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# The Complement of Binary Klein Quadric as a Combinatorial Grassmannian

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## Abstract

Given a hyperbolic quadric of  $PG(5, 2)$ , there are 28 points off this quadric and 56 lines skew to it. It is shown that the  $(28_6, 56_3)$ -configuration formed by these points and lines is isomorphic to the combinatorial Grassmannian of type  $G_2(8)$ . It is also pointed out that a set of seven points of  $G_2(8)$  whose labels share a mark corresponds to a Conwell heptad of  $PG(5, 2)$ . Gradual removal of Conwell heptads from the  $(28_6, 56_3)$ -configuration yields a nested sequence of binomial configurations identical with part of that found to be associated with Cayley-Dickson algebras (arXiv:1405.6888).

**Keywords:** Combinatorial Grassmannian – Binary Klein Quadric – Conwell Heptad

Let  $Q^+(5, 2)$  be a hyperbolic quadric in a five-dimensional projective space  $PG(5, 2)$ . As it is well known (see, e. g., [1, 2]), there are 28 points lying off this quadric as well as 56 lines skew (or, external) to it. If the equation of the quadric is taken in a canonical form  $x_1x_2 + x_3x_4 + x_5x_6 = 0$ , then the 28 off-quadric points are

No.	$x_1$	$x_2$	$x_3$	$x_4$	$x_5$	$x_6$
1	1	1	1	0	0	0
2	1	1	0	0	1	0
3	1	1	0	0	0	1
4	1	1	0	1	0	0
5	1	1	1	0	1	0
6	1	1	1	0	0	1
7	1	1	0	1	1	0
8	1	1	0	1	0	1
9	0	0	1	1	0	0
10	0	0	1	1	1	0
11	0	0	1	1	0	1
12	0	1	1	1	0	0
13	1	0	1	1	1	0
14	1	0	1	1	0	1
15	0	0	0	0	1	1
16	1	0	0	0	1	1
17	0	0	1	0	1	1
18	0	0	0	1	1	1
19	0	1	1	0	1	1
20	0	1	0	1	1	1
21	1	1	1	1	1	1
22	1	1	0	0	0	0
23	1	0	1	1	0	0
24	0	1	1	1	1	0
25	0	1	1	1	0	1
26	0	1	0	0	1	1
27	1	0	1	0	1	1
28	1	0	0	1	1	1

and the 56 external lines read

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	+			+					+																			
2	+						+			+																		
3	+							+			+																	
4	+															+				+								
5	+																	+			+						+	
6		+	+													+												
7		+					+											+										
8		+						+											+		+							
9		+									+									+								+
10		+										+	+															
11			+		+													+										
12			+				+												+		+							
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18				+														+									+	
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21					+									+														+
22						+		+	+																			+
23						+							+															+
24							+	+								+												
25							+							+													+	
26								+					+													+	+	
27									+									+	+								+	+
28									+											+	+					+	+	
29										+	+					+												
30										+					+		+											
31											+		+				+											
32												+					+											+
33												+						+										+
34												+							+		+							+
35												+	+	+														
36	+																										+	+
37		+																						+	+			
38			+																					+				
39				+																					+			+
40					+																					+		+
41						+																				+		+
42							+																			+		+
43								+																		+		+
44									+																	+		+
45										+																+		+
46											+															+		+
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53																		+								+		+
54																			+							+		+
55																				+						+		+
56																				+						+		+

In the latter table, the ‘+’ symbol indicates which point lies on a given line; for example, line 1 consists of points 1, 4 and 9. As it is obvious from this table, each line has three points and through each point there are six lines; hence, these points and lines form a  $(28_6, 56_3)$ -configuration.

Next, a combinatorial Grassmannian  $G_k(X)$  (see, e.g., [3, 4]), where  $k$  is a positive integer and  $X$  is a finite set,  $|X| = N$ , is a point-line incidence structure whose points are  $k$ -element subsets of  $X$  and whose lines are  $(k + 1)$ -element subsets of  $X$ , incidence being inclusion. Obviously,  $G_k(N)$  is a  $\left(\binom{N}{k}_{N-k}, \binom{N}{k+1}_{k+1}\right)$ -configuration; hence, we have another  $(28_6, 56_3)$ -configuration,  $G_2(8)$ .

It is straightforward to see that the two  $(28_6, 56_3)$ -configurations are, in fact, isomorphic. To this end, one simply employs the following bijection between the 28 off-quadric points and the 28 points of  $G_2(8)$  (here, by a slight abuse of notation,  $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$ )

off- $\mathcal{Q}$	$G_2(8)$	off- $\mathcal{Q}$	$G_2(8)$
1	{1, 4}	15	{2, 3}
2	{3, 5}	16	{4, 7}
3	{2, 5}	17	{5, 6}
4	{4, 6}	18	{1, 5}
5	{2, 6}	19	{1, 7}
6	{3, 6}	20	{6, 7}
7	{1, 2}	21	{4, 5}
8	{1, 3}	22	{7, 8}
9	{1, 6}	23	{5, 8}
10	{2, 4}	24	{3, 8}
11	{3, 4}	25	{2, 8}
12	{5, 7}	26	{4, 8}
13	{3, 7}	27	{1, 8}
14	{2, 7}	28	{6, 8}

and verifies step by step that each of the above-listed 56 lines of  $\text{PG}(5, 2)$  is also a line of  $G_2(8)$ ; thus, line 1 of  $\text{PG}(5, 2)$  corresponds to the line {1, 4, 6} of  $G_2(8)$ , line 2 to the line {1, 2, 4}, line 3 to {1, 3, 4}, etc.

This isomorphism entails a very interesting property related to so-called Conwell heptads [5]. Given a  $\mathcal{Q}^+(5, 2)$  of  $\text{PG}(5, 2)$ , a Conwell heptad is a set of seven off-quadric points such that each line joining two distinct points of the heptad is skew to the  $\mathcal{Q}^+(5, 2)$ . There are altogether eight such heptads: any two of them have a unique point in common and each of the 28 points off the quadric is contained in two heptads. The points in the first table are arranged in such a way that, as it is obvious from the bottom part of the second table, the last seven of them represent a Conwell heptad. From the last table we read off that this particular heptad corresponds to those seven points of  $G_2(8)$  whose representatives have mark '8' in common. Clearly, the remaining seven heptads correspond to those septuples of points of  $G_2(8)$  that share one of the remaining seven marks each. Finally, we observe that by removing from our off-quadric  $(28_6, 56_3)$ -configuration the seven points of a Conwell heptad and all the 21 lines defined by pairs of them one gets a  $(21_5, 35_3)$ -configuration isomorphic to  $G_2(7)$ ; gradual removal of additional heptads and the corresponding lines yields the following nested sequence of configurations:

# of heptads removed	Configuration	CG	Remark
0	$(28_6, 56_3)$	$G_2(8)$	
1	$(21_5, 35_3)$	$G_2(7)$	
2	$(15_4, 20_3)$	$G_2(6)$	Cayley-Salmon
3	$(10_3, 10_3)$	$G_2(5)$	Desargues
4	$(6_2, 4_3)$	$G_2(4)$	Pasch
5	$(3_1, 1_3)$	$G_2(3)$	single line
6	$(1_0, 0_3)$	$G_2(2)$	single point
7			empty set

Remarkably, this nested sequence of binomial configurations is identical with part of that found to be associated with Cayley-Dickson algebras [6].

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