \|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f \| \|_1^2 \int_{\mathbb{T}} H'(u)^2 du.$$

To conclude the proof of Theorem 3.3, we show in Section 5 that there exists a sequence of local functions $\{f_k\} \in \mathcal{Q}$ such that

$$\| \|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f_k \| \|_1 \xrightarrow{k \rightarrow \infty} 0,$$

and Section 6 is devoted to prove the second statement of Theorem 3.3.

5 Hilbert space and projections

We now focus on the semi-norm $\| \cdot \|_1$ that was introduced in the previous section by (21). We can easily define from $\| \cdot \|_1$ a semi-inner product on \mathcal{C}_0 through polarization. Denote by \mathcal{N} the kernel of the semi-norm $\| \cdot \|_1$ on \mathcal{C}_0 . Then, the completion of $\mathcal{Q}_0|_{\mathcal{N}}$ denoted by \mathcal{H}_1 is a Hilbert space. Let us explain how the well-known Varadhan's approach is modified. Usually, the Hilbert space on which orthogonal projections are performed is the completion of $\mathcal{C}_0|_{\mathcal{N}}$, in other words it involves all local functions. Then, the standard procedure aims at proving that each element of that Hilbert space can be approximated by a sequence of functions in the range of the generator plus an additional term which is proportional to the current. The crucial steps for obtaining this decomposition consist in: first, controlling the antisymmetric part of the generator by the symmetric one for every cylinder function, and second, proving a strong result on germs of closed forms (see Appendix B). These two key points are not satisfied in our model, but they can be proved when restricted to quadratic functions. It turns out that these weak versions are sufficient, since we are looking for a fluctuation-dissipation approximation that involves quadratic functions only.

In Subsection 5.1, we show that \mathcal{H}_1 is the completion of $\mathcal{S}\mathcal{Q}|_{\mathcal{N}} + \{j_{0,1}^S\}$. In other words, all elements of \mathcal{H}_1 can be approximated by $aj_{0,1}^S + \mathcal{S}g$ for some $a \in \mathbb{R}$ and $g \in \mathcal{Q}$. This is not irrelevant since the symmetric part of the generator preserves the degree of polynomial functions. Moreover, the sum of the two subspaces $\{j_{0,1}^S\}$ and $\overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}}$ is orthogonal, and we denote it by

$$\overline{\mathcal{S}\mathcal{Q}}|_{\mathcal{N}} \oplus^\perp \{j_{0,1}^S\}.$$

Nevertheless, this decomposition is not satisfactory, because we want the fluctuating term to be on the form $\mathcal{L}^m(f_k)$, and not $\mathcal{S}(f_k)$. In order to make this replacement, we need to prove the weak sector condition, that gives a control of $\|A^m g\|_1$ by $\|\mathcal{S}g\|_1$, when g is a quadratic function. The argument is explained in Subsection 5.2 and 5.3, and the weak sector condition is proved in Appendix C. The only trouble is that this new decomposition is not orthogonal any more, so that we can not express the diffusion coefficient as a variational formula, like (35). This problem is solved in Section 6.

5.1 Decomposition according to the symmetric part

We begin this subsection with a table of calculus, very useful in the sequel.

PROPOSITION 5.1. *For all $g \in \mathcal{Q}_0, h \in \mathcal{Q}$,*

$$\begin{aligned} \langle\langle h, \mathcal{S}g \rangle\rangle_1 &= -\langle\langle h, g \rangle\rangle_{1,\star} \\ \langle\langle h, j_{0,1}^S \rangle\rangle_1 &= -\langle\langle h \rangle\rangle_{1,\star\star} \\ \langle\langle j_{0,1}^S, \mathcal{S}h \rangle\rangle_1 &= 0 \\ \|j_{0,1}^S\|_1^2 &= \lambda^2 \chi(1). \end{aligned}$$

Proof. The first two identities are direct consequences of Theorem 4.5 and of Equality (26). The last two ones follow directly. \square

COROLLARY 5.2. *For all $a \in \mathbb{R}$ and $g \in \mathcal{Q}$,*

$$\|aj_{0,1}^S + \mathcal{S}g\|_1^2 = a^2 \lambda \chi(1) + \frac{\lambda}{2} \mathbb{E}_1^\star [(\nabla_{0,1} \Gamma_g)^2] + \frac{\gamma}{2} \mathbb{E}_1^\star [(\nabla_0 \Gamma_g)^2].$$

In particular, the variational formula for $\|h\|_1$, $h \in \mathcal{Q}_0$, writes

$$\|h\|_1^2 = \frac{1}{\lambda \chi(1)} \langle\langle h, j_{0,1}^S \rangle\rangle_1^2 + \sup_{g \in \mathcal{Q}} \{-2 \langle\langle h, \mathcal{S}g \rangle\rangle_1 - \|\mathcal{S}g\|_1^2\}. \quad (30)$$

PROPOSITION 5.3. *We denote by \mathcal{SQ} the space $\{\mathcal{S}h ; h \in \mathcal{Q}\}$. Then,*

$$\mathcal{H}_1 = \overline{\mathcal{SQ}}|_{\mathcal{N}} \oplus^\perp \{j_{0,1}^S\}$$

Proof. We divide the proof into two steps.

(a) The space is well generated – The inclusion $\overline{\mathcal{SQ}}|_{\mathcal{N}} + \{j_{0,1}^S\} \subset \mathcal{H}_1$ is obvious. Moreover, from the variational formula (30) we know that: if $h \in \mathcal{H}_1$ satisfies $\langle\langle h, j_{0,1}^S \rangle\rangle_1 = 0$ and $\langle\langle h, \mathcal{S}g \rangle\rangle_1 = 0$ for all $g \in \mathcal{Q}$, then $\|h\|_1 = 0$.

(b) The sum is orthogonal – This follows directly from the previous proposition and from the fact that: $\langle\langle j_{0,1}^S, \mathcal{S}h \rangle\rangle_1 = 0$ for all $h \in \mathcal{Q}$. \square

5.2 Replacement of \mathcal{S} with \mathcal{L}

In this subsection, we prove identities which mix the antisymmetric and the symmetric part of the generator, which will be used to get the *weak sector condition* (Proposition 5.7).

LEMMA 5.4. *For all $g, h \in \mathcal{Q}$,*

$$\ll \mathcal{S}g, \mathcal{A}^m h \gg_1 = - \ll \mathcal{A}^m g, \mathcal{S}h \gg_1.$$

Proof. This easily follows from the first identity of Proposition 5.1 and from the invariance by translations of the measure \mathbb{P}_1^* :

$$\begin{aligned} \ll \mathcal{S}g, \mathcal{A}^m h \gg_1 &= - \ll g, \mathcal{A}^m h \gg_{1,*} = - \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\tau_x g, \mathcal{A}^m h] = \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\mathcal{A}^m(\tau_x g), h] \\ &= \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\tau_x(\mathcal{A}^m g), h] = \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\mathcal{A}^m g, \tau_{-x} h] = \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\mathcal{A}^m g, \tau_x h] = - \ll \mathcal{A}^m g, \mathcal{S}h \gg_1. \end{aligned}$$

□

LEMMA 5.5. *For all $g \in \mathcal{Q}$,*

$$\ll \mathcal{S}g, j_{0,1}^A \gg_1 = - \ll \mathcal{A}^m g, j_{0,1}^S \gg_1.$$

Proof. By the first identity of Proposition 5.1,

$$\begin{aligned} \ll \mathcal{S}g, j_{0,1}^A \gg_1 &= - \ll g, j_{0,1}^A \gg_{1,*} = - \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[\tau_x g, j_{0,1}^A] = - \sum_{x \in \mathbb{Z}} \mathbb{E}_1^*[g, j_{x,x+1}^A] \\ &= - \sum_{x \in \mathbb{Z}} x \mathbb{E}_1^*[g, j_{x-1,x}^A - j_{x,x+1}^A] = - \sum_{x \in \mathbb{Z}} x \mathbb{E}_1^*[g, \mathcal{A}^m(\omega_x^2)] \\ &= \sum_{x \in \mathbb{Z}} x \mathbb{E}_1^*[\mathcal{A}^m g, \omega_x^2] = - \ll \mathcal{A}^m g, j_{0,1}^S \gg_1. \end{aligned}$$

□

Then, these two lemmas together with the second identity of Proposition 5.1 imply the following:

COROLLARY 5.6. *For all $a \in \mathbb{R}$, $g \in \mathcal{Q}$,*

$$\ll a j_{0,1}^S + \mathcal{S}g, a j_{0,1}^A + \mathcal{A}^m g \gg_1 = 0.$$

We are now in position to state the main result of this subsection.

PROPOSITION 5.7 (Weak sector condition). *(i) There exists two constants $C_0 := C(\gamma, \lambda)$ and $C_1 := C(\gamma, \lambda)$ such that the following inequalities hold for all $f, g \in \mathcal{Q}$:*

$$|\ll \mathcal{A}^m g, \mathcal{S}f \gg_1| \leq C_0 \| \mathcal{S}f \|_1 \| \mathcal{S}g \|_1. \quad (31)$$

$$|\ll \mathcal{A}^m g, \mathcal{S}f \gg_1| \leq C_1 \| \mathcal{S}g \|_1 + \frac{1}{2} \| \mathcal{S}f \|_1. \quad (32)$$

(ii) There exists a positive constant C such that, for all $g \in \mathcal{Q}$,

$$\| \mathcal{A}^m g \|_1 \leq C \| \mathcal{S}g \|_1.$$

Proof. The proof is technical because made of explicit computations for quadratic functions. For that reason, we report it to Appendix C. □

5.3 Decomposition of the Hilbert space

We now deduce from the previous two subsections the expected decomposition of \mathcal{H}_1 .

PROPOSITION 5.8. *We denote by $\mathcal{L}^m \mathcal{Q}$ the space $\{\mathcal{L}^m g ; g \in \mathcal{Q}\}$. Then,*

$$\mathcal{H}_1 = \overline{\mathcal{L}^m \mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}^S\}.$$

Proof. We first prove that \mathcal{H}_1 can be written as the sum of the two subspaces. Then, we show that the sum is direct.

(a) The space is well generated – The inclusion $\overline{\mathcal{L}^m \mathcal{Q}}|_{\mathcal{N}} + \{j_{0,1}^S\} \subset \mathcal{H}_1$ follows from Proposition 4.2. To prove the converse inclusion, let $h \in \mathcal{H}_1$ so that $\ll h, j_{0,1}^S \gg_1 = 0$ and $\ll h, \mathcal{L}^m g \gg_1 = 0$ for all $g \in \mathcal{Q}$. From Corollary 5.3, h can be written as

$$h = \lim_{k \rightarrow \infty} \mathcal{S}g_k$$

for some sequence $\{g_k\} \in \mathcal{Q}$. More precisely, since $\ll \mathcal{S}g_k, \mathcal{A}^m g_k \gg_1 = 0$ by Lemma 5.4,

$$\| \|h\| \|_1^2 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{S}g_k \gg_1 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{L}^m g_k \gg_1.$$

Moreover, we also have by assumption that $\ll h, \mathcal{S}g_k \gg_1 = 0$ for all k , and from Proposition 5.7,

$$\sup_{k \in \mathbb{N}} \| \| \mathcal{L}^m g_k \| \|_1 \leq (C + 1) \sup_{k \in \mathbb{N}} \| \| \mathcal{S}g_k \| \|_1 =: C_h$$

is finite. Therefore,

$$\| \|h\| \|_1^2 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{L}^m g_k \gg_1 = \lim_{k \rightarrow \infty} \ll \mathcal{S}g_k - h, \mathcal{L}^m g_k \gg_1 \leq \lim_{k \rightarrow \infty} C_h \| \| \mathcal{S}g_k - h \| \|_1 = 0.$$

(b) The sum is direct – Let $\{g_k\} \in \mathcal{Q}$ be a sequence such that, for some $a \in \mathbb{R}$,

$$\lim_{k \rightarrow \infty} \mathcal{L}^m g_k = a j_{0,1}^S \quad \text{in } \mathcal{H}_1,$$

By a similar argument,

$$\limsup_{k \rightarrow \infty} \ll \mathcal{S}g_k, \mathcal{S}g_k \gg_1 = \limsup_{k \rightarrow \infty} \ll \mathcal{L}^m g_k, \mathcal{S}g_k \gg_1 = \limsup_{k \rightarrow \infty} \ll \mathcal{L}^m g_k - a j_{0,1}^S, \mathcal{S}g_k \gg_1 = 0,$$

where the last equality comes from the fact that $\ll j_{0,1}^S, \mathcal{S}g_k \gg_1 = 0$ for all k . On the other hand, by Proposition 5.7, $\| \| \mathcal{L}^m g_k \| \|_1 \leq (C + 1) \| \| \mathcal{S}g_k \| \|_1^2$. Then, $a = 0$. This concludes the proof. \square

Recall that $j_{0,1}^S(\mathbf{m}, \omega) = \lambda(\omega_1^2 - \omega_0^2)$. We have obtained the following result.

THEOREM 5.9. *For every $g \in \mathcal{Q}_0$, there exists a unique constant $a \in \mathbb{R}$, such that*

$$g + a(\omega_1^2 - \omega_0^2) \in \overline{\mathcal{L}^m \mathcal{Q}} \quad \text{in } \mathcal{H}_1. \quad (33)$$

In particular, this theorem states that there exists a unique number D , and a sequence of cylinder functions $\{f_k\} \in \mathcal{Q}$ such that

$$\| \| j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f_k \| \|_1 \xrightarrow{k \rightarrow \infty} 0. \quad (34)$$

Let us notice that this convergence also holds with the same constant D and the same sequence f_k if we replace the semi-norm $\| \| \cdot \| \|_1$ with $\| \| \cdot \| \|_\beta$ for any $\beta > 0$ (as a consequence of a standard change of variables argument). This concludes the first statement of Theorem 3.3. We prove the second statement (12) in Proposition 6.5 in Section 6.

6 On the diffusion coefficient

The main goal of this section is to express the diffusion coefficient in several variational formulas. We also prove the second statement of Theorem 3.3. First, recall Definition 2.2, which can be written as

$$D = \lambda^2 + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \mathbb{E} \left[\mathcal{D}_0(\mu_1; \Gamma_f) + 2 \langle j_{0,1}^A - \mathcal{A}^m f, \Gamma_g \rangle_1 - \mathcal{D}_0(\mu_1; \Gamma_g) \right]. \quad (35)$$

From Theorem 5.9, there exists a unique number D such that

$$j_{0,1} - D(\omega_1^2 - \omega_0^2) \in \overline{\mathcal{L}^m \mathcal{Q}} \quad \text{in } \mathcal{H}_1.$$

We are going to obtain a more explicit formula for that D , and relate it to (35), by following the argument in [14]. In Subsection ??, we first rewrite the decomposition of the Hilbert space given in Proposition 5.8, by replacing $j_{0,1}^S$ with $j_{0,1}$. This new statement is based on Corollary 5.6, which gives an orthogonality relation. The second step is to find an other orthogonal decomposition (see (36) below), which will enable us to prove the variational formula (35) for D .

Hereafter, we denote by $\mathcal{L}^{m,*} := \mathcal{S} - \mathcal{A}^m$ the adjoint of the generator in $L^2(\mu_1)$, and

$$j_{0,1}^* := j_{0,1}^S - j_{0,1}^A.$$

LEMMA 6.1. *The following decompositions hold*

$$\mathcal{H}_1 = \overline{\mathcal{L}^m \mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}\} = \overline{\mathcal{L}^{m,*} \mathcal{Q}}|_{\mathcal{N}} \oplus \{j_{0,1}^*\}.$$

Proof. We only sketch the proof of the first decomposition, since it is done in [14]. Let us recall from Proposition 5.8 that $\overline{\mathcal{L}^m \mathcal{Q}}$ has a complementary subspace in \mathcal{H}_1 which is one-dimensional. Therefore, it is sufficient to prove that \mathcal{H}_1 is generated by $\overline{\mathcal{L}^m \mathcal{Q}}$ and the total current. Let $h \in \mathcal{H}_1$ such that $\langle h, j_{0,1} \rangle_1 = 0$ and $\langle h, \mathcal{L}^m g \rangle_1 = 0$ for all $g \in \mathcal{Q}$. By Corollary 5.3, h can be written as

$$h = \lim_{k \rightarrow \infty} \mathcal{S} g_k + a j_{0,1}^S$$

for some sequence $\{g_k\} \in \mathcal{Q}$, and $a \in \mathbb{R}$, and from Corollary 5.6,

$$\|h\|_1^2 = \lim_{k \rightarrow \infty} \langle a j_{0,1}^S + \mathcal{S} g_k, a j_{0,1} + \mathcal{L}^m g_k \rangle_1.$$

Moreover, from Proposition 5.7,

$$\sup_{k \in \mathbb{N}} \|a j_{0,1} + \mathcal{L}^m g_k\|_1^2 \leq 2a^2 \|j_{0,1}\|_1^2 + 2(C+1) \sup_{k \in \mathbb{N}} \|\mathcal{S} g_k\|_1^2 =: C_h$$

is finite. Therefore,

$$\begin{aligned} \|h\|_1^2 &= \lim_{k \rightarrow \infty} \langle a j_{0,1}^S + \mathcal{S} g_k - h, a j_{0,1} + \mathcal{L}^m g_k \rangle_1 \\ &\leq \limsup_{k \rightarrow \infty} C_h \|a j_{0,1}^S + \mathcal{S} g_k - h\|_1^2 = 0. \end{aligned}$$

The same arguments apply to the second decomposition. □

We define bounded linear operators $T, T^* : \mathcal{H}_1 \rightarrow \mathcal{H}_1$ as

$$\begin{aligned} T(a j_{0,1} + \mathcal{L}^m f) &:= a j_{0,1}^S + \mathcal{S} f, \\ T^*(a j_{0,1}^* + \mathcal{L}^{m,*} f) &:= a j_{0,1}^S + \mathcal{S} f. \end{aligned}$$

From the following identity

$$\|aj_{0,1} + \mathcal{L}^m f\|_1^2 = \|aj_{0,1}^* + \mathcal{L}^{m,*} f\|_1^2 = \|aj_{0,1}^S + \mathcal{S}f\|_1^2 + \|aj_{0,1}^A + \mathcal{A}^m f\|_1^2,$$

we can easily see that T^* is the adjoint operator of T and we also have the relations

$$\begin{aligned} \langle\langle Tj_{0,1}^S, j_{0,1}^* \rangle\rangle_1 &= \langle\langle T^* j_{0,1}^S, j_{0,1} \rangle\rangle_1 = \lambda\chi(1) \\ \langle\langle Tj_{0,1}^S, \mathcal{L}^{m,*} f \rangle\rangle_1 &= \langle\langle T^* j_{0,1}^S, \mathcal{L}^m f \rangle\rangle_1 = 0, \text{ for all } f \in \mathcal{Q}. \end{aligned}$$

In particular,

$$\mathcal{H}_1 = \overline{\mathcal{L}^{m,*} \mathcal{Q}}|_{\mathcal{N}} \oplus^\perp \{Tj_{0,1}^S\} \quad (36)$$

and there exists a unique number Q such that

$$j_{0,1}^* - QTj_{0,1}^S \in \overline{\mathcal{L}^{m,*} \mathcal{Q}} \quad \text{in } \mathcal{H}_1.$$

We are going to show that $D = \lambda Q$.

LEMMA 6.2.

$$Q = \frac{\lambda\chi(1)}{\|Tj_{0,1}^S\|_1^2} = \frac{1}{\lambda\chi(1)} \inf_{f \in \mathcal{Q}} \|j_{0,1}^* - \mathcal{L}^{m,*} f\|_1^2. \quad (37)$$

Proof. The first identity follows from the fact that

$$\langle\langle Tj_{0,1}^S, j_{0,1}^* - QTj_{0,1}^S \rangle\rangle_1 = \lambda\chi(1) - Q\|Tj_{0,1}^S\|_1^2 = 0.$$

The second identity is obtained from the following statement

$$\inf_{f \in \mathcal{Q}} \|j_{0,1}^* - QTj_{0,1}^S - \mathcal{L}^{m,*} f\|_1 = 0. \quad (38)$$

□

After an easy computation, we can also prove that $\langle\langle Tg, g \rangle\rangle_1 = \langle\langle Tg, Tg \rangle\rangle_1$ for all $g \in \mathcal{H}_1$. Since $j_{0,1}^S - Tj_{0,1}^S$ is orthogonal to $Tj_{0,1}^S$, we have:

$$j_{0,1}^S - Tj_{0,1}^S \in \overline{\mathcal{L}^{m,*} \mathcal{Q}}.$$

By the fact we obtain the variational formula for $\|Tj_{0,1}^S\|_1$:

PROPOSITION 6.3.

$$\|Tj_{0,1}^S\|_1^2 = \inf_{f \in \mathcal{Q}} \|j_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2. \quad (39)$$

Proof. With a similar argument (in the proof of the previous proposition), we have

$$\inf_{f \in \mathcal{Q}} \|j_{0,1}^S - Tj_{0,1}^S - \mathcal{L}^{m,*} f\|_1 = 0,$$

and

$$\inf_{f \in \mathcal{Q}} \|j_{0,1}^S - Tj_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2 = \inf_{f \in \mathcal{Q}} \|j_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2 - \inf_{f \in \mathcal{Q}} \|Tj_{0,1}^S\|_1^2,$$

which concludes the proof. □

THEOREM 6.4.

$$D = \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \|j_{0,1}^* - \mathcal{L}^{m,*} f\|_1^2 = \frac{\chi(1)}{4 \inf_{f \in \mathcal{Q}} \|j_{0,1}^S - \mathcal{L}^{m,*} f\|_1^2}. \quad (40)$$

Proof. By the definition, $j_{0,1} - D j_{0,1}^S / \lambda \in \overline{\mathcal{L}^{m\mathcal{Q}}}$ and therefore

$$\ll j_{0,1} - j_{0,1}^S \frac{D}{\lambda}, T^* j_{0,1}^S \gg_1 = \lambda \chi(1) - \frac{D}{\lambda} \|T j_{0,1}^S\|_1^2 = 0. \quad (41)$$

So, $D = \lambda Q$, and the variational formula for D can be deduced from the one for Q . \square

REMARK 6.1. We can rewrite the variational formula for D as:

$$\begin{aligned} D &= \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \left\{ \|j_{0,1}^S\|_1^2 + \|Sf\|_1^2 + \|j_{0,1}^A - \mathcal{A}^m f\|_1^2 \right\} \\ &= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \left\{ \|Sf\|_1^2 + \|j_{0,1}^A - \mathcal{A}^m f\|_1^2 \right\} \end{aligned} \quad (42)$$

$$\begin{aligned} &= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \left\{ \|Sf\|_1^2 - 2 \ll j_{0,1}^A - \mathcal{A}^m f, Sg \gg_1 - \|Sg\|_1^2 \right\} \\ &= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \left\{ \ll f, -Sf \gg_{1,*} + 2 \ll j_{0,1}^A - \mathcal{A}^m f, g \gg_{1,*} - \ll g, -Sg \gg_{1,*} \right\} \end{aligned} \quad (43)$$

$$= \lambda + \frac{1}{\chi(1)} \inf_{f \in \mathcal{Q}} \sup_{g \in \mathcal{Q}} \mathbb{E} \left[\mathcal{D}_0(\mu_1; \Gamma_f) + 2 \langle j_{0,1}^A - \mathcal{A}^m f, \Gamma_g \rangle_1 - \mathcal{D}_0(\mu_1; \Gamma_g) \right]. \quad (44)$$

We use the fact that in (42), we can restrict the infimum on functions f satisfying $\ll j_{0,1}^A - \mathcal{A}^m f, j_{0,1}^S \gg_1 = 0$. Let us notice that (43) and (44) recover the variational formula (35).

We are now in position to prove the remaining statement of Theorem 3.3:

PROPOSITION 6.5. For any sequence $\{f_k\} \in \mathcal{Q}$ such that

$$\lim_{k \rightarrow \infty} \|j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f_k\|_1 = 0$$

we have

$$\lim_{k \rightarrow \infty} \mathbb{E} \left\langle \lambda \left(\nabla_{0,1}(\omega_0^2 - \Gamma_{f_k}) \right)^2 + \gamma \left(\nabla_0(\Gamma_{f_k}) \right)^2 \right\rangle_1 = 2D\chi(1).$$

Proof. By assumption,

$$\lim_{k \rightarrow \infty} \|T(j_{0,1} - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^m f_k)\|_1 = 0$$

and therefore

$$\lim_{k \rightarrow \infty} \|j_{0,1}^S - Sf_k\|_1^2 = D^2 \|T(\omega_1^2 - \omega_0^2)\|_1^2.$$

Then, the result follows from

$$D = \lambda Q = \frac{\chi(1)}{\|T(\omega_1^2 - \omega_0^2)\|_1^2}$$

and

$$\|j_{0,1}^S - Sf_k\|_1^2 = \frac{\lambda}{2} \mathbb{E} \left\langle \left(\omega_0^2 - \omega_1^2 - \nabla_{0,1}(\Gamma_{f_k}) \right)^2 \right\rangle_1 + \frac{\gamma}{2} \mathbb{E} \left\langle \left(\nabla_0(\Gamma_{f_k}) \right)^2 \right\rangle_1. \quad (45)$$

\square

7 Green-Kubo formulas

In this section, we study the convergence of the Green-Kubo formula given in (7), and then, we investigate its behavior when the intensity of the exchange noise vanishes.

7.1 Convergence of Green-Kubo formula

Linear response theory predicts that the diffusion coefficient is given by the homogenized *Green-Kubo formula*. Let us define

$$\bar{\kappa}(z) = \lambda + \frac{1}{2} \int_0^{+\infty} dt e^{-zt} \ll j_{0,1}^A(\mathbf{m}, t), j_{0,1}^A(\mathbf{m}, 0) \gg_{1,\star}$$

where $\ll \cdot \gg_{1,\star}$ is the inner product defined by (5). The Laplace transform is defined and is smooth on $(0, +\infty)$, and can be rewritten:

$$\bar{\kappa}(z) = \lambda + \frac{1}{2} \ll j_{0,1}^A, (z - \mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{1,\star}. \quad (46)$$

THEOREM 7.1. *The following limit*

$$\bar{D} := \lim_{\substack{z \rightarrow 0 \\ z > 0}} \bar{\kappa}(z) \quad (47)$$

exists, and is finite.

Proof. We investigate the existence of the limit

$$\lim_{\substack{z \rightarrow 0 \\ z > 0}} \ll j_{0,1}^A, (z - \mathcal{L}^{\mathbf{m}})^{-1} j_{0,1}^A \gg_{1,\star}. \quad (48)$$

The Hilbert space generated by the set of local functions and the inner product $\ll \cdot, \cdot \gg_{1,\star}$ is denoted by \mathbf{L}_\star^2 . We define $h_z := h_z(\mathbf{m}, \omega; 1)$ as the solution of the resolvent equation in \mathbf{L}_\star^2

$$(z - \mathcal{L}^{\mathbf{m}})h_z = j_{0,1}^A. \quad (49)$$

Then we have to prove that

$$L(z) = \frac{1}{2} \ll h_z, j_{0,1}^A \gg_{1,\star}$$

converges as z goes to 0, and that the limit is finite and non-negative. Then, from (46) it will follow that $\bar{D} \geq \lambda > 0$ and \bar{D} is positive. We denote by $\|\cdot\|_1$ the semi-norm corresponding to the symmetric part of the generator due to the flip noise

$$\|f\|_1^2 = \ll f, (-\gamma \mathcal{S}^{\text{flip}})f \gg_{1,\star}$$

and \mathcal{H}_1 is the Hilbert space obtained by the completion of \mathbf{L}_\star^2 w.r.t. that semi-norm. We multiply (49) by h_z and integrate with respect to $\ll \cdot \gg_{1,\star}$ and we get:

$$z \ll h_z, h_z \gg_{1,\star} + \|h_z\|_1^2 + \ll h_z, (-\lambda \mathcal{S}^{\text{exch}})h_z \gg_{1,\star} = \ll h_z, j_{0,1}^A \gg_{1,\star}.$$

Let us notice that $(-\gamma \mathcal{S}^{\text{flip}})(j_{0,1}^A) = 2\gamma j_{0,1}^A$. As a consequence, the Cauchy-Schwarz inequality for the scalar product $\ll \cdot, (-\gamma \mathcal{S}^{\text{flip}}) \cdot \gg_{1,\star}$ on the right-hand side gives

$$\|h_z\|_1^2 \leq C$$

for some positive constant C . Since $\{h_z\}_z$ is a bounded sequence in \mathcal{H}_1 , we can extract a weakly converging subsequence in \mathcal{H}_1 . We continue to denote this subsequence by $\{h_z\}_z$ and we denote by h_0 the limit.

Now we are going to show that the convergence is stronger (see (iv) in Lemma 7.2 below) and that the limit is independent of the subsequence. Since the generator \mathcal{L}^m conserves the degree of homogeneous polynomial functions, we know that the solution of the resolvent equation is expected to be on the form

$$h_z(\omega) = \sum_{x,y \in \mathbb{Z}^2} \varphi_z(x,y) \omega_x \omega_y,$$

where $\varphi_z : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is a square-summable symmetric function. Let $h_z = h_z^- + h_z^\neq$ be the decomposition of h_z according to the two subspaces \mathcal{Q}^- and \mathcal{Q}^\neq , where \mathcal{Q}^- is generated by $\{\omega_x^2, x \in \mathbb{Z}\}$ and \mathcal{Q}^\neq is generated by $\{\omega_x \omega_y, x \neq y\}$. The main point in the following argument is that all gradient terms vanish in \mathbf{L}_*^2 .

First of all, one can easily see how the spaces \mathcal{Q}^- and \mathcal{Q}^\neq are mapped by the generators:

$$\begin{aligned} \mathcal{A}^m : \mathcal{Q}^- &\rightarrow \mathcal{Q}^\neq & \mathcal{A}^m : \mathcal{Q}^\neq &\rightarrow \mathcal{Q} \\ \mathcal{S}^{\text{flip}} : \mathcal{Q}^- &\rightarrow \{0\} & \mathcal{S}^{\text{flip}} : \mathcal{Q}^\neq &\rightarrow \mathcal{Q}^\neq \\ \mathcal{S}^{\text{exch}} : \mathcal{Q}^- &\rightarrow \mathcal{Q}^- & \mathcal{S}^{\text{exch}} : \mathcal{Q}^\neq &\rightarrow \mathcal{Q}^\neq \end{aligned}$$

Moreover, if $f \in \mathcal{Q}^-$, then $\mathcal{A}^m(f)$ is a gradient in \mathcal{Q}^\neq , and $\mathcal{S}^{\text{exch}}(f)$ is a gradient in \mathcal{Q}^- . With all these considerations, (49) rewrites in \mathbf{L}_*^2 as

$$\begin{cases} z h_z^- - \lambda \mathcal{S}^{\text{exch}}(h_z^-) = 0 \\ z h_z^\neq - \lambda \mathcal{S}^{\text{exch}}(h_z^\neq) - \gamma \mathcal{S}^{\text{flip}}(h_z^\neq) - \mathcal{A}^m(h_z^\neq) = j_{0,1}^A. \end{cases}$$

The first equation means that $h_z^- = 0$ in \mathbf{L}_*^2 and therefore the solution h_z of the resolvent equation is an element of \mathcal{Q}^\neq . As a consequence, we can write $(-\gamma \mathcal{S}^{\text{flip}})(h_z) = 2\gamma h_z$, and this remark is one of the key points in the following argument.

LEMMA 7.2. *All the properties below are satisfied:*

- (i) $\lim_{z \rightarrow 0} z \ll h_z, h_z \gg_{1,*} = 0$
- (ii) $\{h_z\}$ weakly converges as z goes to 0 towards h_0 in \mathbf{L}_*^2
- (iii) $\ll j_{0,1}^A, h_0 \gg_{1,*} = \ll h_0, (-\mathcal{S})h_0 \gg_{1,*}$
- (iv) $\ll (h_z - h_0), (-\mathcal{S})(h_z - h_0) \gg_{1,*}$ vanishes as z goes to 0
- (v) the weak limit of $\{h_z\}$ does not depend on the subsequence.

We briefly prove the five points: (i) and (ii) come from the fact that $\ll h_z, h_z \gg_{1,*}$ equals $2\gamma \|h_z\|_1^2$. To get (iii), we multiply (49) by $h_{z'}$ and integrate:

$$z \ll h_{z'}, h_z \gg_{1,*} + \ll h_{z'}, (-\mathcal{S})h_z \gg_{1,*} + \ll h_{z'}, (-\mathcal{A}^m)h_z \gg_{1,*} = \ll h_{z'}, j_{0,1}^A \gg_{1,*}. \quad (50)$$

We first take the limit as $z' \rightarrow 0$ and then as $z \rightarrow 0$, and we use (i) and (ii) to obtain (iii). In the same way, multiplying (49) by h_z gives

$$z \ll h_z, h_z \gg_{1,*} + \ll h_z, (-\mathcal{S})h_z \gg_{1,*} = \ll h_z, j_{0,1}^A \gg_{1,*}.$$

The first term of the left-hand side vanishes as z goes to 0, and the right-hand side converges to $\ll h_0, (-\mathcal{S})h_0 \gg_{1,\star}$. This implies (iv), that is

$$\ll (h_z - h_0), (-\mathcal{S})(h_z - h_0) \gg_{1,\star} \xrightarrow{z \rightarrow 0} 0.$$

The uniqueness of the limit follows by a standard argument with the same idea of (50). We have proved the first part: the limit (48) exists. To obtain its finiteness, we are going to give an upper bound, using the following variational formula:

$$\ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star} = \sup_f \left\{ 2 \ll f, j_{0,1}^A \gg_{1,\star} - \|f\|_{1,z}^2 - \|\mathcal{A}^m f\|_{-1,z}^2 \right\},$$

where the supremum is carried over local functions and the two norms $\|\cdot\|_{\pm 1,z}$ are defined by

$$\|f\|_{\pm 1,z}^2 = \ll f, (z - \mathcal{S})^{\pm 1} f \gg_{1,\star}.$$

For the upper bound, we neglect the term coming from the antisymmetric part $\mathcal{A}^m f$, that gives

$$\ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star} \leq \ll j_{0,1}^A, (z - \mathcal{S})^{-1} j_{0,1}^A \gg_{1,\star}.$$

In the right-hand side we can also neglect the part coming from the exchange symmetric part $\mathcal{S}^{\text{exch}}$, and remind that $\mathcal{S}^{\text{flip}}(j_{0,1}^A) = -2j_{0,1}^A$. This gives an explicit finite upper bound. Then, we have from Lemma 7.2, Property (iii) that the limit

$$\lim_{z \rightarrow \infty} \ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star} = \ll j_{0,1}^A, h_0 \gg_{1,\star} = \ll h_0, (-\mathcal{S})h_0 \gg_{1,\star} \geq 0,$$

and the positiveness is proved. \square

7.2 Equivalence of the Definitions

In this subsection we rigorously prove the equality between the variational formula for the diffusion coefficient and the Green-Kubo formula (see the end of Subsection 2.4).

THEOREM 7.3. *For every $\lambda > 0$ and $\gamma > 0$,*

$$\bar{D} := \lambda + \frac{1}{2} \lim_{\substack{z \rightarrow 0 \\ z > 0}} \ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1} j_{0,1}^A \gg_{1,\star},$$

coincides with the diffusion coefficient D defined in Subsection 6.

Proof. From Subsection 6, we know that the diffusion coefficient can be written different ways. For instance, one can easily check that

$$D = \frac{2}{\|T(\omega_1^2 - \omega_0^2)\|_1^2}.$$

By definition of D , there exists a sequence $\{f_\varepsilon\}_{\varepsilon > 0}$ of functions in \mathcal{Q} such that

$$g_\varepsilon := j_{0,1}^\star - D(\omega_1^2 - \omega_0^2) - \mathcal{L}^{m,\star}(f_\varepsilon)$$

satisfies $\|g_\varepsilon\|_1 \rightarrow 0$ as ε goes to 0. By substitution in the equality above, we get

$$D^{-1} = \frac{1}{2D^2} \ll j_{0,1}^\star - \mathcal{L}^{m,\star} f_\varepsilon - g_\varepsilon, T^\star(j_{0,1}^\star - \mathcal{L}^{m,\star} f_\varepsilon - g_\varepsilon) \gg_1$$

recalling that $\ll Tg, Tg \gg_1 = \ll g, T^*g \gg_1$ for all $g \in \mathcal{H}_1$. Therefore,

$$\begin{aligned} D &= \frac{1}{2} \ll j_{0,1}^* - \mathcal{L}^{m,*}f_\varepsilon - g_\varepsilon, j_{0,1}^S - Sf_\varepsilon - g_\varepsilon \gg_1 \\ &= \frac{1}{2} \ll j_{0,1}^* - \mathcal{L}^{m,*}f_\varepsilon, j_{0,1}^S - Sf_\varepsilon \gg_1 + R_\varepsilon \end{aligned}$$

where R_ε is bounded by $C\|g_\varepsilon\|_1^2$, and then vanishes as ε goes to 0. Finally, from Proposition 5.1, we can write

$$D = \lambda + \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}.$$

The problem is now reduced to prove that

$$\lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*} = \lim_{\substack{z \rightarrow 0 \\ z > 0}} \ll j_{0,1}^A, (z - \mathcal{L}^m)^{-1}j_{0,1}^A \gg_{1,*}. \quad (51)$$

For every $z > 0$ and $\varepsilon > 0$, we have by definition above and (49),

$$j_{0,1}^A = zh_z - \mathcal{L}^m h_z \quad (52)$$

$$j_{0,1}^* = D(\omega_1^2 - \omega_0^2) + g_\varepsilon + \mathcal{L}^{m,*}f_\varepsilon. \quad (53)$$

First, multiply (53) by f_ε and integrate with respect to $\ll \cdot \gg_{1,*}$, keeping in mind that all gradients give no contribution. We get

$$- \ll j_{0,1}^A, f_\varepsilon \gg_{1,*} = \ll f_\varepsilon, g_\varepsilon \gg_{1,*} - \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}$$

and using (52),

$$\ll \mathcal{L}^m h_z, f_\varepsilon \gg_{1,*} - z \ll h_z, f_\varepsilon \gg_{1,*} = \ll f_\varepsilon, g_\varepsilon \gg_{1,*} - \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}.$$

First, let z go to 0, and observe that $z \ll h_z, f_\varepsilon \gg_{1,*}$ vanishes, from the Cauchy-Schwarz inequality together with Statement (i) of Lemma 7.2. The limit of $\ll \mathcal{L}^m h_z, f_\varepsilon \gg_{1,*}$ exists from the weak convergence of $\{h_z\}_z$. Then, take the limit as ε goes to 0, and observe that

$$\begin{aligned} \ll f_\varepsilon, g_\varepsilon \gg_{1,*} &= \ll f_\varepsilon, (-S)(-S)^{-1}g_\varepsilon \gg_{1,*} \leq \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*}^{1/2} \ll g_\varepsilon, (-S)^{-1}f_\varepsilon \gg_{1,*}^{1/2} \\ &\leq C\|g_\varepsilon\|_1 \xrightarrow{\varepsilon \rightarrow 0} 0. \end{aligned}$$

The first equality is justified by the fact that g_ε belongs to \mathcal{Q}_0 , and the last inequality comes from the definition of the semi-norm $\|\cdot\|_1$ given in (21). As a consequence, we have obtained

$$\lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*} = \lim_{\varepsilon \rightarrow 0} \lim_{z \rightarrow 0} \ll -\mathcal{L}^m h_z, f_\varepsilon \gg_{1,*}.$$

In the same way, multiply (53) by h_z and integrate with respect to $\ll \cdot \gg_{\beta,*}$ so that

$$- \ll j_{0,1}^A, h_z \gg_{1,*} = \ll g_\varepsilon, h_z \gg_{1,*} + \ll \mathcal{L}^{m,*}f_\varepsilon, h_z \gg_{1,*}.$$

If we send first z to 0, then $\ll g_\varepsilon, h_z \gg_{1,*}$ converges to $\ll g_\varepsilon, h_0 \gg_{1,*}$ from the weak convergence of $\{h_z\}_z$. With the same argument as before, we write

$$\ll g_\varepsilon, h_0 \gg_{1,*} \leq C\|g_\varepsilon\|_1 \xrightarrow{\varepsilon \rightarrow 0} 0.$$

Therefore,

$$\begin{aligned} \lim_{z \rightarrow 0} \ll j_{0,1}^A, h_z \gg_{1,*} &= \lim_{\varepsilon \rightarrow 0} \lim_{z \rightarrow 0} \ll -\mathcal{L}^{m,*}f_\varepsilon, h_z \gg_{1,*} \\ &= \lim_{\varepsilon \rightarrow 0} \ll f_\varepsilon, (-S)f_\varepsilon \gg_{1,*} \end{aligned}$$

and the claim is proved. \square

7.3 Vanishing exchange noise

With the same ideas of the previous subsection, it can be easily shown that the homogenized Green-Kubo formula also converges if the strength λ of the exchange noise vanishes. The aim of this paragraph is to study the limit of (47) as λ goes to 0. First, we turn (46) into a new definition that highlights the dependence on $\lambda > 0$. For that purpose we introduce new notations: we denote $\mathcal{S}_0 = \gamma \mathcal{S}^{\text{flip}}$, $\mathcal{S}_\lambda = \mathcal{S}_0 + \lambda \mathcal{S}^{\text{exch}}$, and then

$$\begin{cases} \mathcal{L}_0^{\mathbf{m}} = \mathcal{A}^{\mathbf{m}} + \mathcal{S}_0 \\ \mathcal{L}_\lambda^{\mathbf{m}} = \mathcal{A}^{\mathbf{m}} + \mathcal{S}_\lambda = \mathcal{L}_0^{\mathbf{m}} + \lambda \mathcal{S}^{\text{exch}} \end{cases} \quad \text{and} \quad J_0(\mathbf{m})(\omega) = \frac{\omega_0 \omega_1}{\sqrt{m_0 m_1}} = j_{0,1}^{\mathbf{A}}(\mathbf{m}, \omega).$$

Let us introduce the homogenized Green-Kubo formula for both noises:

$$\bar{\kappa}(\lambda, z) := \ll J_0(\mathbf{m}), (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} J_0(\mathbf{m}) \gg_{1, \star} \quad (54)$$

and the homogenized Green-Kubo formula for flip noise only:

$$\bar{\kappa}_0(z) := \ll J_0(\mathbf{m}), (z - \mathcal{L}_0^{\mathbf{m}})^{-1} J_0(\mathbf{m}) \gg_{1, \star}. \quad (55)$$

According to the previous paragraph, we already know that the Green-Kubo formulas (54) and (55) converge as z goes to 0. Then, the following diffusion coefficients are well defined, for all $\lambda > 0$,

$$\begin{cases} \bar{D}(\lambda) := \lambda + \lim_{z \rightarrow 0} \bar{\kappa}(\lambda, z), \\ \bar{D}_0 := \lim_{z \rightarrow 0} \bar{\kappa}_0(z). \end{cases}$$

The main result of this subsection is stated in the following theorem.

THEOREM 7.4. *The function $\lambda \mapsto \bar{D}(\lambda)$ is continuous at 0. More precisely,*

$$\lim_{\lambda \rightarrow 0} \bar{D}(\lambda) = \bar{D}_0.$$

Let us remark that the theorem above does not imply the existence of the hydrodynamics diffusion coefficient $D(0, \gamma)$. This question remains open.

Proof. The proof is divided into two steps. For the sake of readability, we erase the notation \mathbf{m} in $J_0(\mathbf{m})$, and keep in mind its dependence on the disorder.

Step 1 - Convergence of the diffusion coefficient. Let us denote by $h_{z,0}$ and $\bar{h}_{z,\lambda}$ the two solutions of the following resolvent equations in \mathbf{L}_\star^2 :

$$\begin{aligned} (z - \mathcal{L}_0^{\mathbf{m}})h_{z,0} &= J_0, \\ (z - \mathcal{L}_\lambda^{\mathbf{m}, \star})\bar{h}_{z,\lambda} &= J_0. \end{aligned}$$

We look at the following difference, for $\lambda, z > 0$ fixed,

$$\begin{aligned} & \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} J_0 \gg_{1, \star} - \ll J_0, (z - \mathcal{L}_0^{\mathbf{m}})^{-1} J_0 \gg_{1, \star} \right| \\ &= \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} J_0 \gg_{1, \star} - \ll J_0, h_{z,0} \gg_{1, \star} \right| \\ &= \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} \left[(z - \mathcal{L}_0^{\mathbf{m}})h_{z,0} - (z - \mathcal{L}_\lambda^{\mathbf{m}})h_{z,0} \right] \gg_{1, \star} \right| \\ &= \lambda \left| \ll J_0, (z - \mathcal{L}_\lambda^{\mathbf{m}})^{-1} \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1, \star} \right| \\ &= \lambda \left| \ll (z - \mathcal{L}_\lambda^{\mathbf{m}, \star})^{-1} J_0, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1, \star} \right|. \end{aligned}$$

To complete the proof, we are reduced to show that $\lambda \left| \ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,\star} \right|$ vanishes when we first let $z \rightarrow 0$ and then $\lambda \rightarrow 0$. For that purpose, we need more precise information on the two solutions $\bar{h}_{z,\lambda}$ and $h_{z,0}$. Since the generator $\mathcal{L}_\lambda^{\text{m}}$ (resp. \mathcal{L}_0^{m}) conserves the degree of homogeneous polynomial functions, we know that the solution of the resolvent equation $\bar{h}_{z,\lambda}$ (resp. $h_{z,0}$) has to be homogeneous polynomial of degree two, precisely:

$$\bar{h}_{z,\lambda}(\omega) = \sum_{x,y \in \mathbb{Z}} \varphi_{z,\lambda}(\mathbf{m}, x, y) \omega_x \omega_y,$$

where $\varphi_{z,\lambda}(\mathbf{m}, \cdot, \cdot) : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is a square integrable symmetric function. As before, we decompose every degree two function h as $h = h^\# + h^\neq$, where $h^\#$ belongs to $\mathcal{Q}^\#$ and h^\neq belongs to \mathcal{Q}^\neq . We have seen in the proof of Theorem 7.1 that the part belonging to $\mathcal{Q}^\#$ vanishes for the two solutions, in other words, $\bar{h}_{z,\lambda}$ and $h_{z,0}$ are elements of \mathcal{Q}^\neq . As a consequence,

$$\ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,\star} = \ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,\star}$$

which is bounded by

$$\sqrt{\ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,\star}} \sqrt{\ll h_{z,0}, (-\mathcal{S}^{\text{exch}})(h_{z,0}) \gg_{1,\star}}$$

according to the Cauchy-Schwarz inequality for the scalar product $\ll \cdot, (-\mathcal{S}^{\text{exch}}) \cdot \gg_{1,\star}$. We treat separately the two terms into the two lemmas below. We prove that the first term is bounded by $C/\sqrt{\lambda}$, and the second one is uniformly bounded for $\lambda, z > 0$. Here we state the two lemmas:

LEMMA 7.5. *There exists a constant $C > 0$ such that, for all $z, \lambda > 0$,*

$$\ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,\star} \leq \frac{C}{\lambda}.$$

LEMMA 7.6. *There exists a constant $C > 0$ such that, for all $z > 0$,*

$$\ll h_{z,0}, (-\mathcal{S}^{\text{exch}})(h_{z,0}) \gg_{1,\star} \leq C.$$

From these statements we deduce

$$\lambda \left| \ll \bar{h}_{z,\lambda}, \mathcal{S}^{\text{exch}}(h_{z,0}) \gg_{1,\star} \right| \leq C_0 \sqrt{\lambda}$$

where C_0 does not depend on $\lambda, z > 0$, and Theorem 7.4 follows.

Step 2 - Proofs of the two lemmas. We begin with the proof of Lemma 7.5. We recall the resolvent equation in \mathbf{L}_\star^2 :

$$z \bar{h}_{z,\lambda} - (\lambda \mathcal{S}^{\text{exch}} + \mathcal{S}_0 - \mathcal{A}^{\text{m}}) \bar{h}_{z,\lambda} = J_0. \quad (56)$$

We multiply (56) by $h_{z,\lambda}$ and integrate with respect to $\ll \cdot \gg_{1,\star}$, in order to get

$$z \ll \bar{h}_{z,\lambda}, \bar{h}_{z,\lambda} \gg_{1,\star} + \ll \bar{h}_{z,\lambda}, (-\mathcal{S}_0)(\bar{h}_{z,\lambda}) \gg_{1,\star} + \lambda \ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,\star} = \ll J_0, \bar{h}_{z,\lambda} \gg_{1,\star}.$$

The right-hand side rewrites as

$$(2\gamma)^{-1} \ll (-\mathcal{S}_0)(J_0), h_{z,0} \gg_{1,\star}.$$

Cauchy-Schwarz inequality for the scalar product $\ll \cdot, (-\mathcal{S}_0) \cdot \gg_{1,\star}$ on the right-hand side gives

$$z \ll \bar{h}_{z,\lambda}, \bar{h}_{z,\lambda} \gg_{1,\star} \leq C$$

with $C := (2\gamma)^{-1/2} \ll J_0, J_0 \gg_{1,\star}^{1/2}$ and then

$$\lambda \ll \bar{h}_{z,\lambda}, (-\mathcal{S}^{\text{exch}})(\bar{h}_{z,\lambda}) \gg_{1,\star} \leq C.$$

We now turn to Lemma 7.6. We prove a general result, precisely: there exists a constant $C > 0$ such that, for all $g \in \mathcal{Q}^\neq$,

$$\ll g, (-\mathcal{S}^{\text{exch}})g \gg_{1,\star} \leq C \ll g, g \gg_{1,\star}. \quad (57)$$

This fact is proved through explicit computations. Let us write $g \in \mathcal{Q}^\neq$ in the form

$$g(\omega) = \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \phi_{x,k}(\mathbf{m}) \omega_x \omega_{x+k}.$$

A straightforward computation gives that

$$\begin{aligned} \ll g, (-\mathcal{S}^{\text{exch}})g \gg_{1,\star} &= \frac{1}{2} \mathbb{E}_1^* \left[(\nabla_{0,1} \Gamma_g)^2 \right] = \sum_{k \geq 2} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \phi_{z,k}(\tau_{-z} \mathbf{m}) - \phi_{z,k}(\tau_{1-z} \mathbf{m}) \right)^2 \right] \\ &\leq 4 \sum_{k \geq 2} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \phi_{z,k}(\mathbf{m}) \right)^2 \right]. \end{aligned}$$

In the last inequality, we use the fact that the measure \mathbb{P} on the disorder is translation invariant and that $(a - b)^2 \leq 2(a^2 + b^2)$ for all $a, b \in \mathbb{R}$. Besides, one can also check that

$$\ll g, g \gg_{1,\star} = \sum_{k \geq 1} \mathbb{E} \left[\sum_{x,z \in \mathbb{Z}} \phi_{z,k}(\tau_{-z} \mathbf{m}) \phi_{x,k}(\tau_{-x} \mathbf{m}) \right] = \sum_{k \geq 1} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \phi_{z,k}(\mathbf{m}) \right)^2 \right],$$

thanks to the translation invariance of \mathbb{P} . The bound (57) follows directly, with $C = 4$. To prove Lemma 7.6, it remains to show that $\ll h_{z,0}, h_{z,0} \gg_{1,\star}$ is uniformly bounded in z . We recall the resolvent equation in \mathbf{L}_\star^2 :

$$zh_{z,0} - (\mathcal{S}_0 + \mathcal{A}^{\mathbf{m}})h_{z,0} = J_0. \quad (58)$$

Notice that we can write $\mathcal{S}_0(h_{z,0}) = -2\gamma h_{z,0}$. We multiply (58) by $h_{z,0}$ and integrate with respect to $\ll \cdot \gg_{1,\star}$ in order to get

$$z \ll h_{z,0}, h_{z,0} \gg_{1,\star} + 2\gamma \ll h_{z,0}, h_{z,0} \gg_{1,\star} = \ll J_0, h_{z,0} \gg_{1,\star}.$$

As previously, Cauchy-Schwarz inequality for the scalar product $\ll \cdot, (-\mathcal{S}_0) \cdot \gg_{1,\star}$ on the right-hand side gives

$$\ll h_{z,0}, h_{z,0} \gg_{1,\star} \leq C,$$

with $C := (2\gamma)^{-1} \ll J_0, J_0 \gg_{1,\star}^{1/2}$. □

8 The anharmonic chain perturbed by a diffusive noise

In this last main section we say a few words about the anharmonic chain, meaning that the interaction between atoms are non linear, and given by a potential V . As in [14], we assume that the function $V : \mathbb{R} \rightarrow \mathbb{R}_+$ satisfies the following properties:

- (i) $V(\cdot)$ is a smooth symmetric function,
- (ii) there exist δ_- and δ_+ such that $0 < \delta_- \leq V''(\cdot) \leq \delta_+ < +\infty$,
- (iii) $\delta_-/\delta_+ > (3/4)^{1/16}$.

Using the same notations as in the introduction, the configuration $\{p_x, r_x\}$ now evolve according to

$$\begin{cases} \frac{dp_x}{dt} = V'(r_{x+1}) - V'(r_x), \\ \frac{dr_x}{dt} = \frac{p_x}{M_x} - \frac{p_{x-1}}{M_{x-1}}. \end{cases} \quad (59)$$

We define $\pi_x := p_x/\sqrt{M_x}$, and the dynamics on $\{\pi_x, r_x\}$ rewrites:

$$\begin{cases} \frac{d\pi_x}{dt} = \frac{1}{\sqrt{M_x}} [V'(r_{x+1}) - V'(r_x)], \\ \frac{dr_x}{dt} = \frac{\pi_x}{\sqrt{M_x}} - \frac{\pi_{x-1}}{\sqrt{M_{x-1}}}. \end{cases} \quad (60)$$

The total energy

$$\mathcal{E} := \sum_{x \in \mathbb{Z}} \left\{ \frac{\pi_x^2}{2} + V(r_x) \right\}$$

is conserved. The flip and exchange noises have poor ergodicity properties, and can be used for harmonic chains only. For the anharmonic case, we introduce a stronger stochastic perturbation. Now, the total generator of the dynamics writes $\mathcal{L}^m = \mathcal{A}^m + \gamma \mathcal{S}$, where

$$\mathcal{A}^m := \sum_x \frac{1}{\sqrt{m_x}} (X_x - Y_{x,x+1}), \quad \mathcal{S} := \frac{1}{2} \sum_x X_x^2 + Y_{x,x+1}^2, \quad (61)$$

where $Y_{x,y} = \pi_x \partial_{r_y} - V'(r_y) \partial_{\pi_x}$, and $X_x = Y_{x,x}$. For this anharmonic case, the two needed ingredients can be proved directly from [14]. First, notice that the symmetric part of the generator does not depend on the disorder and is exactly the same as in [14]: the proof of the spectral gap is done in Section 12 of this paper. The sector condition can also be proved by inspiring from [14]. After taking into account the random environment and its fluctuation, the same argument of Lemma 8.2, Section 8 can be applied. Indeed, it is mainly based on the fact that the antisymmetric part of the generator can be written in terms of the symmetric one.

9 Hydrodynamic limits

We briefly enlighten the failure in the derivation of the hydrodynamic limits. Let us assume that the initial law for the Markov process $\{\omega(t)\}_{t \geq 0}$ (still generated by $N^2 \mathcal{L}_N^m$), is not the equilibrium measure μ_β^N , but a *local equilibrium measure* (see (63) below). The main goal would be to prove that this property of local equilibrium propagates in time: in other words hydrodynamics limits hold, with an energy profile solution of the diffusion equation with constant coefficient D .

9.1 Statement of the Hydrodynamic Limits Conjecture

The distribution at time t of the Markov chain on \mathbb{T}_N with the generator $N^2 \mathcal{L}_N^m$ and the initial probability measure μ_N is denoted by $\mathbb{P}_{\mu_N, t}^m$. The measure induced by $\mathbb{P}_{\mu_N, t}^m$ on $\mathcal{D}([0, T], \Omega_N)$ is denoted by \mathbb{P}_N^m .

Recall that we denote by \mathcal{M}_1 the set of probability measures on \mathbb{T} , endowed with the weak topology and by $\mathcal{D}([0, T], \mathcal{M}_1)$ the Skorokhod space of trajectories in \mathcal{M}_1 . The measure induced by $\mathbb{P}_N^{\mathbf{m}}$ on $\mathcal{D}([0, T], \mathcal{M}_1)$ is denoted by $\mathcal{Q}_N^{\mathbf{m}} := \mathbb{P}_N^{\mathbf{m}} \circ (\pi^N)^{-1}$, where

$$\pi^N := \frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^2 \delta_{\frac{x}{N}}.$$

CONJECTURE 9.1. *Let $T > 0$ be a time-horizon. Let $\{\mu^N\}_N$ be a sequence of probability measures on Ω_N . Under suitable conditions on the initial law μ^N , for almost every realization of the random environment \mathbf{m} , the measure $\mathcal{Q}_N^{\mathbf{m}}$ weakly converges in $\mathcal{D}([0, T], \mathcal{M}_1)$ to the probability measure concentrated on the path $\{\mathbf{e}(t, u) du\}_{t \in [0, T]}$, where \mathbf{e} is the unique weak solution of the system*

$$\begin{cases} \frac{\partial \mathbf{e}}{\partial t}(t, u) = \mathbf{D} \frac{\partial^2 \mathbf{e}}{\partial u^2}(t, u), & t > 0, u \in \mathbb{T} \\ \mathbf{e}(0, u) = \mathbf{e}_0(u). \end{cases}$$

What we expect as for “suitable assumptions” on the initial law are the common ones in the literature of hydrodynamic limits, when dealing with non compact spaces. The first one is natural and related on the relative entropy:

ASSUMPTION 9.2. *We suppose that there exists a positive constant K_0 such that the relative entropy $H(\mu^N | \mu_\star^N)$ of μ^N with respect to a reference measure μ_\star^N is bounded by $K_0 N$:*

$$H(\mu^N | \mu_\star^N) \leq K_0 N. \quad (62)$$

For instance, if μ^N is defined as a Gibbs local equilibrium state:

$$\prod_{x \in \mathbb{T}_N} \sqrt{\frac{2\pi}{\beta_0(x/N)}} \exp\left(-\frac{\beta_0(x/N)}{2} \omega_x^2\right) d\omega_x \quad (63)$$

for some continuous function $\beta_0 : \mathbb{T} \rightarrow \mathbb{R}_+$, then (62) is satisfied. The second one is related to energy boundness, that has already been a major concern in [18]. More precisely,

ASSUMPTION 9.3. *We assume there exists a positive constant E_0 such that*

$$\limsup_{N \rightarrow \infty} \mu^N \left[\frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^4 \right] \leq E_0. \quad (64)$$

In the derivation of hydrodynamic limits with the usual *entropy method*, we need the following two estimates: first, there exists a positive constant C such that, for any $t > 0$

$$\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[\frac{1}{N} \sum_{x \in \mathbb{T}_N} \omega_x^2(t) \right] \leq C. \quad (65)$$

This can be easily established using (64) and the Cauchy-Schwarz inequality. The second control that we need is

$$\lim_{N \rightarrow \infty} \mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[\int_0^t \frac{1}{N^2} \sum_{x \in \mathbb{T}_N} \omega_x^4(s) ds \right] = 0. \quad (66)$$

If μ^N is a convex combination of Gibbs local equilibrium states, then the same argument of [18] shows that the law of the process remains a convex combination of Gaussian measures, and that (66) holds.

Contrary to the velocity-flip model, we do not need to assume a good control of every energy moment if we expect the usual entropy method to work. This technical need was only due to the relative entropy method.

With Assumptions 9.2 and 9.3 we could try to prove Theorem 9.1 by using the entropy method, which permits to consider more general initial profiles (for example, the profile β_0 can be assumed only bounded, not smooth). The usual technical points of this well-known procedure are the one and two-blocks estimates, as well as tightness. In this model, they are somehow easy to prove because the diffusion coefficient is constant, and there is no need to prove its regularity.

9.2 Replacement of the Current by a Gradient

In this subsection we recall the main steps of the usual entropy method, and explain which ones can be proved for our system. We fix the disorder $\mathbf{m} = \{m_x\}_{x \in \mathbb{T}_N}$ and $T > 0$. For $t \in [0, T]$, we denote by $\mathcal{Z}_{t,\mathbf{m}}^N$ the empirical energy field defined as

$$\mathcal{Z}_{t,\mathbf{m}}^N(H) = \frac{1}{N} \sum_{x \in \mathbb{T}_N} H\left(\frac{x}{N}\right) \omega_x^2(t),$$

where $H : \mathbb{T} \rightarrow \mathbb{R}$ is a smooth function. We rewrite $\mathcal{Z}_{t,\mathbf{m}}^N(H)$ as

$$\mathcal{Z}_{t,\mathbf{m}}^N(H) = \mathcal{Z}_{0,\mathbf{m}}^N(H) + \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) j_{x,x+1}(\mathbf{m}, \omega)(s) ds + \mathfrak{M}_{t,\mathbf{m}}^N(H),$$

where $\mathfrak{M}_{t,\mathbf{m}}^N(H)$ is a martingale. The strategy consists in replacing the current $j_{x,x+1}$ by the linear combination given in Theorem 5.9. For that purpose, for any $f \in \mathcal{Q}$ we rewrite

$$\mathcal{Z}_{t,\mathbf{m}}^N(H) = \mathcal{Z}_{0,\mathbf{m}}^N(H) + \int_0^t D\mathcal{Z}_{s,\mathbf{m}}^N(\Delta_N H) ds + \mathfrak{J}_{t,\mathbf{m},f}^{1,N}(H) + \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) + \mathfrak{M}_{t,\mathbf{m}}^N(H),$$

where

$$\begin{aligned} \mathfrak{J}_{t,\mathbf{m},f}^{1,N}(H) &= \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \left[j_{x,x+1}(\mathbf{m}, \omega)(s) - D(\omega_{x+1}^2 - \omega_x^2)(s) - \mathcal{L}_N^{\mathbf{m}}(\tau_x f)(\mathbf{m}, \omega)(s) \right] ds, \\ \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) &= \int_0^t \sum_{x \in \mathbb{T}_N} \nabla_N H\left(\frac{x}{N}\right) \mathcal{L}_N^{\mathbf{m}}(\tau_x f)(\mathbf{m}, \omega)(s) ds. \end{aligned}$$

Theorem 9.1 would follow from the three lemmas below.

LEMMA 9.4. *For every smooth function $H : \mathbb{T} \rightarrow \mathbb{R}$ and every $\delta > 0$,*

$$\lim_{N \rightarrow \infty} \mathbb{P}_N^{\mathbf{m}} \left[\sup_{[0,T]} \left| \mathfrak{M}_{t,\mathbf{m}}^N(H) \right| > \delta \right] = 0.$$

LEMMA 9.5. *For every $f \in \mathcal{Q}$ and every smooth function $H : \mathbb{T} \rightarrow \mathbb{R}$,*

$$\limsup_{N \rightarrow \infty} \mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[\left| \mathfrak{J}_{t,\mathbf{m},f}^{2,N}(H) \right| \right] = 0.$$

LEMMA 9.6. *There exists a sequence of functions $\{f_k\}_{k \in \mathbb{N}} \in \mathcal{Q}$ such that, for every smooth function $H : \mathbb{T} \rightarrow \mathbb{R}$,*

$$\lim_{k \rightarrow \infty} \lim_{N \rightarrow \infty} \mathbb{E} \left[\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[\left| \mathfrak{J}_{t,\mathbf{m},f_k}^{1,N}(H) \right| \right] \right] = 0.$$

Lemma 9.4 and Lemma 9.5 can be proved, following the same standard arguments given for example in [9, Section 7]. We need the energy moment estimate (66) in Lemma 9.4, in the computation of the quadratic variation of the martingale. The next subsection is devoted to highlight what fails in Lemma 9.6, which should be related to the results of Sections 4, 6 and 5.

REMARK 9.1. Conditioned to proving Lemma 9.6, Theorem 9.1 would follow: recall that $\mathcal{Q}_N^{\mathbf{m}}$ is the distribution on the path space $\mathcal{D}([0, T], \mathcal{M}_1)$ of the process π_t^N . Following the same argument as for the generalized exclusion process in [9, Section 7.6], we can show that the sequence $\{\mathcal{Q}_N^{\mathbf{m}}, N \geq 1\}$ is weakly relatively compact. It remains to prove that every limit point $\mathcal{Q}_*^{\mathbf{m}}$ is concentrated on absolutely continuous paths $\mathbf{e}(t, du) = \mathbf{e}(t, u)du$ whose densities are solutions of the hydrodynamic equations given in Theorem 9.1. It could be seen from Lemma 9.6 by following the proof of [9, Theorem 7.0.1].

9.3 Failed Variance Estimate

In this paragraph we fix the disorder \mathbf{m} , and we erase it whenever no confusion arises. We are going to recall here the usual main steps of the entropy method. We rewrite $\mathfrak{J}_{t, \mathbf{m}, f}^{1, N}(\mathbf{H})$ as

$$\mathfrak{J}_{t, \mathbf{m}, f}^{1, N}(\mathbf{H}) = \int_0^t \sum_{x \in \mathbb{T}_N} \mathbf{G}\left(\frac{x}{N}\right) \tau_x \varphi(\mathbf{m}, \omega)(s) ds,$$

where

$$\begin{cases} \varphi(\mathbf{m}, \omega) := j_{0,1}(\mathbf{m}, \omega) - D(\omega_1^2 - \omega_0^2) - \mathcal{L}_N^{\mathbf{m}}(f)(\mathbf{m}, \omega) \\ \mathbf{G}\left(\frac{x}{N}\right) := \nabla_N \mathbf{H}\left(\frac{x}{N}\right). \end{cases}$$

Entropy inequality – In Lemma 9.6, note that the expectation with respect to the law of the process $\mathbb{P}_N^{\mathbf{m}}$ is taken. There is *a priori* no hope to get any estimate of this expectation, apart from the well-known entropy inequality. More precisely, let us denote by $X_N^f(\omega)$ the following quantity:

$$X_N^f(\omega) := \sum_{x \in \mathbb{T}_N} \mathbf{G}\left(\frac{x}{N}\right) \tau_x \varphi(\omega).$$

From the entropy inequality, we obtain

$$\mathbb{E}_{\mathbb{P}_N^{\mathbf{m}}} \left[\left| \int_0^T X_N^f(\omega)(s) ds \right| \right] \leq \frac{1}{\alpha N} H(\mathbb{P}_N^{\mathbf{m}} | \mu_\beta^N) + \frac{1}{\alpha N} \log \mathbb{E}_{\mu_\beta^N} \left[\exp \left(\alpha N \left| \int_0^T X_N^f(\omega)(s) ds \right| \right) \right],$$

for all $\alpha > 0$. Since the entropy is decreasing in time, we know that, for all disorder field \mathbf{m} , $H(\mathbb{P}_N^{\mathbf{m}} | \mu_\beta^N)$ is bounded. From the arbitrariness of α , we are reduced to investigate the convergence of the second term in the previous right-hand side.

Feynman-Kac formula – Usually, the purpose is to reduce the dynamics problem to the study of the largest eigenvalue for a small perturbation of the generator $N^2 \mathcal{S}_N$. This reduction relies on Feynman-Kac formula and on a variational formula for the largest eigenvalue of a symmetric operator. By Feynman-Kac formula,

$$\mathbb{E}_{\mu_\beta^N} \left[\exp \left\{ N \int_0^T X_N^f(\omega)(s) ds \right\} \right] \leq \exp \left\{ \int_0^T \lambda_N(s) ds \right\}$$

where $\lambda_N(s)$ is the largest eigenvalue of the symmetric operator $N^2 \mathcal{S}_N(\cdot) + NX_N^f(\omega)$. From the variational formula for the largest eigenvalue of an operator in a Hilbert space, we also know that

$$\lambda_N(s) \leq \sup_g \left\{ \langle NX_N^f(\cdot) g(\cdot) \rangle_\beta - N^2 \mathcal{D}_N(\mu_\beta; \sqrt{g}) \right\}$$

where the supremum is taken over all measurable functions g which are densities with respect to μ_β^N . In particular,

$$\frac{1}{N} \log \mathbb{E}_{\mu_\beta^N} \left[\exp \left\{ \int_0^T NX_N^f(\omega)(s) ds \right\} \right] \leq \int_0^T \sup_g \left\{ \langle X_N^f(\omega) g(\omega) \rangle_\beta - N \mathcal{D}_N(\mu_\beta; \sqrt{g}) \right\} ds.$$

Reduction to microscopic blocks – With the same spirit of the one-block estimate presented in [18], it is then crucial to replace microscopic quantities with their spatial averages. Here, with the same ideas of [9], we can replace

$$\begin{aligned} j_{0,1} & \text{ with } \frac{1}{2\ell+1} \sum_{x \in \Lambda_\ell} j_{x,x+1} \\ \omega_0^2 & \text{ with } \frac{1}{2\ell+1} \sum_{x \in \Lambda_\ell} \omega_x^2 \\ \mathcal{L}_N^m(f)(\omega) & \text{ with } \frac{1}{2\ell_f+1} \sum_{x \in \Lambda_{\ell_f}} \mathcal{L}_{s_f+1}^m(\tau_x f) \end{aligned}$$

where $\ell_f = \ell - s_f - 1$ so that $\mathcal{L}_{s_f+1}(\tau_y f)$ is $\mathcal{F}_{\Lambda_{\ell_f}}$ -measurable for every $y \in \Lambda_{\ell_f}$. Let us introduce the following notation

$$W^{f,\ell} := \frac{1}{2\ell'+1} \sum_{y \in \Lambda_{\ell'}} j_{y,y+1} + D \left[\frac{1}{2\ell+1} \sum_{|x| \leq \ell} \omega_x^2 - \frac{1}{2\ell+1} \sum_{|x-1| \leq \ell} \omega_x^2 \right] - \frac{1}{2\ell_f+1} \sum_{y \in \Lambda_{\ell_f}} \mathcal{L}_{s_f+1}(\tau_y f) \quad (67)$$

with $\ell' = \ell - 1$. Finally, thanks to the regularity of the function G and the fact that D is constant, we are able to reduce Lemma 9.6 to Lemma 9.7 below. We also need to perform a cut-off in order to control high energy values, and this is valid thanks to (66).

LEMMA 9.7. *For all $\delta > 0$,*

$$\inf_{f \in \mathcal{Q}} \limsup_{\ell \rightarrow \infty} \limsup_{N \rightarrow \infty} \sup_g \left\{ \langle Y_{N,\ell}^f(\omega) g(\omega) \rangle_\beta - \delta N \mathcal{D}_N(\mu_\beta; \sqrt{g}) \right\} \leq 0, \quad (68)$$

where

$$Y_{N,\ell}^f(\omega) := \sum_{x \in \mathbb{T}_N} G\left(\frac{x}{N}\right) \tau_x W^{f,\ell}(\omega).$$

Reduction to a variance estimate – Then, the challenge is to reduce the proof of Lemma 9.7 to the following result:

$$\inf_{f \in \mathcal{Q}} \lim_{\ell \rightarrow \infty} 2\ell \times \mathbb{E} \left[\langle (-\mathcal{S}_{\Lambda_\ell})^{-1} W^{f,\ell}, W^{f,\ell} \rangle_\beta \right] = 0 \quad (69)$$

This convergence holds, since it is equivalent to the conclusion of Theorem 5.9, where the diffusion coefficient D is defined through the non gradient approach. Here is the main obstacle. If we follow the strategy given in [9, Section 7.3], we can bound the supremum in (68) by the largest eigenvalue of

$\mathcal{S}_{\Lambda_\ell} + \mathbf{b}W^{f,\ell}$ where \mathbf{b} is a small constant. In order to estimate this largest eigenvalue, we usually use a perturbation method which provides a bound on the largest eigenvalue in terms of the variance of $W^{f,\ell}$. This can not be proved, and suggests that the entropy inequality together with the Feynman-Kac formula are not the good tools to prove the hydrodynamic limits for systems which do not have a spectral gap (see the last concluded section).

We conclude this section by explaining why the perturbation theory does not work. Let us try to prove Lemma 9.7. Since μ_β is translation invariant, we may rewrite $\langle Y_N^{f,\ell}(\cdot) g(\cdot) \rangle_\beta$ as

$$\sum_{x \in \mathbb{T}_N} \left\langle G\left(\frac{x}{N}\right) W^{f,\ell}(\omega) \tau_{-x} g(\omega) \right\rangle_\beta.$$

Since the Dirichlet form is convex, the supremum in (68) is bounded from above by

$$\frac{\delta N}{2\ell} \sum_{x \in \mathbb{T}_N} \sup_g \left\{ \mathbf{b} \langle W^{f,\ell} g \rangle_\beta - \mathcal{D}_\ell(\mu_\beta; \sqrt{g}) \right\}, \quad (70)$$

where the constant $\mathbf{b} = \mathbf{b}(x, \ell, \delta, N)$ satisfies

$$|\mathbf{b}| := \left| G\left(\frac{x}{N}\right) \frac{2\ell}{\delta N} \right| \leq \|G\|_\infty \frac{2\ell}{\delta N}.$$

Let us denote by $\lambda_{N,\ell,f}$ this last supremum inside the sum (70), which does not depend on x . We consider a sequence $\{g_k\}_{k \in \mathbb{N}}$ that approaches this supremum, such that

$$\lim_{k \rightarrow \infty} \langle \sqrt{g_k}, (\mathcal{S}_{\Lambda_\ell} + \mathbf{b}W^{f,\ell}) \sqrt{g_k} \rangle_\beta = \lambda_{N,\ell,f}.$$

The idea of the perturbation theory is to expand $\sqrt{g_k}$ around the constant value 1. We write

$$\langle \sqrt{g_k}, (\mathcal{S}_{\Lambda_\ell} + \mathbf{b}W^{f,\ell}) \sqrt{g_k} \rangle_\beta = \mathbf{b} \left(\langle W^{f,\ell} \rangle_\beta + 2 \langle W^{f,\ell} (\sqrt{g_k} - 1) \rangle_\beta + \langle W^{f,\ell} (\sqrt{g_k} - 1)^2 \rangle_\beta \right) - \mathcal{D}_\ell(\mu_\beta; \sqrt{g_k}). \quad (71)$$

We know that $\langle W^{f,\ell} \rangle_\beta = 0$, and we use the Cauchy-Schwarz inequality for the scalar product $\langle \cdot, (-\mathcal{S}_{\Lambda_\ell}) \cdot \rangle_\beta$ in the second term. We obtain that (71) is bounded, for every $A > 0$, by

$$\mathbf{b} \left(\frac{\mathbf{b}}{A} \langle W^{f,\ell}, (-\mathcal{S}_{\Lambda_\ell})^{-1} W^{f,\ell} \rangle_\beta + \frac{A}{\mathbf{b}} \mathcal{D}_\ell(\mu_\beta; \sqrt{g_k}) \right) + \mathbf{b} \langle W^{f,\ell} (\sqrt{g_k} - 1)^2 \rangle_\beta - \mathcal{D}_\ell(\mu_\beta; \sqrt{g_k}).$$

It remains to bound the third term in the expression above. This could be done if we had the following lemma.

LEMMA 9.8. *There exists a constant $C := C(\ell, f, \beta, \gamma, \lambda)$ such that, for every $g \geq 0$,*

$$\langle W^{f,\ell} (\sqrt{g} - 1)^2 \rangle_\beta \leq C \mathcal{D}_\ell(\mu_\beta; \sqrt{g}). \quad (72)$$

As before, we could try to use the fact that $W^{f,\ell}$ is a quadratic function. Even this fact is not helpful, and we give now a counter-example to this last lemma. We denote by H_n the normalized one-variable Hermite polynomial of degree $n \geq 3$ (see Appendix A). Let us consider

$$\begin{cases} \sqrt{g}(\omega) = |H_n(\omega_0)| \\ W^{f,\ell}(\omega) = H_2(\omega_0) = \omega_0^2 - 1. \end{cases}$$

Let us notice that $\langle H_n^2 \rangle_\beta = 1$, and $\langle H_2 \rangle_\beta = 0$, so that the two test functions g and $W^{f,\ell}$ satisfy all expected conditions. By using the recursive relation

$$H_{n+1}(\omega_0) = \omega_0 H_n(\omega_0) - n H_{n-1}(\omega_0),$$

we get for the left-hand side of (72),

$$\begin{aligned} \langle H_2(|H_n| - 1)^2 \rangle_\beta &= \langle \omega_0^2 H_n^2(\omega_0) \rangle_\beta - \langle H_n^2 \rangle_\beta - 2 \langle H_2 |H_n| \rangle_\beta + \langle H_2 \rangle_\beta \\ &= \langle H_{n+1}^2 + 2n H_{n+1} H_{n-1} + n^2 H_{n-1}^2 \rangle_\beta - 1 - 2 \langle H_2 |H_n| \rangle_\beta \\ &= 1 + n^2 - 1 - 2 \langle H_2 |H_n| \rangle_\beta \geq n^2 - 2. \end{aligned}$$

Above the last equality comes from the orthonormality of the polynomial basis, and the last inequality is a consequence of the Cauchy-Schwarz inequality $\langle H_2 |H_n| \rangle_\beta^2 \leq \langle H_2^2 \rangle_\beta \langle H_n^2 \rangle_\beta = 1$. Let us assume that there exists a constant $C > 0$ which does not depend on n such that

$$n^2 - 2 \leq \langle H_2(|H_n| - 1)^2 \rangle_\beta \leq C \mathcal{D}_\ell(\mu_\beta; |H_n|).$$

From the convexity of the Dirichlet form, we have

$$\mathcal{D}_\ell(\mu_\beta; |H_n|) \leq \mathcal{D}_\ell(\mu_\beta; H_n).$$

In the case where n is an even positive integer, the flip noise gives a zero contribution to the Dirichlet form, and then, for all n even, we have

$$\mathcal{D}_\ell(\mu_\beta; H_n) = \frac{\lambda}{2} \langle (H_n(\omega_1) - H_n(\omega_0))^2 \rangle_\beta = \lambda \langle H_n^2 \rangle_\beta - \lambda \langle H_n(\omega_0) H_n(\omega_1) \rangle_\beta = \lambda.$$

In the last equality, we use the fact that H_n is unitary, and that $H_n(\omega_0) H_n(\omega_1)$ constitutes another element of the Hermite polynomial basis, then is orthogonal to the constant polynomial 1. Letting n go to infinity, we obtain a contradiction to (72).

Ergodic decomposition – Another idea would be to use the ergodic decomposition described in Subsection 4.2. The generator \mathcal{S}_ℓ restricted to finite boxes does not have a spectral gap, but it becomes ergodic when restricted to each orbit $\mathcal{G}_x^{N,\beta}$. However, this approach fails, because the space is not compact, and we need to disintegrate the measure μ_β with respect to all energy levels in $(0, +\infty)$. This enforces us to introduce a cut-off in the variational formula giving the largest eigenvalue. In other words, an indicator function $\mathbf{1}\{|\omega_x| \leq E_0\}$ will appear in front of $W^{f,\ell}$. Finally, we will have to deal with functions of the configurations that are not quadratic any more, and we do not know how to prove the convergence result (69) for general functions.

9.4 Conclusion

Even if the non-gradient method can be applied in some cases when the spectral gap does not hold, (and then the diffusion coefficient is well-defined), this does not straightforwardly imply the hydrodynamic limits.

In order to derive the hydrodynamic theorem, we would need to bypass the entropy inequality together with the Feynman-Kac formula. The entropy inequality is however a convenient mean to transform the averages w.r.t. the unknown law μ_t^N into equilibrium averages w.r.t. μ_β^N , which are more easily tractable. The same problem would arise in the relative entropy method, because of the entropy inequality.

A Hermite polynomials and quadratic functions

In the whole section we assume $\beta = 1$. Every result can be restated for the general case after multiplying the process by $\beta^{-1/2}$.

A.1 Hermite polynomials on $\mathbb{R}^{\mathbb{Z}}$

Let χ be the set of positive integer-valued functions $\xi : \mathbb{Z} \rightarrow \mathbb{N}$, such that ξ_x vanish for all but a finite number of $x \in \mathbb{Z}$. The *length* of ξ , denoted by $|\xi|$, is defined as

$$|\xi| := \sum_{x \in \mathbb{Z}} \xi_x.$$

For $\xi \in \chi$, we define the polynomial function on Ω

$$H_\xi(\omega) = \prod_{x \in \mathbb{Z}} h_{\xi_x}(\omega_x),$$

where $\{h_n\}_{n \in \mathbb{N}}$ are the normalized Hermite polynomials w.r.t. the centered one-dimensional Gaussian law with variance 1. The sequence $\{H_\xi\}_{\xi \in \chi}$ forms an orthonormal basis of the Hilbert space $\mathbf{L}^2(\mu_1)$, where μ_1 is the infinite product Gibbs measure defined by (2). As a result, every function $f \in \mathbf{L}^2(\mu_1)$ can be decomposed in the form

$$f(\omega) = \sum_{\xi \in \chi} F(\xi) H_\xi(\omega).$$

Moreover, we can compute the scalar product $\langle f, g \rangle_1$ for $f = \sum_{\xi} F(\xi) H_\xi$ and $g = \sum_{\xi} G(\xi) H_\xi$ as

$$\langle f, g \rangle_1 = \sum_{\xi \in \chi} F(\xi) G(\xi).$$

DEFINITION A.1. We denote by $\chi_n \subset \chi$ the subset sequences of length n , i.e. $\chi_n := \{\xi \in \chi ; |\xi| = n\}$. A function $f \in \mathbf{L}^2(\mu_1)$ is of degree n if its decomposition

$$f = \sum_{\xi \in \chi} F(\xi) H_\xi$$

satisfies: $F(\xi) = 0$ for all $\xi \notin \chi_n$.

In the next paragraph we focus on degree 2 functions, which are by definition on the form

$$\sum_{x \in \mathbb{Z}} \varphi(x, x)(\omega_x^2 - 1) + \sum_{x \neq y} \varphi(x, y) \omega_x \omega_y \quad (73)$$

where $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is a square summable symmetric function.

Local functions – On the set of n -tuples $\mathbf{x} := (x_1, \dots, x_n)$ of \mathbb{Z}^n , we introduce the equivalence relation $\mathbf{x} \sim \mathbf{y}$ if there exists a permutation σ on $\{1, \dots, n\}$ such that $x_{\sigma(i)} = y_i$ for all $i \in \{1, \dots, n\}$. The class of \mathbf{x} for the relation \sim is denoted by $[\mathbf{x}]$ and its cardinal by $c(\mathbf{x})$. Then the set of configurations of χ_n can be identified with the set of n -tuples classes for \sim by the one-to-one application:

$$\begin{aligned} \mathbb{Z}^n / \sim &\rightarrow \chi_n \\ [\mathbf{x}] = [(x_1, \dots, x_n)] &\mapsto \xi^{[\mathbf{x}]} \end{aligned}$$

where for any $y \in \mathbb{Z}$, $(\xi^{[x]})_y = \sum_{i=1}^n \mathbf{1}_{y=x_i}$. We identify $\xi \in \chi_n$ with the occupation numbers of a configuration with n particles, and $[x]$ corresponds to the positions of those n particles. A function $F : \chi_n \rightarrow \mathbb{R}$ is nothing but a symmetric function $F : \mathbb{Z}^n \rightarrow \mathbb{R}$ through the identification of ξ with $[x]$. We denote (with some abuse of notations) by $\langle \cdot, \cdot \rangle$ the scalar product on $\oplus \mathbf{L}^2(\chi_n)$, each χ_n being equipped with the counting measure. Hence, for two functions $F, G : \chi \rightarrow \mathbb{R}$, we have

$$\langle F, G \rangle = \sum_{n \geq 0} \sum_{\xi \in \chi_n} F_n(\xi) G_n(\xi) = \sum_{n \geq 0} \sum_{\mathbf{x} \in \mathbb{Z}^n} \frac{1}{c(\mathbf{x})} F_n(\mathbf{x}) G_n(\mathbf{x}),$$

with F_n, G_n the restrictions of F, G to χ_n .

Dirichlet form – It is not hard to check the following proposition, which is a direct consequence of the fact that h_n has the same parity of the integer n .

PROPOSITION A.1. *If a local function $f \in \mathbf{L}^2(\mu_1)$ is written in the form $f = \sum_{\xi \in \chi} F(\xi) H_\xi$, then*

$$Sf(\omega) = \sum_{\xi \in \chi} (\mathfrak{S}F)(\xi) H_\xi(\omega),$$

where \mathfrak{S} is the operator acting on functions $F : \chi \rightarrow \mathbb{R}$ as

$$\mathfrak{S}F(\xi) = \lambda \sum_{x \in \mathbb{Z}} [F(\xi^{x, x+1}) - F(\xi)] + \gamma \sum_{x \in \mathbb{Z}} ((-1)^{\xi_x} - 1) F(\xi).$$

Here, $\xi^{x, y}$ is obtained from ξ by exchanging ξ_x and ξ_y .

From this result we deduce:

COROLLARY A.2. *For any $f = \sum_{\xi \in \chi} F(\xi) H_\xi \in \mathbf{L}^2(\mu_1)$, we have*

$$\mathcal{D}(f; \mu_1) = \langle f, -Sf \rangle_1 = \sum_{\xi \in \chi} \left\{ \frac{\lambda}{2} \sum_{x \in \mathbb{Z}} (F(\xi^{x, x+1}) - F(\xi))^2 + \gamma \sum_{x \in \mathbb{Z}} ((-1)^{\xi_x} - 1) F^2(\xi) \right\}$$

Quadratic functions – In this paper, we deal with the set of *quadratic* functions f in $\mathbf{L}^2(\mu_1)$, namely degree two functions that are *homogeneous*, i.e. satisfies the algebraic relation

$$\forall \lambda \in \mathbb{R}, \quad f(\lambda \omega) = \lambda^2 f(\omega), \quad \mu_1 - \text{a.s.} \quad (74)$$

We also assume that f has zero average with respect to μ_1 . Therefore, we could also rewrite every f as

$$f(\omega) = \sum_{x \in \mathbb{Z}} \psi(x, x) (\omega_x^2 - \omega_{x+1}^2) + \sum_{x \neq y} \psi(x, y) \omega_x \omega_y,$$

for a suitable function $\psi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ square summable and symmetric, and we recover the form given in (3). We first restrict some variational formula to this class of functions, and then we study sequences of functions that weakly converge in $\mathbf{L}^2(\mu_1)$.

PROPOSITION A.3. *If $f \in \mathbf{L}^2(\mu_1)$ is quadratic in the sense above with zero average w.r.t μ_1 , then the following variational formula*

$$\sup_{g \in \mathbf{L}^2(\mu_1)} \{2 \langle f, g \rangle_1 - \mathcal{D}(g; \mu_1)\}$$

can be restricted over quadratic functions g of zero mean w.r.t μ_1 .

Proof. This fact follows after decomposing g as $\sum_{\xi \in \chi} G(\xi)H_\xi$. Corollary A.2 and the orthogonality of Hermite polynomials imply that we can restrict the supremum over functions g of degree two (73). As a result, g writes as the sum of a quadratic function plus an additional constant term.

Then, notice that the constant term gives a zero contribution in the quantity to maximise: indeed, the Dirichlet form does not change if we add a constant, and the function f is supposed to be centered, so that $\langle f \rangle_1 = 0$. Therefore, we can assume that the supremum is taken over homogeneous functions of degree two, and the same argument shows that g can also be taken with zero average. \square

PROPOSITION A.4. *Let $\{f_n\}_n$ be a sequence of quadratic functions in $\mathbf{L}^2(\mu_1)$. Suppose that $\{f_n\}$ weakly converges to $f \in \mathbf{L}^2(\mu_1)$. Then, f is quadratic.*

Proof. For all $n \in \mathbb{N}$, and $\xi \notin \chi_2$, the scalar product $\langle f_n, H_\xi \rangle_1$ vanishes (by definition). From the weak convergence, we know that

$$\langle f_n, H_\xi \rangle_1 \rightarrow \langle f, H_\xi \rangle_1,$$

as n goes to infinity, for all $\xi \in \chi$. This implies: $\langle f, H_\xi \rangle_1 = 0$ for all $\xi \notin \chi_2$.

Besides, the algebraic relation (74) is still valid after taking the weak limit in $\mathbf{L}^2(\mu_1)$, as well as the zero-average property (with respect to μ_1). This implies that the weak limit $f \in \mathbf{L}^2(\mu_1)$ is quadratic, and of zero mean if every f_n is centered. \square

REMARK A.1. The set denoted by \mathcal{Q} is restricted to *local* quadratic functions. The conclusions of Propositions A.3 and A.4 remain valid if we replace the quadratic functions in $\mathbf{L}^2(\mu_1)$ by elements of \mathcal{Q} .

B A weak version of closed forms results

In that section we prove a theorem that should be thought as a kind of closed forms results, as they are stated in [17] or in [9] (Section A.3.4). We give the link between Theorem B.1 below and closed forms at the end of this paragraph.

B.1 Decomposition of quadratic functions

For the sake of clarity, we erase the dependence on the disorder \mathbf{m} , and consider that the functions are defined on Ω , and square integrable w.r.t. the Gibbs measure μ_1 . We explain how to restate the same result for functions defined on $\Omega_{\mathcal{D}} \times \Omega$ in Remark B.1.

THEOREM B.1. *Let $\{f_n\}_{n \in \mathbb{N}}$ a sequence of quadratic functions in $\mathbf{L}^2(\mu_1)$. Let us define*

$$g_n := \nabla_0(\Gamma_{f_n}) \quad \text{and} \quad h_n := \nabla_{0,1}(\Gamma_{f_n}).$$

If $\{g_n\}$, respectively $\{h_n\}$, weakly converges in $\mathbf{L}^2(\mu_1)$ towards g , respectively h , then there exist $a \in \mathbb{R}$ and $f \in \mathcal{Q}$ such that

$$g(\omega) = \nabla_0(\Gamma_f)(\omega), \tag{75}$$

$$h(\omega) = a(\omega_0^2 - \omega_1^2) + \nabla_{0,1}(\Gamma_f)(\omega). \tag{76}$$

This result remains in force if μ_1 is replaced with the product measure $\mathbb{P}_1^ = \mathbb{P} \otimes \mu_1$, where \mathbb{P} is the law of the disorder. (see Remark B.1)*

Proof. From Proposition A.4, we already know that g and h are quadratic functions in $\mathbf{L}^2(\mu_1)$. Hence, we look for g and h of the form

$$g(\omega) = \sum_{x,y \in \mathbb{Z}} \psi_1(x,y) \omega_x \omega_y \quad (77)$$

$$h(\omega) = \sum_{x,y \in \mathbb{Z}} \psi_2(x,y) \omega_x \omega_y \quad (78)$$

where $\psi_1, \psi_2 : \mathbb{Z}^2 \rightarrow \mathbb{R}$ are square integrable symmetric functions. We are now going to give a list or equalities, being satisfied by the pair of sequences. Let us be more precise. We define, for a pair $(\mathbf{f}^1, \mathbf{f}^2)$ of two $\mathbf{L}^2(\mu_1)$ functions, the following identities, stated in $\mathbf{L}^2(\mu_1)$ sense:

$$\mathbf{(R1)} \quad (\tau_x \mathbf{f}^1)(\omega) + (\tau_x \mathbf{f}^1)(\omega^x) = 0, \text{ for all } x \in \mathbb{Z}.$$

$$\mathbf{(R2)} \quad (\tau_x \mathbf{f}^2)(\omega) + (\tau_x \mathbf{f}^2)(\omega^{x,x+1}) = 0, \text{ for all } x \in \mathbb{Z}.$$

$$\mathbf{(R3)} \quad (\tau_x \mathbf{f}^1)(\omega) + (\tau_x \mathbf{f}^2)(\omega^x) = (\tau_x \mathbf{f}^2)(\omega) + (\tau_{x+1} \mathbf{f}^1)(\omega^{x,x+1}), \text{ for all } x \in \mathbb{Z},$$

It is straightforward to check that, for all $n \in \mathbb{N}$, the pair (g_n, h_n) satisfies identities **(R1–R3)**. Easily, one can show that the latter always take place after passing to the weak limit in $\mathbf{L}^2(\mu_1)$. Precisely, the weak limit (g, h) of $\{g_n, h_n\}$ also satisfy **(R1–R3)**. This follows from the following easy lemma (which is a consequence of the translation invariance of μ_1):

LEMMA B.2. *If $\{g_n\}_n$ weakly converges in $\mathbf{L}^2(\mu_1)$ towards g , then, for all $x \in \mathbb{Z}$,*

$$\begin{aligned} \{g_n(\omega^x)\}_n &\text{ weakly converges towards } g(\omega^x), \\ \{g_n(\omega^{x,x+1})\}_n &\text{ weakly converges towards } g(\omega^{x,x+1}). \end{aligned}$$

Notice that all equalities **(R1–R3)** – now stated for (g, h) – turn into identities for ψ_1 and ψ_2 , defined in (77) and (78). Namely, ψ_1 and ψ_2 have to satisfy

$$\mathbf{(R1)} \quad \begin{cases} \psi_1(x,y) = 0 & \text{if } x \neq 0 \text{ and } y \neq 0, \\ \psi_1(0,0) = 0. \end{cases}$$

$$\mathbf{(R2)} \quad \begin{cases} \psi_2(x,y) = 0 & \text{if } x \notin \{0,1\} \text{ and } y \notin \{0,1\}, \\ \psi_2(x,x) + \psi_2(x+1,x+1) = 0 & \text{for all } x \in \mathbb{Z}. \end{cases}$$

$$\mathbf{(R3)} \quad \begin{cases} 2\psi_2(x,0) = \psi_1(x-1,0) - \psi_1(x,0) & \text{if } x \notin \{0,1\}, \\ \psi_1(-1,0) = \psi_1(1,0). \end{cases}$$

The first two identities imply that g writes on the form

$$g(\omega) = \sum_{x \neq 0} \psi_1(x,0) \omega_x \omega_0, \quad (79)$$

and h rewrites as

$$h(\omega) = \sum_{x \neq 0,1} \psi_2(x,0) [\omega_1 \omega_x - \omega_0 \omega_x] + \psi_2(0,0) (\omega_0^2 - \omega_1^2), \quad (80)$$

whereas the final equality makes a connection between g and h . In view of (75) and (76), we are going to need the following straightforward lemma:

LEMMA B.3. Let $f \in \mathbf{L}^2(\mu_1)$ be of the form

$$f(\omega) = \sum_{x,y \in \mathbb{Z}} \varphi(x,y) \omega_x \omega_y,$$

where $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is a symmetric function. Then,

$$\nabla_0(\Gamma_f) = -4 \sum_{\substack{z \in \mathbb{Z} \\ x \neq 0}} \varphi(x+z, z) \omega_0 \omega_x, \quad (81)$$

$$\nabla_{0,1}(\Gamma_f) = 2 \sum_{\substack{z \in \mathbb{Z} \\ x \neq 0,1}} [\varphi(x+z, z) - \varphi(x+z, z+1)] (\omega_1 \omega_x - \omega_0 \omega_x). \quad (82)$$

Confronting (81)–(82) with (79)–(80), and keeping in mind the expected result of Theorem B.1, we are now looking for a symmetric function $\varphi : \mathbb{Z}^2 \rightarrow \mathbb{R}$ which is square-summable on \mathbb{Z}^2 and satisfies

$$\begin{cases} \sum_{z \in \mathbb{Z}} \varphi(x+z, z) = -\frac{1}{4} \psi_1(x, 0) & \text{for all } x \neq 0, \\ \sum_{z \in \mathbb{Z}} [\varphi(x+z, z) - \varphi(x+z, z+1)] = \frac{1}{2} \psi_2(x, 0) & \text{for all } x \notin \{0, 1\}. \end{cases}$$

Such a function φ exists if and only if, for all $x \notin \{0, 1\}$,

$$2\psi_2(x, 0) = \psi_1(x-1, 0) - \psi_1(x, 0).$$

This last equality is true according to (R3), and the result is proved, with $a = \psi_2(0, 0)$ and $f \in \mathcal{Q}$ defined as

$$f(\omega) = \sum_{x,y \in \mathbb{Z}} \varphi(x,y) \omega_x \omega_y.$$

□

REMARK B.1. In the whole section, every result that involves the Gibbs measure μ_1 can be translated into the same result involving the product measure \mathbb{P}_1^* . By instance, the decomposition in the Hilbert space turns into the following: every $f \in \mathbf{L}^2(\mathbb{P}_1^*)$ can be written as

$$f(\mathbf{m}, \omega) = \sum_{\xi \in \mathcal{X}} F(\mathbf{m}, \xi) H_\xi(\omega).$$

If f is quadratic, it rewrites

$$f(\mathbf{m}, \omega) = \sum_{x,y \in \mathbb{Z}} \varphi(\mathbf{m}, x, y) \omega_x \omega_y,$$

where, for all $\mathbf{m} \in \Omega_{\mathcal{D}}$, $\varphi(\mathbf{m}, \cdot, \cdot)$ is a symmetric function on \mathbb{Z}^2 , square summable and integrable w.r.t. \mathbb{P} . Moreover, the translation operator τ_x that is involved in identities (R1–R3) should translate also the disorder environment, as it is defined at the beginning of Subsection 2.3. The result follows since \mathbb{P}_1^* is space translation invariant.

B.2 Connection with closed forms results

Let us briefly explain the connection between Theorem B.1 and the closed forms as they are defined for example in [17]. For that purpose, we are going to reformulate identities (R1–R3). First, interpret $\mathbf{f}_x^1(\omega)$, respectively $\mathbf{f}_x^2(\omega)$, as the price to change the configuration $\omega \in \Omega$ into ω^x , respectively to change ω into $\omega^{x,x+1}$. In particular,

- the price to flip ω_x when the configuration is ω should be equal to $-\mathbf{f}_x^1(\omega^x)$: this is **(R1)**,
- the price to exchange ω_x and ω_{x+1} when the configuration is ω should also be equal to $-\mathbf{f}_x^2(\omega^{x,x+1})$: this is **(R2)**.

In the context of interacting particle systems, *closed forms* are expected to give the same price for any 2-step path with equal end points. In our setting, the last equality **(R3)** can be translated into: “The quantity at site x is flipped, and then exchanged with the quantity at site $x + 1$. Equally, the quantities at site x and $x + 1$ are exchanged first, and then the quantity at site $x + 1$ is flipped.” There are three other such paths, that we do not need to prove our statement:

- two quantities are exchanged at sites $x, x + 1$, and also independently at sites $y, y + 1$, with $\{x, x + 1\} \cap \{y, y + 1\} = \emptyset$,
- two quantities are flipped independently at sites x and y , with $x \neq y$,
- the quantity at site x is flipped, and then the quantities at sites y and $y + 1$ are exchanged, for $y \notin \{x, x + 1\}$, and the converse is also possible.

Recall that we have defined $\Omega := \mathbb{R}^{\mathbb{Z}}$. We denote by \mathcal{B} the space of real-valued functions

$$\mathcal{B} := \{f : \Omega \rightarrow \mathbb{R}\}. \quad (83)$$

We are now interested in the space of *forms*, which are defined as $(\mathbf{f}_x^1, \mathbf{f}_x^2)_{x \in \mathbb{Z}}$ where $\mathbf{f}_x^1 \in \mathcal{B}$, and $\mathbf{f}_x^2 \in \mathcal{B}$, for every $x \in \mathbb{Z}$. To each function $F : \Omega \rightarrow \mathbb{R}$ is associated a form:

DEFINITION B.1. A form $\mathbf{f} = (\mathbf{f}_x^1, \mathbf{f}_x^2)_{x \in \mathbb{Z}}$ is an exact form if there exists a continuous function $F : \Omega \rightarrow \mathbb{R}$ such that

$$\forall x \in \mathbb{Z}, \forall \omega \in \Omega, \begin{cases} \mathbf{f}_x^1(\omega) = F(\omega^x) - F(\omega), \\ \mathbf{f}_x^2(\omega) = F(\omega^{x,x+1}) - F(\omega). \end{cases} \quad (84)$$

Easily, one can prove that all exact forms are closed forms. We now present two examples of closed forms that play a central role.

EXAMPLE B.1. We denote by $\mathbf{a} = (\mathbf{a}^1, \mathbf{a}^2)$ the closed form defined by

$$\begin{cases} \mathbf{a}_x^1(\omega) = 0, \\ \mathbf{a}_x^2(\omega) = \omega_x^2 - \omega_{x+1}^2, \end{cases}$$

for all $x \in \mathbb{Z}$ and configurations $\omega \in \Omega$. This closed form corresponds to the formal function $F(\omega) = \sum_x x \omega_x^2$, but this is not an exact form.

EXAMPLE B.2. Let h be a cylinder function. Let us recall that we denote by Γ_h the formal sum $\sum_x \tau_x h$, and define $\mathbf{u}_h = (\mathbf{u}_h^1, \mathbf{u}_h^2)$ as

$$\begin{cases} (\mathbf{u}_h^1)_x(\omega) = \Gamma_h(\omega^x) - \Gamma_h(\omega), \\ (\mathbf{u}_h^2)_x(\omega) = \Gamma_h(\omega^{x,x+1}) - \Gamma_h(\omega), \end{cases}$$

for all $x \in \mathbb{Z}$, and configurations $\omega \in \Omega$. Though $\sum_x \tau_x h$ is a formal sum, these two equalities are well defined. Let us notice that \mathbf{u}_h is a closed form that is not exact, unless h is constant.

These two examples show that closed forms on Ω are not always exact forms. Let us introduce the notion of a germ of a closed form.

DEFINITION B.2. A pair of continuous functions $f = (f^1, f^2)$, where $f^i : \Omega \rightarrow \mathbb{R}$, is a germ of closed form if $\mathbf{f} = (\tau_x f)_{x \in \mathbb{Z}}$ is a closed form.

Examples B.1 and B.2 provide two types of germs of closed forms. Consider the cylinder function $\mathfrak{a}(\omega) = (0, \omega_0^2 - \omega_1^2)$. The collection $(\tau_x \mathfrak{a})_{x \in \mathbb{Z}}$ is the closed form \mathfrak{a} of Example B.1. For a cylinder function h , the collection $(\nabla_x \Gamma_h, \nabla_{x,x+1} \Gamma_h)_{x \in \mathbb{Z}}$ obtained through translations of the cylinder function $(\nabla_0 \Gamma_h, \nabla_{0,1} \Gamma_h)$ is the closed form of Example B.2. For a pair of $\mathbf{L}^2(\mathbb{P}_1^*)$ -functions $f = (f^1, f^2)$, we called it a *germ of closed form* if $\mathbf{f} = (\tau_x f)_{x \in \mathbb{Z}}$ satisfies all of conditions as a closed form in $\mathbf{L}^2(\mathbb{P}_1^*)$ -sense. Usually, Theorem B.1 is replaced with a similar result that concerns every germ of closed form in $\mathbf{L}^2(\mathbb{P}_1^*)$: see [17, Theorem 5.1] or [9, Theorem A.3.4.14].

C Proof of the weak sector condition

In this section we prove Proposition 5.7 that we recall here for the sake of clarity.

PROPOSITION C.1 (Weak Sector condition). *(i) There exists two constants $C_0(\gamma, \lambda)$ and $C_1(\gamma, \lambda)$ such that the following inequality hold for all $f, g \in \mathcal{Q}$:*

$$\begin{aligned} || \ll \mathcal{A}^m g, \mathcal{S}f \gg_\beta || &\leq C_0 || \mathcal{S}f ||_\beta || \mathcal{S}g ||_\beta. \\ || \ll \mathcal{A}^m g, \mathcal{S}f \gg_\beta || &\leq C_1 || \mathcal{S}g ||_\beta + \frac{1}{2} || \mathcal{S}f ||_\beta. \end{aligned}$$

(ii) There exists a positive constant $C(\beta)$ such that, for all $g \in \mathcal{Q}$,

$$|| \mathcal{A}^m g ||_\beta \leq C(\beta) || \mathcal{S}g ||_\beta.$$

Proof. We prove (i). We assume that

$$\begin{aligned} g(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} \psi_{x,0}(\mathbf{m})(\omega_{x+1}^2 - \omega_x^2) + \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \psi_{x,k}(\mathbf{m}) \omega_x \omega_{x+k} \\ f(\mathbf{m}, \omega) &= \sum_{x \in \mathbb{Z}} \varphi_{x,0}(\mathbf{m})(\omega_{x+1}^2 - \omega_x^2) + \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \varphi_{x,k}(\mathbf{m}) \omega_x \omega_{x+k}. \end{aligned}$$

We denote by $\nabla^m \psi$ the discrete Laplacian in the variable \mathbf{m} , that is

$$\nabla^m \psi(\mathbf{m}) = 2\psi(\mathbf{m}) - \psi(\tau_1 \mathbf{m}) - \psi(\tau_{-1} \mathbf{m}),$$

and $\tau_x \nabla^m$ is the operator defined as

$$(\tau_x \nabla^m) \psi(\mathbf{m}) := \nabla^m \psi(\tau_x \mathbf{m}).$$

Straightforward computations show that

$$\begin{aligned}
\|Sg\|_\beta^2 &= \frac{\gamma}{2} \mathbb{E}_\beta^* \left[(\nabla_0 \Gamma_g)^2 \right] + \frac{\lambda}{2} \mathbb{E}_\beta^* \left[(\nabla_{0,1} \Gamma_g)^2 \right] \\
&= \frac{4\gamma}{\beta^2} \sum_{\substack{x \in \mathbb{Z} \\ k \geq 1}} \mathbb{E}[\psi_{x,k}^2] + \frac{2\lambda}{\beta^2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_x (\nabla^m \psi_{x,0}) \right)^2 \right] \\
&\quad + \frac{\lambda}{\beta^2} \sum_{k \geq 2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} [\tau_{-x}(\psi_{x,k}) - \tau_{1-x}(\psi_{x,k})] \right)^2 \right],
\end{aligned} \tag{85}$$

$$\|Sf\|_\beta^2 \geq \|S^{\text{flip}} f\|_\beta^2 = \frac{\gamma}{2} \mathbb{E}_\beta^* \left[\left(2 \sum_{\substack{z \in \mathbb{Z} \\ k \geq 1}} \varphi_{z,k}(\mathbf{m}) \omega_0 \omega_k \right)^2 \right] = \frac{2\gamma}{\beta^2} \sum_{k \geq 1} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \varphi_{z,k}(\mathbf{m}) \right)^2 \right]. \tag{86}$$

Now we deal with $\ll \mathcal{A}^m g, Sf \gg_\beta$. From Proposition 5.1, and by definition,

$$\begin{aligned}
\ll \mathcal{A}^m g, Sf \gg_\beta &= - \sum_{z \in \mathbb{Z}} \mathbb{E}_\beta^* [f, \tau_z(\mathcal{A}^m g)] \\
&= - \sum_{x,z \in \mathbb{Z}} \mathbb{E} \left[\varphi_{x,0}(\mathbf{m}) \langle \omega_{x+1}^2 - \omega_x^2, \tau_z(\mathcal{A}^m g) \rangle_\beta \right] - \sum_{\substack{x,z \in \mathbb{Z} \\ k \geq 1}} \mathbb{E} \left[\varphi_{x,k}(\mathbf{m}) \langle \omega_x \omega_{x+k}, \tau_z(\mathcal{A}^m g) \rangle_\beta \right] \\
&= \frac{2}{\beta^2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[\frac{\tau_x(\nabla^m \psi_{x,0})}{\sqrt{m_x m_{x+1}}} \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,1}) \right] \\
&\quad + \frac{1}{\beta^2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[\left(\frac{\tau_1 \psi_{x,1}}{\sqrt{m_x m_{x+1}}} - \frac{\psi_{x,1}}{\sqrt{m_{x+1} m_{x+2}}} \right) \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,2}) \right] \\
&\quad + \frac{1}{\beta^2} \sum_{k \geq 2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[\left(\frac{\tau_1 \psi_{x,k}}{\sqrt{m_x m_{x+1}}} - \frac{\psi_{x,k}}{\sqrt{m_{x+k} m_{x+k+1}}} \right) \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,k+1}) \right] \\
&\quad + \frac{1}{\beta^2} \sum_{k \geq 2} \sum_{x \in \mathbb{Z}} \mathbb{E} \left[\left(\frac{\tau_{-1} \psi_{x,k}}{\sqrt{m_x m_{x+1}}} - \frac{\psi_{x,k}}{\sqrt{m_{x+k} m_{x+k-1}}} \right) \sum_{z \in \mathbb{Z}} \tau_{-z}(\varphi_{z,k-1}) \right].
\end{aligned}$$

From Cauchy-Schwarz inequality, and recalling $1/\sqrt{m_0 m_1} \leq C$ (\mathbb{P} -a.s.), we obtain the following bound:

$$\| \ll \mathcal{A}^m g, Sf \gg_\beta \| \leq \frac{2C}{\beta^2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_x(\nabla^m \psi_{x,0}) \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,1} \right)^2 \right]^{1/2} \tag{87}$$

$$+ \frac{3C}{\beta^2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_1 \psi_{x,1} - \psi_{x,1} \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,2} \right)^2 \right]^{1/2} \tag{88}$$

$$+ \frac{3C}{\beta^2} \sum_{k \geq 2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_1 \psi_{x,k} - \psi_{x,k} \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,k+1} \right)^2 \right]^{1/2} \tag{89}$$

$$+ \frac{3C}{\beta^2} \sum_{k \geq 2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_{-1} \psi_{x,k} - \psi_{x,k} \right)^2 \right]^{1/2} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \tau_{-z} \varphi_{z,k-1} \right)^2 \right]^{1/2}. \tag{90}$$

Now we are going to use two times the trivial inequality $\sqrt{ab} \leq a/\varepsilon + \varepsilon b/2$ for a particular choice of $\varepsilon > 0$: in (87) we take $\varepsilon = \gamma/C$ and in (88) we take $\varepsilon = 2\gamma/(3C)$. This trick gives the final bound

$$\begin{aligned} \|\ll \mathcal{A}^{\mathbf{m}}g, \mathcal{S}f \gg_{\beta}\| &\leq \frac{2C^2}{\gamma\beta^2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_x (\nabla^{\mathbf{m}} \psi_{x,0}) \right)^2 \right] + \frac{2\gamma}{\beta^2} \sum_{k \geq 1} \mathbb{E} \left[\left(\sum_{z \in \mathbb{Z}} \varphi_{z,k} (\tau_{-z} \mathbf{m}) \right)^2 \right] \\ &\quad + \frac{9C^2}{\gamma\beta^2} \sum_{k \geq 2} \mathbb{E} \left[\left(\sum_{x \in \mathbb{Z}} \tau_1 \psi_{x,k} - \psi_{x,k} \right)^2 \right]. \end{aligned}$$

Recalling (86), we obtain

$$\|\ll \mathcal{A}^{\mathbf{m}}g, \mathcal{S}f \gg_{\beta}\| \leq \frac{9C^2}{\gamma\lambda} \|\mathcal{S}g\|_{\beta}^2 + \frac{1}{2} \|\mathcal{S}f\|_{\beta}^2.$$

If we use the Cauchy-Schwarz inequality, we get:

$$\ll \mathcal{A}^{\mathbf{m}}g, \mathcal{S}f \gg_{\beta}^2 \leq \frac{18C^2}{\gamma\lambda} \|\mathcal{S}g\|_{\beta}^2 \|\mathcal{S}f\|_{\beta}^2.$$

We have proved (i) with $C_0 = \sqrt{18C^2/(\gamma\lambda)}$ and $C_1 = 9C^2/(\gamma\lambda)$. Now we turn to (ii). From Lemma 5.5 and Statement (i),

$$\ll \mathcal{A}^{\mathbf{m}}g, j_{0,1}^{\mathcal{S}} \gg_{\beta} = \ll \mathcal{S}g, j_{0,1}^{\mathcal{A}} \gg_{\beta} \leq \|\mathcal{S}g\|_{\beta} \|j_{0,1}^{\mathcal{A}}\|_{\beta}.$$

Moreover, from Statement (i), we also get, for all $f \in \mathcal{Q}_0$,

$$-2 \ll \mathcal{A}^{\mathbf{m}}g, \mathcal{S}f \gg_{\beta} \leq \|\mathcal{S}f\|_{\beta}^2 + \frac{2C}{\gamma\lambda} \|\mathcal{S}g\|_{\beta}^2.$$

As a result, the variational formula (30) for $\|\mathcal{A}^{\mathbf{m}}g\|_{\beta}^2$ gives:

$$\|\mathcal{A}^{\mathbf{m}}g\|_{\beta}^2 \leq \frac{1}{\lambda\chi(\beta)} \ll \mathcal{A}^{\mathbf{m}}g, j_{0,1}^{\mathcal{S}} \gg_{\beta}^2 + \frac{9C^2}{\gamma\lambda} \|\mathcal{S}g\|_{\beta}^2 \leq \left(\frac{\|j_{0,1}^{\mathcal{A}}\|_{\beta}^2}{\lambda\chi(\beta)} + \frac{9C^2}{\gamma\lambda} \right) \|\mathcal{S}g\|_{\beta}^2.$$

The result is proved. \square

D Tightness

In this section we prove the tightness of the sequence $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$. First, let us recall that the space \mathcal{M}_1 is equipped with the norm defined as

$$\|\mathcal{Y}\|_1 := \sup_{\|H\|_{\infty} \leq 1} \|\mathcal{Y}(H)\|,$$

where the supremum is taken over all smooth functions $H : \mathbb{T} \rightarrow \mathbb{R}$ bounded by 1.

THEOREM D.1. *For almost all realization of the disorder $\mathbf{m} \in \Omega_{\mathcal{D}}$, the sequence $\{\mathfrak{Y}_{\mathbf{m}}^N\}_{N \geq 1}$ is tight in $\mathcal{D}([0, T], \mathcal{M}_1)$.*

Proof. The tightness of the sequence $\{\mathcal{Y}_{\mathbf{m}}^N\}$ follows from two conditions (see [9], page 299):

$$\lim_{A \rightarrow \infty} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_\beta^N} \left[\sup_{0 \leq t \leq T} \|\mathcal{Y}_{t,\mathbf{m}}^N\|_1 > A \right] = 0 \quad (91)$$

$$(92)$$

$$\lim_{\delta \rightarrow 0} \limsup_{N \rightarrow \infty} \mathbb{P}_{\mu_\beta^N} \left[w(\mathcal{Y}_{\mathbf{m}}^N, \delta) > \varepsilon \right] = 0, \quad \text{for all } \varepsilon > 0, \quad (93)$$

where the modulus of continuity $w(\mathcal{Y}, \delta)$ is defined by

$$w(\mathcal{Y}, \delta) = \sup_{\substack{\|t-s\| < \delta \\ 0 \leq s \leq t \leq T}} \|\mathcal{Y}_t - \mathcal{Y}_s\|_1.$$

Let us remind the decomposition of $\mathcal{Y}_{t,\mathbf{m}}^N$ given in (10):

$$\mathcal{Y}_{t,\mathbf{m}}^N(\mathbf{H}) = \mathcal{Y}_{0,\mathbf{m}}^N(\mathbf{H}) + \int_0^t D\mathcal{Y}_{s,\mathbf{m}}(\Delta_N \mathbf{H}) ds + \mathfrak{M}_{t,\mathbf{m},f_k}^{1,N}(\mathbf{H}) + Z_{t,\mathbf{m},f_k}^N(\mathbf{H}),$$

where $\mathfrak{M}_{t,\mathbf{m},f_k}^{1,N}(\mathbf{H})$ is the martingale defined in Subsection 3.3, and $Z_{t,\mathbf{m},f_k}^N(\mathbf{H})$ is defined as the sum of the remaining terms in the decomposition. On the first hand,

$$\mathbb{E}_{\mu_\beta^N} \left[\sup_{0 \leq t \leq T} \left(Z_{t,\mathbf{m},f_k}^N(\mathbf{H}) \right)^2 \right]$$

can be estimated by the proof of Lemma 3.2 and Theorem 3.3. On the other hand,

$$\mathbb{E}_{\mu_\beta^N} \left[\left(\mathfrak{M}_{t,\mathbf{m},f_k}^{1,N}(\mathbf{H}) \right)^2 \right]$$

can be computed explicitly. □

References

- [1] O. Ajanki and F. Huveneers, *Rigorous scaling law for the heat current in disordered harmonic chain*, Comm. Math. Phys. **301** (2011), no. 3, 841–883.
- [2] C. Bernardin, *Thermal conductivity for a noisy disordered harmonic chain*, J. Stat. Phys. **133** (2008), no. 3, 417–433.
- [3] C. Bernardin and F. Huveneers, *Small perturbation of a disordered harmonic chain by a noise and an anharmonic potential*, Probab. Theory Related Fields **157** (2013), no. 1-2, 301–331.
- [4] C. Bernardin and G. Stoltz, *Anomalous diffusion for a class of systems with two conserved quantities*, Nonlinearity **25** (2012), no. 4, 1099–1133.
- [5] A. Dhar, V. Kannan, and J. L. Lebowitz, *Heat conduction in disordered harmonic lattices with energy conserving noise*, Phys. Rev. E **83** (2011), no. 021108.
- [6] S. N. Ethier and T. G. Kurtz, *Markov processes*, Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics, John Wiley & Sons Inc., New York, 1986, Characterization and convergence.
- [7] A. Faggionato and F. Martinelli, *Hydrodynamic limit of a disordered lattice gas*, Probab. Theory Related Fields **127** (2003), no. 4, 535–608.

- [8] M. Jara and C. Landim, *Quenched non-equilibrium central limit theorem for a tagged particle in the exclusion process with bond disorder*, Ann. Inst. Henri Poincaré Probab. Stat. **44** (2008), no. 2, 341–361.
- [9] C. Kipnis and C. Landim, *Scaling limits of interacting particle systems*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 320, Springer-Verlag, Berlin, 1999.
- [10] K. Komoriya, *Hydrodynamic limit for asymmetric mean zero exclusion processes with speed change*, Ann. Inst. H. Poincaré Probab. Statist. **34** (1998), no. 6, 767–797.
- [11] T. Komorowski, C. Landim, and S. Olla, *Fluctuations in Markov processes*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 345, Springer, Heidelberg, 2012, Time symmetry and martingale approximation.
- [12] C. Landim and H. T. Yau, *Fluctuation-dissipation equation of asymmetric simple exclusion processes*, Probab. Theory Related Fields **108** (1997), no. 3, 321–356.
- [13] M. Mourragui and E. Orlandi, *Lattice gas model in random medium and open boundaries: hydrodynamic and relaxation to the steady state*, J. Stat. Phys. **136** (2009), no. 4, 685–714.
- [14] S. Olla and M. Sasada, *Macroscopic energy diffusion for a chain of anharmonic oscillators*, arXiv:1109.5297v3 (2013).
- [15] J. Quastel, *Bulk diffusion in a system with site disorder*, Ann. Probab. **34** (2006), no. 5, 1990–2036.
- [16] M. Reed and B. Simon, *Methods of modern mathematical physics. I. Functional analysis*, Academic Press, New York, 1972.
- [17] M. Sasada, *Hydrodynamic limit for exclusion processes with velocity*, Markov Process. Related Fields **17** (2011), no. 3, 391–428.
- [18] M. Simon *Hydrodynamic limit for the velocity-flip model*, Stochastic Processes and their Applications **123** (2013), 3623–3662.
- [19] S. R. S. Varadhan, *Nonlinear diffusion limit for a system with nearest neighbor interactions. II*, Asymptotic problems in probability theory: stochastic models and diffusions on fractals (Sanda/Kyoto, 1990), Pitman Res. Notes Math. Ser., vol. 283, Longman Sci. Tech., Harlow, 1993, pp. 75–128.