Stochastic maximum principle for optimal control of SPDEs driven by white noise

Marco Fuhrman, Ying Hu, Gianmario Tessitore

▶ **To cite this version:**

Marco Fuhrman, Ying Hu, Gianmario Tessitore. Stochastic maximum principle for optimal control of SPDEs driven by white noise. 2014. hal-01064475v1

HAL Id: hal-01064475
https://hal.science/hal-01064475v1

Preprint submitted on 16 Sep 2014 (v1), last revised 8 Jun 2017 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Stochastic maximum principle for optimal control of SPDEs driven by white noise

Marco Fuhrman
Dipartimento di Matematica, Politecnico di Milano
via Bonardi 9, 20133 Milano, Italy
e-mail: marco.fuhrman@polimi.it

Ying Hu
IRMAR, Université Rennes 1
Campus de Beaulieu, 35042 Rennes Cedex, France
e-mail: ying.hu@univ-rennes1.fr

Gianmario Tessitore
Dipartimento di Matematica e Applicazioni, Università di Milano-Bicocca
Via Cozzi 53, 20125 Milano, Italy
e-mail: gianmario.tessitore@unimib.it

Abstract

We prove the maximum principle of Pontryagin's type for the optimal control of a stochastic partial differential equation driven by a white noise under the hypothesis that the control domain is convex.

1 Introduction

In this paper, we consider the following infinite-dimensional stochastic optimal control problem. Let H be a Hilbert space, A be the infinitesimal generator of a C_0 semigroup, and W a cylindrical Wiener process.

The state equation is

$$d_t X_t = AX_t dt + F(X_s, u_s)ds + G(X_s, u_s)dW_s, \quad X_0 = x_0 \tag{1.1}$$

where $x_0 \in H$ and the solution will evolve in H.

The cost function is

$$J(x, u) = \mathbb{E}\int_0^T L(X_s, u_s)ds + \mathbb{E}\Phi(X_T). \tag{1.2}$$

The precise hypothesis will be given in the next section.

Our goal in this paper is to give a necessary condition for an optimal control. Such necessary condition called stochastic maximum principle (SMP) is extensively studied in finite dimensional case, see the seminar paper by Peng [16] which gives the general SMP. Concerning the infinite

dimensional case, to our best knowledge, only the SMP for the state equation driven by trace-class noise is established, see [1, 11, 5].

Hence in this paper, we will establish SMP for SPDEs driven by white noise (i.e. cylindrical Wiener process). On the other hand, we suppose in this paper that the control domain is convex, which allows us to apply convex perturbation instead of spike pertubation.

This paper is organized as follows: we introduce our control problem in the next section. In Section 3, we apply the convex perturbation, the first-order variation equation and the dual equation. Particular attention will be paid to the regularity of the adjoint process. And the last section is devoted to the SMP and an example will be given.

2 Formulation of the optimal control problem

Assumption 2.1. *(i) A is the generator of a C_0 semigroup e^{tA}, $t \geq 0$, in H. Moreover $\forall s > 0$:*

$$e^{sA} \in L_2(H) \text{ with } |e^{sA}|_{L_2(H)} \leq Ls^{-\alpha}; \text{for suitable } L > 0,\ \alpha \in [0, 1/2).$$

where $L_2(H)$ is the (Hilbert) space of Hilbert Schmidt operators in H.

(ii) U is a bounded convex subset of a separable Banach space U_0

(iii) $F : H \times U \to H$ is Lipschitz in both variables

(iv) $G : H \times U \to L(H)$ verifies for all $s > 0$, $t \in [0, T]$, $X, Y \in H$, $u, v \in U$,

$$\begin{aligned} &|e^{sA}G(t, 0, u)|_{L_2(H)} \leq L\ s^{-\alpha}, \\ &|e^{sA}G(t, X, u) - e^{sA}G(t, Y, v)|_{L_2(H)} \leq L\ s^{-\alpha}(|X - Y| + |u - v|), \end{aligned} \tag{2.1}$$

for some constants $L > 0$ and $\alpha \in [0, 1/2)$.

(v) $F(\cdot, \cdot)$ is Gateaux differentiable $H \times U \to H$,
for all $s > 0$, $e^{sA}G(\cdot, \cdot)$ is Gateaux differentiable $H \times U \to L_2(H)$.

(vi) $L(\cdot, \cdot)$ and $\Phi(\cdot)$ are bounded Lipschitz and differentiable.

(vii) For all $\Xi \in H$ the map $u \to G(X, u)\Xi$ is Gateaux differentiable and

$$|\nabla_u G(X, u)\Xi|_{\mathcal{L}(U_0, H)} \leq cost|\Xi|_H \quad \text{recall } (U_0 \subset L^\infty)$$

Under the above assumptions the state equation (formulated in mild sense):

$$X_t = e^{tA}x_0 + \int_0^t e^{(t-s)A}F(X_s, u_s)ds + \int_0^t e^{(t-s)A}G(X_s, u_s)dW_s$$

admits a unique solution $X \in L^p_W(\Omega, C([0, T], H))$, see [2].

Remark 2.2. *If we perturb the control by spike variation that is we consider solution of*

$$X^\epsilon_t = e^{tA}x_0 + \int_0^t e^{(t-s)A}F(X_s, u^\epsilon_s)ds + \int_0^t e^{(t-s)A}G(X_s, u^\epsilon_s)dW_s$$

where $u^\epsilon_s = u_s I_{[t_0, t_0+\epsilon]^c}(s) + v_0 I_{[t_0, t_0+\epsilon]}(s)$ for fixed $t_0 \in [0, T]$, $v_0 \in U$ then

$$|X^\epsilon(t_0 + \delta) - X(t_0 + \delta)|_{L^2(\Omega, \mathbb{P}, H)} \approx \delta^{(1/2-\alpha)}$$

2.1 An example

Let $(\mathcal{W}(t,x))$, $t \geq 0$, $x \in [0,1]$ be a space time white noise
$(\mathcal{F}_t)_{t\geq 0}$ denotes its natural (completed) filtration.

The set of admissible control actions U is a convex subset of $L^\infty([0,1])$.
A control u is a (progressive) process with values in U.

The controlled state equation is the following SPDE: for $t \in [0,T]$, $x \in [0,1]$,

$$\begin{cases} dX_t(x) &= \dfrac{\partial^2}{\partial x^2} X_t(x)\, dt + b(x, X_t(x), u_t(x))\, dt + \sigma(x, X_t(x), u_t(x)) d\mathcal{W}(t,x), \\ X_t(0) &= X_t(1) = 0, \quad t \in [0,T] \\ X_0(x) &= x_0(x), \quad x \in [0,1] \end{cases}$$

where $b(x,r,u), \sigma(x,r,u) : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are given,
we assume they are C^1 and Lipschitz with respect to r and u;
for fixed r and u we suppose $b(\cdot, r, u) \in L^2([0,1])$, $\sigma(\cdot, r, u) \in L^\infty([0,1])$ bdd.

We also introduce the cost functional:

$$J(u) = \mathbb{E} \int_0^T \int_{\mathcal{O}} l(x, X_t(x), u_t(x))\, dx\, dt + \mathbb{E} \int_{\mathcal{O}} h(x, X_T(x))\, dx$$

where $l(x,r,u) : \mathcal{O} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $h(x,r) : \mathcal{O} \times \mathbb{R} \to \mathbb{R}$ are given bounded functions, we assume that they are C^1 with bounded derivatives with respect to r and u;

The noise is reformulated as a $L^2([0,1])$ valued cylindrical Wiener process (W_t)

$$\mathbb{E} < W_t, x >_{L^2} < W_s, y >_{L^2} = (t \wedge s) < x, y >_{L^2}, \quad \forall x, y \in H = L^2([0,1])$$

A is the realization of the second derivative operator in H with Dirichlet boundary conditions. So it is an unbounded operator with domain $H_0^2([0,1]) \subset H = L^2([0,1])$.

For all $X, V \in H$, $x \in [0,1]$ the non linearities are defined by

$$F(X,u)(x) = b(x, X(x), u(x)), \qquad [G(X,u)V](x) = \sigma(x, X(x), u(x))V(x),$$

$$L(X,u)(x) = \int_{\mathcal{O}} l(X(x), u(x)) dx, \quad \Phi(X)(x) = \int_{\mathcal{O}} h(X(x)) dx$$

The state equation written in abstract form becomes

$$d_t X_t = AX_t dt + F(X_s, u_s) ds + G(X_s, u_s) dW_s, \quad X_0 = x_0$$

where $x_0 \in H$ and the solution will evolve in H.

The cost becomes

$$J(x,u) = \mathbb{E} \int_0^T L(X_s, u_s) ds + \mathbb{E}\Phi(X_T)$$

3 First Variation Equation

Let $(\bar{X}, \bar{u})$ be an optimal pair, fix any other bounded. U-valued progressive control v

Let $u_t^\epsilon = \bar{u}_t + \epsilon(v_t - \bar{u}_t)$ and X_t^ϵ the corresponding solution of the state equation.

Finally denote $(\delta u)_t = v_t - \bar{u}_t$

Since we are not considering spike variations things are easy at this level

$$X_t^\epsilon = \bar{X}_t + \epsilon Y_t + o(\epsilon).$$

$$\begin{cases} dY_t &= \Big[AY_t + \nabla_X F(\bar{X}_t, \bar{u}_t)Y_t + \nabla_u F(\bar{X}_t, \bar{u}_t)(\delta u)_t\Big]\, dt \\ & \quad \nabla_X G(\bar{X}_t, \bar{u}_t)Y_t\, dW_t + \nabla_u G(\bar{X}_t, \bar{u}_t)(\delta u)_t,\, dW_t \\ Y_0^\epsilon &= 0 \end{cases}$$

By [2] the above equation admits an unique mild solution with

$$\mathbb{E}(\sup_{t\in[0,T]} |Y_t|^2) < +\infty,$$

Moreover

$$J(x, u^\epsilon) = J(x, \bar{u}) + \epsilon I(v) + o(\epsilon)$$

with

$$I(v) = \mathbb{E}\int_0^T \left[\langle \nabla_X L(\bar{X}_t, \bar{u}_t), Y_t\rangle + \langle \nabla_u L(\bar{X}_t, \bar{u}_t), (\delta u)_t\rangle\right] dt + \mathbb{E}\langle \nabla_X \Phi(\bar{X}_T), Y_T\rangle$$

We fix a basis $(e_i)_{i\in\mathbb{N}} \in H$ and assume that for all $i \in \mathbb{N}$ the map $X \to G(X, u)e_i$ is Gateaux differentiable $H \to H$.

We notice that in our concrete case for all $V \in H$

$$[\nabla_X (G(X, u)e_i)V](x) = \frac{\partial}{\partial X}\sigma(x, X_t(\xi), u_t(\xi))e_i(\xi)V(\xi)$$

So it is enough to choose $e_i \in L^\infty([0, 1])$

We denote $\nabla_X (G(\bar{X}_t, \bar{u}_t)e_i)V = C_i(t)V$.

Recall that gradients ∇_X are with respect to variables $X \in H = L^2([0, 1])$

For simplicity we let $F = 0$ from now on.

The equation for the first variation becomes

$$\begin{cases} dY_t(x) &= AY_t dt + \sum_{i=1}^\infty C_i(t)Y_t\, d\beta_t^i + \nabla_u G(\bar{X}_t, \bar{u}_t)(\delta u)_t\, dW_t \\ Y_0 &= 0 \end{cases}$$

where $\beta_t^i = \langle e_i, W_t\rangle$ and we have:

1. $|C_i(t)|_{L(H)} \le c, \quad \mathbb{P} - \text{a.s. for all } t \in [0, T]$
2. $\sum_{i=1}^\infty |e^{tA}C_i(s)v|^2 \le ct^{-2\alpha}|v|_H^2 \quad \text{for all } t \ge 0,\, s \ge 0, \quad (\alpha < 1/2)$
3. $\sum_{i=1}^\infty |e^{tA}e_i|^2 \le ct^{-2\alpha} \quad \text{for all } t \ge \quad (\alpha < 1/2),$

4 Linear BSDE driven by white noise and duality

The main result of this paper is the well solvability of linear BSDE driven by white noise and its duality with linear FSDE. In this section, we first give some precise estimates concerning the solution of linear FSDE and then we study the existence and uniqueness of solution of BSDE.

4.1 Linear Forward SDE

Let us first give some precise estimates concerning the solution of the linear FSDE. Consider the following linear FSDE:

$$\begin{cases} d_t X_t &= AX_t\,dt + \sum_{i=1}^{\infty} C_i(t)X_t\,d\beta_t^i + \sum_{i=1}^{\infty} C_i(t)\gamma_t\,d\beta_t^i \\ & \quad + \sum_{i=1}^{\infty} \Gamma_t e_i\,d\beta_t^i + \rho_t dt \\ X_s &= x. \end{cases} \tag{4.1}$$

Assumption 4.1. *1.* $C_i(t) \in L(H)$; $\quad |C_i(t)|_{L(H)} \leq c$.

2. $\sum_{i=1}^{\infty} |e^{tA}C_i x|^2 \leq ct^{-2\alpha}|x|^2$
where (e^{tA}) *is the* C_0*-semigroup generated by* A *and* $\alpha < \frac{1}{2}$, $t \leq 1$.

3. (e^{tA}) *is hilbert-Schmidt with*

$$||e^{tA}||^2_{L_2(H)} \leq ct^{-2\alpha}, \quad t \leq 1, \quad \alpha < \frac{1}{2}.$$

4. C_i *are strongly measurable processes.*

Theorem 4.2. *Assume that* $x \in L^p(\Omega, \mathcal{F}_s, \imath, H)$, $\rho \in L^p_{\mathcal{P}}(\Omega, L^1([s,T], H))$, $\gamma \in L^{\infty}_{\mathcal{P}}([s,T] \times \Omega, H)$, $\Gamma \in L^{\infty}_{\mathcal{P}}([s,T] \times \Omega, L(H))$ *(*Γ *stronly progressively measurable), p large enought. Then there exists a unique solution X for (4.1) with*

$$\mathbb{E}(\sup_{t\in[s,T]} |X_t|^p) \leq c(1 + \mathbb{E}|x|^p + |\Gamma|^p_{L^{\infty}} + |\gamma|^p_{L^{\infty}} + \mathbb{E}\left(\int_s^T |\rho_t|dt\right)^p).$$

Moreover if we replace any of the above sums by a finite sium then the corresponding solution converges to the above on the norm

$$\mathbb{E}(\sup_{t\in[s,T]} |X_t - X_t^n|^p) \to 0$$

Proof. The proof of this theorem is similar to [6] and we omit it. □

Proposition 4.3. *If* γ_t *is Hilbert-Schmidt valued then*

$$\mathbb{E}(\sup_{t\in[s,T]} |X_t|^2) \leq c[\mathbb{E}|x|^2 + \mathbb{E}\left(\int_s^T |\rho_t|dt\right)^2 + |\gamma|^2_{L^{\infty}} + \mathbb{E}\int_s^T |\Gamma_t|^2_{L_2(H)}dt].$$

Proof. It suffices to apply directly Gronwall's lemma. □

Proposition 4.4.

$$\mathbb{E}|X_t|^2 \leq c[\mathbb{E}|x|^2 + \int_s^t (t_l)^{-2\alpha}\mathbb{E}|\gamma_l|_H^2 dl + \int_s^t (t-l)^{-2\alpha}|\Gamma_l|_{L(H)}^2 dl + \mathbb{E}\left(\int_s^t |\rho_l| dl\right)^2].$$

Proof. Let

$$u(t) = \mathbb{E}|X_t|^2,$$
$$v(t) = c\mathbb{E}(|\Gamma_l|_{L(H)}^2 + |\gamma_l|^2),$$
$$K = c\mathbb{E}|x|^2 + c\mathbb{E}\left(\int_s^t |\rho_l| dl\right)^2.$$

Then

$$u(t) \leq K + \int_s^t (t-l)^{-2\alpha} v(l) dl + c\int_s^t (t-l)^{-2\alpha} u(l) dl.$$

let

$$e(t) = K + \int_s^t (t-l)^{-2\alpha} v(l) dl,$$

then

$$u(t) \leq e(t) + c\int_s^t (t-l)^{-2\alpha} u(l) dl.$$

By [9],

$$u(t) \leq e(t) + c\int_s^t (t-l)^{-2\alpha} e(l) dl.$$

Now

$$\begin{aligned} & \int_s^t (t-l)^{-2\alpha} e(l) dl \\ = \; & K\int_s^t (t-l)^{-2\alpha} dl + c\int_s^t (t-l)^{-2\alpha} \int_s^l (l-\zeta)^{-2\alpha} v(\zeta) d\zeta \\ \leq \; & K(t-s)^{1-2\alpha} + c\int_s^t v(\zeta) \int_\zeta^t (t-l)^{-2\alpha}(l-\zeta)^{-2\alpha} dl. \end{aligned}$$

But

$$\begin{aligned} & \int_\zeta^t (t-l)^{-2\alpha}(l-\zeta)^{-2\alpha} dl = \int_0^{t-\zeta} (t-\zeta-s)^{-2\alpha} s^{-2\alpha} ds \\ = \; & \int_0^1 (1-z)^{-2\alpha} z^{-2\alpha} (t-\zeta)^{-4\alpha+1} dz \\ = \; & c(t-\zeta)^{-4\alpha+1} \leq c(t-\zeta)^{-2\alpha}. \end{aligned}$$

Consequently,

$$u(t) \leq cK + c\int_s^t (t-l)^{-2\alpha} v(l) dl.$$

□

Remark 4.5. *From the above proposition, we have also:*

$$\mathbb{E}\int_s^T |X_l|^2 dl \leq c[\mathbb{E}|x|^2 + \int_s^T \mathbb{E}(|\Gamma_l|_{L(H)}^2 + |\gamma_l|_H^2) dl + \mathbb{E}(\int_s^T |\rho_l| dl)^2].$$

4.2 Existence of Linear BSDE

In this subsection, we study the following linear BSDE:

$$\begin{cases} -dP_t &= [A^*P_t + \sum_{i=1}^{\infty} C_i^*(t)Q_t e_i]\,dt + f_t\,dt - \sum_{i=1}^{\infty} Q_t e_i\,d\beta_t^i \\ P_T &= \eta. \end{cases} \tag{4.2}$$

where $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, $f \in L^2_{\mathcal{P}}(\Omega \times [0,T], H)$, and $\{e_i\}$ is a fixed orthonormal basis in H.

Theorem 4.6. *There exists at least one solution (P,Q) for (4.2) such that*

4.2.1 Approximating Equation

The first step to prove the existence is to study the approximating equation.

$$\begin{cases} -dP_t^N &= [A^*P_t^N + \sum_{i=1}^{N} C_i^*(t)Q_t^N e_i]\,dt + f_t\,dt - \sum_{i=1}^{\infty} Q_t^N e_i\,d\beta_t^i \\ P_T^N &= \eta. \end{cases} \tag{4.3}$$

Proposition 4.7. *There exists a unique (P^N, Q^N) with*

$$P^N \in L^2_{\mathcal{P}}(\Omega, C([0,T],H), \quad Q^N \in L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(H))$$

verifying (4.3) in the following mild sense:

$$P_t^N = e^{(T-t)A^*}\eta + \int_t^T e^{(s-t)A^*}[\sum_{i=1}^{N} C_i^*(s)Q_s^N e_i + f_s]\,ds - \sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*}Q_s^N e_i\,d\beta_s^i.$$

Proof. It is a direct consequence in [10]. □

Proposition 4.8.

$$\begin{aligned} &\mathbb{E}\langle P_t^N, x\rangle + \mathbb{E}\int_t^T \langle P_s^N, \rho_s\rangle ds + \mathbb{E}\int_t^T \langle Q_s^N, \Gamma_s\rangle_{L_2(H)} ds + \mathbb{E}\int_t^T \sum_{i=1}^{M}\langle Q_s^N e_i, C_i^*(s)\gamma_s\rangle ds \\ = \;& \mathbb{E}\langle \eta, X_T^{N,M}\rangle + \mathbb{E}\int_t^T \langle f_s, X_s^{N,M}\rangle ds, \end{aligned}$$

where

$$dX_s^{N,M} = (AX_s^{N,M} + \rho_s)ds + \sum_{i=1}^{N} C_i(s)X_s^{N,M}d\beta_s^i + \sum_{i=1}^{\infty}\Gamma_s e_i d\beta_s^i + \sum_{i=1}^{M} C_i(s)\gamma_s d\beta_s^i.$$

Proof. First approximate A with Yosida approximations and then pass to the limit. We notice that all terms are well defined. □

4.2.2 Weak Convergence

The second step to prove the existence is to define (P, Q) as weak limit of (P^N, Q^N) in some Hilbert space.

Corollary 4.9. *We have*

1. *P^N converges weakly to an element in $L^2([0,T]\times\Omega, H)$;*
2. *for any t, P_t^N converges weakly to an element P_t in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}, H)$;*
3. *Q^N converges weakly to an element Q in $L^2([0,T]\times\Omega, L_2H))$.*

Moreover, we have

$$\begin{aligned} & \mathbb{E}\langle \tilde{P}_t, x\rangle + \mathbb{E}\int_t^T \langle P_s, \rho_s\rangle ds + \mathbb{E}\int_t^T \langle Q_s, \Gamma_s\rangle_{L_2(H)} ds + \mathbb{E}\int_t^T \sum_{i=1}^M \langle Q_s e_i, C_i^*(s)\gamma_s\rangle ds \\ = \;& \mathbb{E}\langle \eta, X_T^M\rangle + \mathbb{E}\int_t^T \langle f_s, X_s^M\rangle ds, \end{aligned}$$

where

$$dX_s^M = (AX_s^M + \rho_s)ds + \sum_{i=1}^\infty C_i(s)X_s^M d\beta_s^i + \sum_{i=1}^\infty \Gamma_s e_i d\beta_s^i + \sum_{i=1}^M C_i(s)\gamma_s d\beta_s^i.$$

Proof. The result follows from the fact that if $\Gamma \in L^\infty(\Omega\times[0,T], L(H))$, then

$$\mathbb{E}(\sup_{t\in[0,T]} 6X_t^{N,M} - X_t^M|^p) \to 0.$$

Moreover by the previous estimates the map $(\Gamma, x) \mapsto X^{N,M}$ are uniformly bounded as maps from $L^2(\Omega, \mathcal{F}_s, \mathbb{P}) \times L^2_{\mathcal{P}}(\Omega\times[0,T])$. Hence by, for instance, $\mathbb{E}\int_t^T \langle Q_s^N, \Gamma\rangle_{L_2} ds$, we define a uniformy bounded and weakly convergenging linear functional on $L^2(\Omega\times[0,T], L_2(H))$. The limiting functional

$$\mathbb{E}\int_t^T \langle Q, \Gamma\rangle_{L_2} ds = \mathbb{E}\int_t^T \langle X_s^M, f_s\rangle ds + \langle X_T^M, \eta\rangle$$

can be extended to the whole $L^2(\Omega\times[0,T], L^2)$ by duality. □

Proposition 4.10. *The map $t \mapsto P_t$ is weakly continuous $[0,T] \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$.*

Proof. Choose $\rho = 0$, $\gamma = 0$ and $\gamma = 0$ in the definition of $X^{x,s}$ (where s, x are starting time and position, $x \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$).

fix $x \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ and remark that

$$\mathbb{E}\langle P_s, x\rangle = \mathbb{E}\langle P_s, x_s\rangle = \langle < X_T^{x_s,s}, \eta\rangle + \mathbb{E}\int_s^T \langle X_l^{x_s,s}, f_l\rangle dl,$$

where $x_s = \mathbb{E}(x|\mathcal{F}_s)$.

Notice that

$$\sup_{l\in[s,T]} \mathbb{E}|X_l^{s,x_s}|^2 \le c.$$

The weak continuity is proved if we show that the map $s \mapsto X_l^{s,x_s}$ is continuous in $\bar{s}$, $\forall l > \bar{s}$. On the one hand, assume that $s_n \downarrow \bar{s}$. Then

$$||X_l^{s_n,x_{s_n}} - X_l^{\bar{s},x_{\bar{s}}}||_{L^2}^2 \leq c||X_l^{s_n,x_{s_n}} - X_l^{s_n,x_{\bar{s}}}||_{L^2}^2 + ||X_l^{s_n,x_{\bar{s}}} - X_l^{\bar{s},x_{\bar{s}}}||_{L^2}^2.$$

The first term is controlled by $\mathbb{E}|x_{s_n} - x_{\bar{s}}|^2$. And the second term is $||X_l^{s_n,x_{\bar{s}}} - X_l^{s_n,X_{s_n}^{\bar{s},X_{\bar{s}}}}||_{L^2}^2$ and is controlled by

$$\begin{aligned}
& \mathbb{E}|X_{s_n}^{\bar{s},X_{\bar{s}}} - x_{\bar{s}}|^2 \\
\leq\ & c\mathbb{E}|e^{(s_n-\bar{s})A}x_{\bar{s}} - x_{\bar{s}}|^2 + c\mathbb{E}|\sum_{i=1}^{\infty}\int_{\bar{s}}^{s_n} e^{(s_n-l)A}C_i(l)X_l^{\bar{s},x_{\bar{s}}}d\beta_t^i| \\
\leq\ & c\mathbb{E}|e^{(s_n-\bar{s})A}x_{\bar{s}} - x_{\bar{s}}|^2 + c\int_{\bar{s}}^{s_n}(s_n-l)^{-2\alpha}dl\sup_l \mathbb{E}|X_l^{\bar{s},x_{\bar{s}}}|^2,
\end{aligned}$$

which converges to 0.

On the other hand, if $s_n \uparrow s$,

$$X_l^{x_n,s_n} - X_l^{s,\bar{s}} = X_l^{X_s^{x_n,s_n},\bar{s}} - X_l^{s,\bar{s}}.$$

Hence it is enough to prove that

$$X_s^{x_n,s_n} \to x.$$

But

$$X_s^{x_n,s_n} - x = e^{(s-s_n)A}x_n - x + \int_{s_n}^{s}\sum_{i=1}^{\infty} C_i(l)X_l^{x_n,s_n}d\beta_l^i.$$

Noticing that $\mathbb{E}|X_l^{x_n,s_n}|^2 \leq c$, and that

$$e^{(s-s_n)A}x_n - x = e^{(s-s_n)A}(x_n - x) + e^{(s-s_n)A}x - x,$$

we conclude the proof. □

Corollary 4.11. *P is a progessively measurable process.*

Proof. Fix t and choose a basis $\{\varphi_m\}$ in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}, H)$. We have

$$P_l = \sum_{m=1}^{\infty}\langle P_l, \varphi_m\rangle\varphi_m.$$

□

Proposition 4.12.

$$\tilde{P} = P.$$

Proof. We have that $P \in L^2_{\mathcal{P}}(\Omega \times [0,T], H)$. Notice that we have fixed a version. Moreover notice that since $\mathbb{E}|P_t^N|^2 \leq c$, we obtain

$$\mathbb{E}\int_0^T \langle P_l^N, \rho_l\rangle dl \to \int_0^T \langle P_l, \rho_l\rangle dl, \quad \forall \rho \in L^2_{\mathcal{P}}(\Omega \times [0,T], H).$$

□

4.2.3 Regularity of Q

We have, letting $x = 0$,$\rho = 0$, and $\gamma = 0$,

$$\mathbb{E}\int_0^T \langle Q_s, \sum_{i=1}^M C_i(s)\gamma(s)\rangle ds = \mathbb{E}\langle X_T^{M,\gamma}, \eta\rangle + \mathbb{E}\int_0^T \langle X_s^{M,\gamma}, f_s\rangle ds.$$

Consider now the functionals

$$\tilde{\gamma} \mapsto \mathbb{E}\langle X_T^{M,\gamma}, \eta\rangle + \mathbb{E}\int_0^T \langle X_s^{M,\gamma}, f_s\rangle ds,$$

where $\gamma_s = (T-s)^\alpha \tilde{\gamma}_s$. By the previous estimates such functionals are equibounded and converge $(\forall \tilde{\gamma} \in L^\infty)$ to

$$\mathbb{E}\langle X_T^{\gamma}, \eta\rangle + \mathbb{E}\int_0^T \langle X_s^{\gamma}, f_s\rangle ds.$$

clearly the convergence still holds now $\forall \tilde{\gamma} \in L^2([0,T]\times\Omega, H)$ if we extend the functionals by continuity. Consequently there exists $L \in L^2([0,T]\times\Omega, H)$ such that for any $\tilde{\gamma} \in L^2([0,T]\times\Omega, H)$,

$$\int_0^T \langle (T-s)^\alpha \sum_{i=1}^M C_i^*(s) Q_s e_i, \tilde{\gamma}_s\rangle ds \to \mathbb{E}\int_0^T \langle L_s, \tilde{\gamma}_s\rangle ds.$$

By definition we put $L = (T-\cdot)^\alpha \sum_{i=1}^\infty C_i^*(\cdot)Q.e_i$ and we have $(T-\cdot)^\alpha \sum_{i=1}^M C_i^*(\cdot)Q.e_i$ converges weakly to $(T-\cdot)^\alpha \sum_{i=1}^\infty C_i^*(\cdot)Q.e_i$ in $L^2([0,T]\times\Omega, H)$.

Lemma 4.13. *$(T-\cdot)^\alpha \sum_{i=1}^N C_i^*(\cdot)Q_\cdot^N e_i$ converges weakly to $(T-\cdot)^\alpha \sum_{i=1}^\infty C_i^*(\cdot)Q.e_i$ in $L^2([0,T]\times\Omega, H)$.*

Proof. We have the following duality relation:

$$\mathbb{E}\int_0^T \langle (T-s)^\alpha \sum_{i=1}^N C_i^*(s) Q_s^N e_i, \gamma_s\rangle ds = \mathbb{E}\langle X_T^N, \eta\rangle + \mathbb{E}\int_0^T \langle X_s^N, f_s\rangle ds,$$

where

$$\begin{cases} d_t X_t^N &= A X_t^N\, dt + \sum_{i=1}^N C_i(t) X_t^N\, d\beta_t^i + \sum_{i=1}^N C_i(t)\tilde{\gamma}_t\, d\beta_t^i \\ X_0^N &= 0, \end{cases}$$

and $\tilde{\gamma} = (T-s]^\alpha \gamma$.

On the other hand,

$$\mathbb{E}\int_0^T \langle (T-s)^\alpha \sum_{i=1}^\infty C_i^*(s) Q_s e_i, \gamma_s\rangle ds = \mathbb{E}\langle X_T, \eta\rangle + \mathbb{E}\int_0^T \langle X_s, f_s\rangle ds,$$

where

$$\begin{cases} d_t X_t &= A X_t\, dt + \sum_{i=1}^\infty C_i(t) X_t\, d\beta_t^i + \sum_{i=1}^\infty C_i(t)\tilde{\gamma}_t\, d\beta_t^i \\ X_0^N &= 0. \end{cases}$$

And the claim follows by Proposition 4.4 and Theorem 4.2. □

Now we notice that for any fixed t, $f \mapsto \int_t^T (T-s)^{-2\alpha} e^{(s-t)A^*} f_s ds$ is a bounded linear functional $L^2([0,T]\times\Omega, H) \to L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$. Hence

$$\int_t^T \sum_{i=1}^N e^{(s-t)A^*} C_i^*(s) Q_s^N ds = \int_t^T (T-s)^{-\alpha} \left[\sum_{i=1}^N (T-s)^\alpha e^{(s-t)A^*} C_i^*(s) Q_s^N e_i\right] ds$$

converges weakly to $\int_t^T (T-s)^{-\alpha} \left[\sum_{i=1}^\infty (T-s)^\alpha e^{(s-t)A^*} C_i^*(s) Q_s e_i\right] ds$. Similarly $\sum_{i=1}^\infty \int_t^T e^{(s-t)A^*} Q_s^N e_i d\beta_s^i$ converges weakly to $\sum_{i=1}^\infty \int_t^T e^{(s-t)A^*} Q_s e_i d\beta_s^i$, since Q^N converges weakly to Q in $L^2(\Omega\times[0,T], L_2(H))$. Consequenly passing to the limit in equation () we get

$$\begin{aligned} P_t &= \int_t^T e^{(s-t)A^*} f_s ds + e^{(T-t)A^*}\eta + \int_t^T e^{(s-t)A^*} Q_s e_i d\beta_s^i \\ &\quad + \int_t^T (T-s)^{-\alpha} e^{(s-t)A^*} \left[\sum_{i=1}^N (T-s)^\alpha e^{(s-t)A^*} C_i^*(s) Q_s^N e_i\right] ds. \end{aligned}$$

Remark 4.14. *In the same way, $\sum_{i=1}^N \int_t^T e^{(s-t)A^*} C_i^*(s) Q_s ds$ converges weakly to $\int_t^T (T-s)^{-\alpha} \left[\sum_{i=1}^\infty (T-s)$*

Notice that Q_ζ being compace can be written

$$Q_\zeta = \sum_{j=1}^\infty a_j(\zeta) f_j(\zeta) \langle g_j(\zeta), \cdot\rangle,$$

where $a_j \in \mathbb{R}$, (f_j), (g_j) are orthonormal basis in H. The choice can be done in a measurable way. Let

$$\gamma_\zeta^N = (\sum_{j=1}^N sgn(a_j(\zeta) f_j(\zeta) \langle g_j(\zeta), \cdot\rangle)\varphi(\zeta),$$

where φ is a real valued measurable function. We notice that $|\Gamma_s^N|_{L(H)} \le \varphi(\zeta)$.

By the duality relation,

$$\mathbb{E}\int_0^T \langle Q_\zeta, \Gamma_\zeta^N\rangle = \mathbb{E}\langle X_T^N, \eta\rangle + \int_0^t \langle X_s^N, f_s\rangle ds,$$

we deduce that

$$\mathbb{E}\int_0^T (\sum_{j=1}^N |a_j(\zeta)|)\varphi(\zeta) d\zeta \le c(\mathbb{E}\eta^2)^{\frac{1}{2}} \left(\int_0^T (T-l)^{-2\alpha}\varphi^2(l) dl\right)^{\frac{1}{2}} + c\left(\int_0^T |f_s|^2 ds\right)^{\frac{1}{2}} \left(\int_0^T (T-l)^{1-2\alpha}\varphi^2(l) dl\right)^{\frac{1}{2}}$$

Letting $N \to +\infty$ in the above estimate, we get

$$\mathbb{E}\int_0^T |\mathbb{Q}_\zeta|_{L_1}\varphi(\zeta) d\zeta \le c_{f,\eta}\left(\int_0^T (T-l)^{-2\alpha}\varphi^2(l) dl\right)^{\frac{1}{2}}.$$

Letting $\tilde{\varphi}(l) = 5T-l)^{-\alpha}\varphi(l)$, we deduce

$$\mathbb{E}\int_0^T |\mathbb{Q}_\zeta|_{L_1(H)}(T-l)^\alpha\tilde{\varphi}(\zeta) d\zeta \le c_{f,\eta}|\tilde{\varphi}|_{L^2([0,T])}.$$

Corollary 4.15.

$$\mathbb{E}\int_0^T (T-l)^{2\alpha}|Q_l|^2_{L_1(H)}dl \le c.$$

Corollary 4.16. *$\int_t^T \sum_{i=1}^N e^{(s-t)A^*}C_i^*(s)Q_s e_i ds$ converges strongly in $L^1(H)$. In this case it obviouly converges to $\int_t^T (T-s)^{-\alpha}\left[(T-s)^\alpha e^{(s-t)A^*}\sum_{i=1}^\infty C_i^*(s)Q_s e_i\right]ds$.*

Proof. Here we assume that

$$\sum_{i=1}^\infty |e^{sA^*}C_i^*(\zeta)x|^2 \le cs^{-2\alpha}|x|^2.$$

This is the case for heat equation in $[0,1]$.

Let

$$Q_s = \sum_{j=1}^\infty a_j(s)f_j(s)\langle g_j(s),\cdot\rangle.$$

Then

$$\begin{aligned}
&\sum_{i=1}^N \mathbb{E}|\int_t^T e^{(s-t)A^*}C_i^*(s)Q_s e_i ds| \\
\le\ & \sum_{i=1}^N \mathbb{E}\int_t^T |e^{(s-t)A^*}C_i^*(s)Q_s e_i|ds \\
=\ & \mathbb{E}\int_t^T \sum_{i=1}^N |e^{(s-t)A^*}C_i^*(s)\sum_{j=1}^\infty a_j(s)f_j(s)\langle g_j(s),e_i\rangle|ds \\
\le\ & \mathbb{E}\int_t^T \sum_{i=1}^N\sum_{j=1}^\infty |a_j(s)|\ |e^{(s-t)A^*}C_i^*(s)f_j(s)|\ |\langle g_j(s),e_i\rangle|ds \\
\le\ & \mathbb{E}\int_t^T \sum_{j=1}^\infty |a_j(s)|\left(\sum_{i=1}^\infty |e^{(s-t)A^*}C_i^*(s)f_j(s)|^2\right)^{\frac12}\left(\sum_{i=1}^\infty \langle g_j(s),e_i\rangle^2\right)^{\frac12} \\
\le\ & \mathbb{E}\int_t^T \sum_{j=1}^\infty |a_j(s)|c(s-t)^{-\alpha}ds \\
\le\ & c\mathbb{E}\int_t^T |Q_s|_{L_1}(s-t)^{-\alpha}ds \\
=\ & c\mathbb{E}\int_t^T |Q_s|_{L_1}(T-s)^\alpha(s-t)^{-\alpha}(T-s)^{-\alpha}ds \\
\le\ & c\left(\int_t^T |Q_s|^2_{L_1}(T-s)^{2\alpha}ds\right)^{\frac12}\left(E\int_t^T (s-t)^{-2\alpha}(T-s)^{-2\alpha}ds\right)^{\frac12} \\
\le\ & \infty.
\end{aligned}$$

We have consequently proved that $\sum_{i=1}^N \int_t^T e^{(s-t)A^*}C_i^*(s)Q_s e_i ds$ converges strongly in $L^1(\Omega,\mathcal{F}_T,\mathbb{P},H)$

to $\sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*}C_i^*(s)Q_se_ids$. We also have that for any $\xi\in L^2(\Omega,\mathcal{F}_T,\mathbb{P},H)$,

$$\mathbb{E}\langle\sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*}C_i^*(s)Q_se_ids,\xi\rangle=\mathbb{E}\langle\int_t^T(T-s)^{-\alpha}\left[(T-s)^{\alpha}e^{(s-t)A^*}\sum_{i=1}^{\infty}\infty_i^*(s)Q_se_i\right]ds,\xi\rangle.$$

The right hand is defined before by weak convergence. so the two definitions concur. □

4.3 Uniqueness of Linear BSDE

We prove the uniqueness in the following couple of processes (P,Q).

1. (1) $P\in L^2_{\mathcal{P}}(\Omega\times[0,T],H)$, $Q\in L^2_{\mathcal{P}}(\Omega\times[0,T],L_2(H))$.
2. (2) $\mathbb{E}\int_0^T(T-s)^{2\alpha}|Q_s|^2_{L_1(H)}ds<\infty$.
3. (3) $\sup_n\mathbb{E}\int_0^T(T-s)^{2\alpha}|\sum_{i=1}^n C_i^*(s)Q(s)e_i|^2ds<\infty$.
4. (4) for any $t\in[0,T]$,

$$P_t=e^{(T-t)A^*}\eta+\int_t^T e^{(s-t)A^*}\sum_{i=1}^{\infty}C_i^*(s)Q_se_ids+\int_t^T e^{(s-t)A^*}f_sds-\sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*}Q_se_id\beta_s^i.$$

We notice that under assumption (2) $\sum_{i=1}^n\int_t^T e^{(s-t)A^*}C_i^*(s)Q_se_i|^2ds$ converges stronly in L^1 so the second term of the right hand side is well defined. moreover, if $(N_n)_n$ is any sequence such that $(T-\cdot)^{\alpha}\sum_{i=1}^n C_i^*(s)Q_se_i$ converges weakly in $L^2(\Omega\times[0,T])$ to an element say L, then $\sum_{i=1}^{N_n}\int_t^T e^{(s-t)A^*}C_i^*(s)Q_se_ids$ converges weakly to $\int_t^T(T-s)^{-\alpha}e^{(s-t)A^*}L_sds$. Hence

$$\sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*}C_i^*(s)Q_se_ids=\int_t^T(T-s)^{-\alpha}e^{(s-t)A^*}L_sds.$$

Theorem 4.17. *The mild solution is unique.*

Proof. Let $(N_n)_n$ be ang sequence such that $(T-\cdot)^{\alpha}\sum_{i=1}^n C_i^*(s)Q_se_i$ converges weakly in $L^2(\Omega\times[0,T],H)$ to L, then

$$P_t=e^{(T-t)A^*}\eta+\sum_{i=1}^{N_n}\int_t^T e^{(s-t)A^*}C_i^*(s)Q_se_ids+\int_t^T e^{(s-t)A^*}(T-s)^{-\alpha}\varphi_s^nds-\sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*}Q_se_id\beta_s^i,$$

where

$$\varphi_s^n=(T-s)^{\alpha}\left[L_s-\sum_{i=1}^{N_n}C_i^*(s)Q_se_ids\right].$$

We know that φ^n converges weakly to 0 in $L^2(\Omega\times[0,T],H)$.

We need the following lemma:

Lemma 4.18. *Assume that ξ is a progressively measurable process with $\mathbb{E}\int_0^T (T-s)^{2\alpha}|\xi|^2 ds < \infty$. Then for any $N \in \mathbb{N}$, there exists a unique couple of processes (P,Q) with $P \in C([0,T], L^2(\Omega, H))$ and $Q \in L^2_{\mathcal{P}}(\Omega \times [0,T], L_2(H))$ such that*

$$P_t = e^{(T-t)A^*}\eta + \int_t^T e^{(s-t)A^*} \sum_{i=1}^{N_n} C_i^*(s) Q_s e_i ds + \int_t^T e^{(s-t)A^*}\xi_s ds - \sum_{i=1}^{\infty}\int_t^T e^{(s-t)A^*} Q_s e_i d\beta_s^i.$$

Moreover, letting X^n be the solution of the FSDE:

$$\left\{ \begin{array}{rcl} dX_s^n &=& AX_s^n\, ds + \sum_{i=1}^{N_n} C_i(s) X_s^n\, d\beta_s^i + \Gamma dW_s \\ X_t^{N_n} &=& x, \end{array} \right.$$

with $\Gamma \in L^\infty((\Omega \times [0,T], L_2(H))$. Then we have the following duality relation:

$$\begin{aligned} \mathbb{E}\langle P_t, x\rangle + \mathbb{E}\int_t^T \langle Q_s, \Gamma_s\rangle_{L_2(H)} ds &= \mathbb{E}\langle X_T^n, \eta\rangle + \mathbb{E}\int_t^T \langle X_s^n, \xi_s\rangle_{L_2(H)} ds \\ &= \mathbb{E}\langle X_T^n, \eta\rangle + \mathbb{E}\int_t^T \langle (T-s)^{-\alpha} X_s^n, (T-s)^{\alpha}\xi_s\rangle_{L_2(H)} ds. \end{aligned}$$

Proof. Notice that $\mathbb{E}|X_s^{N_n}|^2 \le c$, so the last integral is well defined. The proof of this lemma is relegated to Appendix. □

End of Proof of Theorem 4.17. Let (P,Q) and (P',Q') two solutions and let $\bar{P} = P - P'$ and $\bar{Q} = Q - Q'$. Then by the above duality formula,

$$\mathbb{E}\langle \bar{P}_t, x\rangle + \mathbb{E}\int_t^T \langle \bar{Q}_s, \Gamma_s\rangle_{L_2(H)} ds = \mathbb{E}\int_t^T \langle (T-s)^{-\alpha} X_s^n, \bar{\varphi}_s^n\rangle_{L_2(H)} ds,$$

where

$$\bar{\varphi}_s^n = (T-s)^{\alpha}\left[\bar{L}_s - \sum_{i=1}^{N_n} C_i^*(s)\bar{Q}_s e_i\right],$$

and L is the weak limit of $\sum_{i=1}^{N_n} C_i^*(s)\bar{Q}_s e_i$.

finally we notice that there exists a constant c such that for any $n \in \mathbb{N}$, and $s \in [t,T]$,

$$\mathbb{E}|X_s^n|^2 \le c,$$

and moreover

$$\mathbb{E}|X_s^n - X_s|^2 \to 0, \quad \forall s \in [t,T].$$

Since $\bar{\varphi}^n$ converges weakly to 0 in $L^2(\Omega \times [0,T]$, we obtain

$$\mathbb{E}\langle \bar{P}_t, x\rangle + \mathbb{E}\int_t^T \langle \bar{Q}_s, \Gamma_s\rangle_{L_2(H)} ds = 0,$$

which concludes the proof of uniqueness. □

5 Stochastic Maximum Principle

Summarizing all what we have

$$J(x,u^\epsilon)-J(x,\bar u) \;=\; \epsilon\,\mathbb{E}\int_0^T \langle (\delta u)_s, \left[\nabla_u F(\bar X_s,\bar u_s)\right]^* p_s\rangle ds + \epsilon\mathbb{E}\int_0^T \langle \nabla_u L(\bar X_s,\bar u_s),(\delta u)_s\rangle ds$$
$$+\epsilon\mathbb{E}\int_0^T \mathrm{Tr}\left[\left(\nabla_u G(\bar X_s,\bar u_s)(\delta u)_s\right) Q_s\right] ds + o(\epsilon)$$

And we now know that all the terms in the above formula are well defined.

Recall we are assuming that $|\nabla_u G(\bar X_s,\bar u_s)v_s|_{\mathcal{L}(\mathcal{H})} \le$ cost and we have just proved that $Q \in L_1(H)$, $\mathbb{P}\otimes dt$, a.s..

So we con conclude (by the usual localization - Lebesgue differentiation procedure) that

Theorem 5.1. *$\forall v \in U$ it holds $\mathbb{P}\otimes dt$ a.s.*

$$\langle v-\bar u_s, \left[\nabla_u F(\bar X_s,\bar u_s)\right]^* p_s\rangle + \langle \nabla_u L(\bar X_s,\bar u_s), v-\bar u_s\rangle + Tr\left[\left(\nabla_u G(\bar X_s,\bar u_s)v_s\right) Q_s\right] \ge 0$$

6 Appendix

6.1 Schatten - Von Neumann Operators

We denote by $L_2(H)$ the Hilbert space of Hilbert Schmidt operators $H\to H$ endowed with the scalar product $\langle L,M\rangle_2 = \sum_{i=1}^\infty \langle Le_i, Me_i\rangle_H$

Given $L\in L_2(H)$ there exists a sequence $(a_n^L)_{n\in\mathbb{N}} \in \ell_2$ and a couple of orthonormal bases $(e_n^L)_{n\in\mathbb{N}}$, $(f_n^L)_{n\in\mathbb{N}}$ in H such that

$$L=\sum_{n=1}^\infty a_n^L f_n^L \langle e_n^L,\cdot\rangle \text{ and } |L|_2 = \sum_n (a_n^L)^2.$$

If $t\to L_t$ is a L_2 valued process then the above objects can be selected with the same measurability properties as L.

Define $L_1(H)=\{L\in L_2(H) : |L|_1<\infty\}$ where

$$|L|_1 := \sup\left\{\langle B,L\rangle_2 : B\in L_2(H),\, |B|_{\mathcal{L}(H)}\le 1\right\}$$

1. If $B\in\mathcal{L}(H)$ and $L\in L_1(H)$ then LB, BL are in $L_1(H)$ moreover
   $|LB|_1 \le |L|_1|B|_{\mathcal{L}(H)}$, $|BL|_1\le |L|_1|B|_{\mathcal{L}(H)}$
2. If $L\in L_1(H)$ the trace $\mathrm{Tr}(L) := \sum_{i=1}^\infty \langle e_i, Le_i\rangle$ converges absolutely and its value is independent on the choice of the basis $(e_i)_{i\in\mathbb{N}}$
3. $|L|_1 = \sum_{n=1}^\infty |a_n^L|$, $\mathrm{Tr}(L)=\sum_{n=1}^\infty a_n^L$ consequently $|\mathrm{Tr}(L)|\le |L|_1$

6.2 Linear BSDE: some classical results

Proposition 6.1. *Assume that*

$$\mathbb{E}\int_0^T (T-s)^{2\alpha}|\varphi_s|^2 ds < \infty.$$

then the BSDE

$$-dY_t = (A^* Y_t + \varphi_t)dt - Q_t dW_t$$

admits a unque solution with

$$\mathbb{E}\int_t^T |Q_s|^2 ds + \sup_{s\in[t,T]} \mathbb{E}|Y_s|^2 \leq c(T-t)^{1-2\alpha}\int_t^T (T-s)^{2\alpha}\mathbb{E}|\varphi_s|^2 ds.$$

Proof. The uniqueness is clear. Let us put

$$Y_t = \int_t^T \mathbb{E}(\varphi_s|\mathcal{F}_t)ds.$$

Moreover, by the martingale representation theorem,

$$\mathbb{E}(\varphi_s|\mathcal{F}_t) = \varphi_s - \int_t^s g(s,l)dW_l.$$

Notice that

$$\mathbb{E}\int_t^\rho |g(\rho,\sigma)|^2 d\sigma \leq \mathbb{E}|\varphi_\rho|^2 + \mathbb{E}|\mathbb{E}(\varphi_\rho 6\mathcal{F}_t)|^2 \leq 2\mathbb{E}|\varphi_\rho|^2.$$

From the above two equations, we have:

$$\begin{aligned} Y_t &= \int_t^T \mathbb{E}(\varphi_s|\mathcal{F}_t)ds = \int_t^T \varphi_s ds - \int_t^T \left(\int_t^s g(s,l)dW_l\right) ds \\ &= \int_t^T \varphi_s ds - \int_t^T \left(\int_l^T g(s,l)ds\right) dW_l. \end{aligned}$$

So there exists a unique couple (Y,Q) and

$$Q_l = \int_l^T g(s,l)ds.$$

Let us now establish the estimates. As $Q_\sigma = \int_\sigma^T g(\rho,\sigma)d\rho$,

$$\mathbb{E}|Q_\sigma|^2 = \mathbb{E}|\int_\sigma^T (T-\rho)^{-\alpha}(T-\rho)^\alpha g(\rho,\sigma)d\rho|^2 \leq (T-t)^{1-2\alpha}\int_\sigma^T (T-\rho)^{2\alpha}\mathbb{E}|g(\rho,\sigma)|^2 d\rho.$$

Thus

$$\int_t^T \mathbb{E}|Q_\sigma|^2 d\sigma \leq (T-t)^{1-2\alpha}\int_t^T\int_\sigma^T (T-\rho)^{2\alpha}\mathbb{E}|g(\rho,\sigma)|^2 d\rho d\sigma$$

$$\leq (T-t)^{1-2\alpha}\int_t^T (T-\rho)^{2\alpha}\left(\int_t^\rho E|g(\rho,\sigma)|^2 d\sigma\right)d\rho$$
$$\leq (T-t)^{1-2\alpha}\int_t^T (T-\rho)^{2\alpha}\mathbb{E}|\varphi(\rho)|^2 d\rho.$$

on the other hand,

$$\mathbb{E}|Y_t|^2 = \mathbb{E}|\int_t^T \mathbb{E}(\varphi_s|\mathcal{F}_t)ds|^2$$
$$= \mathbb{E}|\int_t^T (T-s)^{-\alpha}(T-s)^{\alpha}\mathbb{E}(\varphi_s|\mathcal{F}_t)ds|^2$$
$$\leq (T-t)^{1-2\alpha}\int_t^T (T-s)^{2\alpha}\mathbb{E}|\varphi(s)|^2 ds.$$

□

Proof of Lemma. The existence and uniqueness follow from the above estimate. It suffices to establish the duality by some truncation argument. □

References

[1] A. Bensoussan. Stochastic maximum principle for distributed parameter systems. J. Franklin Inst. 315 (1983), no. 5-6, 387–406.

[2] G. Da Prato, J. Zabczyk. Stochastic equations in infinite dimensions. Encyclopedia of Mathematics and its Applications, 44. Cambridge University Press, Cambridge, 1992.

[3] K. Du, Q. Meng. Stochastic maximum principle for infinite dimensional control systems. Preprint arXiv:1208.0529.

[4] K. Du, Q. Meng. A General Maximum Principle for Optimal Control of Stochastic Evolution Equations. Preprint arXiv:1206.3649.

[5] M. Fuhrman, Y. Hu, G. Tessitore. Stochastic maximum principle for optimal control of SPDEs. C. R. Math. Acad. Sci. Paris 350 (2012), no. 13-14, 683–688.

[6] M. Fuhrman, G. Tessitore. Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. Ann. Probab. 30 (2002), no. 3, 1397–1465.

[7] G. Guatteri. Stochastic maximum principle for SPDEs with noise and control on the boundary. Systems Control Lett. 60 (2011), no. 3, 198–204.

[8] G. Guatteri, G. Tessitore. On the Backward Stochastic Riccati Equation in Infinite Dimensions. SIAM J. Control and Optimization, 44 (2005), no. 1, 159–194.

[9] Henry.

[10] Y. Hu, S. Peng. Adapted solution of a backward semilinear stochastic evolution equation. Stochastic Anal. Appl. 9 (1991), no. 4, 445–459.

[11] Y. Hu, S. Peng. Maximum principle for semilinear stochastic evolution control systems. Stochastics Stochastics Rep. 33 (1990), no. 3-4, 159–180.

[12] A. Lunardi. Analytic semigroups and optimal regularity in parabolic problems. Progress in Nonlinear Differential Equations and their Applications, 16. Birkhauser, Basel, 1995.

[13] Q. Lü, X. Zhang. General Pontryagin-type stochastic maximum principle and backward stochastic evolution equations in infinite dimensions. Preprint arXiv:1204.3275.

[14] B. Øksendal, A. Sulem, T. Zhang. Optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. Adv. in Appl. Probab. 43 (2011), no. 2, 572–596.

[15] A. Pazy. Semigroups of linear operators and applications to partial differential equations. Applied Mathematical Sciences, 44. Springer, New York, 1983.

[16] S. Peng. A general stochastic maximum principle for optimal control problems. SIAM J. Control Optim. 28 (1990), no. 4, 966–979.

[17] S. Tang, X. Li. Maximum principle for optimal control of distributed parameter stochastic systems with random jumps. Differential equations, dynamical systems, and control science, 867–890, Lecture Notes in Pure and Appl. Math., 152, Dekker, New York, 1994.

[18] G. Tessitore. Some remarks on the Riccati equation arising in an optimal control problem with state- and control-dependent noise. SIAM J. Control Optim. 30 (1992), no. 3, 717–744.

[19] G. Tessitore. Existence, uniqueness and space regularity of the adapted solutions of a backward SPDE. Stochastic Anal. Appl. 14 (1996), no 4, 461–486.

[20] J. M. A. M. van Neerven, M. C. Veraar, L. Weis. Stochastic integration in UMD Banach spaces. Ann. Probab. 35 (2007), no. 4, 1438–1478.

[21] J. Yong, X.Y. Zhou. Stochastic controls. Hamiltonian systems and HJB equations. Applications of Mathematics (New York), 43. Springer, New York, 1999.

[22] X.Y. Zhou. On the necessary conditions of optimal controls for stochastic partial differential equations. SIAM J. Control Optim. 31 (1993), no. 6, 1462–1478.