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The Complement of Binary Klein Quadric as a Combinatorial Grassmannian

Metod Saniga

Institute for Discrete Mathematics and Geometry, Vienna University of Technology
Wiedner Hauptstraße 8–10, A-1040 Vienna, Austria
(metod.saniga@tuwien.ac.at)

and

Astronomical Institute, Slovak Academy of Sciences
SK-05960 Tatranská Lomnica, Slovak Republic
(msaniga@astro.sk)

Abstract

Given a hyperbolic quadric of $\text{PG}(5, 2)$, there are 28 points off this quadric and 56 lines skew to it. It is shown that the $(28_6, 56_3)$ -configuration formed by these points and lines is isomorphic to the combinatorial Grassmannian of type $G_2(8)$. It is also pointed out that a set of seven points of $G_2(8)$ whose labels share a mark corresponds to a Conwell heptad of $\text{PG}(5, 2)$. Gradual removal of Conwell heptads from the $(28_6, 56_3)$ -configuration yields a nested sequence of binomial configurations identical with part of that found to be associated with Cayley-Dickson algebras (arXiv:1405.6888).

Keywords: Combinatorial Grassmannian – Binary Klein Quadric – Conwell Heptad

Let $\mathcal{Q}^+(5, 2)$ be a hyperbolic quadric in a five-dimensional projective space $\text{PG}(5, 2)$. As it is well known (see, e. g., [1, 2]), there are 28 points lying off this quadric as well as 56 lines skew (or, external) to it. If the equation of the quadric is taken in a canonical form $x_1x_2 + x_3x_4 + x_5x_6 = 0$, then the 28 off-quadric points are

No.	x_1	x_2	x_3	x_4	x_5	x_6
1	1	1	1	0	0	0
2	1	1	0	0	1	0
3	1	1	0	0	0	1
4	1	1	0	1	0	0
5	1	1	1	0	1	0
6	1	1	1	0	0	1
7	1	1	0	1	1	0
8	1	1	0	1	0	1
9	0	0	1	1	0	0
10	0	0	1	1	1	0
11	0	0	1	1	0	1
12	0	1	1	1	0	0
13	1	0	1	1	1	0
14	1	0	1	1	0	1
15	0	0	0	0	1	1
16	1	0	0	0	1	1
17	0	0	1	0	1	1
18	0	0	0	1	1	1
19	0	1	1	0	1	1
20	0	1	0	1	1	1
21	1	1	1	1	1	1
22	1	1	0	0	0	0
23	1	0	1	1	0	0
24	0	1	1	1	1	0
25	0	1	1	1	0	1
26	0	1	0	0	1	1
27	1	0	1	0	1	1
28	1	0	0	1	1	1

and the 56 external lines read

No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28
1	+			+					+																			
2	+						+			+																		
3	+							+			+																	
4	+															+			+									
5	+																	+			+							+
6		+	+													+												
7		+					+											+										
8		+						+											+									
9		+									+									+								+
10		+										+	+															
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14			+									+			+													
15				+	+					+																		
16				+		+					+																	
17				+												+					+							
18				+														+										+
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24							+	+								+												
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26								+					+															
27									+									+	+									
28								+												+	+							
29									+	+						+						+	+					
30										+					+		+											
31											+		+				+											
32												+					+					+						+
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36	+																									+	+	
37		+																						+	+			
38			+																						+			
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40					+																						+	+
41						+																					+	+
42							+																				+	+
43								+																			+	+
44									+																		+	+
45										+																	+	+
46											+																+	+
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52																	+										+	+
53																		+									+	+
54																			+								+	+
55																				+							+	+
56																					+						+	+

In the latter table, the ‘+’ symbol indicates which point lies on a given line; for example, line 1 consists of points 1, 4 and 9. As it is obvious from this table, each line has three points and through each point there are six lines; hence, these points and lines form a $(28_6, 56_3)$ -configuration.

Next, a combinatorial Grassmannian $G_k(X)$ (see, e.g., [3, 4]), where k is a positive integer and X is a finite set, $|X| = N$, is a point-line incidence structure whose points are k -element subsets of X and whose lines are $(k + 1)$ -element subsets of X , incidence being inclusion. Obviously, $G_k(N)$ is a $\left(\binom{N}{k}_{N-k}, \binom{N}{k+1}_{k+1}\right)$ -configuration; hence, we have another $(28_6, 56_3)$ -configuration, $G_2(8)$.

It is straightforward to see that the two $(28_6, 56_3)$ -configurations are, in fact, isomorphic. To this end, one simply employs the following bijection between the 28 off-quadric points and the 28 points of $G_2(8)$ (here, by a slight abuse of notation, $X = \{1, 2, 3, 4, 5, 6, 7, 8\}$)

off- \mathcal{Q}	$G_2(8)$	off- \mathcal{Q}	$G_2(8)$
1	$\{1, 4\}$	15	$\{2, 3\}$
2	$\{3, 5\}$	16	$\{4, 7\}$
3	$\{2, 5\}$	17	$\{5, 6\}$
4	$\{4, 6\}$	18	$\{1, 5\}$
5	$\{2, 6\}$	19	$\{1, 7\}$
6	$\{3, 6\}$	20	$\{6, 7\}$
7	$\{1, 2\}$	21	$\{4, 5\}$
8	$\{1, 3\}$	22	$\{7, 8\}$
9	$\{1, 6\}$	23	$\{5, 8\}$
10	$\{2, 4\}$	24	$\{3, 8\}$
11	$\{3, 4\}$	25	$\{2, 8\}$
12	$\{5, 7\}$	26	$\{4, 8\}$
13	$\{3, 7\}$	27	$\{1, 8\}$
14	$\{2, 7\}$	28	$\{6, 8\}$

and verifies step by step that each of the above-listed 56 lines of $\text{PG}(5, 2)$ is also a line of $G_2(8)$; thus, line 1 of $\text{PG}(5, 2)$ corresponds to the line $\{1, 4, 6\}$ of $G_2(8)$, line 2 to the line $\{1, 2, 4\}$, line 3 to $\{1, 3, 4\}$, etc.

This isomorphism entails a very interesting property related to so-called Conwell heptads [5]. Given a $\mathcal{Q}^+(5, 2)$ of $\text{PG}(5, 2)$, a Conwell heptad is a set of seven off-quadric points such that each line joining two distinct points of the heptad is skew to the $\mathcal{Q}^+(5, 2)$. There are altogether eight such heptads: any two of them have a unique point in common and each of the 28 points off the quadric is contained in two heptads. The points in the first table are arranged in such a way that, as it is obvious from the bottom part of the second table, the last seven of them represent a Conwell heptad. From the last table we read off that this particular heptad corresponds to those seven points of $G_2(8)$ whose representatives have mark ‘8’ in common. Clearly, the remaining seven heptads correspond to those septuples of points of $G_2(8)$ that share one of the remaining seven marks each. Finally, we observe that by removing from our off-quadric $(28_6, 56_3)$ -configuration the seven points of a Conwell heptad and all the 21 lines defined by pairs of them one gets a $(21_5, 35_3)$ -configuration isomorphic to $G_2(7)$; gradual removal of additional heptads and the corresponding lines yields the following nested sequence of configurations:

# of heptads removed	Configuration	CG	Remark
0	$(28_6, 56_3)$	$G_2(8)$	Cayley-Salmon Desargues Pasch single line single point empty set
1	$(21_5, 35_3)$	$G_2(7)$	
2	$(15_4, 20_3)$	$G_2(6)$	
3	$(10_3, 10_3)$	$G_2(5)$	
4	$(6_2, 4_3)$	$G_2(4)$	
5	$(3_1, 1_3)$	$G_2(3)$	
6	$(1_0, 0_3)$	$G_2(2)$	
7			

Remarkably, this nested sequence of binomial configurations is identical with part of that found to be associated with Cayley-Dickson algebras [6].

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