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# STRESSES IN MIXTURE THEORY

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**Abstract**—The foundations of mixture theory are formulated using a geometrical approach. In order to model diffusion, configurations of mixtures are allowed in which the various constituents may occupy different regions in space in addition to the usual relaxation of the impenetrability axiom. Forces on a mixture are defined as continuous linear functional on the space of virtual displacements of the mixture. This implies the existence of partial stresses without any further assumptions. The notions of the total force and the total stress are critically reviewed and introduced through the definition of a simple body model of a mixture. Some examples of such models are given.

## 1. INTRODUCTION

Mixture theory is the branch of continuum mechanics that describes the mechanics of diffusion. In contradiction with the axiom of material impenetrability that is usually assumed in mechanics, mixture theory allows a number of bodies, called constituents, to penetrate into one another while at the same time the subbodies of each particular constituent do not interpenetrate.

The main notions regarding forces introduced in mixture theory are those of partial stresses and the total stress. For the case of statics, to which we confine ourselves in this paper, the total stress is the sum of the partial stresses. These notions are typical to the dual character of mixture theory. On the one hand we wish to regard each constituent as a body, hence a stress is associated with it, and on the other hand we wish to regard the whole mixture as a body, hence the total stress is associated with it.

This tendency in mixture theory was first stated formally by Truesdell [1] in the *Three Metaphysical Principles*:

- (1) *All properties of the mixture must be mathematical consequences of the properties of the constituents.*
- (2) *So as to describe the motion of a constituent, we may in imagination isolate it from the rest of the mixture, provided we allow properly for the actions of the other constituents upon it.*
- (3) *The motion of the mixture is governed by the same equations as is a single body.*

It seems to us that the first principle can be regarded as a definition of a mixture and the other two are the two faces of mixture theory mentioned above.

Trying to proceed along the point of view that each constituent behaves as a single body and deduce the existence of partial stresses, one faces a difficulty: The form of forces in terms of body forces and surface forces and the state of equilibrium that are usually assumed in continuum mechanics in proving the existence of stresses may no longer be applicable for a constituent.

Several attempts to overcome this difficulty were made. The most recent one, Sampaio [3], presents an axiomatic theory of mixtures in which the existence of the partial stresses and various other features of mixture theory are obtained on the basis of assumptions regarding the external forces and the mutual forces between the constituents (see also [4–7]). The following describe roughly the assumptions made by Sampaio.

- (1) Forces are bounded linear functionals on the space of velocity fields with respect to the norm of uniform convergence.

- (2) The force on each body is composed of an external force and forces exerted by the other constituents.
- (3) The mutual forces between the constituents and the external forces are biadditive.
- (4) The force exerted by the  $i$ th constituent of the body  $B$  on the  $j$ th constituent of the body  $A$  is equal in magnitude and opposite in direction to the force exerted by the  $j$ th constituent of the body  $A$  on the  $i$ th constituent of the body  $B$ .
- (5) The sum of all forces and moments acting on each constituent is zero. In this sum Sampaio includes the external forces, the interaction with the various constituents outside the body and the interaction with other constituents within the body.
- (6) Both the interactions between the constituents and the external forces are local in the sense that interactions at a distance are excluded.
- (7) By the previous assumptions, in particular by assumption (1), forces can be represented by measures. It is assumed that the singular parts of these measures are concentrated on the boundaries of the bodies. This assumption is made for the external forces, the interactions between the constituents of different bodies, and the interactions between the various constituents of the same body. Moreover, these singular parts of the forces are assumed to be absolutely continuous with respect to the common area measure of the interacting bodies.
- (8) Using assumption (7) the author proves that forces are composed of body forces and surface forces. In order to prove the stress principle he further assumes that the body forces are essentially bounded with respect to the volume measure and the surface forces are essentially bounded with respect to the area measure.

We note that the fact that in Sampaio's and most other formulations of mixture theory all constituents occupy the same region in space means that there is no diffusion at the boundaries. If the velocities of the various constituents differ at the boundary, the constituents will occupy different regions in space after any finite time interval. This restriction seems even stronger when we notice that if subbodies of the various constituents that occupy the same region in space at one time will occupy the same region in space at all times, no diffusion at the boundary of any subbody will occur. In other words, either the assumption regarding the constituents does not hold for the subconstituents or diffusion is prohibited altogether.

Here, we present another approach that is based on a general definition of a force in continuum mechanics as an element of the cotangent bundle of an appropriate configuration manifold [8, 9]. This approach will be reviewed in Section 2 and in Section 3 we will apply it to the statics of mixtures.

In Section 4 we try to formalize Truesdell's third principle and the notion of total stress to which it leads. It seems to us that the total stress has meaning only in some limited cases. For example, if we have a vessel filled with a mixture of gases we expect the traction on the walls to be that induced by the sum of the partial stresses. However, if some of the gases can diffuse through the walls of the vessel, the traction acting on the walls will be different than that induced by the "total stress". Roughly, the notion of a total stress will not have meaning when the constituents act independently of one another.

Thus, rather than assuming that the mixture behaves as a single body, we introduce simple body models of a mixture, the validity of which is decided upon by the particular case at hand. For a few examples of simple body models of a mixture, we show the relation between the partial stresses and the total stress.

## 2. FORCES ON SIMPLE BODIES

In this section we review the basic structure of the mechanics of simple bodies, i.e. bodies having only one constituent.

**DEFINITION 2.1.** A *simple body* is the closure of a bounded open subset of  $\mathcal{R}^3$  having a smooth boundary. The *physical space* is the Euclidean space  $\mathcal{R}^3$ . A body  $P$  which is a subset of the body  $B$  is a *subbody* of  $B$ . Given a body  $B$  we will denote the collection of its subbodies by  $\mathbf{B}$ .

DEFINITION 2.2. A *configuration* of the body  $B$  is a  $C^1$  embedding  $\kappa: B \rightarrow \mathcal{R}^3$ .

This definition of a configuration of a body is clearly in agreement with the principle of impenetrability which is traditionally postulated in continuum mechanics.

PROPOSITION 2.3. The configuration space  $\mathcal{Q}_B$  of the body  $B$ , i.e. the set of all configurations of  $B$ , is an open subset of the Banach space  $C^1(B, \mathcal{R}^3)$  of continuously differentiable mappings  $B \rightarrow \mathcal{R}^3$  equipped with the norm

$$\|u\| = \max \left\{ \sup_{X \in B} |u(X)|, \sup_{X \in B} |Du(X)| \right\}.$$

For the proof of this proposition see [10]. We note that if we use the norm of uniform convergence proposition 2.3 does not hold and for any configuration  $\kappa$  there exist mappings which are arbitrarily close to  $\kappa$  that violate the principle of impenetrability.

Proposition 2.3 implies that as an open subset of the Banach space  $C^1(B, \mathcal{R}^3)$ ,  $\mathcal{Q}_B$  has a differentiable structure of a Banach manifold and that its tangent space  $T_\kappa \mathcal{Q}_B$  at any configuration  $\kappa$  in  $\mathcal{Q}_B$  is naturally isomorphic to  $C^1(B, \mathcal{R}^3)$ . We recall that the elements of  $T_\kappa \mathcal{Q}_B$  can be thought of as Lagrangian *velocity fields* or *virtual displacements*.

DEFINITION. 2.4. A *force* on the simple body  $B$  is an element of the cotangent bundle  $T^* \mathcal{Q}_B$ . Thus, a force on  $B$  is a continuous linear functional defined on  $T_\kappa \mathcal{Q}_B$  for some  $\kappa \in \mathcal{Q}_B$ , or in other words, forces can be naturally identified with elements of the dual space  $C^1(B, \mathcal{R}^3)^*$ . Given a force  $f$  and a virtual displacement  $u$ , the evaluation  $f(u)$  will be referred to as the virtual work performed by the force  $f$  through the virtual displacement  $u$ .

Let  $C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))$  denote the space of continuous mappings

$$B \rightarrow \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3),$$

where  $L(\mathcal{R}^3, \mathcal{R}^3)$  is the vector space of linear transformations  $\mathcal{R}^3 \rightarrow \mathcal{R}^3$ , equipped with the norm of uniform convergence. We define the mapping

$$D^1: C^1(B, \mathcal{R}^3) \rightarrow C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))$$

by  $D^1(u) = (u, Du)$ , where  $Du$  is the derivative of  $u$ .

PROPOSITION 2.5. Any force  $f \in C^1(B, \mathcal{R}^3)^*$  is given in the form  $f = D^{1*}(S)$ , for some  $S \in C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))^*$ , where  $D^{1*}$  is the adjoint map of  $D^1$ . Conversely, any element  $S \in C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))^*$  represents a unique force  $f \in C^1(B, \mathcal{R}^3)^*$  by the equation  $f = D^{1*}(S)$ .

PROOF. The mapping  $D^1$  is linear, injective, continuous and norm preserving. Thus, it is an isomorphism of  $C^1(B, \mathcal{R}^3)$  onto a closed subspace of  $C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))$ . It follows that the mapping  $D^{1*}: C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))^* \rightarrow C^1(B, \mathcal{R}^3)^*$  is onto so that every force is of the required form.

An element  $S \in C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))^*$  will be referred to as a *stress representation* and if  $f = D^{1*}(S)$ , for  $f \in C^1(B, \mathcal{R}^3)^*$ , we say that  $S$  represents  $f$ . Thus,  $S$  represents  $f$  if

$$f(u) = S(u, Du) \quad \text{for all } u \in C^1(B, \mathcal{R}^3).$$

Identifying  $C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))$  with  $C^0(B, \mathcal{R})^{12}$ , we can clearly identify the space  $C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))^*$  with  $C^0(B, \mathcal{R})^{*12}$  which by Riesz representation theorem is the space of collections of 12 Radon measures over the body. Thus, for each stress representation  $S$  there exist 12 Radon measures  $\sigma_i, \tau_{il}, i, l = 1, 2, 3$ , such that for any element  $v$  of  $C^0(B, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))$  whose components are  $v_i, v_{il}$ ,

$$S(v) = \int_B v_i d\sigma_i + \int_B v_{il} d\tau_{il},$$

where we use the summation convention. Using proposition 2.5 we have proven the following proposition.

PROPOSITION 2.6. Every force acting on the body  $B$  can be represented by 12 Radon measures  $\sigma_i, \tau_{il}$ , called the *stress measures*, in the form

$$F(u) = \int_B u_i d\sigma_i + \int_B u_{i,l} d\tau_{il},$$

$u \in C^1(B, \mathcal{R}^3)$ , where  $u_i$  are the components of  $u$  and a comma denotes partial differentiation. Conversely, any collection of 12 measures represents by the equation above a unique force on  $B$ .

Since the image of the mapping  $D^1$  is not a dense subset of  $C^0(B, \mathcal{R})^{12}$  its adjoint is not injective. This fact explains the lack of uniqueness in the relation between forces and stresses.

We note that any given collection of stress measures can be restricted to subbodies of  $B$ . Given a subbody  $P$  of  $B$ , the restrictions of the stress measures to  $P$  induce a unique force  $f_P$  on  $P$  that is represented in the form

$$f_P(w) = \int_P w_i d\sigma_i + \int_P w_{i,l} d\tau_{il},$$

for any  $w \in C^1(P, \mathcal{R}^3)$ . This possibility to restrict forces to subbodies is the reason for the representation of forces by stresses.

PROPOSITION 2.7. Assume that the stress measures are given in terms of the differentiable densities  $s_i, T_{ij}$ , defined on  $\kappa(B)$  and called *Cauchy's stress densities*, in the form

$$\int_B u_i d\sigma_i + \int_B u_{i,l} d\tau_{il} = \int_{\kappa(B)} ((u \circ \kappa^{-1})_i s_i + (u \circ \kappa^{-1})_{i,j} T_{ij}) dv.$$

Then, the force on any subbody  $P$  is given by a body force field  $b : \kappa(P) \rightarrow \mathcal{R}^3$  and a surface force field  $t : \partial\kappa(P) \rightarrow \mathcal{R}^3$  in the form

$$f_P(w) = \int_{\kappa(P)} (w \circ \kappa^{-1})_i b_i dv + \int_{\partial\kappa(P)} (w \circ \kappa^{-1})_i t_i da.$$

In addition, in such a case, the stress  $S$  represents the force  $f_P$  if and only if

$$\begin{aligned} T_{ij,j} + b_i &= s_i \quad \text{in } \kappa(P), \\ T_{ij} n_j &= t_i \quad \text{on } \partial\kappa(P), \end{aligned}$$

where  $n_j$  are the components of the unit normal to the boundary of  $\partial\kappa(P)$ . Here,  $v$  is the volume measure on  $\kappa(P)$  and  $a$  is the area measure on  $\partial\kappa(P)$ .

We note that by the properties of the configuration  $\kappa$ , the measures are completely continuous with respect to the volume measure in  $\kappa(B)$  if and only if they are completely continuous with respect to the volume measure in  $B$ . In addition, since  $u_{i,l} = u_{i,j} \kappa_{j,l}$ , the introduction of  $u_{i,j}$  instead of  $u_{i,l}$  in the above equation is insignificant. The proposition can be easily proven by integrating by parts the term  $u_{i,j} T_{ij}$  in the above equation and using Gauss' theorem.

NOTATION. Using the notation  $\sigma$  and  $\tau$  for the collections  $\{\sigma_i\}, \{\tau_{il}\}$  respectively, we write the expression on the right hand side of the equation in Proposition 2.6 as

$$\int_B u \cdot d\sigma + \int_B Du \cdot d\tau.$$

Denoting the collections of densities  $\{s_i\}$  and  $\{T_{ij}\}$  by  $s$  and  $T$ , respectively, we write

$$(u \circ \kappa^{-1})_i \circ s_i + (u \circ \kappa^{-1})_{i,j} T_{ij}$$

as

$$u \circ \kappa^{-1} \cdot s + (Du \circ \kappa^{-1} D\kappa^{-1}) \cdot T,$$

where

$$D(u \circ \kappa^{-1}) = Du \circ \kappa^{-1} D(\kappa^{-1}) = Du \circ \kappa^{-1} D\kappa^{-1} \circ \kappa^{-1}.$$

Here and in Section 4 we use the following convention: (1) Composition of functions precedes all other operations and the operation denoted by a dot (“ $\cdot$ ”) is preceded by all other operation except for addition. (2) If  $A$  and  $B$  are two functions defined in the same domain whose values are linear mappings,  $AB$  is the mapping whose value at any point is the composition of the values of  $A$  and  $B$ . If  $u$  is a vector field, then  $Au$  will denote the vector field obtained by pointwise evaluation. As we have mentioned these operations are preceded by composition of functions. (3)  $D\kappa^{-1}$  is the function defined on  $B$  whose value at any point is the inverse of the value of  $D\kappa$  at that point.

#### REMARKS

- (1) We note that with the suggested structure the existence of stresses as entities that allow the restriction of forces to subbodies is an immediate result of the definition of forces. Stresses may be as irregular as measures and the condition  $f = D^{1*}(S)$  is the most general form of the “equilibrium equations” in continuum mechanics.
- (2) The traditional assumption that forces are composed of body forces and surface forces was obtained mathematically using the additional mild assumption of smoothness of the representing measures. Cauchy’s formula and the equilibrium equation, with the additional term  $s_i$ , was obtained without using any equilibrium assumption.
- (3) Assuming that each subbody is in equilibrium one can easily prove that  $s_i = 0$  and that  $T_{ij}$  is symmetric.
- (4) The structure that we introduced here may be generalized to the geometry of differentiable manifolds and to the case of continuum mechanics of grade higher than one (see [9]).

**DEFINITION 2.8.** A *force system* is a mapping  $\mathbf{F}: \mathbf{B} \rightarrow T^*\mathbf{Q}$ . We say that a force system is *consistent* with the stress measures  $\sigma_i, \tau_{ij}$  if the force on each subbody  $P$  is represented by the restrictions of the stress measures to  $P$ . It can be shown (see [9]) that there is at most one collection of stress measures  $\sigma_i, \tau_{ij}$  that is consistent with a force system and so there is a one to one relation between consistent force systems and stresses.

**DEFINITION 2.9.** A *loading* is a section  $\varphi: \mathbf{Q} \rightarrow T^*\mathbf{Q}$ , i.e., it is a mapping that assigns a force to any configuration. A *constitutive relation* is a mapping

$$\psi: \mathbf{Q} \rightarrow C^0(B, \mathbb{R}^3 \times L(\mathbb{R}^3, \mathbb{R}^3))^*$$

and it gives the stress measures  $\sigma_i, \tau_{ij}$ , corresponding to any configuration. We can now formulate the problem of continuum mechanics. Given  $\varphi$  and  $\psi$  find  $\kappa \in \mathbf{Q}$  such that the force represented by  $\psi(\kappa)$  is  $\varphi(\kappa)$ . Clearly, a solution of the problem will give us the force acting on each subbody of  $B$  at the configuration  $\kappa$ .

### 3. STRESSES IN MIXTURES

As we mentioned in the introduction the basic idea that we follow is that with the definition of a force as an element of the cotangent bundle of the configuration space, the properties of forces follow from the kinematical properties of the system. For example the choice of the topology for the configuration space for simple bodies which lead to all the properties of forces and stresses was a result of the principle of impenetrability. It seems to us that it is the first instance where the principle of impenetrability is directly related with the stress theory. Similarly, we will show here how some properties of forces and stresses in mixtures, in particular, the existence of partial stresses, are related to the fact that the various constituents may interpenetrate.

**DEFINITION 3.1.** A *mixture of sets* is a set  $M$  together with a surjective mapping  $\mu: M \rightarrow A$  where  $A$  is called the *indexing set* and  $\mu$  is called the *constituent mapping*. For  $\alpha \in A$ ,  $\mu^{-1}(\alpha)$  is called the  $\alpha$  *constituent* of the mixture. It may happen that the various constituents are subsets of the same set ( $\mathbb{R}^3$  for example) but clearly, they are always disjoint subsets of  $M$ .

Alternatively, a mixture could have been defined as a partition of  $M$  (which might be generated by an equivalence class on  $M$ ). The restriction of a function  $t$  defined on  $M$  to the  $\alpha$  constituent will be called the  $\alpha$  component of the function and will be denoted by  $t_\alpha$ . Clearly, functions defined on the constituents of a mixture induce a unique function on the mixture. It follows that the set  $C^M$  of all functions  $t: M \rightarrow C$  ( $C$  is an arbitrary set) is isomorphic to the product  $\prod_{\alpha \in A} C^{\mu^{-1}(\alpha)}$ .

If for a mixture  $M$ , the constituents are topological spaces, the collection of open sets of all the constituents can serve as a basis for a topology on  $M$  that will be referred to as the *mixture topology*. The mixture topology is clearly the finest topology that makes all the inclusions of the various constituents in  $M$  continuous. A function defined on  $M$  is continuous with respect to the mixture topology if its restrictions to the various constituents are continuous. Conversely, a collection of continuous functions on the constituents defines a continuous function on the mixture.

**DEFINITION 3.2.** A *mixture of bodies* is a mixture of sets  $M$  whose constituents are simple bodies and whose indexing set is  $\{1, 2, \dots, a\}$ . We will denote the  $\alpha$  constituent of  $M$  and by  $B_\alpha$  and the configuration space of  $B_\alpha$  by  $\mathcal{Q}_\alpha$ . If  $N, M \supset N$ , is a mixture of bodies such that for each  $\alpha$ , the  $\alpha$  constituent of  $N$  is a subbody of the  $\alpha$  constituent of  $M$ , we say that  $N$  is a *submixture* of  $M$ . The collection of all submixtures of  $M$  will be denoted by  $\mathbf{M}$ .

We note that the mixture topology for a mixture of bodies induces on  $M$  a structure of a 3-dimensional compact differentiable manifold with a boundary which is not a subset of  $\mathcal{R}^3$ .

**DEFINITION 3.3.** A *mixed configuration* of  $M$  is a mapping  $\kappa: M \rightarrow \mathcal{R}^3$  whose components are configurations of the constituents. We will denote the  $\alpha$  component of  $\kappa$  by  $\kappa_\alpha$ . Thus, the configuration space of the mixture,  $\mathcal{Q}_M$ , can be identified with  $\mathcal{Q}_1 \times \mathcal{Q}_2 \times \dots \times \mathcal{Q}_a$  endowed with the product topology. we will denote by  $\pi_\alpha$  the projection  $\mathcal{Q}_M \rightarrow \mathcal{Q}_\alpha$ .

Note that the basic property of mixtures is that we do not exclude mixed configurations such that for distinct  $\alpha, \beta \in \{1, \dots, a\}$ ,  $\kappa_\alpha(B_\alpha) \cap \kappa_\beta(B_\beta) \neq \emptyset$ . Since each  $\mathcal{Q}_\alpha$  is an open subset of  $C^1(B_\alpha, \mathcal{R}^3)$  it follows that  $\mathcal{Q}_M$  is an open subset of  $\prod_\alpha C^1(B_\alpha, \mathcal{R}^3)$  and thus it has the induced differentiable structure. In particular, for any mixed configuration  $\lambda$ ,  $T_\lambda \mathcal{Q}_M$ , can be identified with  $\prod_\alpha C^1(B_\alpha, \mathcal{R}^3)$ .

The previous definition does not require that all the constituents will occupy the same region in space as for example is assumed in [3]. As we mentioned in the introduction such a requirement is inconsistent with processes in which diffusion occur.

**DEFINITION 3.4.** A *force* on a mixture of bodies  $M$  is an element of the cotangent bundle  $T^* \mathcal{Q}_M$ .

It follows that forces on the mixture can be identified with elements of

$$\{\prod_\alpha C^1(B_\alpha, \mathcal{R}^3)\}^* = \prod_\alpha C^1(B_\alpha, \mathcal{R}^3)^*.$$

**PROPOSITION 3.5.** Any force  $g \in T^* \mathcal{Q}_M$  can be represented in the form

$$g(u) = \sum_\alpha \int_{B_\alpha} u_\alpha \cdot d\sigma_\alpha + \int_{B_\alpha} D u_\alpha \cdot d\tau_\alpha,$$

where  $u_\alpha$  is the virtual displacement of the  $\alpha$  constituent (the  $\alpha$  component of  $u$ ),  $\sigma_\alpha$  is a collection of three Radon measures defined on  $B_\alpha$  and  $\tau_\alpha$  is a collection of 9 Radon measures defined on  $B_\alpha$ ,  $\alpha = 1, \dots, a$ . The measures  $\sigma_\alpha, \tau_\alpha$  are the *partial stress measures* of the  $\alpha$  constituent.

**PROOF.** Since  $\{\prod_\alpha C^1(B_\alpha, \mathcal{R}^3)\}^* = \prod_\alpha C^1(B_\alpha, \mathcal{R}^3)^*$  via the relation

$$g(u) = \sum_\alpha g_\alpha(u_\alpha),$$

where  $g_\alpha \in C^1(B_\alpha, \mathcal{R}^3)^*$  and  $u_\alpha$  is the virtual displacement of the  $\alpha$  constituent, the assertion follows by the results of the previous section.

As in the case of simple bodies, the stress measures induce forces on the submixtures by

restricting the stresses to the subconstituents, i.e. by

$$g_P(u) = \sum_{\alpha} \int_{P_{\alpha}} u_{\alpha} \cdot d\sigma_{\alpha} + \int_{P_{\alpha}} Du_{\alpha} \cdot d\tau_{\alpha},$$

**PROPOSITION 3.6:** Assume that the partial stress measures  $\sigma_{\alpha}, \tau_{\alpha}$  are given in terms of the differentiable densities  $s_{\alpha} : \kappa_{\alpha}(B_{\alpha}) \rightarrow \mathcal{R}^3$  and  $T_{\alpha} : \kappa_{\alpha}(B_{\alpha}) \rightarrow L(\mathcal{R}^3, \mathcal{R}^3)$  in the form

$$\int_{B_{\alpha}} u_{\alpha} \cdot d\sigma_{\alpha} + \int_{B_{\alpha}} Du_{\alpha} \cdot d\tau_{\alpha} = \int_{\kappa_{\alpha}(B_{\alpha})} (u_{\alpha} \circ \kappa_{\alpha}^{-1} \cdot s_{\alpha} + D(u_{\alpha} \circ \kappa_{\alpha}^{-1}) \cdot T_{\alpha}) dv.$$

Then, the force  $g$  on any submixture  $N = \{P_{\alpha}; B_{\alpha} \supset P_{\alpha}, \alpha = 1, \dots, a\}$  is given by the partial body force fields  $b_{\alpha} : \kappa_{\alpha}(P_{\alpha}) \rightarrow \mathcal{R}^3$  and the partial surface force fields  $t_{\alpha} : \partial\kappa_{\alpha}(P_{\alpha}) \rightarrow \mathcal{R}^3$  in the form

$$g_P(w) = \sum_{\alpha} \int_{\kappa_{\alpha}(P_{\alpha})} w_{\alpha} \circ \kappa_{\alpha}^{-1} \cdot b_{\alpha} dv + \int_{\partial\kappa_{\alpha}(P_{\alpha})} w_{\alpha} \circ \kappa_{\alpha}^{-1} \cdot t_{\alpha} da.$$

In addition, a necessary and sufficient condition that the partial stresses  $\{\sigma_{\alpha}, \tau_{\alpha}\}$  represent the force  $g$  is that the equations

$$\begin{aligned} \operatorname{div} T_{\alpha} + b_{\alpha} &= s_{\alpha} \quad \text{in } \kappa_{\alpha}(P_{\alpha}) \\ T_{\alpha}(n_{\alpha}) &= t_{\alpha} \quad \text{on } \partial\kappa_{\alpha}(P_{\alpha}) \end{aligned}$$

hold for each  $\alpha$  ( $n_{\alpha}$  is the unit normal to the boundary of  $\kappa_{\alpha}(P_{\alpha})$ ).

Again, this proposition can be proven easily by integrating by parts the terms involving the derivative  $D(u_{\alpha} \circ \kappa_{\alpha}^{-1})$  of the Eulerian partial virtual displacements, and using Gauss' theorem on  $\kappa_{\alpha}(P_{\alpha})$  for each  $\alpha$ .

**DEFINITION 3.7.** A force system for the mixture is a mapping  $\mathbf{F}_M : \mathbf{M} \rightarrow T^*Q_M$ . We say that a force system is *consistent* with the collection of partial stresses  $\{\sigma_{\alpha}, \tau_{\alpha}\}$  if the force on any submixture is represented by the restrictions of the stresses to the submixture. Again, it is clear from the result for simple bodies that there is at most one collection of partial stresses that is consistent with a given force system.

**DEFINITION 3.8.** A *loading* for the mixture is a section of  $T^*Q_M$ , i.e. it gives the force on the mixture in terms of the force on the various constituents for any configuration of the mixture. We note that it is possible to have coupling between the configuration of the constituent  $\alpha$  and the force acting on the constituent  $\beta$ .

As in the case of a simple body a *constitutive relation* for a mixture is a mapping

$$\psi : Q_M \rightarrow \Pi_{\alpha} C^0(B_{\alpha}, \mathcal{R}^3 \times L(\mathcal{R}^3, \mathcal{R}^3))^*$$

and it gives the stress measures  $\{\sigma_{\alpha}, \tau_{\alpha}\}$  corresponding to any configuration. The *problem of continuum mechanics* is formulated in analogy with the case of a simple body as finding, for given loading and constitutive relation, a configuration  $\kappa$  such that the stress given by the constitutive relation at  $\kappa$  represents the force given by the loading at  $\kappa$ .

An important problem that one faces when attempting to solve the problem of continuum mechanics for a mixture is that the interaction forces and boundary conditions for the various constituents are not known. Clearly, these boundary conditions reflect the mechanism of the possible diffusion between the various constituents at the boundary and are of constitutive nature. As such, their specification would be made on the basis of additional physical assumptions describing the particular case at hand. Such a model and the boundary conditions that follow are given for a class of physical situations by Rajagopal *et al.* [11] on the basis of an assumption regarding the thermodynamic processes involved.

#### 4. SIMPLE BODY MODELS OF A MIXTURE

In this section we formulate mathematically the process of modeling a mixture by a simple body. Traditionally, the notion of a total stress is introduced in mixture theory in order to



assign to the mixtures some overall properties that can be measured or properties that are analogous to those of simple bodies. The total stress is usually shown to be the sum of the partial stresses by adding up the equilibrium equations for the various constituents.

We would like to mention that the addition of the stress fields does not make sense if the stress fields are not defined on the same domain. In addition, the forces acting on the various constituents are members of distinct vector spaces ( $f_\alpha \in T^*\mathcal{Q}_\alpha$ ) and their addition is not possible. Hence, for diffusive mixtures, it is questionable whether the notion of a total stress makes sense. We treat the representation of a mixture by a simple body as a model that may approximate in some sense the behavior of the mixture and whose validity should be examined in each particular application at hand according to the assumptions that we will make.

**DEFINITION 4.1.** Let  $M$  be a mixture. We say that the body  $B$  is a *simple body model* of  $M$  if the following structure is given.

- (1) There is a mapping  $n: \mathbf{B} \rightarrow \mathbf{M}$  that will be called the *subbody mapping*, satisfying  $n(B) = M$ , and  $n(P) \supset n(P')$  for  $P \supset P'$ . We will denote by  $n_\alpha$  the component of  $n$  that assigns to  $P$  the  $\alpha$  constituent of  $n(P)$ .
- (2) There is a differentiable injective mapping

$$m_B: \mathcal{Q}_B \rightarrow \mathcal{Q}_M$$

called the *configuration mapping* which is *local* in the sense that if  $\kappa|P = \kappa'|P$  for some subbody  $P$  then  $m_B(\kappa)|n(P) = m_B(\kappa')|n(P)$ .

The mapping  $m_B$  induces similar mappings  $m_P: \mathcal{Q}_P \rightarrow \mathcal{Q}_{n(P)}$ , for the various subbodies by  $m_P(\kappa) = m_B(\kappa')|n(P)$ , where  $\kappa'$  is any configuration of  $B$  extending  $\kappa$ .  $m_P(\kappa)$  is independent of the extension  $\kappa'$  by the locality assumption. Conversely, if for every subbody  $P$  of  $B$  there is a differentiable injective mapping

$$m_P: \mathcal{Q}_P \rightarrow \mathcal{Q}_{n(P)}$$

satisfying  $m_P(\kappa)|n(P') = m_P(\kappa|P')$ , for any subbody  $P'$  of  $P$ , then  $m_B$  satisfies the locality assumption above. Thus, we may omit the index  $P$  for local configuration mappings. We will use the notation  $m_\alpha$  for  $\pi_\alpha \circ m$ .

While the general theory presented in the previous section allowed arbitrary diffusion of the constituents, i.e. arbitrary motion of any constituent with respect to the others, a simple body model allows diffusion in specific modes only: The configurations of the various constituents are all related to that of the body model.

We note that a simple body model induces the tangent mapping

$$Tm_P: T\mathcal{Q}_P \rightarrow T\mathcal{Q}_{n(P)}$$

for every subbody  $P$  and again the velocity fields of the constituents are not independent. Since  $m_P$  is assumed to be injective, the same holds for  $(Tm_P)_\kappa$  for each  $\kappa \in \mathcal{Q}_P$ . From condition 2 above it follows immediately that  $(Tm_P)_\kappa(u)|n(P') = (Tm_{P'})_{\kappa|P'}(u|P')$ .

In addition, the simple body model induces the mapping

$$T^*m_P: T^*(\mathcal{Q}_{n(P)})|m(\mathcal{Q}_P) \rightarrow T^*\mathcal{Q}_P$$

that assigns forces on the simple body model to forces acting on the mixture. Here,  $T^*(\mathcal{Q}_{n(P)})|m(\mathcal{Q}_P)$  is the restriction of the cotangent bundle to the image of  $m$ . Thus, if  $f = T^*m_P(g)$ , we have

$$f(u) = \sum_\alpha g_\alpha(T(m_P)_\alpha(u))$$

and we call  $f$  the total force acting on the simple body model corresponding to the force  $g$  on the mixture. In analogy, a force system  $\mathbf{F}$  on  $\mathbf{B}$  corresponds to a force system  $\mathbf{F}_M$  on  $\mathbf{M}$  if  $\mathbf{F}(P) = T^*m_P(\mathbf{F}_M(n(P)))$  for every  $P$ . We see again that the notion of total force has meaning only when the motions of the constituents are not independent.

Let  $\sigma_B, \tau_B$  be measures that represent the force  $f = T^*m_P(g)$ , where  $g$  is represented by the

measures  $\sigma_\alpha, \tau_\alpha$ . Then,

$$\int_P u \cdot d\sigma_B + \int_P Du \cdot d\tau_B = \sum_\alpha \left( \int_{n_\alpha(P)} D(m_P)_\alpha(u) \cdot d\sigma_\alpha + \int_{n_\alpha(P)} D(D(m_P)_\alpha(u)) \cdot d\tau_\alpha \right)$$

In the general setting presented so far it is impossible to write a more explicit relation between the measures  $\sigma_B, \tau_B$  and  $\sigma_\alpha, \tau_\alpha$ .

**EXAMPLE 4.2.** Let  $M$  be a mixture and  $B$  a simple body. A Mapping  $J: M \rightarrow B$  whose components are embeddings induces a subbody mapping for the mixture as follows. For any subbody  $P \in \mathbf{B}$  we set  $n(P) = J^{-1}(\{P\})$ . Since the components of  $J$  are embeddings  $J_\alpha^{-1}(P)$  is indeed a subbody of  $B_\alpha$  for each  $\alpha$ . In such a case the relation between stress measures can be rewritten as

$$\int_P u \cdot d\sigma_B + \int_P Du \cdot d\tau_B = \sum_\alpha \left\{ \int_{J_\alpha^{-1}(P)} (D(m_P)_\alpha(u)) \circ J_\alpha^{-1} \cdot dJ_\alpha(\sigma_\alpha) + \int_{J_\alpha^{-1}(P)} D(D(m_P)_\alpha(u)) \circ J_\alpha^{-1} \cdot dJ_\alpha(\tau_\alpha) \right\},$$

where  $J_\alpha(\sigma_\alpha)$  and  $J_\alpha(\tau_\alpha)$  are the measures on  $P$  induced by  $J_\alpha$  and the stress measures for the mixture.

**EXAMPLE 4.3.** Let  $J$  be as in the previous example. For any subbody  $P \in \mathbf{B}$  set  $m(\kappa) = \kappa \circ J$  for all  $\kappa \in \mathcal{Q}_P$ . Again,  $m$  is well defined by the requirement that the components of  $J$  are embeddings. It follows immediately that  $Dm_\kappa(u) = u \circ J$ , and for the relation between the measures representing  $g$  and  $f = T^*m(g)$ , we have

$$\int_P u \cdot d\sigma_B + \int_P Du \cdot d\tau_B = \int_P u \cdot \sum_\alpha dJ_\alpha(\sigma_\alpha) + \int_P Du \cdot \sum_\alpha DJ_\alpha \circ J_\alpha^{-1} dJ_\alpha(\tau_\alpha)$$

It follows that the force  $f = T^*m_P(g)$  may be represented by the measures

$$\sigma_B = \sum_\alpha J_\alpha(\sigma_\alpha) \quad \text{and} \quad \tau_B = \sum_\alpha DJ_\alpha \circ J_\alpha^{-1} J_\alpha(\tau_\alpha).$$

We note that if a force system  $\mathbf{F}$  on  $\mathbf{B}$  corresponds to a force system  $\mathbf{F}_M$  on  $\mathbf{M}$ , the equation above holds for each subbody, the uniqueness of the relation between force systems and stress measures implies that  $\mathbf{F}$  is induced by the stress measures

$$\sum_\alpha J_\alpha(\sigma_\alpha) \quad \text{and} \quad \sum_\alpha DJ_\alpha \circ J_\alpha^{-1} J_\alpha(\tau_\alpha).$$

**EXAMPLE 4.4.** Consider the case where  $B_\alpha = B$  and  $n_\alpha$  is the identity. Let  $m$  be given in terms of a collection of  $C^2$  diffeomorphisms  $I = (I_1, \dots, I_a): \mathcal{R}^3 \rightarrow \mathcal{R}^3$  in the form  $m(\kappa) = I \circ \kappa$ . In this case we have  $Dm_\kappa(u)(X) = DI_{\kappa(X)}(u(X))$  or alternatively  $Dm_\kappa(u) = DI \circ \kappa u$ . Using the notation of tangent mappings and vector fields we can write  $Tm_\kappa(u) = TI_\kappa \circ u$ . Thus, for any  $y \in \mathcal{R}^3$

$$D(Dm_\kappa(u))_X(y) = D^2I_{\kappa(X)}(y, u(X)) + DI_{\kappa(X)}Du_X(y)$$

so that  $D(Dm_\kappa(u)) = D^2I \circ \kappa u + DI \circ \kappa Du$  and with the notation of tangent mappings  $T(Tm_\kappa(u)) = T^2I_\kappa \circ Tu$ .

It follows that the relation between the stress measures assumes the form

$$\int_P u \cdot d\sigma_B + \int_P Du \cdot d\tau_B = \sum_\alpha \left[ \int_P DI_\alpha \circ \kappa u \cdot d\sigma_\alpha + \int_P (D^2I_\alpha \circ \kappa u + DI_\alpha \circ \kappa Du) \cdot d\tau_\alpha \right].$$

If a force system  $\mathbf{F}$  on  $\mathbf{B}$  corresponds to a force system  $\mathbf{F}_M$  on  $\mathbf{M}$ , the equation above holds for each subbody and we have

$$\sigma_B = \sum_\alpha (\sigma_\alpha DI_\alpha \circ \kappa + \tau_\alpha D^2I_\alpha \circ \kappa), \quad \tau_B = \sum_\alpha \tau_\alpha DI_\alpha \circ \kappa.$$

**EXAMPLE 4.5.** We now consider the case where the stresses are given in terms of the densities  $s, T, s_\alpha, T_\alpha$  as in Propositions 2.7 and 3.6.

For Example 4.2, the relation between the stress densities assumes the form

$$\begin{aligned} & \int_P (u \circ \kappa^{-1} \cdot s) \circ \kappa |D\kappa| dV + \int_P (Du \circ \kappa^{-1} D(\kappa^{-1}) \cdot T) \circ \kappa |D\kappa| dV \\ &= \sum_{\alpha} \int_P (u_{\alpha} \circ \kappa_{\alpha}^{-1} \cdot s_{\alpha}) \circ \kappa_{\alpha} \circ J_{\alpha}^{-1} |D\kappa_{\alpha}| \circ J_{\alpha}^{-1} |D(J_{\alpha}^{-1})| dV \\ & \quad + \sum_{\alpha} \int_P (Du_{\alpha} \circ \kappa_{\alpha}^{-1} D(\kappa_{\alpha}^{-1}) \cdot T_{\alpha}) \circ \kappa_{\alpha} \circ J_{\alpha}^{-1} |D\kappa_{\alpha}| \circ J_{\alpha}^{-1} |D(J_{\alpha}^{-1})| dV \end{aligned}$$

For Example 4.3 the relation between the stress densities specializes to

$$s = \sum_{\alpha} s_{\alpha}, \quad TD(\kappa^{-1})^T = \sum_{\alpha} T_{\alpha} D(\kappa^{-1})^T (DJ_{\alpha})^T \circ J_{\alpha}^{-1} \circ \kappa^{-1}$$

and for Example 4.4 the relation will be given by

$$\begin{aligned} v \cdot s &= \sum_{\alpha} \{ (DI_{\alpha} v) \cdot s_{\alpha} \circ I_{\alpha} + (D^2 I_{\alpha} v) (D(\kappa^{-1}) \circ I_{\alpha}^{-1} D(I_{\alpha}^{-1})) \circ I_{\alpha} \cdot T_{\alpha} \circ I_{\alpha} \} |DI_{\alpha}| \\ A \cdot T &= \sum_{\alpha} DI_{\alpha} A D(I_{\alpha}^{-1}) \circ I_{\alpha} \cdot T_{\alpha} \circ I_{\alpha} |DI_{\alpha}|, \end{aligned}$$

where  $v$  is any vector field and  $A$  is any tensor field defined on  $\mathcal{R}^3$ .

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