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ON CONTINUUM KINEMATICS OF VOLUMETRIC GROWTH

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A continuum model for a body whose elements are distinguishable although its mass is not conserved is defined using a balance law for a property that characterizes it. A variational formulation of the balance law for the characterizing property is interpreted in various ways and is used in order to define growth formally.

Keywords. Continuum mechanics, growth, organisms, extensive properties.

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1 Introduction

The work presented here is an attempt to formulate a continuum theory of growth. The description of growth within the framework of classical continuum mechanics presents difficulties because body elements are associated with the conservation of mass. Hence, adding mass to a body or removing mass from it would change the material structure of the body. The addition and removal of material points, the basic notion of continuum mechanics, will prevent the usual definition of central notions like deformations and velocity fields.

In previous works ([4], [7], and [5]) growth of bodies was described using two approaches. In the first (see [4] and [7]) organisms were introduced as sets whose elements are distinguishable even though the material points that make them may be added or removed. Such “organisms” represent bodies like those of humans whose points are ideally identifiable although the material points that make them may change during the growth process. This was done by the introduction of a *content* of an organism as a mapping assigning material

points to elements of the organism. The material points in that case were elements of a material manifold containing all the “virtual” material points.

In a second approach (see [7] and [5]), in order to model phenomena like surface growth, body points are generated on known submanifolds so the body structure is not conserved.

We note that it is possible to define generalized versions of the notions of velocity fields, forces and stresses even if the material structure is not available. This is done using a global formulation where the basic mathematical object is the configuration space (e.g. [8]).

Here we try to motivate the notion of an organism. We do this by defining the body elements without relating them to the notion of mass so they might exist even if mass is added to a body or removed from it. To that end we make use of an idea presented in [1] where “material elements” are defined as integral lines of the velocity fields. This is modified in the present work by considering body elements that are integral lines of the flow field associated with any extensive property satisfying the Cauchy postulates. Actually, the Cauchy postulates assumed are more general than the traditional in that we assume production of the property rather than balance. The usual tetrahedron argument will lead to the existence of a flow field that generalizes the velocity field.

Further generalization is suggested after studying the variational version of the balance law and replacing the Cauchy postulates by the requirement that the growth is a continuous linear functional on the space of differentiable functions. This requirement serves then as a formal definition of growth.

2 The General Balance Law for a Control Volume

Consider a fixed region R in the three dimensional Euclidean space \mathbb{R}^3 that serves to model the physical space. The general balance law is concerned with the balance of an extensive property. That is, we consider the situation where for each subregion P of R there is a real number I_P given by

$$I_P = \int_P \beta dV + \int_{\partial P} \tau dA,$$

where β is a scalar field over P that is interpreted as the time rate of the density of the property associated with I , τ is a scalar field over the boundary of P interpreted as the flux of the property out of the subregion.

In general, the fields β and τ depend on the subregion P under consideration. The Cauchy postulate assumes that in fact β does not depend on P and

that the value $\tau(x)$ at x on the boundary of P , depends on P only through its tangent plane, or equivalently the unit normal, to the boundary at x .

A second assumption made in the formulation of the general balance law is that I_P for every subregion P is bounded by its volume, i.e., there is a scalar field s over R , such that

$$I_P = \int_P s dV.$$

The field s is interpreted as the production rate of the of the property.

Cauchy's theorem for the general balance law states that there is a vector field \mathbf{u} such

$$\tau = \mathbf{u} \cdot \mathbf{n},$$

so that the dependence of the flux on the normal is linear. We will refer to \mathbf{u} as the *flow* of the property associated with I . This relation implies that if τ (say as a part of I_P) is given for every subregion P , there is at most one vector field \mathbf{u} that satisfies $\tau = \mathbf{u} \cdot \mathbf{n}$ for every the boundary of every P . Various continuity assumptions imply the continuity of \mathbf{u} . In addition, using Cauchy's theorem the balance of I assumes the form

$$\text{Div } \mathbf{u} + \beta = s.$$

In general, the fields τ for the various subbodies are time dependent and so is the vector field \mathbf{u} .

3 Body Elements

The vector field \mathbf{u} has integral lines by the existence theorem for the solutions of time dependent ordinary differential equations. While this is not the usual practice (as a body element is usually taken as a primitive notion), the situation suggests that we make the following definitions. A *body element* is an integral line of the flow \mathbf{u} . The collection of body elements is the *universal body* and will be denoted by \mathcal{B} . A *motion* of the universal body is the mapping

$$\phi: \mathcal{B} \times \mathbb{R} \rightarrow \mathbb{R}^3$$

induced by the flow. Since any integral line is associated with a unique point in \mathbb{R}^3 that serves as an initial condition, we will often identify a body element with its initial condition to which we will refer as the reference configuration of that element. Indeed, the initial conditions may serve as to put a topological and differentiable structure on \mathcal{B} . We will use ϕ_t to denote the restriction of ϕ to $\mathcal{B} \times t$. Such a mapping of the universal body into space will be referred

to as a universal configuration. A compact three dimensional submanifold B with boundary of \mathcal{B} is a *body* and the restriction of the motion to a body is a *configuration* of the body.

3.1 A Simple Example of Volumetric Growth

Consider a simple one dimensional situation where we can measure mass density, mass production and mass flow. We denote the mass density by ρ and consider it as a function of position in the physical space and time. The mass production, i.e., the density of the mass produced with respect to the volume as a function of location in space and time, is denoted by p . We set

$$\beta = \frac{\dot{\rho}}{\rho}, \quad s = \frac{\dot{p}}{p}.$$

Assume in addition that the flow is given by $\mathbf{u}(x, t) = x$ so that $x = x_0 e^t$.

The described setting allows in this case the definition of body elements although mass is not conserved. Note that for the case where $s = 0$ we get

$$\rho = \rho_0 e^{-t}.$$

For the other extreme situation where $\beta = 0$, we get

$$p = p_0 e^t.$$

4 The Variational Principle Associated with the General Balance Law

If we multiply the differential version of the generalized balance law $\text{Div } \mathbf{u} + \beta = s$ by a function w and integrate over the region R , we obtain (using the divergence theorem and Cauchy's theorem) the variational form of the balance law. Specifically we deduce that for any function w we have

$$\int_{\partial R} \tau w \, dA + \int_R \beta w \, dV = \int_R \mathbf{u} \cdot \nabla w \, dV + \int_R s w \, dV.$$

We will refer to this equation as the variational principle associated with the generalized balance law. Either side of this equation represents a linear functional defined on the space of mappings on R . We denote this functional by Φ and note that it is continuous if we use the C^1 metric on the space of continuously differentiable mappings $C^1(R)$. Thus we have

$$\Phi(w) = \int_{\partial R} \tau w \, dA + \int_R \beta w \, dV = \int_R \mathbf{u} \cdot \nabla w \, dV + \int_R s w \, dV.$$

Functionals on spaces of continuously differentiable functions and their applications in continuum mechanics were considered in [2],[3] and [6]. The results presented in these works may be applied to the situation discussed here to conclude the following.

Consider a subregion P of \mathfrak{R}^3 that is a smooth three dimensional sub-manifold with boundary and let \mathcal{Q} be an open subset of $C^1(P)$. We regard \mathcal{Q} as a differentiable manifold whose tangent space $T_u\mathcal{Q} = C^1(P)$. We consider elements of the cotangent bundle to \mathcal{Q} . The basic result is that any element of the cotangent bundle i.e., a linear continuous functional defined on $C^1(P)^*$ can be represented by measure μ_0 over P and an \mathfrak{R}^3 valued measure μ_1 in the form

$$\Phi(w) = \int_P w d\mu_0 + \int_P \nabla w \cdot d\mu_1,$$

for every element w of $T_u\mathcal{Q}$.

The measures μ_0 and μ_1 are not determined uniquely by Φ . However, if we are given a functional Φ_P for every subregion P of \mathfrak{R}^3 , the representing measures are unique.

In case the representing measures μ_0 and μ_1 are given in terms of densities s and \mathbf{u} with respect to the volume measure on \mathfrak{R}^3 , respectively, then the procedure of obtaining the variational principle from the balance equation and Cauchy's postulate may be reversed in order to prove the balance differential equation, the existence of β and τ , and Cauchy's postulate. These are proved only on the basis of the definition of Φ , the assumption of the differentiability of the measures with respect to the volume measure, and the differentiability of the density \mathbf{u} . Thus, we may replace the differential balance law and accompanying assumptions by the definition of the functional Φ as an element of $C^1(P)^*$.

Consider the role of the balance law in the definition of body elements as discussed above. The possibility that the flow be a measure, so that integral lines cannot be defined in the usual way, suggests a generalization of the notion of a body element.

5 The Interpretation of the Variational Formulation of the General Balance Law

The interpretation of the variational formulation of the general balance law in the mechanical context is the traditional principle of virtual work. Here, \mathcal{Q} is the collection of configurations of the body in the physical space (embeddings of P in \mathfrak{R}^3), w is a generalized velocity vector field and Φ is the force functional producing mechanical power $\Phi(w)$. The quantity β is a vector field which is

interpreted as the body force, τ is the surface force, \mathbf{u} is a tensor field—the stress tensor—and s is the self force field present when equilibrium is not assumed.

We wish to suggest an interpretation for the variational formulation of the general balance law. While the basic variables u , and w are intuitively clear for the force balance law, it is not immediately obvious how should these variables be interpreted in the context of the general balance law.

We regard w as the distribution of a potential energy density in space per unit of the property associated with I . Thus, the product sw is the power density associated with the production term s . The gradient ∇w is interpreted as a force and its contraction with the velocity of the material elements is the associated power. Thus, $\Phi(w)$ is the power associated with generation and motion of material.

One could also assign interpretation of the mathematical setting which is outside the field of mechanics. For example, one can regard w as a *benefit* field defined on space. Then, growth, regarded as the production and motion of matter as given by s and \mathbf{u} , produce the rate of income $\Phi(w)$.

This also suggests that the growth of the organism will follow a constitutive relation giving the fields \mathbf{u} and s as functions of the potential field w . One would speculate that such constitutive laws can model various aspects of ecology.

An interpretation in the field of thermodynamics is offered by identifying w with the *coldness* (i.e., the reciprocal of the temperature field) and $\Phi(w)$ with the contribution of heat to the entropy growth rate. Here, the fields \mathbf{u} and s are the heat flow and heat source distribution, respectively. A constitutive relation will clearly relate the heat flow to the temperature field as expected.

6 A Formal Definition of Growth

One can turn the order of things around and start with a potential field over space as a primitive notion. Thus, for every P of R we consider the Banach space $C^1(P)$ of differentiable potential fields. We define a *growth* Φ as an element of the dual space $C^1(P)^*$. It is noted that unlike the analogous continuum theory of forces, the original space is a Banach space and there is no need to go to the tangent bundle in order to define linear functionals.

Thus, the basic representation property of C^1 functionals implies that there is a real valued measure μ_0 and an \mathfrak{R}^3 valued measure μ_1 over P such that

$$\Phi(w) = \int_P w d\mu_0 + \int_P \nabla w \cdot d\mu_1,$$

for every potential field w .

In case the measures μ_0 and μ_1 representing Φ can be represented by densities s and \mathbf{u} with respect to the volume measure on R , respectively, then

$$\Phi(w) = \int_{\partial P} \tau w dA + \int_P \beta w dV = \int_P \mathbf{u} \cdot \nabla w dV + \int_P s w dV,$$

where

$$\text{Div } \mathbf{u} + \beta = s, \quad \text{on } P, \quad \text{and} \quad \tau = \mathbf{u} \cdot \mathbf{n}, \quad \text{on } \partial P.$$

Thus for example, with the interpretation of w as the coldness distribution and $\Phi(w)$ as the temperature related entropy production, this result implies the existence of the heat “growth” by heat a flow vector field and a heat production field without further assumptions.

In the case of representation by smooth densities, the growth may be associated with a body containing the integral lines of \mathbf{u} .

7 Conclusions

The foregoing formalism allows the definition of a body or an organism with respect to any particular extensive property. Unlike the traditional approach where the mass property is conserved, the extensive properties considered here may grow due to a source term. In addition, depending on the structure of the singularities of the flow field, material elements may be added or removed from the organism. In the case where a number of such extensive properties are measured, each will determine its own material structure. The various properties are compatible if they induce the same material structure. Otherwise, diffusion occurs in the body.

Generalizations of the foregoing discussions may follow various directions. One may consider functionals on spaces of “potential” distributions with degrees of differentiability different than 1. We note that if one chooses the C^0 space of potential fields, the flow vector fields would be missing and we would have only the source term. If on the other hand we consider degree of differentiability greater than 1 we could obtain more complex interactions in terms of boundary terms and one could speculate that by integration of these higher order fields we would obtain a structure for the body elements which is more complex than merely points in \mathbb{R}^3 .

Another way in which the theory could be generalized is by considering “potential fields” valued in ranges more complex than \mathbb{R} , e.g., vector potentials and potentials valued on manifolds. This may allow the generalization of the body element to body vector etc. as is the situation for materials with microstructure.

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