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Stochastic Maximum Principle for Optimal Control of Partial Differential Equations Driven by White Noise

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Abstract

We prove a stochastic maximum principle of Pontryagin's type for the optimal control of a stochastic partial differential equation driven by white noise in the case when the set of control actions is convex. Particular attention is paid to well-posedness of the adjoint backward stochastic differential equation and the regularity properties of its solution with values in infinite-dimensional spaces.

1 Introduction

In this paper we consider an infinite-dimensional stochastic optimal control problem for a system evolving in a Hilbert space H and described by a state equation of the form

$$dX_t = AX_t dt + F(X_t, u_t) dt + G(X_t, u_t) dW_t, \quad X_0 = x \in H, \quad (1.1)$$

where A is the infinitesimal generator of a strongly continuous semigroup e^{tA} of linear operators, W a cylindrical Wiener process in H , F and G are suitable drift and diffusion coefficients, with values in H and $\mathcal{L}(H)$ respectively, depending on a control process u taking values in a set \mathcal{U} contained in another Hilbert space U . The cost function is

$$J(x, u) = \mathbb{E} \int_0^T L(X_t, u_t) dt + \mathbb{E} \Phi(X_T), \quad (1.2)$$

for suitable real-valued functions L , Φ (more precise assumptions will be given later).

Our goal in this paper is to give a necessary condition for existence of an optimal control. This condition, called stochastic maximum principle (SMP) in the sense of Pontryagin, was extensively studied in the finite dimensional case, especially after the seminal paper by Peng [17] which gives a very general form of the SMP. There is at present a great interest in generalizations to the infinite dimensional case, that was started in [1], with particular emphasis on the application to optimal control of stochastic partial differential equations (SPDEs): see [4], [5], [6], [8], [12], [14], [15], [19], [24]. The main limitation of the present state of the art is perhaps that known results only deal with the case when the Wiener process driving the equation is finite-dimensional or has a trace-class covariance operator whereas, to our best knowledge, there is no result for a cylindrical Wiener process W .

In this paper our purpose is to establish the SMP for an evolution equation driven by a cylindrical Wiener process, in a form suitable for direct application to controlled SPDEs driven by white noise. On the other hand, we will suppose in this paper that the control domain \mathcal{U} is convex, which allows us to apply convex perturbation arguments instead of spike variation in the deduction of the necessary optimality condition. Our main result is as follows: under suitable conditions, in particular differentiability conditions on F, G, L, Φ , any optimal control \bar{u} and the corresponding trajectory \bar{X} must satisfy the SMP inequality

$$\langle \nabla_u [F(\bar{X}_t, \bar{u}_t)](v - \bar{u}_t), p_t \rangle + \langle \nabla_u L(\bar{X}_t, \bar{u}_t), v - \bar{u}_t \rangle + \text{Tr} [q_t^* (\nabla_u [G(\bar{X}_t, \bar{u}_t)](v - \bar{u}_t))] \geq 0, \quad v \in \mathcal{U}.$$

Here ∇ denotes the gradient operator, (p, q) is a pair of stochastic processes taking values, respectively, in H and in the space of Hilbert-Schmidt operators on H , which we characterize as the unique solution to the so-called adjoint equation, a linear backward stochastic differential equation (BSDE) of the form

$$\begin{cases} -dp_t &= \left[A^* p_t + \sum_{i=1}^{\infty} C_i^*(t) q_t e_i + \nabla_x L(\bar{X}_t, \bar{u}_t) \right] dt - \sum_{i=1}^{\infty} q_t e_i d\beta_t^i, \\ p_T &= \nabla_x \Phi(\bar{X}_T), \end{cases} \quad (1.3)$$

where e_i is an orthonormal basis of H , $\beta_t^i = \langle W_t, e_i \rangle$ are independent scalar Brownian motions, and C_i are processes in $\mathcal{L}(H)$ defined as $C_i(t) = \nabla_x [G(\bar{X}_t, \bar{u}_t) e_i]$. There are some technical difficulties that we have to face. The first one is the occurrence of the first series in (1.3), which needs to be interpreted in a suitable way and for which convergence holds in a weak sense in general; we call this class of BSDEs ‘singular’. The second one is the occurrence of the trace operator in the SMP: for this term to be well defined we need to prove that the process q satisfies additional regularity results, in particular it takes values in the space of trace-class operators on H , for which the trace is meaningful.

This paper is organized as follows: in the next section we introduce our control problem and formulate the main assumptions; then we prove that they are verified in the reference case of the nonlinear controlled heat equation perturbed by noise. In Section 3 we apply the convex perturbation argument, we deduce an equation for the first-order variation process and, under some more assumptions, we prove the SMP taking for granted some results on singular BSDEs. In Section 4 we make a careful study of a general class of linear singular BSDEs, proving in particular the trace-class regularity and useful duality relations. In the Appendices we conclude with some reminders and some auxiliary results.

2 Formulation of the Optimal Control Problem

Given two real separable Hilbert spaces X and Y by $\mathcal{S}_2(X, Y)$ we denote the Hilbert space of Hilbert-Schmidt operators from X to Y and by $\mathcal{S}_1(X, Y)$ we denote the Banach space of trace class operators (these are instances of the Schatten-von Neumann classes of operators: some reminders are collected in Appendix 5.1). We write $\mathcal{S}_1(X)$, $\mathcal{S}_2(X)$ instead of $\mathcal{S}_1(X, X)$, $\mathcal{S}_2(X, X)$.

Given two Banach spaces D and E , we say that a mapping $f : D \rightarrow E$ is of class $\mathcal{G}^1(D, E)$ if it is Gâteaux differentiable and $\nabla f : D \rightarrow \mathcal{L}(D, E)$ is strongly continuous (that is, continuous with respect to the strong operator topology).

Finally H is a separable Hilbert space and $(W_t)_{t \geq 0}$ is an H -valued cylindrical Wiener process. We denote $(\mathcal{F}_t)_{t \geq 0}$ the corresponding Brownian filtration, completed in the usual way, that verifies the usual conditions.

By $L_{\mathcal{P}}^p([0, T] \times \Omega, H)$ we denote the Banach space of H -valued progressively measurable processes X with $\mathbb{E} \int_0^T |X_s|^p ds < \infty$ and by $L_{\mathcal{P}}^p(\Omega, C([0, T], H))$ the subspace of H -valued progressively measurable processes X with continuous trajectories satisfying $\mathbb{E} \sup_{s \in [0, T]} |X_s|^p < \infty$. Here and below we use the symbol $|\cdot|$ to denote a norm when the corresponding space is clear from the context, otherwise we use a subscript.

Assumption 2.1 (i) A is the generator of a strongly continuous semigroup e^{tA} , $t \geq 0$, of bounded linear operators in H .

(ii) \mathcal{U} is a convex subset of a separable Banach space U .

(iii) $F : H \times U \rightarrow H$ is a map of class $\mathcal{G}^1(H \times U, H)$. Moreover, denoting its gradient on $H \times U$ as $(\nabla_x F, \nabla_u F)$, there exist constants $L, C \geq 0$ such that

$$|\nabla_x F(x, u)|_{\mathcal{L}(H)} + |\nabla_u F(x, u)|_{\mathcal{L}(U, H)} \leq L, \quad |F(0, u)| \leq C, \quad x \in H, u \in \mathcal{U}.$$

(iv) $G : H \times U \rightarrow L(H)$ satisfies $e^{sA}G(x, u) \in \mathcal{S}_2(H)$ for all $s > 0$, $x \in H$, $u \in U$, and the map $(x, u) \rightarrow e^{sA}G(x, u)$ is of class $\mathcal{G}^1(H \times U, \mathcal{S}_2(H))$. Moreover there exist constants $L, C \geq 0$ and $\alpha \in [0, 1/2)$ such that

$$\begin{aligned} |\nabla_x [e^{sA}G(x, u)]|_{\mathcal{L}(H, \mathcal{S}_2(H))} + |\nabla_u [e^{sA}G(x, u)]|_{\mathcal{L}(U, \mathcal{S}_2(H))} &\leq Ls^{-\alpha}, \\ |e^{sA}G(0, u)|_{\mathcal{S}_2(H)} &\leq Cs^{-\alpha}, \end{aligned}$$

for all $x \in H$, $u \in \mathcal{U}$.

(v) $L : H \times U \rightarrow \mathbb{R}$ and $\Phi : H \rightarrow \mathbb{R}$ are bounded, Lipschitz and of class \mathcal{G}^1 .

(vi) There exists an orthonormal basis $(e_i)_{i \in \mathbb{N}} \in H$ such that, for all $i \in \mathbb{N}$ and all $u \in U$, the map $x \rightarrow G(x, u)e_i$ is of class $\mathcal{G}^1(H, H)$. Moreover there exists a constant $L \geq 0$ such that

$$|\nabla_x [G(x, u)e_i]|_{\mathcal{L}(H)} \leq L, \quad i \in \mathbb{N}, x \in H, u \in U.$$

(vii) For all $x, y \in H$ the map $u \rightarrow G(x, u)y$ is of class $\mathcal{G}^1(U, H)$ and there exists a constant $C \geq 0$ such that

$$|\nabla_u [G(x, u)y]v|_H \leq C|y|_H|v|_U, \quad x, y \in H, u, v \in U.$$

Any progressively measurable \mathcal{U} -valued process u will be called an admissible control.

Under the above assumptions, for every admissible control, the state equation (formulated in mild sense):

$$X_t = e^{tA}x + \int_0^t e^{(t-s)A}F(X_s, u_s)ds + \int_0^t e^{(t-s)A}G(X_s, u_s)dW_s \quad (2.1)$$

admits a unique solution $X \in L^p_p(\Omega, C([0, T], H))$ for every $p \geq 1$, see [2]. Moreover the cost associated to the initial datum x and control u is the well defined real number

$$J(x, u) = \mathbb{E} \int_0^T L(X_t, u_t)dt + \mathbb{E}\Phi(X_T).$$

Remark 2.2

1. Combining points (iv) and (vii) of Assumption 2.1 it is easy to check that for every $s > 0$, $x, h \in H$, $u \in U$ and any direction $v \in U$ we have

$$(\nabla_u[e^{sA}G(x, u)]v)h = e^{sA}\nabla_u[G(x, u)h]v. \quad (2.2)$$

Similarly, $\nabla_x[e^{sA}G(x, u)]e_i = e^{sA}\nabla_x[G(x, u)e_i]$.

2. For further use we need to introduce a suitable approximation of the derivative operator $(\nabla_u[e^{sA}G(x, u)]v)$. Let us denote Π_n the orthogonal projection in $\mathcal{L}(H)$ onto the linear span of e_1, \dots, e_n . Next, for fixed $x \in H$, $u, v \in U$, let us define operators in $\mathcal{L}(H)$ setting, for every $y \in H$,

$$\Gamma(x, u, v)y = \nabla_u[G(x, u)y]v, \quad \Gamma^n(x, u, v)y = \nabla_u[G(x, u)(\Pi_n y)]v = \sum_{i=1}^n \nabla_u[G(x, u)e_i]v \langle y, e_i \rangle_H.$$

Note that by (vi) we have

$$|\Gamma(x, u, v)|_{\mathcal{L}(H)} \leq C|v|, \quad |\Gamma^n(x, u, v)|_{\mathcal{L}(H)} \leq C|v|\sqrt{n}, \quad \Gamma^n(x, u, v)y \rightarrow \Gamma(x, u, v)y$$

in H as $n \rightarrow \infty$. Moreover, recalling (2.2), we have for $s > 0$

$$|e^{sA}\Gamma^n(x, u, v) - (\nabla_u[e^{sA}G(x, u)]v)|_{\mathcal{S}_2(H)}^2 = \sum_{i>n} |e^{sA}\nabla_u[G(x, u)e_i]v|_H^2 \downarrow 0, \quad (2.3)$$

since the series $\sum_{i=1}^\infty |e^{sA}\nabla_u[G(x, u)e_i]v|_H^2 = |\nabla_u[e^{sA}G(x, u)]v|_{\mathcal{S}_2(H)}^2$ is convergent. Finally, given a trace class operator $Q \in \mathcal{S}_1(H)$ it is easy to check that

$$|\text{Tr}(Q\Gamma^n)| \leq C|v|_U |Q|_{\mathcal{S}_1(H)}, \quad \text{Tr}(Q\Gamma^n) \rightarrow \text{Tr}(Q\Gamma). \quad (2.4)$$

2.1 The Reference Example

Consider the following controlled stochastic heat equation in $[0, 1]$:

$$\begin{cases} dX_t(\xi) &= \frac{\partial^2}{\partial x^2} X_t(\xi) dt + b(\xi, X_t(\xi), u_t(\xi)) dt + \sigma(\xi, X_t(\xi), u_t(\xi))d\mathcal{W}(t, \xi), \\ X_t(0) &= X_t(1) = 0, \quad t \in [0, T], \\ X_0(\xi) &= x(\xi), \quad \xi \in [0, 1], \end{cases}$$

where b and $\sigma : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$ are given Borel measurable functions. We assume that $b(t, \cdot, \cdot)$ and $\sigma(t, \cdot, \cdot)$ are of class C^1 , Lipschitz uniformly with respect to t , and that $b(\cdot, 0, 0)$ and $\sigma(\cdot, 0, 0)$ are bounded. In the above equation $(\mathcal{W}(t, \xi))$, $t \geq 0$, $\xi \in [0, 1]$ is a space time white noise and by $(\mathcal{F}_t)_{t \geq 0}$ we denote its natural (completed) filtration. The set of admissible control actions \mathcal{U} is a convex subset of $U := L^2([0, 1])$ and we assume that $\mathcal{U} \subset L^\infty([0, 1])$. A control u is a (progressive) process with values in \mathcal{U} . We also introduce the cost functional:

$$J(u) = \mathbb{E} \int_0^T \int_0^1 l(\xi, X_t(\xi), u_t(\xi)) d\xi dt + \mathbb{E} \int_0^1 h(\xi, X_T(\xi)) d\xi,$$

where $l : [0, 1] \times \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$, $h : [0, 1] \times \mathbb{R} \rightarrow \mathbb{R}$ are given bounded, Borel measurable functions. We assume that, for a.e. $\xi \in [0, 1]$, $b(\xi, \cdot, \cdot)$ and $h(\xi, \cdot)$ are of class C^1 with bounded derivatives (uniformly with respect to ξ).

To reformulate the problem in our general framework we have set $H = L^2([0, 1])$ and consider an H -valued cylindrical Wiener process $(W_t)_{t \geq 0}$.

A is the realization of the second derivative operator in H with Dirichlet boundary conditions. So $\mathcal{D}(A) = H^2([0, 1]) \cap H_0^1([0, 1])$ and $A\phi = \phi''$ for all $\phi \in \mathcal{D}(A)$.

Finally for $x, y \in L^2([0, 1])$, $u, v \in L^\infty([0, 1])$,

$$F(x, u)(\xi) = b(\xi, x(\xi), u(\xi)), \quad [G(x, u)y](\xi) = \sigma(\xi, x(\xi), u(\xi))y(\xi),$$

$$L(x, u) = \int_0^1 l(\xi, x(\xi), u(\xi))d\xi, \quad \Phi(x) = \int_0^1 h(\xi, X(\xi))d\xi.$$

The state equation written in abstract form is

$$dX_t = AX_t dt + F(X_t, u_t)dt + G(X_t, u_t)dW_t, \quad X_0 = x,$$

where $x \in H$ and the solution will evolve in H . Moreover the cost becomes

$$J(x, u) = \mathbb{E} \int_0^T L(X_s, u_s)ds + \mathbb{E}\Phi(X_T).$$

It is well known (see [3]) that Assumptions 2.1 (i)-(v) are satisfied. Concerning Assumption 2.1 (vi) we notice that in our concrete case for all $y \in L^2([0, 1])$,

$$[\nabla_x(G(x, u)e_i)y](\xi) = \frac{\partial \sigma}{\partial x}(\xi, x(\xi), u(\xi))e_i(\xi)y(\xi).$$

So it is enough to choose an orthonormal basis in $L^2([0, 1])$ with $\sup_i \sup_\xi |e_i(\xi)| < \infty$, for instance a trigonometrical basis.

Finally concerning Assumption 2.1 (vii) we notice that for all $y \in L^2([0, 1])$ and all $v \in L^\infty([0, 1])$:

$$[\nabla_u((G(x, u)y)v)](\xi) = \frac{\partial \sigma}{\partial u}(\xi, x(\xi), u(\xi))y(\xi)v(\xi),$$

and $||[\nabla_u((G(x, u)y)v)](\xi)||_{L^2([0, 1])} \leq L_\sigma |y|_{L^2([0, 1])} |v|_{L^\infty([0, 1])}$ where L_σ is the Lipschitz constant of σ with respect to u .

3 The Stochastic Maximum Principle

3.1 First Variation

In this section we perturb a given admissible control, that eventually will be the optimal one, and compute the corresponding expansion of the cost.

In the following u and u' are two admissible controls and we assume that $\delta u := u' - u$ is bounded in U . Moreover we denote by X^ϵ the state (e.g. the solution to equation (2.1) corresponding to the control $u^\epsilon := (1 - \epsilon)u + \epsilon u' = u + \epsilon \delta u$).

Regular dependence on parameters of the mild solution to forward SDEs gives us the first order expansion of the state:

Theorem 3.1 *The map $\epsilon \rightarrow X^\epsilon$ is of class \mathcal{G}^1 from $[0, 1)$ to $L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ and its derivative in $\epsilon = 0$ is given by the unique mild solution $Y \in L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ of the linear equation*

$$\begin{cases} dY_t &= \left[AY_t + \nabla_x F(X_t, u_t)Y_t + \nabla_u F(X_t, u_t)\delta u_t \right] dt \\ &+ \left[\nabla_x G(X_t, u_t)Y_t + \nabla_u G(X_t, u_t)\delta u_t \right] dW_t, \\ Y_0 &= 0. \end{cases} \quad (3.1)$$

Explicitly, we have \mathbb{P} -a.s.

$$\begin{aligned} Y_t &= \int_0^t e^{(t-s)A} \nabla_x F(X_s, u_s) Y_s ds + \int_0^t e^{(t-s)A} \nabla_u F(X_s, u_s) \delta u_s ds \\ &+ \int_0^t \nabla_x [e^{(t-s)A} G(X_s, u_s)] Y_s dW_s + \int_0^t \nabla_u [e^{(t-s)A} G(X_s, u_s)] \delta u_s dW_s, \quad t \in [0, T]. \end{aligned} \quad (3.2)$$

Proof. The proof will follow arguments similar to the ones exposed in [7], see the proof of Proposition 3.3. We limit ourselves to proving the claim in the case $F = 0$, the general case being a straightforward extension.

Consider the mapping Φ from $L^p_{\mathcal{P}}(\Omega, C([0, T], H)) \times [0, 1)$ to $L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ given by

$$\Phi(\Xi, \epsilon)_t = e^{tA} x + \int_0^t e^{(t-l)A} G(\Xi_l, u_l + \epsilon \delta u_l) dW_l.$$

Clearly X^ϵ is a solution to the state equation (2.1) with u replaced by u^ϵ if and only if it is a fixed point of $\Phi(\cdot, \epsilon)$.

In [7] it is shown that, if $\beta > 0$ is large enough then $\Phi(\cdot, \epsilon)$ is a contraction, uniformly with respect to ϵ , in $L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ endowed with the equivalent norm $(\mathbb{E} \sup_{t \in [0, T]} e^{\beta t} |\Xi_t|^p)^{1/p}$. Moreover $\Phi(\cdot, \epsilon)$ is of class \mathcal{G}^1 from $L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ to $L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ with derivative, in the direction N , given by:

$$(\nabla_{\Xi} \Phi(\Xi, \epsilon) N)_t = \int_0^t e^{(t-l)A} \nabla_x G(\Xi_l, u_l + \epsilon \delta u_l) N_l dW_l.$$

Concerning the dependence on ϵ we have

$$\begin{aligned} I_t^h &:= \frac{\Phi(\Xi, \epsilon + h)_t - \Phi(\Xi, \epsilon)_t}{h} - \int_0^t \nabla_u \left(e^{(t-l)A} G(\Xi_l, u_l) \delta u_l \right) dW_l \\ &= \int_0^t \left\{ \int_0^1 \left[\nabla_u \left(e^{(t-l)A} G(\Xi_l, u_l^\epsilon + \zeta h \delta u_l) \right) \delta u_l - \nabla_u \left(e^{(t-l)A} G(\Xi_l, u_l^\epsilon) \right) \delta u_l \right] d\zeta \right\} dW_l. \end{aligned}$$

By the factorization method (see the proof of Proposition 3.2 in [7]) we get for $1/p < \gamma < 1/2 - \alpha$:

$$|I^h|_{L^p_{\mathcal{P}}(\Omega, C([0, T], H))} \leq c_p \mathbb{E} \int_0^T |V_l^h|^p dl,$$

where

$$V_l^h = \int_0^l (l-\sigma)^{-\gamma} \left\{ \int_0^1 \left[\nabla_u \left(e^{(l-\sigma)A} G(\Xi_\sigma, u_\sigma^\epsilon + \zeta h \delta u_\sigma) \right) \delta u_\sigma - \nabla_u \left(e^{(l-\sigma)A} G(\Xi_\sigma, u_\sigma^\epsilon) \right) \delta u_\sigma \right] d\zeta \right\} dW_\sigma.$$

By the Burkholder-Davis-Gundy inequality

$$\mathbb{E}|V_l^h|^p \leq c_p \mathbb{E} \left\{ \int_0^l (l-\sigma)^{-2\alpha} \left[\int_0^1 \left| \nabla_u \left(e^{(l-\sigma)A} G(\Xi_\sigma, u_\sigma^\epsilon + \zeta h \delta u_\sigma) \right) \delta u_\sigma - \nabla_u \left(e^{(l-\sigma)A} G(\Xi_\sigma, u_\sigma^\epsilon) \right) \delta u_\sigma \right|_{S_2(K, H)}^2 d\zeta \right] d\sigma \right\}^{p/2}.$$

Thus by Hypothesis 2.1-(iv):

$$\mathbb{E}|V_l^h|^p \leq c_p \left\{ |\delta u|_{L^\infty(\Omega \times [0, T], U)}^2 \int_0^l (l-\sigma)^{-2(\gamma+\alpha)} d\sigma \right\}^{p/2} \leq c_p l^{p/2-p(\gamma+\alpha)} |\delta u|_{L^\infty(\Omega \times [0, T], U)}^p.$$

The continuity of $\nabla_u (e^{(s-\sigma)A} G(\Xi, \cdot))$ and Dominated Convergence Theorem yield: $\mathbb{E}|Y_s^h|^p \rightarrow 0$ and consequently that $|I^h|_{L^p_{\mathcal{P}}(\Omega, C([0, T], H))} \rightarrow 0$. We can therefore conclude that Φ is differentiable with respect to ϵ as well, with

$$\nabla_\epsilon \Phi(\Xi, \epsilon) = \int_0^t \nabla_u \left(e^{(t-l)A} G(\Xi_l, u_l) \right) \delta u_l dW_l.$$

The continuity of $\nabla_\Xi \Phi(\Xi, \epsilon)$ and $\nabla_\epsilon \Phi(\Xi, \epsilon)$ with respect to Ξ and ϵ can be proved in a similar way.

Summing up, Φ is a mapping of class \mathcal{G}^1 on $L^p_{\mathcal{P}}(\Omega, C([0, T], H)) \times [0, 1)$. The parameter depending contraction principle (see [7] Proposition 2.4) yields that the map $[0, 1) \ni \epsilon \rightarrow X^\epsilon \in L^p_{\mathcal{P}}(\Omega, C([0, T], H))$ that relates the parameter to the fixed point is of class \mathcal{G}^1 (in this case just differentiable with continuous derivative). Moreover its derivative satisfies:

$$\nabla_\epsilon X^\epsilon = \nabla_\Xi \Phi(X^\epsilon, \epsilon) \nabla_\epsilon X^\epsilon + \nabla_\epsilon \Phi(X^\epsilon, \epsilon).$$

Plugging in the above relation the expressions for $\nabla_\Xi \Phi$ and $\nabla_\epsilon \Phi$ we get,

$$\nabla_\epsilon X_t^\epsilon = \int_0^t e^{(t-l)A} \nabla_x G(X_l^\epsilon, u_l + \epsilon \delta u_l) \nabla_\epsilon X_l^\epsilon dW_l + \int_0^t e^{(t-l)A} \nabla_u G(X_l^\epsilon, u_l + \epsilon \delta u_l) \delta u_l dW_l,$$

and the claim follows letting $\epsilon = 0$ and denoting Y_t the limit. \square

As a consequence of the above result we have the following expansion of the cost:

Proposition 3.2 *With the above notation, we have:*

$$J(x, u^\epsilon) = J(x, u) + \epsilon I(\delta u) + o(\epsilon),$$

where

$$I(\delta u) = \mathbb{E} \int_0^T \left[\langle \nabla_x L(X_t, u_t), Y_t \rangle + \langle \nabla_u L(X_t, u_t), \delta u_t \rangle \right] dt + \mathbb{E} \langle \nabla_x \Phi(X_T), Y_T \rangle.$$

Proof. By the above theorem, if $R_t^\epsilon := \epsilon^{-1}[X_t^\epsilon - X_t - \epsilon Y_t]$ then $|R^\epsilon|_{L^p(\Omega, C([0, T], H))} \rightarrow 0$. Moreover

$$\begin{aligned}
J(x, u^\epsilon) - J(x, u) &= \mathbb{E} \int_0^T [L(X_t + \epsilon(Y_t + R_t^\epsilon), u_t + \epsilon \delta u_t) - L(X_t, u_t)] dt \\
&\quad + \mathbb{E} [\Phi(X_T + \epsilon(Y_T + R_T^\epsilon)) - \Phi(X_T)] \\
&= \epsilon \mathbb{E} \int_0^T \int_0^1 \nabla_x L(X_t + \lambda \epsilon(Y_t + R_t^\epsilon), u_t + \lambda \epsilon \delta u_t)(Y_t + R_t^\epsilon) d\lambda dt \\
&\quad + \epsilon \mathbb{E} \int_0^T \int_0^1 \nabla_u L(X_t + \lambda \epsilon(Y_t + R_t^\epsilon), u_t + \lambda \epsilon \delta u_t) \delta u_t d\lambda dt \\
&\quad + \epsilon \mathbb{E} \int_0^1 \nabla_x \Phi(X_T + \lambda \epsilon(Y_T + R_T^\epsilon))(Y_T + R_T^\epsilon) d\lambda,
\end{aligned}$$

taking into account the continuity and boundedness of $\nabla_x L$, $\nabla_u L$ and $\nabla_x \Phi$, applying the dominated convergence theorem it is then easy to show that

$$\begin{aligned}
&\mathbb{E} \int_0^T \int_0^1 \nabla_x L(X_t + \lambda \epsilon(Y_t + R_t^\epsilon), u_t + \lambda \epsilon \delta u_t) Y_t d\lambda dt \rightarrow \mathbb{E} \int_0^T \nabla_x L(X_t, u_t) Y_t dt, \\
&\mathbb{E} \int_0^T \int_0^1 \nabla_x L(X_t + \lambda \epsilon(Y_t + R_t^\epsilon), u_t + \lambda \epsilon \delta u_t) R_t^\epsilon d\lambda dt \rightarrow 0, \\
&\mathbb{E} \int_0^T \int_0^1 \nabla_u L(X_t + \lambda \epsilon(Y_t + R_t^\epsilon), u_t + \lambda \epsilon \delta u_t) \delta u_t d\lambda dt \rightarrow \mathbb{E} \int_0^T \nabla_u L(X_t, u_t) \delta u_t dt, \\
&\mathbb{E} \int_0^1 \nabla_x \Phi(X_T + \lambda \epsilon(Y_T + R_T^\epsilon)) Y_T d\lambda \rightarrow \mathbb{E} \nabla_x \Phi(X_T) Y_T, \\
&\mathbb{E} \int_0^1 \nabla_x \Phi(X_T + \lambda \epsilon(Y_T + R_T^\epsilon)) R_T^\epsilon d\lambda \rightarrow 0,
\end{aligned}$$

and the proof is completed. \square

3.2 Stochastic Maximum Principle

We will prove the stochastic maximum principle under the following additional assumption:

Assumption 3.3 *We have $e^{tA} \in \mathcal{S}_2(H)$ for all $t > 0$ and there exist $c > 0$ and $\alpha < 1/2$ such that*

$$|e^{tA}|_{\mathcal{S}_2(H)} \leq ct^{-\alpha}, \quad t \in (0, T].$$

In order to state the stochastic maximum principle we assume that an optimal control \bar{u} exists and we denote \bar{X} is the corresponding state. Next we need to introduce the dual process (p_t, q_t) , with values in $H \times \mathcal{S}_2(H)$. To this end we fix an orthonormal basis $\{e_i\}_{i \in \mathbb{N}}$ in H such that point (vi) in Assumption 2.1 holds and we define $C_i(t)h = \nabla_x [G(\bar{X}_t, \bar{u}_t)e_i]h$ for $h \in H$. Then we have

$$|C_i(t)|_{\mathcal{L}(H)} \leq L. \quad (3.3)$$

Recalling Remark 2.2-1 and taking into account Hypothesis 2.1-(iv) we also obtain

$$\sum_{i=1}^{\infty} |e^{tA} C_i(s)h|^2 = \sum_{i=1}^{\infty} |\nabla_x [e^{tA} G(\bar{X}_s, \bar{u}_s)e_i]h|^2 = |\nabla_x [e^{tA} G(\bar{X}_s, \bar{u}_s)]h|_{\mathcal{S}_2(H)}^2 \leq L^2 t^{-2\alpha} |h|_H^2, \quad (3.4)$$

for all $t > 0$, $s \geq 0$, $h \in H$.

Next we introduce the adjoint equation for the unknown process (p, q) , written formally

$$\begin{cases} -dp_t &= \left[A^* p_t + \sum_{i=1}^{\infty} C_i^*(t) q_t e_i + \nabla_x L(\bar{X}_t, \bar{u}_t) \right] dt - \sum_{i=1}^{\infty} q_t e_i d\beta_t^i, \\ p_T &= \nabla_x \Phi(\bar{X}_T), \end{cases} \quad (3.5)$$

where $\beta_t^i = \langle e_i, W_t \rangle$, $i = 1, 2, \dots$ is a family of independent Brownian motions. The precise notion of (mild) solution to this equation is as follows: we say that a pair (p, q) with $p \in L^2_{\mathcal{P}}(\Omega \times [0, T], H)$, $q \in L^2_{\mathcal{P}}(\Omega \times [0, T], \mathcal{S}_2(H))$ is a mild solution to equation (3.5) if, for any $t \in [0, T]$, we have

$$\begin{aligned} p_t &= e^{(T-t)A^*} \nabla_x \Phi(\bar{X}_T) + \sum_{i=1}^{\infty} \int_t^T e^{(s-t)A^*} C_i^*(s) Q_s e_i ds \\ &\quad + \int_t^T e^{(s-t)A^*} \nabla_x L(\bar{X}_s, \bar{u}_s) ds - \sum_{i=1}^{\infty} \int_t^T e^{(s-t)A^*} Q_s e_i d\beta_s^i, \quad \mathbb{P} - a.s., \end{aligned} \quad (3.6)$$

where, for fixed t , the series $\sum_{i=1}^{\infty} \int_t^T e^{(s-t)A^*} C_i^*(s) Q_s e_i ds$ is required to converge weakly in the space $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$ [in Proposition 4.18 below we will also give additional conditions that guarantee that the series converges in a stronger sense]. We refer the reader to the next section, in particular to Definition 4.1 and Remark 4.2, for a more precise discussion of this notion of solution. There we will also prove the following result.

Proposition 3.4 *Under Assumptions 2.1 and 3.3 there exists a unique solution (p, q) to equation (3.5). Moreover, $q_t \in \mathcal{S}_1(H)$ $d\mathbb{P} \otimes dt$ -a.s. and*

$$\mathbb{E} \int_0^T (T-t)^{2\alpha} |q_t|_{\mathcal{S}_1(H)}^2 dt < \infty.$$

Proof. This follows from Theorem 4.14 and Proposition 4.15. Hypothesis 4.1, needed for these statements to hold, is verified due to Assumption 3.3 and inequalities (3.3) and (3.4). \square

The final ingredient in the proof of the stochastic maximum principle is a duality relation involving the first variation process Y solution to equation (3.1) with $u = \bar{u}$ and $X = \bar{X}$. In the present notation this equation (to be understood in the mild form (3.2)) can be written formally as

$$\begin{cases} dY_t &= \left[AY_t + \nabla_x F(\bar{X}_t, \bar{u}_t) Y_t + \nabla_u F(\bar{X}_t, \bar{u}_t) \delta u_t \right] dt \\ &\quad + \sum_{i=1}^{\infty} C_i(t) Y_t d\beta_t^i + \sum_{i=1}^{\infty} \nabla_u [G(\bar{X}_t, \bar{u}_t) e_i] \delta u_t d\beta_t^i, \\ Y_0 &= 0. \end{cases} \quad (3.7)$$

In the next section we will prove the following duality relation.

Proposition 3.5 *With the previous assumptions and notations, suppose that $\rho : [0, T] \times \Omega \rightarrow H$ and $\Gamma : [0, T] \times \Omega \rightarrow \mathcal{S}_2(H)$ are progressively measurable and bounded and \mathcal{Y} denotes the unique mild solution to the equation*

$$\begin{cases} d\mathcal{Y}_t &= \left[A\mathcal{Y}_t + \nabla_x F(\bar{X}_t, \bar{u}_t) \mathcal{Y}_t + \rho_t \right] dt + \sum_{i=1}^{\infty} C_i(t) \mathcal{Y}_t d\beta_t^i + \sum_{i=1}^{\infty} \Gamma_t e_i d\beta_t^i, \\ \mathcal{Y}_0 &= 0. \end{cases} \quad (3.8)$$

Then

$$\mathbb{E} \int_0^T \langle p_t, \rho_t \rangle dt + \mathbb{E} \int_0^T \langle q_t, \Gamma_t \rangle_{\mathcal{S}_2(H)} dt = \mathbb{E} \langle \nabla_x \Phi(\bar{X}_T), \mathcal{Y}_T \rangle + \mathbb{E} \int_0^T \langle \nabla_x L(\bar{X}_t, \bar{u}_t), \mathcal{Y}_t \rangle dt. \quad (3.9)$$

Proof. This is exactly formula (4.13) of Corollary 4.8, where we put $s = 0$, $x = 0$, $\gamma = 0$, $\eta = \nabla_x \Phi(\bar{X}_T)$, $f_t = \nabla_x L(\bar{X}_t, \bar{u}_t)$ and note that in this case $\tilde{\mathcal{Y}}^{\infty, M}$ coincides with the solution \mathcal{Y} to (3.8). \square

Now we are ready to state and prove the main result of this paper.

Theorem 3.6 *Suppose that Assumptions 2.1 and 3.3 hold and that an optimal pair (\bar{u}, \bar{X}) exists. Then for every $v \in \mathcal{U}$ we have, $d\mathbb{P} \otimes dt$ -a.s.,*

$$\langle \nabla_u [F(\bar{X}_t, \bar{u}_t)](v - \bar{u}_t), p_t \rangle + \langle \nabla_u L(\bar{X}_t, \bar{u}_t), v - \bar{u}_t \rangle + \text{Tr} [q_t^* (\nabla_u [G(\bar{X}_t, \bar{u}_t)](v - \bar{u}_t))] \geq 0,$$

where (p, q) is the unique mild solution to equation (3.5).

Proof. *Step 1:* we prove the duality formula

$$\begin{aligned} & \mathbb{E} \int_0^T \langle p_t, \nabla_u F(\bar{X}_t, \bar{u}_t) \delta u_t \rangle dt + \mathbb{E} \int_0^T \text{Tr} [q_t^* (\nabla_u [G(\bar{X}_t, \bar{u}_t)] \delta u_t)] dt \\ &= \mathbb{E} \langle \nabla_x \Phi(\bar{X}_T), Y_T \rangle + \mathbb{E} \int_0^T \langle \nabla_x L(\bar{X}_t, \bar{u}_t), Y_t \rangle dt, \end{aligned} \quad (3.10)$$

where Y is the first variation process solution to (3.7).

We define

$$\rho_t = \nabla_u F(\bar{X}_t, \bar{u}_t) \delta u_t, \quad \Gamma_t h = \nabla_u [G(\bar{X}_t, \bar{u}_t) h] \delta u_t,$$

for every $h \in H$. Since we take δu to be a bounded process, it follows that ρ is also bounded, by Assumption 2.1-(iii). Heuristically, we note that with this choice the equations (3.7) and (3.8) coincide, so that $Y = \mathcal{Y}$ and (3.10) coincides with (3.9). However such argument is not correct, as we can not directly apply Proposition 3.5 above, since Γ is not a bounded process with values in $\mathcal{S}_2(H)$, so we have to revert to an approximation procedure.

Let us denote Π_n the orthogonal projection in $\mathcal{L}(H)$ onto the linear span of e_1, \dots, e_n and define

$$\Gamma_t^n h = \nabla_u [G(\bar{X}_t, \bar{u}_t) (\Pi_n h)] \delta u_t = \sum_{i=1}^n \nabla_u [G(\bar{X}_t, \bar{u}_t) e_i] \delta u_t \langle h, e_i \rangle_H, \quad h \in H.$$

Each Γ^n is a bounded process in $\mathcal{L}(H)$ (by Assumption 2.1-(vii) and since δu is bounded) and since it has finite rank it is also bounded in $\mathcal{S}_2(H)$. Let \mathcal{Y}^n be the unique mild solution to the equation

$$\begin{cases} d\mathcal{Y}_t^n &= [A\mathcal{Y}_t^n + \nabla_x F(\bar{X}_t, \bar{u}_t) \mathcal{Y}_t^n + \rho_t] dt + \sum_{i=1}^{\infty} C_i(t) \mathcal{Y}_t^n d\beta_t^i + \sum_{i=1}^{\infty} \Gamma_t^n e_i d\beta_t^i, \\ \mathcal{Y}_0^n &= 0. \end{cases} \quad (3.11)$$

We can now apply Proposition 3.5 and obtain the duality relation

$$\mathbb{E} \int_0^T \langle p_t, \rho_t \rangle dt + \mathbb{E} \int_0^T \langle q_t, \Gamma_t^n \rangle_{\mathcal{S}_2(H)} dt = \mathbb{E} \langle \nabla_x \Phi(\bar{X}_T), \mathcal{Y}_T^n \rangle + \mathbb{E} \int_0^T \langle \nabla_x L(\bar{X}_t, \bar{u}_t), \mathcal{Y}_t^n \rangle dt. \quad (3.12)$$

Now we let $n \rightarrow \infty$. It is convenient at this point to recall the notation introduced in Remark 2.2-2, namely the operators $\Gamma(x, u, v)$ and $\Gamma^n(x, u, v)$: indeed we have

$$\Gamma_t = \Gamma(\bar{X}_t, \bar{u}_t, \delta u_t), \quad \Gamma_t^n = \Gamma^n(\bar{X}_t, \bar{u}_t, \delta u_t),$$

and it follows from (2.3) that for $s > 0$

$$|e^{sA}\Gamma_t^n - (\nabla_u[e^{sA}G(\bar{X}_t, \bar{u}_t)]\delta u_t)|_{\mathcal{S}_2(H)}^2 = |e^{sA}\Gamma_t^n(\bar{X}_t, \bar{u}_t, \delta u_t) - (\nabla_u[e^{sA}G(\bar{X}_t, \bar{u}_t)]\delta u_t)|_{\mathcal{S}_2(H)}^2 \downarrow 0, \quad (3.13)$$

$d\mathbb{P} \otimes dt$ -a.s., and from (2.4) that

$$|\mathrm{Tr}(q_t^*\Gamma_t^n)| \leq C |\delta u_t|_U |q_t|_{\mathcal{S}_1(H)} \leq c |q_t|_{\mathcal{S}_1(H)}, \quad \mathrm{Tr}(q_t^*\Gamma_t^n) \rightarrow \mathrm{Tr}(q_t^*\Gamma_t), \quad d\mathbb{P} \otimes dt - a.s., \quad (3.14)$$

since we know that $q_t \in \mathcal{S}_1(H)$ $d\mathbb{P} \otimes dt$ -a.s. Writing down the mild form of the equations for Y and \mathcal{Y}^n and subtracting we obtain (compare (3.2)),

$$\begin{aligned} \mathcal{Y}_t^n - Y_t &= \int_0^t e^{(t-s)A} \nabla_x F(X_s, u_s) (\mathcal{Y}_s^n - Y_s) ds + \int_0^t \nabla_x [e^{(t-s)A} G(X_s, u_s)] (\mathcal{Y}_s^n - Y_s) dW_s \\ &\quad + \int_0^t \left(e^{(t-s)A} \Gamma_t^n - \nabla_u [e^{(t-s)A} G(X_s, u_s)] \delta u_s \right) dW_s. \end{aligned}$$

It follows from (3.13) that the last integral tends to zero in $L^2(\Omega, \mathcal{F}, \mathbb{P}, H)$. By standard estimates it also follows that $\sup_{t \in [0, T]} \mathbb{E} |\mathcal{Y}_t^n - Y_t|_H^2 \rightarrow 0$, so that we can pass to the limit in the right-hand side of (3.12). Next we note that

$$\mathbb{E} \int_0^T \langle q_t, \Gamma_t^n \rangle_{\mathcal{S}_2(H)} dt = \mathbb{E} \int_0^T \mathrm{Tr}(q_t^* \Gamma_t^n) dt \rightarrow \mathbb{E} \int_0^T \mathrm{Tr}(q_t^* \Gamma_t) dt$$

by dominated convergence, as it follows from (3.14) and the fact that

$$\mathbb{E} \int_0^T |q_t|_{\mathcal{S}_1(H)} dt \leq \left(\int_0^T (T-t)^{-2\alpha} dt \right)^{1/2} \left(\mathbb{E} \int_0^T (T-t)^{2\alpha} |q_t|_{\mathcal{S}_1(H)}^2 dt \right)^{1/2} < \infty$$

by Proposition 3.4 and the assumption that $\alpha < 1/2$. Passing to the limit in (3.12) we finish the proof of Step 1.

Step 2: conclusion. It follows from Proposition 3.2 and the duality formula of Step 1 that

$$\begin{aligned} J(x, u^\epsilon) - J(x, \bar{u}) &= \epsilon \mathbb{E} \int_0^T \langle \nabla_u F(\bar{X}_t, \bar{u}_t) \delta u_t, p_t \rangle dt + \epsilon \mathbb{E} \int_0^T \langle \nabla_u L(\bar{X}_t, \bar{u}_t), \delta u_t \rangle dt \\ &\quad + \epsilon \mathbb{E} \int_0^T \mathrm{Tr} [q_t^* (\nabla_u [G(\bar{X}_t, \bar{u}_t)] \delta u_t)] dt + o(\epsilon) \end{aligned}$$

Since \bar{u} is optimal, we have $J(x, u^\epsilon) - J(x, \bar{u}) \geq 0$ and the proof can be concluded by standard arguments based on localization and the Lebesgue differentiation theorem, see, e.g., [17, 23]. \square

4 Singular Infinite-dimensional BSDEs

The main purpose of this section is to give a complete proof of Propositions 3.4 and 3.5 that were used in an essential way to prove the stochastic maximum principle. They both refer to properties of the dual (backward) equation (3.5). To simplify the notation we will present our results in the case when $F = 0$, the general case being essentially the same. On the other hand, we will address a class of backward equations which are otherwise more general, namely of the form

$$\begin{cases} -dP_t &= [A^*P_t + \sum_{i=1}^{\infty} C_i^*(t)Q_t e_i dt + f_t] dt - \sum_{i=1}^{\infty} Q_t e_i d\beta_t^i, \\ P_T &= \eta, \end{cases} \quad (4.1)$$

where $\beta_t^i = \langle e_i, W_t \rangle$, $i = 1, 2, \dots$ is a family of independent Brownian motions, $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, $f \in L^2_{\mathcal{P}}(\Omega \times [0, T], H)$. The unknown process is the pair denoted (P, Q) and takes values in $H \times \mathcal{S}_2(H)$. We will work under the following assumptions, which are assumed to hold throughout this section.

Hypothesis 4.1

1. e^{tA} , $t \geq 0$, is a strongly continuous semigroup of bounded linear operators in H . Moreover, $e^{tA} \in \mathcal{S}_2(H)$ for all $t > 0$ and there exist constants $c > 0$ and $\alpha \in [0, 1/2)$ such that $|e^{tA}|_{\mathcal{S}_2(H)} \leq ct^{-\alpha}$ for all $t \in (0, T]$.
2. The processes C_i are strongly progressively measurable with values in $\mathcal{L}(H)$. Moreover we have $|C_i(t)|_{\mathcal{L}(H)} \leq c$, \mathbb{P} -a.s. for all $t \in [0, T]$ and $i \in \mathbb{N}$.
3. $\sum_{i=1}^{\infty} |e^{tA} C_i(s) h|^2 \leq ct^{-2\alpha} |h|_H^2$ for all $t \in (0, T]$, $s \geq 0$, $h \in H$.

Strongly progressively measurable processes means that they are progressively measurable with values in the space $\mathcal{L}(H)$ endowed with the Borel sets of the strong operator topology.

We notice that the sum $\sum_{i=1}^{\infty} C_i^*(t)Q_t e_i$ in equation 4.1 is not convergent in general, even when Q_t is Hilbert-Schmidt. Moreover the semigroup does not seem to be directly helpful since under Hypothesis 4.1 it is not clear whether the sum $\sum_{i=1}^{\infty} e^{sA^*} C_i^*(t)Q_t e_i$ converges or not. The main result of this paper is the proof of well-posedness of such class of linear BSDEs which are driven by white noise and involve a ‘singular’ infinite sum.

We give the following notion of (mild) solution.

Definition 4.1 *We say that a pair of processes (P, Q) with $P \in L^2_{\mathcal{P}}(\Omega \times [0, T], H)$, $Q \in L^2_{\mathcal{P}}(\Omega \times [0, T], \mathcal{S}_2(H))$ is a mild solution to equation (4.1) if the following holds:*

1. Denoting $S^M(s) := \sum_{i=1}^M (T-s)^{\alpha} C_i^*(s) Q_s e_i$, $s \in [0, T]$ then the sequence (S^M) converges weakly in $L^2_{\mathcal{P}}(\Omega \times [0, T], H)$.
2. For any $s \in [0, T]$,

$$P_s = e^{(T-s)A^*} \eta + \sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l e_i dl + \int_s^T e^{(l-s)A^*} f_l dl - \sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} Q_l e_i d\beta_l^i, \quad \mathbb{P}\text{-a.s.} \quad (4.2)$$

Remark 4.2 Notice that, for any fixed s , the map $g \mapsto \int_s^T (T-l)^{-\alpha} e^{(l-s)A^*} g_l dl$ is a bounded linear functional from $L^2_{\mathcal{P}}([0, T] \times \Omega, H)$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, hence weakly continuous. So if condition 1 above holds then, for all fixed $s \in [0, T]$, the sum

$$\sum_{i=1}^M \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l dl = \int_s^T (T-l)^{-\alpha} e^{(l-s)A^*} S^M(l) dl$$

converges, weakly in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, to a limit that we denote $\sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l e_i dl$ and that appears in (4.2).

4.1 Linear Forward SDEs

We will study equation (4.1) exploiting duality arguments. To this end we start by collecting precise estimates on the solutions of a suitable family of linear forward SDEs. Namely, given any starting time $s \in [0, T]$, we consider the equation on the time interval $[s, T]$:

$$\begin{cases} d\mathcal{Y}_t &= A\mathcal{Y}_t dt + \sum_{i=1}^{\infty} C_i(t) \mathcal{Y}_t d\beta_t^i + \sum_{i=1}^{\infty} C_i(t) \gamma_t d\beta_t^i + \sum_{i=1}^{\infty} \Gamma_t e_i d\beta_t^i + \rho_t dt, \\ \mathcal{Y}_s &= x, \end{cases} \quad (4.3)$$

together with the approximating equations, for $N, M \in \mathbb{N} \cup \{\infty\}$:

$$\begin{cases} d\tilde{\mathcal{Y}}_t^{N,M} &= A\tilde{\mathcal{Y}}_t^{N,M} dt + \sum_{i=1}^N C_i(t) \tilde{\mathcal{Y}}_t^{N,M} d\beta_t^i + \sum_{i=1}^M C_i(t) \gamma_t d\beta_t^i + \sum_{i=1}^{\infty} \Gamma_t e_i d\beta_t^i + \rho_t dt, \\ \tilde{\mathcal{Y}}_s^{N,M} &= x. \end{cases} \quad (4.4)$$

In the above equation we always assume that $x : \Omega \rightarrow H$ is bounded and \mathcal{F}_s measurable, $\rho, \gamma : [s, T] \times \Omega \rightarrow H$ are progressively measurable and bounded, $\Gamma : [s, T] \times \Omega \rightarrow \mathcal{S}_2(H)$ is also progressively measurable and bounded.

For further use we note that $\tilde{\mathcal{Y}}^{N,M}$ clearly does not depend on M when $\gamma = 0$.

We start from a standard estimate on this SDE. Its proof coincides with the one of Proposition 3.2 in [7] and will be omitted.

Theorem 4.3 *For all $p \in [2, \infty)$ and $N, M \in \mathbb{N} \cup \{\infty\}$, in the space $L^p_{\mathcal{P}}(\Omega, C([s, T], H))$ there exists a unique solution \mathcal{Y} to equation (4.3) and a unique solution $\tilde{\mathcal{Y}}^{N,M}$ to equation (4.4); note that $\tilde{\mathcal{Y}}^{\infty, \infty} = \mathcal{Y}$. Moreover, if p is large enough, the following estimate holds:*

$$\mathbb{E} \left(\sup_{t \in [s, T]} |\mathcal{Y}_t|^p \right) \leq c_p \left(1 + \mathbb{E}|x|^p + |\Gamma|_{L^\infty([s, T] \times \Omega, \mathcal{L}(H))}^p + |\gamma|_{L^\infty([s, T] \times \Omega, H)}^p + \mathbb{E} \left(\int_s^T |\rho_t| dt \right)^p \right).$$

We also have

$$\tilde{\mathcal{Y}}^{N,M} \rightarrow \tilde{\mathcal{Y}}^{N, \infty}, \quad \tilde{\mathcal{Y}}^{N, \infty} \rightarrow \tilde{\mathcal{Y}}^{\infty, \infty}, \quad \tilde{\mathcal{Y}}^{M, M} \rightarrow \mathcal{Y}, \quad \tilde{\mathcal{Y}}^{N, \infty} \rightarrow \mathcal{Y}, \quad \tilde{\mathcal{Y}}^{\infty, M} \rightarrow \mathcal{Y} \quad (4.5)$$

in the norm of $L^p_{\mathcal{P}}(\Omega, C([s, T], H))$, namely $\mathbb{E}(\sup_{t \in [s, T]} |\tilde{\mathcal{Y}}_t^{N,M} - \tilde{\mathcal{Y}}_t^{N, \infty}|^p) \rightarrow 0$ etc.

In addition, the above estimate holds for $\tilde{\mathcal{Y}}^{N,M}, \tilde{\mathcal{Y}}^{N, \infty}, \tilde{\mathcal{Y}}^{\infty, M}$ uniformly with respect to N and M .

Finally, the first (respectively, the second) convergence result in (4.5) holds true uniformly with respect to N (respectively, to M).

The next estimate involves the Hilbert-Schmidt norm of Γ .

Proposition 4.4 *Under the above assumptions and notations it holds*

$$\sup_{t \in [s, T]} \mathbb{E} |\mathcal{Y}_t|^2 \leq c \mathbb{E} \left[|x|^2 + \left(\int_s^T |\rho_t| dt \right)^2 + |\gamma|_{L^\infty([s, T] \times \Omega, H)}^2 + \int_s^T |\Gamma_t|_{\mathcal{S}_2(H)}^2 dt \right].$$

Proof. Writing equation (4.3) in the mild form, namely

$$\mathcal{Y}_t = e^{(t-s)A} x + \int_s^t e^{(t-l)A} \rho_l dl + \sum_{i=1}^{\infty} \int_s^t e^{(t-l)A} [C_i(l)(\mathcal{Y}_l + \gamma_l) + \Gamma_l e_i] d\beta_l^i,$$

and taking into account Hypothesis 4.1 we get

$$\begin{aligned} \mathbb{E} |\mathcal{Y}_t|^2 &\leq c \mathbb{E} \left[|x|^2 + \left(\int_s^t |\rho_l| dl \right)^2 + \int_s^t (t-l)^{-2\alpha} |\gamma_l|^2 dl + \int_s^t |e^{(t-l)A} \Gamma_l|_{\mathcal{S}_2(H)}^2 dl \right] \\ &\quad + c \int_s^t (t-l)^{-2\alpha} \mathbb{E} |\mathcal{Y}_l|^2 dl. \end{aligned} \quad (4.6)$$

We obtain

$$\mathbb{E} |\mathcal{Y}_t|^2 \leq c \mathbb{E} \left[|x|^2 + \left(\int_s^T |\rho_l| dl \right)^2 + |\gamma|_{L^\infty}^2 + \int_s^T |\Gamma_l|_{\mathcal{S}_2(H)}^2 dl \right] + c \int_s^t (t-l)^{-2\alpha} \mathbb{E} |\mathcal{Y}_l|^2 dl,$$

and the claim then follows applying a variant of the Gronwall Lemma, see Lemma 7.1.1 in [10]. \square

The final estimate will be an important tool in the rest of the paper and again exploits a special version of the Gronwall Lemma.

Proposition 4.5 *For $s \leq t \leq T$,*

$$\mathbb{E} |\mathcal{Y}_t|^2 \leq c \mathbb{E} \left[|x|^2 + \int_s^t (t-l)^{-2\alpha} |\gamma_l|_H^2 dl + \int_s^t (t-l)^{-2\alpha} |\Gamma_l|_{\mathcal{L}(H)}^2 dl + \left(\int_s^t |\rho_l| dl \right)^2 \right]. \quad (4.7)$$

Proof. We first note that

$$|e^{(t-l)A} \Gamma_l|_{\mathcal{S}_2(H)} \leq |e^{(t-l)A}|_{\mathcal{S}_2(H)} |\Gamma_l|_{\mathcal{L}(H)} \leq c(t-l)^{-\alpha} |\Gamma_l|_{\mathcal{L}(H)}.$$

Consequently, letting

$$u(t) := \mathbb{E} |\mathcal{Y}_t|^2, \quad v(t) := c \mathbb{E} (|\Gamma_t|_{\mathcal{L}(H)}^2 + |\gamma_t|^2), \quad w(t) := c \mathbb{E} |x|^2 + c \mathbb{E} \left(\int_s^t |\rho_l| dl \right)^2,$$

it follows from (4.6) that

$$u(t) \leq w(t) + \int_s^t (t-l)^{-2\alpha} v(l) dl + c \int_s^t (t-l)^{-2\alpha} u(l) dl = e(t) + c \int_s^t (t-l)^{-2\alpha} u(l) dl,$$

where we set $e(t) = w(t) + \int_s^t (t-l)^{-2\alpha} v(l) dl$. Using again the Gronwall Lemma in [10] Lemma 7.1.1 we obtain

$$u(t) \leq e(t) + c \int_s^t (t-l)^{-2\alpha} e(l) dl. \quad (4.8)$$

Next we note that

$$\begin{aligned} \int_s^t (t-l)^{-2\alpha} e(l) dl &= \int_s^t w(l)(t-l)^{-2\alpha} dl + c \int_s^t (t-l)^{-2\alpha} \int_s^l (l-\zeta)^{-2\alpha} v(\zeta) d\zeta \\ &\leq w(t) \int_s^t (t-l)^{-2\alpha} dl + c \int_s^t (t-l)^{-2\alpha} \int_s^l (l-\zeta)^{-2\alpha} v(\zeta) d\zeta \\ &\leq cw(t) + c \int_s^t v(\zeta) \int_\zeta^t (t-l)^{-2\alpha} (l-\zeta)^{-2\alpha} dl d\zeta. \end{aligned}$$

But since

$$\begin{aligned} \int_\zeta^t (t-l)^{-2\alpha} (l-\zeta)^{-2\alpha} dl &= \int_0^{t-\zeta} (t-\zeta-s)^{-2\alpha} s^{-2\alpha} ds = \int_0^1 (1-z)^{-2\alpha} z^{-2\alpha} (t-\zeta)^{-4\alpha+1} dz \\ &= c(t-\zeta)^{-4\alpha+1} \leq c(t-\zeta)^{-2\alpha}, \end{aligned}$$

from (4.8) we obtain the required conclusion:

$$u(t) \leq cw(t) + c \int_s^t (t-l)^{-2\alpha} v(l) dl.$$

□

Corollary 4.6 *Integrating the inequality (4.7) with respect to t we get*

$$E \int_s^T |\mathcal{Y}_l|^2 dl \leq c\mathbb{E} \left[|x|^2 + \int_s^T \left(|\Gamma_l|_{L(H)}^2 + |\gamma_l|^2 \right) dl + \left(\int_s^T |\rho_l| dl \right)^2 \right]. \quad (4.9)$$

4.2 Existence of a Solution to the Singular BSDE

We will proceed by approximation. Namely, we consider the following BSDE in infinite dimensions where the singular sum in the drift has been truncated:

$$\begin{cases} -dP_s^N &= [A^* P_s^N + \sum_{i=1}^N C_i^*(s) Q_t^N e_i + f_s] ds - \sum_{i=1}^\infty Q_s^N e_i d\beta_s^i, \\ P_T^N &= \eta. \end{cases} \quad (4.10)$$

We still assume that $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, $f \in L^2_{\mathcal{P}}(\Omega \times [0, T], H)$, so the above equation fits in the classical theory of Hilbert valued BSDEs, see [11]. In particular it holds:

Proposition 4.7 *Assume that $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, $f \in L^2_{\mathcal{P}}(\Omega \times [0, T], H)$. Then there exists a unique (P^N, Q^N) with*

$$P^N \in L^2_{\mathcal{P}}(\Omega, C([0, T], H)), \quad Q^N \in L^2_{\mathcal{P}}(\Omega \times [0, T], \mathcal{S}_2(H))$$

verifying (4.10) in the following mild sense:

$$P_s^N = e^{(T-s)A^*} \eta + \int_s^T e^{(l-s)A^*} \left[\sum_{i=1}^N C_i^*(l) Q_l^N e_i + f_l \right] dl - \sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} Q_l^N e_i d\beta_l^i. \quad (4.11)$$

Moreover choosing as before $x : \Omega \rightarrow H$ bounded and \mathcal{F}_s measurable, ρ, γ from $[s, T] \times \Omega$ to H progressively measurable and bounded and $\Gamma : [s, T] \times \Omega \rightarrow \mathcal{S}_2(H)$ progressively measurable and bounded it holds:

$$\begin{aligned} & \mathbb{E} \langle P_s^N, x \rangle + \mathbb{E} \int_s^T \langle P_l^N, \rho_l \rangle dl + \mathbb{E} \int_s^T \langle Q_l^N, \Gamma_l \rangle_{\mathcal{S}_2(H)} dl + \mathbb{E} \int_s^T \sum_{i=1}^M \langle Q_l^N e_i, C_i(l) \gamma_l \rangle dl \\ &= \mathbb{E} \langle \eta, \tilde{\mathcal{Y}}_T^{N,M} \rangle + \mathbb{E} \int_s^T \langle f_l, \tilde{\mathcal{Y}}_l^{N,M} \rangle dl, \end{aligned} \quad (4.12)$$

where $\tilde{\mathcal{Y}}^{N,M}$ is the solution to equation (4.4).

We now define a candidate solution (P, Q) as the weak limit of (P^N, Q^N) in some Hilbert space.

Corollary 4.8 *We have*

1. P^N converges weakly to an element P in $L_{\mathcal{P}}^2([0, T] \times \Omega, H)$;
2. for any t , P_t^N converges weakly to an element \tilde{P}_t in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}, H)$;
3. Q^N converges weakly to an element Q in $L_{\mathcal{P}}^2([0, T] \times \Omega, \mathcal{S}_2(H))$.

Moreover choosing $\eta, f, x, \rho, \gamma, \Gamma$ as in Proposition 4.7 it holds, for all $M \in \mathbb{N}$:

$$\begin{aligned} & \mathbb{E} \langle \tilde{P}_s, x \rangle + \mathbb{E} \int_s^T \langle P_l, \rho_l \rangle dl + \mathbb{E} \int_s^T \langle Q_l, \Gamma_l \rangle_{\mathcal{S}_2(H)} dl + \mathbb{E} \int_s^T \sum_{i=1}^M \langle Q_l e_i, C_i(l) \gamma_l \rangle dl \\ &= \mathbb{E} \langle \eta, \tilde{\mathcal{Y}}_T^{\infty, M} \rangle + \mathbb{E} \int_s^T \langle f_l, \tilde{\mathcal{Y}}_l^{\infty, M} \rangle dl. \end{aligned} \quad (4.13)$$

Proof. Let us consider the processes \mathcal{Y} and $\tilde{\mathcal{Y}}^{N,M}$, defined as solutions to equations (4.3) and (4.4). In the first part of the proof we take $s = 0, x = 0$ and $\gamma = 0$ in these equations and we recall that the processes $\tilde{\mathcal{Y}}^{N,M}$ do not depend on M , so that in particular $\tilde{\mathcal{Y}}^{\infty, M} = \mathcal{Y}$.

By the estimates in Proposition 4.4 the maps

$$\mathcal{T} : (\rho, \Gamma) \mapsto (\tilde{\mathcal{Y}}_T^{\infty, M}, \tilde{\mathcal{Y}}^{\infty, M}), \quad \mathcal{T}_N : (\rho, \Gamma) \mapsto (\tilde{\mathcal{Y}}_T^{N, M}, \tilde{\mathcal{Y}}^{N, M})$$

can be extended to bounded linear maps from the space $L_{\mathcal{P}}^2(\Omega \times [0, T], H) \times L_{\mathcal{P}}^2(\Omega \times [0, T], \mathcal{S}_2(H))$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H) \times L_{\mathcal{P}}^2(\Omega \times [0, T], H)$. We denote by $\mathcal{T}^*, \mathcal{T}_N^*$ their Hilbert space adjoints. Given arbitrary $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, $f \in L_{\mathcal{P}}^2(\Omega \times [0, T], H)$ and setting $(P, Q) = \mathcal{T}^*(\eta, f)$, we see that

$$\mathbb{E} \int_0^T \langle P_l, \rho_l \rangle dl + \mathbb{E} \int_0^T \langle Q_l, \Gamma_l \rangle_{\mathcal{S}_2(H)} dl = \mathbb{E} \langle \eta, \mathcal{Y}_T^M \rangle + \mathbb{E} \int_0^T \langle f_l, \mathcal{Y}_l^M \rangle dl,$$

while (4.12) (with $s = 0, x = 0, \gamma = 0$) shows that $(P^N, Q^N) = \mathcal{T}_N^*(\eta, f)$. Since $\tilde{\mathcal{Y}}^{N,M} \rightarrow \tilde{\mathcal{Y}}^{\infty, M}$ as specified in Theorem 4.3, it follows easily that $P^N \rightarrow P$ weakly in $L_{\mathcal{P}}^2([0, T] \times \Omega, H)$ and $Q^N \rightarrow Q$ weakly in $L_{\mathcal{P}}^2([0, T] \times \Omega, \mathcal{S}_2(H))$.

Now, for arbitrary $s \in [0, T]$ and $x \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$ it follows from (4.12) (still with $\gamma = 0$) that $\mathbb{E}\langle P_s^N, x \rangle$ has a limit as $N \rightarrow \infty$, equal to

$$-\mathbb{E} \int_s^T \langle P_l^N, \rho_l \rangle dl - \mathbb{E} \int_s^T \langle Q_l^N, \Gamma_l \rangle_{S_2(H)} dl + \mathbb{E} \langle \eta, \tilde{\mathcal{Y}}_T^{N,M} \rangle + \mathbb{E} \int_s^T \langle f_l, \tilde{\mathcal{Y}}_l^{N,M} \rangle dl.$$

This shows that P_s^N converges weakly in $L^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$, and we denote by \tilde{P}_s its limit.

Finally, coming back to general γ , equality (4.13) follows from (4.12) letting $N \rightarrow \infty$. \square

We then need to get some regularity property of the process (\tilde{P}_s) which, for the moment, we can not identify with (P_s) and has only been defined \mathbb{P} -a.s. for any fixed $s \in [0, T]$.

Proposition 4.9 *The map $s \mapsto \tilde{P}_s$ is weakly continuous from $[0, T]$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$.*

Proof. In order to stress its dependence on the initial time and state, given $s \in [0, T]$ and $x \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$, denote by $\mathcal{Y}^{x,s}$ the solution of equation (4.3) with $\rho = \Gamma = \gamma = 0$, namely:

$$d\mathcal{Y}_t^{\xi,s} = A\mathcal{Y}_t^{\xi,s} ds + \sum_{i=1}^{\infty} C_i(s)\mathcal{Y}_t^{\xi,s} d\beta_s^i, \quad \mathcal{Y}_s^{\xi,s} = x. \quad (4.14)$$

We know by [2] that the above equation admits a unique mild solution with $\sup_{t \in [s, T]} \mathbb{E}|\mathcal{Y}_t^{x,s}|^2 \leq c_T(1 + \mathbb{E}|x|^2)$, moreover if $x' \in L^2(\Omega, \mathcal{F}_s, \mathbb{P}, H)$ then $\sup_{t \in [s, T]} \mathbb{E}|\mathcal{Y}_t^{x,s} - \mathcal{Y}_t^{x',s}|^2 \leq c_T(\mathbb{E}|x - x'|^2)$.

For fixed $x \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, let us denote $x_s := \mathbb{E}(x|\mathcal{F}_s)$. Then equation (4.13) yields:

$$\mathbb{E}\langle \tilde{P}_s, x \rangle = \mathbb{E}\langle \tilde{P}_s, x_s \rangle = \mathbb{E}\langle \mathcal{Y}_T^{x_s, s}, \eta \rangle + \mathbb{E} \int_s^T \langle \mathcal{Y}_l^{x_s, s}, f_l \rangle dl.$$

Since

$$\sup_{s \in [0, T]} \sup_{t \in [s, T]} \mathbb{E}|\mathcal{Y}_t^{s, x_s}|^2 \leq \sup_{s \in [0, T]} c_T(1 + \mathbb{E}|x_s|^2) \leq c,$$

the weak continuity is proved if we show that, for all $t > s$, the map $s \mapsto \mathcal{Y}_t^{s, x_s}$ is continuous in the norm of $L^2(\Omega, \mathcal{F}, \mathbb{P}, H)$.

If $s_n \downarrow s$ then $\mathbb{E}|\mathcal{Y}_t^{s_n, x_{s_n}} - \mathcal{Y}_t^{s, x_s}|^2 \leq 2\mathbb{E}|\mathcal{Y}_t^{s_n, x_{s_n}} - \mathcal{Y}_t^{s_n, x_s}|^2 + 2\mathbb{E}|\mathcal{Y}_t^{s_n, x_s} - \mathcal{Y}_t^{s, x_s}|^2$. The first term is controlled by $\mathbb{E}|x_{s_n} - x_s|^2 = \mathbb{E}|\mathbb{E}(x|\mathcal{F}_{s_n}) - \mathbb{E}(x|\mathcal{F}_s)|^2 \rightarrow 0$.

The second term is $\mathbb{E}|\mathcal{Y}_t^{s_n, x_s} - \mathcal{Y}_t^{s_n, \mathcal{Y}_{s_n}^{s, x_s}}|^2$ and is controlled by

$$\begin{aligned} \mathbb{E}|\mathcal{Y}_{s_n}^{s, x_s} - x_s|^2 &\leq 2\mathbb{E}|e^{(s_n-s)A}x_s - x_s|^2 + 2\mathbb{E}\left|\sum_{i=1}^{\infty} \int_s^{s_n} e^{(s_n-t)A}C_i(t)\mathcal{Y}_t^{s, x_s} d\beta_t^i\right|^2 \\ &\leq 2\mathbb{E}|e^{(s_n-s)A}x_s - x_s|^2 + c \sup_{l \in [s, T]} \mathbb{E}|\mathcal{Y}_l^{s, x_s}|^2 \int_s^{s_n} (s_n - t)^{-2\alpha} dt, \end{aligned}$$

which converges to 0 as $s_n \downarrow s$.

On the other hand, if $s_n \uparrow s$, $\mathcal{Y}_t^{s_n, x_{s_n}} - \mathcal{Y}_t^{s, x} = \mathcal{Y}_t^{s, \mathcal{Y}_s^{s_n, x_{s_n}}} - \mathcal{Y}_t^{s, x}$. Hence it is enough to prove that $\mathcal{Y}_s^{s_n, x_{s_n}} \rightarrow x$ in the norm of $L^2(\Omega, \mathcal{F}, \mathbb{P}, H)$. But, proceeding as before, we have

$$\mathbb{E}|\mathcal{Y}_s^{x_{s_n}, s_n} - x|^2 \leq 2\mathbb{E}|e^{(s-s_n)A}x_{s_n} - x|^2 + c \int_{s_n}^s (s-l)^{-2\alpha} \mathbb{E}|\mathcal{Y}_l^{x_{s_n}, s_n}|^2 dl.$$

The claim follows recalling that $\sup_n \sup_{l \in [s_n, s]} \mathbb{E} |\mathcal{Y}^{x_{s_n}, s_n}|^2 \leq c_T \sup_n (1 + \mathbb{E} |x_{s_n}|^2) < \infty$ and that $|e^{(s-s_n)A} x_{s_n} - x| \leq c |x_{s_n} - x| + |e^{(s-s_n)A} x - x|$. \square

We are now in a position to prove that \tilde{P} and P coincide.

Proposition 4.10 \tilde{P} is a progressively measurable process and $\tilde{P} = P$, $dt \otimes d\mathbb{P}$ -a.s.

Proof. We first prove progressive measurability. Fixed an arbitrary $t \in [0, T]$ we choose a basis $\{\varphi_m\}$ in $L^2(\Omega, \mathcal{F}_t, \mathbb{P}, H)$. We have $\tilde{P}_l = \sum_{m=1}^{\infty} (\mathbb{E} \langle \tilde{P}_l, \varphi_m \rangle_H) \varphi_m$, $\forall l \leq t$. Since $\mathbb{E} \langle \tilde{P}_l, \varphi_m \rangle_H$ is a continuous function we immediately deduce that \tilde{P} restricted to $[0, t]$ is $\mathcal{B}[0, t] \otimes \mathcal{F}_t$ measurable.

To show that (\tilde{P}) and (P) coincide choose $x = \gamma = \Gamma = 0$ and an arbitrary bounded progressively measurable process ρ in (4.3). By definition of \tilde{P} , for all $t \in [0, T]$, $\mathbb{E} \langle P_t^N, \rho_t \rangle \rightarrow \mathbb{E} \langle \tilde{P}_t, \rho_t \rangle$ so, by dominated convergence theorem (exploiting the measurability of (\tilde{P})) $\int_0^T \mathbb{E} \langle P_l^N, \rho_l \rangle dl \rightarrow \int_0^T \mathbb{E} \langle \tilde{P}_l, \rho_l \rangle dl$. But we already know, see Corollary 4.8, that $\int_0^T \mathbb{E} \langle P_l^N, \rho_l \rangle dl \rightarrow \int_0^T \mathbb{E} \langle P_l, \rho_l \rangle dl$ and the claim is proved. \square

In the following when we refer to the process P we will always refer to its version \tilde{P} .

We come now to the study of process Q .

Lemma 4.11 Setting $S^M(s) = \sum_{i=1}^M (T-s)^\alpha C_i^*(s) Q_s e_i$, S^M converges weakly in $L^2_{\mathcal{P}}(\Omega \times [0, T]; H)$. The limit will be denoted by $\sum_{i=1}^{\infty} (T - \cdot)^\alpha C_i^*(\cdot) Q \cdot e_i$.

Proof. Given an arbitrary bounded progressively measurable process γ in H , let $\hat{\mathcal{Y}}^{M, \gamma}$ be the mild solution of the equation

$$\begin{cases} d\hat{\mathcal{Y}}_t^{M, \gamma} &= A\hat{\mathcal{Y}}_t^{M, \gamma} dt + \sum_{i=1}^{\infty} C_i(t) \hat{\mathcal{Y}}_t^{M, \gamma} d\beta_t^i + \sum_{i=1}^M C_i(t) (T-t)^\alpha \gamma_t d\beta_t^i, \\ \hat{\mathcal{Y}}_0^{M, \gamma} &= 0. \end{cases} \quad (4.15)$$

Similarly, let $\hat{\mathcal{Y}}^\gamma$ be the mild solution of equation of

$$\begin{cases} d\hat{\mathcal{Y}}_t^\gamma &= A\hat{\mathcal{Y}}_t^\gamma dt + \sum_{i=1}^{\infty} C_i(t) \hat{\mathcal{Y}}_t^\gamma d\beta_t^i + \sum_{i=1}^{\infty} C_i(t) (T-t)^\alpha \gamma_t d\beta_t^i, \\ \hat{\mathcal{Y}}_0^\gamma &= 0. \end{cases} \quad (4.16)$$

We see that equation (4.15) (respectively, (4.16)) coincide with equation (4.4) (respectively, (4.3)) with $s = x = \Gamma = \rho = 0$, $N = \infty$ and γ replaced by $(T - \cdot)^\alpha \gamma$.

The equality (4.13) reads:

$$\mathbb{E} \int_0^T \langle S^M(l), \gamma_l \rangle dl = \mathbb{E} \int_0^T \langle (T-l)^\alpha \sum_{i=1}^M C_i^*(l) Q_l e_i, \gamma_l \rangle dl = \mathbb{E} \langle \hat{\mathcal{Y}}_T^{M, \gamma}, \eta \rangle + \mathbb{E} \int_0^T \langle \hat{\mathcal{Y}}_l^{M, \gamma}, f_l \rangle dl. \quad (4.17)$$

By the estimate (4.7), we have $|\mathbb{E} \int_0^T \langle S^M(l), \gamma_l \rangle dl| \leq c |\gamma|_{L^2_{\mathcal{P}}([0, T] \times \Omega, H)}$. Since the set of bounded elements γ is dense in $L^2_{\mathcal{P}}([0, T] \times \Omega, H)$ it follows that $|S^M|_{L^2_{\mathcal{P}}([0, T] \times \Omega, H)} \leq c$. Since, by Theorem 4.3, the right-hand side of (4.17) converges as $M \rightarrow \infty$ (to the limit $\mathbb{E} \langle \hat{\mathcal{Y}}_T^\gamma, \eta \rangle + \mathbb{E} \int_0^T \langle \hat{\mathcal{Y}}_l^\gamma, f_l \rangle dl$) when γ is bounded, we conclude that S^M converges weakly in $L^2_{\mathcal{P}}([0, T] \times \Omega, H)$. \square

If we replace Q by the approximating operators Q^M we obtain the same limit:

Lemma 4.12 $(T - \cdot)^\alpha \sum_{i=1}^M C_i^*(\cdot) Q_i^M e_i$ converges weakly to $(T - \cdot)^\alpha \sum_{i=1}^\infty C_i^*(\cdot) Q_i e_i$ in $L^2_{\mathcal{P}}([0, T] \times \Omega, H)$.

Proof. The proof of the existence of the weak limit of the sequence $(T - \cdot)^\alpha \sum_{i=1}^M C_i^*(\cdot) Q_i^M e_i$ follows the same argument as in the proof of the above Lemma replacing equation (4.15) by

$$\begin{cases} d\hat{\mathcal{Y}}_t^{M, \gamma} &= A\hat{\mathcal{Y}}_t^{M, \gamma} dt + \sum_{i=1}^M C_i(t)\hat{\mathcal{Y}}_t^{M, \gamma} d\beta_t^i + \sum_{i=1}^M C_i(t)(T-t)^\alpha \gamma_t d\beta_t^i, \\ \hat{\mathcal{Y}}_0^{M, \gamma} &= 0. \end{cases} \quad (4.18)$$

and replacing the second equality in (4.17) by

$$E \int_0^T \langle (T-l)^\alpha \sum_{i=1}^M C_i^*(l) Q_l^M e_i, \gamma_l \rangle dl = \mathbb{E} \langle \hat{\mathcal{Y}}_T^{M, \gamma}, \eta \rangle + \mathbb{E} \int_0^T \langle \hat{\mathcal{Y}}_l^{M, \gamma}, f_l \rangle dl,$$

which follows from (4.12). The proof that the limit equals $(T - \cdot)^\alpha \sum_{i=1}^\infty C_i^*(\cdot) Q_i e_i$ comes from the observation that $\mathbb{E} \sup_{s \in [0, T]} |\hat{\mathcal{Y}}_s^{M, \gamma} - \tilde{\mathcal{Y}}_s^{M, \gamma}|^p \rightarrow 0$, which corresponds to the convergence $\tilde{\mathcal{Y}}^{M, M} \rightarrow \mathcal{Y}$ in Theorem 4.3. \square

We are now in a position to prove existence of a solution to the singular BSDE (4.1). For the uniqueness part we need the following Lemma on linear BSDEs with unbounded forcing term which proof will be postponed to the Appendix.

Lemma 4.13 Assume that ξ is a progressively measurable process in H with $\mathbb{E} \int_0^T (T-l)^{2\alpha} |\xi_l|^2 dl < \infty$. Then for any $N \in \mathbb{N}$ and any $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ there exists a unique pair of processes (p, q) with p progressively measurable in H , with $s \mapsto p_s$ continuous from $[0, T]$ to $L^2(\Omega, \mathcal{F}, H)$, and $q \in L^2_{\mathcal{P}}(\Omega \times [0, T], \mathcal{S}_2(H))$ such that:

$$p_s = e^{(T-s)A^*} \eta + \int_s^T e^{(l-s)A^*} \sum_{i=1}^N C_i^*(l) q_l e_i dl + \int_s^T e^{(l-s)A^*} \xi_l dl - \sum_{i=1}^\infty \int_s^T e^{(l-s)A^*} q_l e_i d\beta_l^i. \quad (4.19)$$

Moreover, letting $\gamma = \rho = 0$, $M = \infty$ and $\Gamma \in L^\infty(\Omega \times [0, T], \mathcal{S}_2(H))$ in equation (4.4) the following duality relation holds:

$$\mathbb{E} \langle p_s, x \rangle + \mathbb{E} \int_s^T \langle q_l, \Gamma_l \rangle_{\mathcal{S}_2(H)} ds = \mathbb{E} \langle \tilde{\mathcal{Y}}_T^N, \eta \rangle + \mathbb{E} \int_s^T \langle (T-l)^{-\alpha} \tilde{\mathcal{Y}}_l^N, (T-l)^\alpha \xi_l \rangle dl.$$

Notice that $\sup_{l \in [s, T]} \mathbb{E} |\tilde{\mathcal{Y}}_l^N|^2 < +\infty$ and that $\alpha < 1/2$, so the last integral is well defined.

Proof. The proof of this lemma is postponed to the Appendix. \square

Theorem 4.14 The pair (P, Q) constructed in Corollary 4.8 is the unique mild solution to the singular BSDEs (4.1).

Proof. *Existence:* As noticed in Remark 4.2, for any fixed s , the map $g \mapsto \int_s^T (T-l)^{-\alpha} e^{(l-s)A^*} g_l dl$ is weakly continuous from $L^2_{\mathcal{P}}([0, T] \times \Omega, H)$ to $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$.

Hence, by Lemma 4.11 and Lemma 4.12 the sequence $\sum_{i=1}^N \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l^N dl$ converges, weakly in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}, H)$, to $\sum_{i=1}^\infty \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l e_i dl$.

Similarly $\sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} Q_l^N e_i d\beta_l^i$ converges weakly to $\sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} Q_l e_i d\beta_l^i$, since Q^N converges weakly to Q in $L^2(\Omega \times [0, T], \mathcal{S}_2(H))$.

The fact that (P, Q) is a mild solution of the singular BSDEs (4.1) follows by passing to the limit in equation (4.11).

Uniqueness: Let (P, Q) and (P', Q') two solutions and let $\bar{P} = P - P'$ and $\bar{Q} = Q - Q'$. Moreover let \bar{L} be the weak limit in $L^2_{\mathcal{P}}(\Omega \times [0, T])$, as $N \rightarrow \infty$, of $(T - \cdot)^{\alpha} \sum_{i=1}^N C_i^*(s) \bar{Q}_s e_i$ and define:

$$\bar{\varphi}_s^N = \bar{L}_s - (T - s)^{\alpha} \sum_{i=1}^N C_i^*(s) \bar{Q}_s e_i.$$

Then equation (4.2) yields:

$$\bar{P}_s = - \sum_{i=1}^{\infty} \int_s^T e^{(l-s)A^*} \bar{Q}_l d\beta_l^i + \sum_{i=1}^N \int_t^T e^{(l-s)A^*} C_i^*(l) \bar{Q}_l e_i dl + \int_s^T e^{(l-s)A^*} (T - l)^{-\alpha} \bar{\varphi}_l^N dl.$$

thus, for any fixed N , (\bar{P}, \bar{Q}) is the unique mild solution of equation (4.19) with $\xi_l = (T - l)^{-\alpha} \bar{\varphi}_l^N$. By Lemma 4.13 we obtain that, for all \mathcal{S}_2 -valued bounded predictable Γ :

$$\mathbb{E} \langle \bar{P}_s, x \rangle + \mathbb{E} \int_s^T \langle \bar{Q}_l, \Gamma_l \rangle_{\mathcal{S}_2(H)} dl = \mathbb{E} \int_s^T \langle (T - l)^{-\alpha} \tilde{\mathcal{Y}}_l^{N, \Gamma}, \bar{\varphi}_l^N \rangle_{L_2(H)} dl,$$

where again $\tilde{\mathcal{Y}}^{N, \Gamma}$ is the mild solution of equation (4.4) with $\gamma = \rho = 0$. By Theorem 4.3 we have, in particular: $\sup_{l \in [t, T], N \in \mathbb{N}} \mathbb{E} |\tilde{\mathcal{Y}}_s^{N, \Gamma}|^2 \leq c$, and $\mathbb{E} |\tilde{\mathcal{Y}}_l^{N, \Gamma} - \mathcal{Y}_l^{\Gamma}|^2 \rightarrow 0$, $\forall l \in [s, T]$, where \mathcal{Y}^{Γ} is the mild solution of equation (4.4) with $\gamma = \rho = 0$. In particular $\tilde{\mathcal{Y}}^{N, \Gamma}$ converges strongly to \mathcal{Y}^{Γ} in $L^2_{\mathcal{P}}(\Omega \times [s, T], H)$.

Since $\bar{\varphi}^N$ converges weakly to 0, we obtain:

$$\mathbb{E} \langle \bar{P}_s, x \rangle + \mathbb{E} \int_s^T \langle \bar{Q}_l, \Gamma_l \rangle_{L_2(H)} dl = 0,$$

which concludes the proof of uniqueness. □

4.3 Trace Class Regularity

We will now prove that the martingale term Q enjoys a *trace class regularity* that will be essential to formulate the maximum principle.

Proposition 4.15 *If (P, Q) is the unique mild solution of equation (4.1), then*

$$\mathbb{E} \int_0^T (T - l)^{2\alpha} |Q_l|_{\mathcal{S}_1(H)}^2 dl \leq c \left(\mathbb{E} |\eta|^2 + \mathbb{E} \int_0^T |f_l|^2 dl \right),$$

where the constant c only depends on the constants in Hypothesis 4.1.

Proof. Since Q_l is, \mathbb{P} -a.s. of class $\mathcal{S}_2(H)$ and therefore compact, it can be written (see Appendix 1) as:

$$Q_l = \sum_{j=1}^{\infty} a_j(l) h_j(l) \langle g_j(l), \cdot \rangle,$$

where $a_j(l) \in \mathbb{R}$ and $(h_j(l))_j, (g_j(l))_j$ are orthonormal bases in H . Moreover, we can choose the processes $a_j(l), h_j(l), g_j(l)$ ($l \in [0, T]$) to be progressively measurable. Let

$$\Gamma_l^n = \varphi(l) \sum_{j=1}^n \text{sgn}(a_j(l)) h_j(l) \langle g_j(l), \cdot \rangle,$$

where φ is an arbitrary positive real-valued bounded progressively measurable process. We note that $|\Gamma_l^n|_{\mathcal{L}(H)} \leq \varphi(l)$ and, being of rank n , the process Γ^n is also bounded in $\mathcal{S}_2(H)$.

By (4.13) we have

$$\mathbb{E} \int_0^T \langle Q_l, \Gamma_l^n \rangle_{\mathcal{S}_2(H)} dl = \mathbb{E} \langle \mathcal{Y}_T^n, \eta \rangle + \mathbb{E} \int_0^T \langle \mathcal{Y}_l^n, f_l \rangle dl,$$

where \mathcal{Y}^n is now the solution to equation (4.3) with $s = x = \gamma = \rho = 0$ and $\Gamma = \Gamma^n$. If we compute $\langle Q_l, \Gamma_l^n \rangle_{\mathcal{S}_2(H)}$ and we estimate $\mathbb{E} |\mathcal{Y}_T^n|^2$ by (4.7) and $\mathbb{E} \int_0^T |\mathcal{Y}_l^n|^2 dl$ by (4.9) we obtain:

$$\begin{aligned} \mathbb{E} \int_0^T \sum_{j=1}^n |a_j(l)| \varphi(l) dl &\leq c (\mathbb{E} |\eta|^2)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T (T-l)^{-2\alpha} \varphi(l)^2 dl \right)^{\frac{1}{2}} \\ &\quad + c \left(\int_0^T \mathbb{E} |f_l|^2 dl \right)^{\frac{1}{2}} \left(\mathbb{E} \int_0^T \varphi(l)^2 dl \right)^{\frac{1}{2}}. \end{aligned}$$

If now $n \rightarrow \infty$, recalling that $\sum_{j=1}^{\infty} |a_j| = |Q|_{\mathcal{S}_1(H)}$ (see Appendix 1 below) we get:

$$\mathbb{E} \int_0^T \varphi(l) |Q_l|_{\mathcal{S}_1} dl \leq c_{f,\eta}^{\frac{1}{2}} \left(\mathbb{E} \int_0^T (T-l)^{-2\alpha} \varphi(l)^2 dl \right)^{\frac{1}{2}},$$

where $c_{f,\eta} = c \left(\mathbb{E} |\eta|^2 + \mathbb{E} \int_0^T |f_l|^2 dl \right)$. Denoting $\tilde{\varphi}(l) = (T-l)^{-\alpha} \varphi(l)$, we can rewrite the last estimate as:

$$\mathbb{E} \int_0^T \tilde{\varphi}(l) [(T-l)^\alpha |Q_l|_{L_1(H)}] dl \leq c_{f,\eta}^{\frac{1}{2}} |\tilde{\varphi}|_{L_p^2(\Omega, \times [0, T])}$$

and the claim follows from the arbitrariness of φ . \square

We end this section by proving that, under the following additional assumption, the weak limit that defines the term $\sum_{i=1}^N \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l e_i dl$ in Definition 4.1 is indeed a strong limit in L^1 .

Hypothesis 4.16 *We have $\sum_{i=1}^{\infty} |e^{sA^*} C_i^*(l) x|^2 \leq cs^{-2\alpha} |x|^2$ for all $s > 0, l \geq 0, x \in H$, with $\alpha < 1/2$.*

Remark 4.17 In the example in Section 2.1 this requirement coincides with Hypothesis 4.1-2 since A and C_i are self adjoint.

Proposition 4.18 *If Hypothesis 4.16 holds, in addition to Hypothesis 4.1, then the sequence $\sum_{i=1}^N \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l e_i dl$ converges strongly in $L^1(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ as $N \rightarrow \infty$. The limit obviously coincides with the weak limit in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ introduced in Definition 4.1.*

Proof. As above we expand Q as $Q_l = \sum_{j=1}^{\infty} a_j(l)h_j(l)\langle g_j(l), \cdot \rangle$. Then

$$\begin{aligned}
\sum_{i=1}^N \mathbb{E} \left| \int_s^T e^{(l-s)A^*} C_i^*(l) Q_l e_i dl \right| &\leq \mathbb{E} \int_s^T \sum_{i=1}^N \left| e^{(l-s)A^*} C_i^*(l) \sum_{j=1}^{\infty} a_j(l)h_j(l)\langle g_j(l), e_i \rangle \right| dl \\
&\leq \mathbb{E} \int_s^T \sum_{i=1}^N \sum_{j=1}^{\infty} |a_j(l)| |e^{(l-s)A^*} C_i^*(l)h_j(l)| |\langle g_j(l), e_i \rangle| dl \\
&\leq \mathbb{E} \int_s^T \sum_{j=1}^{\infty} |a_j(l)| \left(\sum_{i=1}^{\infty} |e^{(l-s)A^*} C_i^*(l)h_j(l)|^2 \right)^{\frac{1}{2}} \left(\sum_{i=1}^{\infty} \langle g_j(l), e_i \rangle^2 \right)^{\frac{1}{2}} \\
&\leq c \mathbb{E} \int_s^T \sum_{j=1}^{\infty} |a_j(l)| (l-s)^{-\alpha} dl \\
&= c \mathbb{E} \int_s^T |Q_l|_{\mathcal{S}_1(H)} (l-s)^{-\alpha} dl \\
&\leq c \left(\int_s^T |Q_l|_{\mathcal{S}_1(H)}^2 (T-l)^{2\alpha} dl \right)^{\frac{1}{2}} \left(E \int_s^T (l-s)^{-2\alpha} (T-l)^{-2\alpha} ds \right)^{\frac{1}{2}}.
\end{aligned}$$

Strong convergence in $L^1(\Omega; H)$ follows from Proposition 4.15. The coincidence of the two limits is evident by testing them against any bounded \mathcal{F}_s -measurable H -valued random variable. \square

5 Appendix

5.1 Trace Class Operators

We collect here some basic facts on Hilbert-Schmidt and trace class operators. For a detailed treatment and for the proof of the results stated below see [18].

Given a real separable Hilbert space H we denote by $\mathcal{S}_2(H)$ the Hilbert space of Hilbert Schmidt operators $H \rightarrow H$ endowed with the scalar product $\langle L, M \rangle_{\mathcal{S}_2(H)} = \sum_{i=1}^{\infty} \langle L e_i, M e_i \rangle_H$ where (e_i) is any orthonormal basis in H .

If $L \in \mathcal{S}_2(H)$ then there exists a sequence $(a_j^L)_{j \in \mathbb{N}} \in \ell_2$ and a pair of orthonormal bases $(e_j^L)_{j \in \mathbb{N}}$, $(h_j^L)_{j \in \mathbb{N}}$ in H such that $L = \sum_{j=1}^{\infty} a_j^L h_j^L \langle e_j^L, \cdot \rangle$. Moreover $|L|_{\mathcal{S}_2(H)} = \sum_{j=1}^{\infty} (a_j^L)^2$. Finally if $t \rightarrow L_t$ is a $\mathcal{S}_2(H)$ -valued measurable process then the above objects can be selected with the same measurability property.

We define trace class operators in the following way: $\mathcal{S}_1(H) = \{L \in \mathcal{S}_2(H) : |L|_{\mathcal{S}_1} < \infty\}$, where

$$|L|_{\mathcal{S}_1} := \sup \{ \langle B, L \rangle_{\mathcal{S}_2} : B \in \mathcal{S}_2(H), |B|_{\mathcal{L}(H)} \leq 1 \}.$$

The following results are true:

1. If $B \in \mathcal{L}(H)$ and $L \in \mathcal{S}_1(H)$ then LB, BL are in $\mathcal{S}_1(H)$, and moreover $|LB|_{\mathcal{S}_1(H)} \leq |L|_{\mathcal{S}_1(H)} |B|_{\mathcal{L}(H)}$, $|BL|_{\mathcal{S}_1(H)} \leq |L|_{\mathcal{S}_1(H)} |B|_{\mathcal{L}(H)}$.
2. If $L \in \mathcal{S}_1(H)$ and (e_i) is an arbitrary orthonormal basis, the trace $\text{Tr}(L) := \sum_{i=1}^{\infty} \langle e_i, L e_i \rangle$ converges absolutely and its value is independent of the choice of the basis (e_i) .

3. Using the expansion introduced above $|L|_{\mathcal{S}_1(H)} = \sum_{j=1}^{\infty} |a_j^L|$, $\text{Tr}(L) = \sum_{j=1}^{\infty} a_j^L$ consequently $|\text{Tr}(L)| \leq |L|_{\mathcal{S}_1(H)}$.

5.2 Linear, Infinite-dimensional BSDEs with Unbounded Terms

We prove here that infinite dimensional BSDEs can be well-posed even if they include an unbounded term. The following proposition is an extension of the results in [12] and its proof follows the same lines.

Proposition 5.1 *Under the same assumptions and notations of Section 2, let $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; H)$ and let ϕ be a progressively measurable process in H satisfying*

$$\mathbb{E} \int_0^T (T-s)^{2\alpha} |\varphi_s|^2 ds < \infty,$$

then the BSDE

$$-dp_t = (A^* p_t + \varphi_t) dt - q_t dW_t, \quad p_T = \eta$$

admits a unique mild solution, that is a unique pair of processes (p, q) with $p \in L^2_{\mathcal{P}}(\Omega, C([0, T], H))$, $q \in L^2_{\mathcal{P}}(\Omega \times [0, T], \mathcal{S}_2(H))$ verifying

$$p_t = e^{(T-t)A^*} \eta + \int_t^T e^{(l-t)A^*} \varphi_l dl - \int_t^T e^{(l-t)A^*} q_l dW_l.$$

Moreover the following estimate holds:

$$\mathbb{E} \int_t^T |q_s|_{\mathcal{S}_2(H)}^2 ds + \sup_{s \in [t, T]} \mathbb{E} |p_s|^2 \leq c \mathbb{E} |\eta|^2 + c(T-t)^{1-2\alpha} \int_t^T (T-s)^{2\alpha} \mathbb{E} |\varphi_s|^2 ds.$$

Proof. The uniqueness follows directly from [12] since the difference (\bar{p}, \bar{q}) of two solutions satisfies:

$$-d\bar{p}_t = A^* \bar{p}_t dt - \bar{q}_t dW_t, \quad \bar{p}_T = 0.$$

Concerning existence, let us set:

$$p_t = e^{(T-t)A^*} \mathbb{E}(\eta | \mathcal{F}_t) + \int_t^T e^{(s-t)A^*} \mathbb{E}(\varphi_s | \mathcal{F}_t) ds.$$

Moreover, by the martingale representation theorem,

$$\mathbb{E}(\varphi_s | \mathcal{F}_t) = \varphi_s - \int_t^s g(s, l) dW_l, \quad \mathbb{E}(\eta | \mathcal{F}_t) = \eta - \int_t^T h(l) dW_l.$$

Notice that

$$\mathbb{E} \int_t^\rho |g(\rho, \sigma)|^2 d\sigma \leq \mathbb{E} |\varphi_\rho|^2 + \mathbb{E} |\mathbb{E}(\varphi_\rho | \mathcal{F}_t)|^2 \leq 2\mathbb{E} |\varphi_\rho|^2, \quad \mathbb{E} \int_t^\rho |h(\sigma)|^2 d\sigma \leq 2\mathbb{E} |\eta|^2.$$

From the above two equations, we have:

$$\begin{aligned}
p_t &= \int_t^T e^{(s-t)A^*} \varphi_s ds - \int_t^T e^{(s-t)A^*} \left(\int_t^s g(s, l) dW_l \right) ds + e^{(T-t)A^*} \eta - \int_t^T e^{(T-t)A^*} h(l) dW_l \\
&= \int_t^T e^{(s-t)A^*} \varphi_s ds - \int_t^T e^{(l-t)A^*} \left(\int_l^T e^{(s-l)A^*} g(s, l) ds \right) dW_l \\
&\quad + e^{(T-t)A^*} \eta - \int_t^T e^{(l-t)A^*} e^{(T-l)A^*} h(l) dW_l.
\end{aligned}$$

So setting

$$q_l = \int_l^T e^{(s-l)A^*} g(s, l) ds + e^{(T-l)A^*} h(l),$$

we deduce that (P, Q) is the unique solution

Let us now establish the estimates.

$$\begin{aligned}
\mathbb{E}|q_\sigma|^2 &\leq c \left(\int_\sigma^T (T-\rho)^{-\alpha} (T-\rho)^\alpha \mathbb{E}|g(\rho, \sigma)| d\rho \right)^2 + c \mathbb{E}|h(\sigma)|^2 d\sigma \\
&\leq (T-t)^{1-2\alpha} \int_\sigma^T (T-\rho)^{2\alpha} \mathbb{E}|g(\rho, \sigma)|^2 d\rho + c \mathbb{E}|h(\sigma)|^2 d\sigma.
\end{aligned}$$

Thus

$$\begin{aligned}
\int_t^T \mathbb{E}|q_\sigma|^2 d\sigma &\leq (T-t)^{1-2\alpha} \int_t^T \int_\sigma^T (T-\rho)^{2\alpha} \mathbb{E}|g(\rho, \sigma)|^2 d\rho d\sigma + c \int_t^T \mathbb{E}|h(\sigma)|^2 d\sigma \\
&\leq (T-t)^{1-2\alpha} \int_t^T (T-\rho)^{2\alpha} \left(\int_t^\rho \mathbb{E}|g(\rho, \sigma)|^2 d\sigma \right) d\rho + c \int_t^T \mathbb{E}|h(\sigma)|^2 d\sigma \\
&\leq (T-t)^{1-2\alpha} \int_t^T (T-\rho)^{2\alpha} \mathbb{E}|\varphi(\rho)|^2 d\rho + c \mathbb{E}|\eta|^2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathbb{E}|p_t|^2 &\leq c \mathbb{E} \left| \int_t^T \mathbb{E}(\varphi_s | \mathcal{F}_t) ds \right|^2 + \mathbb{E}|\mathbb{E}(\eta | \mathcal{F}_t)|^2 = \mathbb{E} \left[\int_t^T (T-s)^{-\alpha} (T-s)^\alpha |\mathbb{E}(\varphi_s | \mathcal{F}_t)| ds \right]^2 + \mathbb{E}|\eta|^2 \\
&\leq (T-t)^{1-2\alpha} \int_t^T (T-s)^{2\alpha} \mathbb{E}|\varphi(s)|^2 ds + \mathbb{E}|\eta|^2,
\end{aligned}$$

which concludes the proof. \square

Proof of Lemma 4.13. Using Proposition 5.1, existence and uniqueness follows by a standard contraction argument (see [12]). The final duality property is established by a simple truncation argument. \square

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