The Minimal Controllability Problem for structured systems

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***Abstract*— This paper considers the Minimal Controllability Problem (MCP), *i.e.* the problem of controlling a linear system with an input vector having as few non-zero entries as possible. We focus on structured systems which represent an interesting class of parameter dependent linear systems and look for structural controllability properties based on the sparsity pattern of the input vector. We show first that the MCP is solvable when a rank condition is satisfied and show that generically one non-zero entry in the input vector is sufficient to achieve controllability when there is no specific system structure. We give, according to the fixed zero/non-zero pattern of the state matrix entries, the minimum number and the possible location of non-zero entries in the input vector to ensure generic controllability. The analysis based on graph tools provides with a simple polynomial MCP solution and highlights the structural mechanisms that make it useful to act on some variables to ensure controllability.**

I. INTRODUCTION

The contemporary science is faced to the emergence of very large scale complex systems such as power systems, traffic systems, networked systems, distributed systems, or biological systems. A major challenge is to control such systems while acting on them in a parsimonious way. This general problem may be declined into many variants which lead to numerous works in the recent literature, see for example [1], [2], [3], [4], [5]. When the system is represented by state space equations, we look for inputs which make the whole system controllable in the usual sense. Due to complexity and cost constraints it is of interest to look for control laws acting on a reasonably small number of state variables. Such input selection strategies have been reported for controlling large scale systems or for leader selection of multi-agent systems [1], [6], [7]. In [8], the authors analysed three variants for the problem of minimizing the number of control inputs:

- when each input acts on a single state,
- when each input acts on an arbitrary number of states,
- when inputs belong to a pre-specified set of control inputs (acting in a given way on some states).

In this spirit, another interesting variant has been recently considered in [9], *i.e.* the problem of controlling a linear system with an unique input vector having as few non-zero entries as possible. It is proved that this Minimal Controllability Problem (MCP) is NP-hard.

Here we address the solvability of this MCP focusing on the framework of structured systems which are linear parameterized systems with a given structure, *i.e.* the entries of the state space matrix are either free parameters or fixed zeroes [10]. The structural controllability (controllability for structured systems) was introduced by Lin [11], who proved that the system is structurally controllable if and only if a connection condition and a rank condition are both satisfied.

We look first at the solvability condition of the MCP for linear systems (in the standard acceptance). It turns out that the problem is solvable when a cyclicity condition is satisfied. It happens that generically one non-zero entry in the input vector is sufficient to achieve controllability when there is no specific system structure, *i.e.* there is no fixed zero entry in the state matrix. Things are different when there is some structure, *i.e.* when there is some fixed zero/non-zero pattern for the state matrix entries. In the presence of structure, which usually comes from the physical nature of the process, we give the minimum number and the possible location of non-zero entries in the input vector to ensure generic controllability. The analysis based on graph tools provides with a simple polynomial MCP solution and highlights the structural mechanisms that make it useful to act on some variables to ensure controllability.

The organization of the paper is as follows. We formulate in Section 2 the Minimal Controllability Problem (MCP) for usual linear systems and show that MCP is solvable generically with one non-zero entry in the input vector when there are no fixed zeroes in the state matrix. Section 3 introduces the linear structured systems and refines some results on structural controllability. The MCP for linear structured systems is investigated in Section 4, the solvability condition and the minimum number of non-zero entries in the input vector for structural controllability are given. Simple examples illustrate the approach. Some concluding remarks end the paper.

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II. PROBLEM FORMULATION

In this paper, we consider the linear system Σ defined by (1)

$$\Sigma : \quad \dot{x}(t) = Ax(t) + bu(t) \quad , \tag{1}$$

where $x(t) \in R^n$ is the state vector and $u(t) \in R$ the input signal. We study the controllability of this system, *i.e.* the possibility to drive in finite time the state from the origin to any point in the state space. Controllability is known to be equivalent to the fact that the Kalman matrix is full rank, *i.e.*:

$$rank(K) = n. \tag{2}$$

where $K = [b, Ab, \ldots, A^{n-1}b]$

Definition 1: The Minimal Controllability Problem (MCP) is defined as follows: given the $n \times n$ matrix A, find an $n \times 1$ input vector b such that:

- the pair (A, b) is controllable,
- the number ν of non zero entries of b is minimum, this minimum number is denoted by ν^*.

We first look at the solvability of the MCP, *i.e.* the possibility of finding a b such that (A, b) is controllable. This is indeed a basic result of linear algebra [12]. Recall first that a $n \times n$ matrix Q is said to be cyclic if there exists a vector v such that $\text{rank}[v, Qv, \ldots, Q^{n-1}v] = n$.

Lemma 1: The MCP of Definition 1 has a solution if and only if the matrix A is cyclic.

The result follows from the definition of cyclicity. Notice that cyclicity of A is also characterized by the fact that its minimal polynomial is equal to its characteristic polynomial. It is also equivalent to the fact that identical eigenvalues of A belong to the same Jordan block.

We give now a simple corollary of Lemma 1 which will be useful in the sequel.

Corollary 1: The MCP of Definition 1 has a solution only if $\text{rank}(A) \geq (n-1)$.

If $\text{rank}(A) < (n-1)$, since vectors $Ab, \ldots, A^{n-1}b$ belong to $\text{Image}(A)$, then $\text{rank}[Ab, \ldots, A^{n-1}b] < n-1$ and $\text{rank}[b, Ab, \ldots, A^{n-1}b] < n$, therefore (A, b) is not controllable.

Let us now state a result which is valid for almost any matrix A. Before, let us recall that a property depending on k parameters, is said to be generic (or structural), if it is true for all values of the parameters (*i.e.* any $\lambda \in R^k$) outside a proper algebraic variety of the parameter space, [13] [14]. An algebraic variety is defined by the common zeroes of a finite set of polynomials. The variety is proper when it is not the whole parameter set.

Lemma 2: For a system of type (1), when considering as parameter set the n^2 entries of the matrix A, the pair (A, b) is generically controllable with $b = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$. Then ν^* is generically one.

Proof: With this b vector, the matrix K is composed of the first columns of matrices $I_n, A, A^2, \ldots, A^{n-1}$ and $det(K)$ is a polynomial in the entries a_{ij} of A. Since (A, b) is controllable if and only if $det(K) \neq 0$, the a_{ij}'s for which the system is not controllable belong to an algebraic variety. Moreover taking $A = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ 1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}$, leads to $K = I_n$, which proves that the variety is proper and ends the proof. ■

This result is somewhat puzzling since the main result of [9] says that the Minimal Controllability Problem is NP-hard [15]. This means that, indeed the difficulties in solving the problem will result from very special structural properties of A. In the following we will prove that the problem remains easy to solve (*i.e.* in polynomial time) generically when the structure of A consists in the presence of zeros in fixed locations.

III. Linear structured systems and structural controllability

We consider a linear system with parameterized entries denoted by Σ_Λ.

$$\Sigma_\Lambda : \quad \dot{x}(t) = A_\Lambda x(t) + b_\Lambda u(t) \quad , \tag{3}$$

where $x(t) \in R^n$ is the state vector and $u(t) \in R$ the input signal. A_Λ and b_Λ are matrices of appropriate dimensions. This system is called a linear structured system if the entries of the composite matrix $J_\Lambda = [A_\Lambda, b_\Lambda]$ are either fixed zeros or independent parameters (not related by algebraic equations), [10].

For such systems, one can study generic properties in the sense of the previous section, *i.e.* properties which are true for almost any value of the parameters. A matrix Q_Λ with parameterized entries is called a structured matrix and its generic rank will be denoted g-rank(Q_Λ).

A directed graph $G(\Sigma_\Lambda) = (Z, W)$ can be associated with the structured system Σ_Λ of type (3):

- the vertex set is $Z = X \cup U$ where X and U are the state and input sets given by $\{x_1, x_2, \ldots, x_n\}$ and $\{u\}$ respectively,
- the edge set is $W = \{(x_i, x_j) | a_{\Lambda ji} \neq 0\} \cup \{(u, x_j) | b_{\Lambda j} \neq 0\}$, where $a_{\Lambda ji}$ denotes the entry (j, i) of the matrix A_Λ and $b_{\Lambda j}$ the jth entry of b_Λ.

Recall that a path in $G(\Sigma_\Lambda)$ from a vertex i_0 to a vertex i_q is a sequence of edges, $(i_0, i_1), (i_1, i_2), \ldots, (i_{q-1}, i_q)$, such that $i_t \in Z$ for $t = 0, 1, \ldots, q$ and $(i_{t-1}, i_t) \in W$ for $t = 1, 2, \ldots, q$. If $i_0 \in U$ and, $i_q \in X$, the path is called an input-state path. The system Σ_Λ is said to be *input-connected* if for any state vertex x_i, there exists an input-state path with end vertex x_i.

The structural controllability was introduced by Lin who proved the following result.

Theorem 1: [11] Let Σ_Λ be the linear structured system defined by (3) with associated graph $G(\Sigma_\Lambda)$. The system is structurally controllable if and only if:

1) The system Σ_Λ is input-connected,
2) g-rank$[A_\Lambda, b_\Lambda]$=n.

In the following, the conditions 1 and 2 of Theorem 1 will be referred to as the input connection condition and the rank condition, respectively.

A. Structural controllability: refined analysis

1) Input connection condition: Let Σ_Λ be the linear structured system defined by (3) with its associated graph $G(\Sigma_\Lambda)$. Two vertices v_i and v_j of $G(\Sigma_\Lambda)$ are said to be equivalent if there exists a path from v_i to v_j and a path from v_j to v_i. In this context v_i is assumed to be equivalent to itself. The equivalent classes corresponding with this equivalence relation are called the strongly connected components of $G(\Sigma_\Lambda)$. The input vertex u is a strongly connected component composed of a unique vertex. The strongly connected components can be endowed with a natural partial order. The strongly connected components C_i and C_j are such that $C_i \preceq C_j$ if there exists an edge (v_i, v_j) where $v_i \in C_i$ and $v_j \in C_j$. The infimal elements with this order are the strongly connected components with no ingoing edge. Notice that the input vertex is such an infimal element.

Definition 2: An infimal strongly connected component of $G(\Sigma_\Lambda)$ which is not an input vertex, is called a *Critical Connection Component* (CCC). The number of Critical Connection Components is called the *connection defect* of Σ_Λ and denoted by $d_c(\Sigma_\Lambda)$.

One has the following result:

Proposition 1: [16] Let Σ_Λ be the linear structured system defined by (3) with associated graph $G(\Sigma_\Lambda)$. Σ_Λ is input connected if and only if $G(\Sigma_\Lambda)$ has no Critical Connection Component, or equivalently $d_c(\Sigma_\Lambda) = 0$.

2) Rank condition: We will characterize the rank condition, *i.e.* g-rank$[A_\Lambda, b_\Lambda] = n$. This generic rank will be computed using a bipartite graph associated with the system Σ_Λ.

Introduce now the bipartite graph $V(\Sigma_\Lambda)$ as follows.

The bipartite graph associated with the system Σ_Λ is $V(\Sigma_\Lambda) = (V^+, V^-; W')$ where the sets V^+ and V^- are two disjoint vertex sets and W' is the edge set. The vertex set V^+ is given by $X^+ \cup U$, the vertex set V^- is given by X^-, with $X^+ = \{x_1^+, \ldots, x_n^+\}$ the first set of state vertices, $X^- = \{x_1^-, \ldots, x_n^-\}$ the second set of state vertices and $U = \{u\}$ the input vertex. Notice that here we have split each state vertex x_i of $G(\Sigma_\Lambda)$ into two vertices x_i^+ and x_i^-. The edge set W' is described by $W_A \cup W_b$ with $W_A = \{(x_j^+, x_i^-) | a_{\Lambda ij} \neq 0\}$ and $W_b = \{(u, x_i) | b_{\Lambda i} \neq 0\}$. In the latter, for instance $a_{\Lambda ij} \neq 0$ means that the (i, j)-th entry of the matrix A_Λ is a parameter (structurally nonzero).

A *matching* in a bipartite graph $V = (V^+, V^-; W')$ is an edge set $M \subseteq W'$ such that the edges in M have no common vertex. The cardinality of a matching, *i.e.* the number of edges it consists of, is also called its size. A matching M is called maximum if its cardinality is maximum. The maximum matching problem is the problem of finding such a matching of maximal cardinality. Recall the follwing proposition:

Proposition 2: [16] Let Σ_Λ be the linear structured system defined by (3) with associated bipartite graph $V(\Sigma_\Lambda)$. The generic rank of $[A_\Lambda, b_\Lambda]$ is equal to the size of a maximal matching in $V(\Sigma_\Lambda)$. In particular, g-rank$[A_\Lambda, b_\Lambda] = n$ if and only if there exists a size n matching in $V(\Sigma_\Lambda)$.

A useful tool to parameterize all the maximal matchings in a bipartite graph is the Dulmage-Mendelsohn decomposition which will be presented now.

The DM-Decomposition allows to decompose a bipartite graph $V = (V^+, V^-; W')$ into a uniquely defined family of bipartite subgraphs $V_i = (V_i^+, V_i^-; W_i')$, $i = 0, 1, \ldots, r, \infty$ called the DM-components, where V_i^+ (resp. V_i^-; W_i') is a partition of V^+ (resp. V^-; W'). V_0 is called minimal inconsistent part, V_∞ is called maximal inconsistent part and the rest consistent parts. These components have the following properties:

Proposition 3: [14] Let $V = (V^+, V^-; W')$ be a bipartite graph and its DM-Decomposition with $V_i = (V_i^+, V_i^-; W_i')$, $i = 0, 1, \ldots, r, \infty$ its DM-Components. One has the following properties:

1. A maximum matching on V is a union of maximum matchings on the DM-Components V_i, $i = 0, 1, \ldots, r, \infty$.
2. A vertex $v \in V_0^-$ (or V_i^+, V_i^-, $i = 1, \ldots, r$ or V_∞^+) is covered by any maximum matching on V.
3. A vertex $v \in V^+$ belongs to the minimal inconsistent part V_0^+ if and only if there exists a maximum matching on V that does not cover v.
4. A vertex $v \in V^-$ belongs to V_∞^- if and only if there exists a maximum matching on V that does not cover v.

The rank condition can then be expressed using only the maximal inconsistent part of the DM decomposition as follows:

Proposition 4: [17] Let Σ_Λ be the linear structured system defined by (3) with associated bipartite graph $V(\Sigma_\Lambda)$ and the corresponding DM-decomposition. One has $n-$g-rank$[A_\Lambda, b_\Lambda] = card(V_\infty^-(\Sigma_\Lambda)) - card(V_\infty^+(\Sigma_\Lambda))$. In particular, g-rank$[A_\Lambda, b_\Lambda] = n$ if and only if $V_\infty(\Sigma_\Lambda) = \emptyset$.

The result follows from point 4 of Proposition 3.

IV. Application to the Minimal Controllability Problem

A. Structural effect of an input addition

We start with the structured matrix A_Λ with associated graph $G(A_\Lambda)$ and associated bipartite graph $V(A_\Lambda)$. We will study the effect of adding an input u on the input connection and rank conditions.

State now two technical propositions which are similar to results of [8].

Proposition 5: Concerning input connection, adding the input u will produce the following effect:

The Critical Connection Components of $G(\Sigma_\Lambda)$ are the same as in $G(A_\Lambda)$ except those for which there exists an edge from u to this CCC, and $d_c(\Sigma_\Lambda) = d_c(A_\Lambda) - \alpha$, where α is the number of CCC's of $G(A_\Lambda)$ such that there exists an edge (u, x_i), where x_i belongs to a CCC of $G(A_\Lambda)$.

Proof: For any Critical Connection Component C_j such that there exists an edge (u, x_i), where $x_i \in C_j$, C_j is no longer a Critical Connection Component. The other Critical Connection Components are unchanged and the result follows.■

Proposition 6: Concerning the rank condition, adding the input vertex u and the associated edges (u, x_i) to $G(A_\Lambda)$, when g-rank$(A_\Lambda) = n - 1$, will produce the following effect:

If there is an edge (u, x_i) in $G(A_\Lambda)$, where x_i is such that x_i^- belongs to the maximal inconsistent part $V_\infty(A_\Lambda)$ of the DM-decomposition of $V(A_\Lambda)$, then $V_\infty(\Sigma_\Lambda) = \emptyset$ and g-rank$[A_\Lambda, b_\Lambda] = n$. If there is no such edge then g-rank$[A_\Lambda, b_\Lambda] = n - 1$.

Proof: Note first that adding input u and associated edges in $V(A_\Lambda)$ cannot decrease the rank.

If there is an edge (u, x_i), where x_i is such that x_i^- belongs to the maximal inconsistent part $V_\infty(A_\Lambda)$ of the DM-decomposition of $V(A_\Lambda)$, from point 4 of Proposition 3 there exists a maximum matching on $V(A_\Lambda)$ that does not cover x_i^-. This matching together with (u, x_i^-) provides with a matching of size n in $V(A_\Lambda, b_\Lambda)$, then g-rank$[A_\Lambda, b_\Lambda] = n$.

Suppose now that there exists a size n matching in $V([A_\Lambda, b_\Lambda])$. This matching contains an edge (u, x_i^-). Removing this edge induces a size $n-1$ (then maximal) matching in $V(A_\Lambda)$. From point 4 of Proposition 3 this implies that x_i^- belongs to the maximal inconsistent part $V_\infty(A_\Lambda)$. Therefore if there is no edge (u, x_i), where x_i is such that x_i^- belongs to the maximal inconsistent part $V_\infty(A_\Lambda)$ of the DM-decomposition, g-rank$[A_\Lambda, b_\Lambda] = n - 1$.■

It follows from the Propositions 5 and 6 that the minimal number of non-zero entries of b_Λ to ensure both input connection and rank conditions is less than or equal to $d_c(A_\Lambda) + 1$.

Denote by $\bar{V}_\infty$ the set of x_i's such that x_i^- belongs to the maximal inconsistent part $V_\infty(A_\Lambda)$ of the DM-decomposition of $V(A_\Lambda)$ and $\bar{C} = \cup C_j$ for $j = 1, \ldots, d_c$ the union of Critical Connection Components. We are now able to state the main result of this paper:

Theorem 2: Let A_Λ be a $n \times n$ structured matrix with associated graph $G(A_\Lambda)$ and associated bipartite graph $V(A_\Lambda)$. The Minimal Controllability Problem is solvable for A_Λ if and only if g-rank$(A_\Lambda) \geq n - 1$. Moreover:

- when g-rank$(A_\Lambda) = n$, then $\nu^* = d_c(A_\Lambda)$,
- when g-rank$(A_\Lambda) = n-1$, if $\bar{C} \cap \bar{V}_\infty = \emptyset$ then $\nu^* = d_c(A_\Lambda)+1$ else $\nu^* = d_c(A_\Lambda)$.

Proof: From Corallary 1, g-rank$(A_\Lambda) \geq (n - 1)$ is a necessary condition for the solvability of the MCP.

When g-rank$(A_\Lambda) = n$, we have only to check the input connection condition. From Proposition 5, the minimal number of non-zero entries of b_Λ is $\nu^* = d_c(A_\Lambda)$.

When g-rank$(A_\Lambda) = n - 1$, there are two cases. If there is no vertex x_i of a Critical Connection Component of $G(A_\Lambda)$ such that x_i^- belongs to the maximal inconsistent part, $V_\infty^-(A_\Lambda)$, the $d_c(A_\Lambda)$ edges needed to satisfy the input connection condition will be of no use for the rank condition and another edge is needed for the rank condition, therefore the minimal number of non-zero entries of b_Λ is $\nu^* = d_c(A_\Lambda) + 1$. In the other case, there is an edge (u, x_i) such that x_i^- belongs to the maximal inconsistent part $V_\infty(A_\Lambda)$ of the DM-decomposition, and from Proposition 6 the rank condition is also satisfied, then $\nu^* = d_c(A_\Lambda)$.■

Notice that this Theorem also gives the possible locations for the non-zero entries of b_Λ to ensure the controllability in the MCP context.

Solving the MCP mainly implies the computation of the Critical Connection Components of $G(A_\Lambda)$ and of the DM-decomposition of $V(A_\Lambda)$. The decomposition of a graph into strongly connected components is a standard combinatorial problem for which very efficient polynomial algorihms are available. The DM-decomposition essentially implies the determination of a particular maximal matching [18] completed by an alternate chain technique, see the details in [14]. It follows that solving the MCP for structured systems is a polynomial problem.

Example 1: As mentioned above, Olshevsky has shown in [9] that it is NP-hard to compute a sparsest b for the MCP of Definition 1. As an illustration, consider the example given by Olshevsky where the A matrix of system (1) is

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -7/2 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & -5/2 \\ 3/4 & 1/2 & 0 & 4 & 0 & 0 & 0 & 13/8 \\ 0 & 3/4 & 1/2 & 0 & 5 & 0 & 0 & 11/8 \\ 5/4 & 0 & 3/4 & 0 & 0 & 6 & 0 & 3/2 \\ 3/2 & 5/4 & 1 & 0 & 0 & 0 & 7 & 9/4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 8 \end{pmatrix}.$$

A is full rank so the MCP has a solution. It is shown in [9] that

$\nu^* = 3$. There exist several minimal solutions for b that make the system controllable, they all have the b_8 entry different from zero. Now look at a structured system of the form (3) which has the same zero/non-zero structure as the above matrix A. The corresponding graph $G(A_\Lambda)$ is depicted in Figure 1. The existing loops covering all state vertices imply the vacuity of the maximal inconsistent part of the DM decomposition of $V(A_\Lambda)$ and the full generic rank of A_Λ, then $\nu^* = d_c(A_\Lambda)$. There is only one Critical Connection Component $\{x_8\}$, so $d_c(A_\Lambda) = 1$. The solution b_Λ to the MCP for the structured matrix A_Λ is then $b_{\Lambda 8} \neq 0$ and $b_{\Lambda i} = 0$ for $i = 1, \ldots, 7$. It follows that for almost any value of the parameters with the structure corresponding with A_Λ, MCP is solvable in polynomial time in the strucured case with $\nu^* = 1$ and that state x_8 must be impacted by the input. In the standard (non structured) case, to ensure the controllability, for some particular numerical values of the entries of A, two other entries of b must be non-zero.

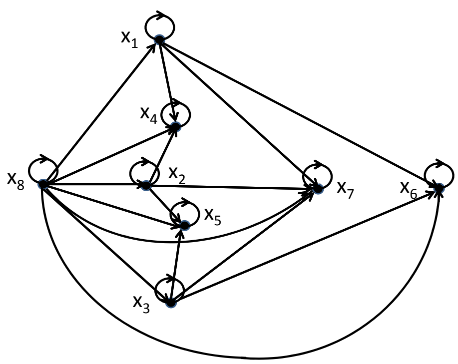

Fig. 1. Example of Olshevsky

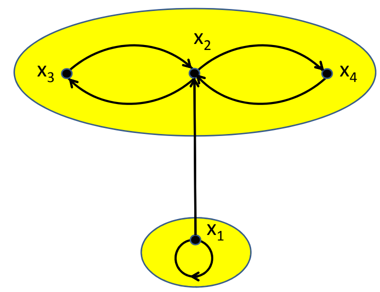

Fig. 2. Graph of Example 2

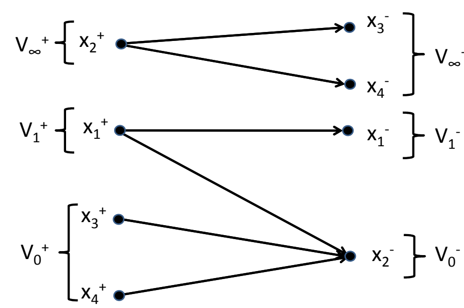

Fig. 3. DM decomposition of Example 2

Example 2: Let A_Λ be a (4×4) structured matrix for which the associated graph $G(A_\Lambda)$ is depicted in Figure 2. $G(A_\Lambda)$ possesses two strongly connected components $\{x_1\}$ and $\{x_2, x_3, x_4\}$. $\{x_1\}$ is the Critical Connection Component then $d_c(A_\Lambda) = 1$. The DM decomposition of $V(A_\Lambda)$ is shown in Figure 3. The maximal size of a matching in $V(A_\Lambda)$ is 3, then the generic rank of A_Λ is equal to 3 and the MCP is solvable. Since there is no vertex x_i of the Critical Connection Component $\{x_1\}$ such that x_i^- is in $V_\infty^-(A_\Lambda) = \{x_3^-, x_4^-\}$, then by Theorem 2, $\nu^* = d_c(A_\Lambda) + 1 = 2$. The non-zero entries of b_Λ to ensure structural controllability are $b_{\Lambda 1}$ for the input connection condition and either $b_{\Lambda 3}$ or $b_{\Lambda 4}$ for the rank condition. A MCP solution for this example is given in Figure 4.

Example 3: Consider now another example, illustrated in Figure 5, which differs from the previous one by the addition of the edge (x_1, x_4). It turns out that the Critical Connection Component $\{x_1\}$ is unchanged. The DM decomposition of $V(A_\Lambda)$ is shown in Figure 6. The maximal size of a matching in $V(A_\Lambda)$ is 3, then the MCP is solvable. The vertex x_1 of the Critical Connection Component $\{x_1\}$ is such that x_1^- is in $V_\infty^-(A_\Lambda) = \{x_1^-, x_3^-, x_4^-\}$, then by Theorem 2, $\nu^* = d_c(A_\Lambda) = 1$. In this case a unique non-zero entry $b_{\Lambda 1}$ of b_Λ is sufficient to ensure both input connection and rank conditions.

V. Concluding remarks

This paper presents a structural analysis of the Minimal Controllability Problem (MCP). This problem comes down to controlling a linear system with an input vector having as few non-zero entries as possible. We proved that MCP is solvable when a rank condition is satisfied and gave the minimum number and the possible location of non-zero entries in the input vector to ensure generic controllability. A simple polynomial MCP solution is given for structured systems. This contrasts with MCP for standard linear systems where the problem turns out to be NP-hard [9] and where additional non-zero entries in the input vector may be necessary for controllability. The proposed graph approach allows to visualize the structural mechanisms that make it useful to act on some variables to ensure generic controllability.

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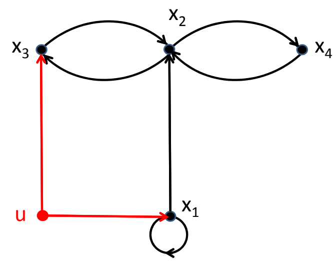

Fig. 4. A solution for Example 2

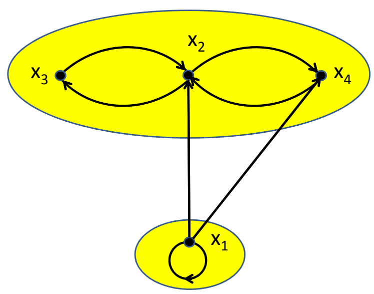

Fig. 5. Graph of Example 3

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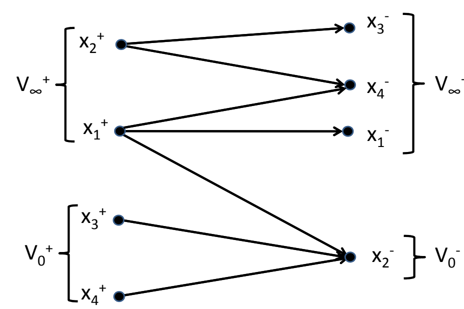

Fig. 6. DM decomposition of Example 3