On hyperbolic analogues of some classical theorems in spherical geometry

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ON HYPERBOLIC ANALOGUES OF SOME CLASSICAL THEOREMS IN SPHERICAL GEOMETRY

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Abstract. We give the hyperbolic analogues of some classical theorems in spherical geometry due to Menelaus, Euler, Lexell, Ceva and Lambert. Some of the spherical results are also made more precise.

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1. Introduction

We give the hyperbolic analogues of several theorems in spherical geometry. The first theorem is due to Menelaus and is contained in his *Spherics* (cf. [13] [14] [4] [15]). The second one is due to Euler [1]. The third result was poved by Euler [2] and by his student Lexell [8]. We shall elaborate in the corresponding sections on the importance and the impact of each of these theorems. We also include a proof of the hyperbolic version of the well-known Euclidean theorem of Ceva, because it is in the same spirit as Euler's theorem (although the proof is easier), and we prove a hyperbolic version of a theorem of Lambert, as an application of the hyperbolic version of a theorem of Euler. We also give more precise versions of some of the results in spherical geometry. Our goal is to go through some works of some great mathematicians of the past centuries and to put some of their results in a modern perspective. Furthermore, putting together results in the three geometries and highlighting the analogies is mathematically appealing.[1]

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[1]Since this paper is motivated by classical theorems, a few words on the history are in order. No Greek manuscript of Menelaus (1st-2nd c. A.D) survives, but only Arabic translations. This work remained rather unknown (except for the classical "Menelaus theorem" which was quoted by Ptolemy) until very recently.[2] Between the times of Menelaus and of Euler, no progress was made in the field of spherical geometry. Euler wrote nearly twelve papers on spherical geometry and in fact he revived the subject. Several of his young collaborators and disciples followed him in this field (see the survey [9]). For the work done before Euler on hyperbolic geometry, Lexell refers to Theodosius, who lived two centuries before Menelaus, and whose work is much less interesting. He writes in the introduction to his paper [8]: "From that time in which the Elements of Spherical Geometry of Theodosius had been put on the record, hardly any other questions are found, treated by the geometers, about further perfection of the theory of figures drawn on spherical surfaces, usually treated in the Elements of Spherical Trigonometry and aimed to be used

2. A result on right triangles

We start with a result on right triangles which gives a relation between the hypothenuse of a right triangle and a cathetus, in terms of the angle they make (Theorem 2.1). This is a non-Euclidean analogue of the fact that in the Euclidean case, the ratio of the two corresponding lengths is the cosine of the angle they make. The result in the hyperbolic case is motivated by a similar result of Menelaus in the spherical case contained in his *Spherics.* This result is of major importance from the historical point of view, because Menelaus gave only a sketch of a proof, and writing a complete proof of it gave rise to several mathematical developments by Arabic mathematicians between the 9th and the 13th centuries. These developments include the discovery of polarity theory and in particular the definition of the polar triangle in spherical geometry, as well as the introduction of an invariant spherical cross ratio. It is also probable that the invention of the sine rule was motivaed by this result. All this is discussed in the two papers [13] and [14], which contains a report on the proof of Menelaus' theorem completed by several Arabic mathematician is reported on.

The proof that we give in the hyperbolic case works as well in the spherical case, with a modification which amounts to replacing the hyperbolic sine and cosine functions by the sine and cosine functions. (See Remark 2.3 at the end of this section.) Thus, in particular, we get a very short proof of Menelaus' Theorem.

In the statement of this theorem, we refer to Figure 1.

Theorem 2.1. *In the hyperbolic plane, consider two geodesics L_1, L_2 starting from a point A and making an acute angle α at that point. Consider two points C and E on L_1, with C between A and E, and the two perpendiculars CB and ED onto L_2. Then, we have:*

$$\frac{\sinh(AC+AB)}{\sinh(AC-AB)} = \frac{1+\cos\alpha}{1-\cos\alpha}.$$

In particular, we have

$$\frac{\sinh(AC+AB)}{\sinh(AC-AB)} = \frac{\sinh(AE+AD)}{\sinh(AE-AD)}.$$

which is the form in which Menelaus stated his theorem in the spherical case (where sinh is replaced by sin).

To prove Theorem 2.1, we use the following lemma.

Lemma 2.2. *In the triangle ABC, let $a = BC$, $b = AC$ and $c = AB$. Then we have:*

$$\tanh b = \cos\alpha \cdot \tanh c.$$

Proof. The formula is a corollary of the cosine and sine formulae for hyperbolic triangles. We provide it for completeness.

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in the solution of spherical triangles." Lexell and Euler were not aware of the work of Menelaus, except for his results that were quoted by Ptolemy. A French version of the work of Theodosius is available [19].

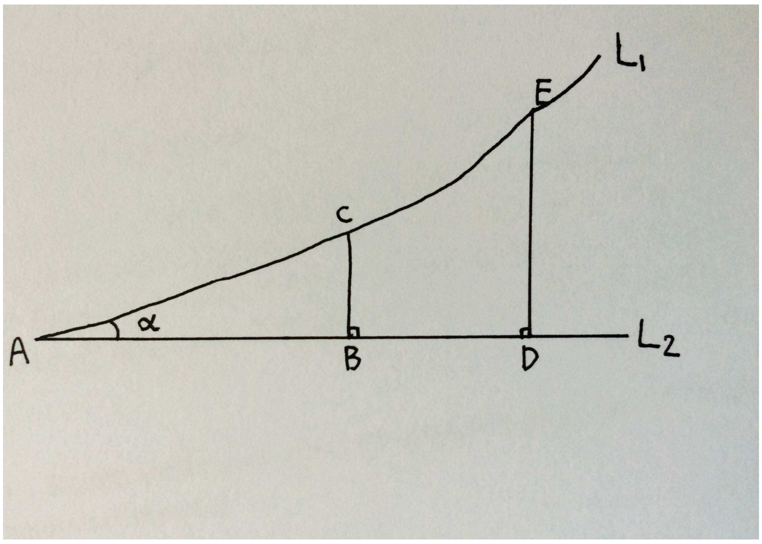

FIGURE 1. The right triangles ABC and ADE.

From the hyperbolic cosine formula, we have (using the fact that the angle $\widehat{ABC}$ is right)

$$\cosh c = \cosh a \cdot \cosh b.$$

By the hyperbolic sine formula, we have

$$\sinh c = \frac{\sinh a}{\sin \alpha}.$$

As a result, we have

$$\begin{aligned}\sin^2 \alpha &= \frac{(\sinh a)^2}{(\sinh c)^2} = \frac{(\cosh a)^2 - 1}{(\sinh c)^2} \\ &= \frac{\frac{(\cosh c)^2}{(\cosh b)^2} - 1}{(\sinh c)^2} = \frac{(\cosh c)^2 - (\cosh b)^2}{(\cosh b)^2 \cdot (\sinh c)^2}.\end{aligned}$$

Then we have

$$\begin{aligned}\cos^2 \alpha &= 1 - \frac{(\cosh c)^2 - (\cosh b)^2}{(\cosh b)^2 \cdot (\sinh c)^2} \\ &= \frac{(\cosh b)^2 \cdot (\sinh c)^2 - (\cosh c)^2 + (\cosh b)^2}{(\cosh b)^2 \cdot (\sinh c)^2} \\ &= \frac{(\cosh b)^2 \cdot (\cosh c)^2 - (\cosh c)^2}{(\cosh b)^2 \cdot (\sinh c)^2} \\ &= \frac{(\sinh b)^2 \cdot (\cosh c)^2}{(\cosh b)^2 \cdot (\sinh c)^2}.\end{aligned}$$

Since $\cos \alpha > 0$, we get

$$\cos \alpha = \frac{\sinh b \cdot \cosh c}{\cosh b \cdot \sinh c}.$$

□

To prove Theorem 2.1, it suffices to write the ratio $\dfrac{\sinh(c+b)}{\sinh(c-b)}$ as

$$\frac{\sinh b\cdot\sinh c+\cosh b\cdot\sinh c}{\cosh b\cdot\sinh c-\cosh c\cdot\sinh b}=\frac{\frac{\sinh b}{\cosh b}+\frac{\sinh c}{\cosh c}}{\frac{\sinh c}{\cosh c}-\frac{\sinh b}{\cosh b}}.$$

Using Lemma 2.2, the above ratio becomes

$$\frac{1+\cos\alpha}{1-\cos\alpha}.$$

Remark 2.3. An analogous proof works for the spherical case, and it gives the following more precise result of Menelaus' theorem:

$$\frac{\sin(AC+AB)}{\sin(AC-AB)}=\frac{1+\cos\alpha}{1-\cos\alpha}.$$

3. Euler's ratio-sum formula for hyperbolic triangles

Euler, in his memoir [1],[3] proved the following:

Theorem 3.1. *Let ABC be a triangle in the plane and let D,E,F be points on the sides BC,AC,AB respectively. If the lines AD,BE,CF intersect at a common point O, then we have*

$$\frac{AO}{OD}\cdot\frac{BO}{OE}\cdot\frac{CO}{OF}=\frac{AO}{OD}+\frac{BO}{OE}+\frac{CO}{OF}+2. \tag{1}$$

The following notation will be useful for generalization: Setting $\alpha=\dfrac{AO}{OD},\beta=\dfrac{BO}{OE},\gamma=\dfrac{CO}{OF}$, Equation (1) is equivalent to

$$\alpha\beta\gamma=\alpha+\beta+\gamma+2. \tag{2}$$

Euler also gave the following construction which is a converse of Theorem 3.1:

> Given three segments AOD,BOE,COF meeting at a common point O and satisfying (1), we can construct a triangle ABC such that the points D,E,F are as in the theorem.

After the Euclidean case, Euler proved a version of Theorem 3.1 for spherical triangles. Using the above notation, we state Euler's theorem:

Theorem 3.2. *Let ABC be a spherical triangle and let D,E,F be points on the sides BC,AC,AB respectively. If the lines AD,BE,CF intersect at a common point O, then*

$$\alpha\beta\gamma=\alpha+\beta+\gamma+2 \tag{3}$$

where $\alpha=\dfrac{\tan AO}{\tan OD}$, $\beta=\dfrac{\tan BO}{\tan OE}$ *and* $\gamma=\dfrac{\tan CO}{\tan OF}$.

We now prove an analogous result for hyperbolic triangles:

[3]The memoir was published in 1815, that is, 22 years after Euler's death. For various reasons, there was sometimes a long span of time between the moment Euler wrote his papers and the moment they were published; see [9] for comments on this matter.

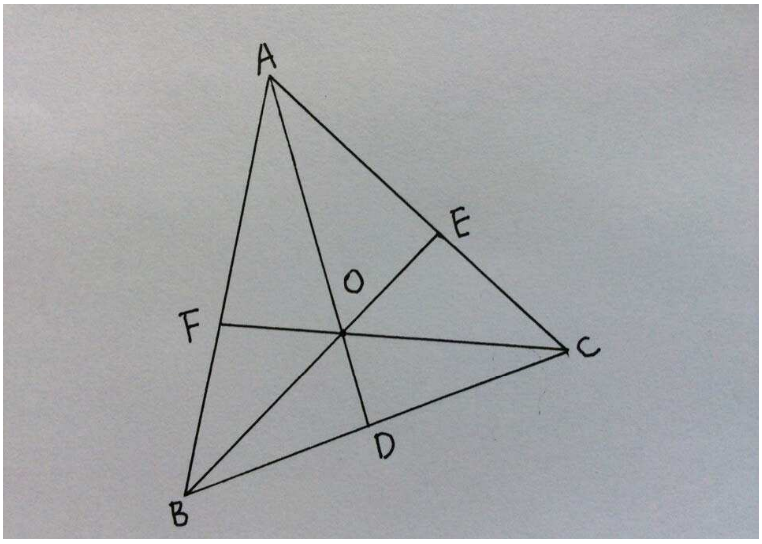

FIGURE 2. Triangle.

Theorem 3.3. *Let ABC be a triangle in the hyperbolic plane and let D, E, F be points on the lines joining the sides BC, AC, AB, respectively. If the lines AD, BE, CF intersect at a common point O, then*

$$\alpha\beta\gamma = \alpha + \beta + \gamma + 2, \tag{4}$$

where $\alpha = \dfrac{\tanh AO}{\tanh OD}$, $\beta = \dfrac{\tanh BO}{\tanh OE}$ *and* $\gamma = \dfrac{\tanh CO}{\tanh OF}$.

3.1. **Proof of Theorem 3.3.** Let ABC be a triangle in the hyperbolic plane. Suppose that the lines AD, BE and CF intersect at O.

We shall use the cosine and sine formulae in the hyperbolic triangle AFO:

$$\cos \widehat{AFO} = \frac{\cosh AF \cdot \cosh OF - \cosh AO}{\sinh AF \cdot \sinh OF}. \tag{5}$$

$$\frac{\sinh AF}{\sin \widehat{AOF}} = \frac{\sinh AO}{\sin \widehat{AFO}}. \tag{6}$$

Lemma 3.4. *We have*

$$\sin \widehat{BOD} = \tanh OF \cdot \left(\frac{\sin \widehat{BOF}}{\tanh AO} + \frac{\sin \widehat{AOF}}{\tanh BO} \right).$$

Proof. Combining (5) with (6), we have

$$\begin{aligned} \tan \widehat{AFO} &= \frac{\sin \widehat{AFO}}{\cos \widehat{AFO}} = \frac{\sinh AO \cdot \sinh OF \cdot \sin \widehat{AOF}}{\cosh AO \cdot \cosh OF - \cosh AO} \\ &= \frac{\sinh AO \cdot \sinh OF \cdot \sin \widehat{AOF}}{\left(\cosh AO \cdot \cosh OF - \sinh AO \cdot \sinh OF \cdot \cos \widehat{AOF}\right) \cdot \cosh OF - \cosh AO}, \end{aligned}$$

where we have replaced $\cosh AC$ by

$$\cosh AO \cdot \cosh OF - \sinh AO \cdot \sinh OF \cdot \cos \widehat{AOF}.$$

Using the identity,

$$\cosh^2 x - 1 = \sinh^2 x$$

we simplify the above equality to get

$$\tan \widehat{AFO} = \frac{\sinh AO \cdot \sin \widehat{AOF}}{\cosh AO \cdot \sinh OF - \sinh AO \cdot \cosh OF \cdot \cos \widehat{AOF}}. \tag{7}$$

In the same way, we have

$$\tan \widehat{BFO} = \frac{\sinh BO \cdot \sin \widehat{BOF}}{\cosh BO \cdot \sinh OF - \sinh BO \cdot \cosh OF \cdot \cos \widehat{BOF}}. \tag{8}$$

Since $\widehat{AFO} + \widehat{BFO} = \pi$, $\tan \widehat{AFO} + \tan \widehat{AFO} = 0$. Then (7) and (8) imply:

$$\begin{aligned} & \sinh AO \cdot \cosh BO \cdot \sinh OF \cdot \sin \widehat{AOF} \\ - & \sinh AO \cdot \sinh BO \cdot \cosh \cdot \sin \widehat{AOF} \cdot \cos \widehat{BOF} \\ + & \sinh BO \cdot \cosh AO \cdot \sinh OF \cdot \sin \widehat{BOF} \\ - & \sinh BO \cdot \sinh AO \cdot \cosh OF \cdot \sin \widehat{BOF} \cdot \cos \widehat{AOF} = 0. \end{aligned}$$

Equivalently, we have

$$\begin{aligned} & \sinh AO \cdot \cosh BO \cdot \sinh OF \cdot \sin \widehat{AOF} + \sinh BO \cdot \cosh AO \cdot \sinh OF \cdot \sin \widehat{BOF} \\ = & \sinh AO \cdot \sinh BO \cdot \cosh OF \cdot \left(\sin \widehat{AOF} \cdot \cos \widehat{BOF} + \sin \widehat{BOF} \cdot \cos \widehat{AOF} \right) \\ = & \sinh AO \cdot \sinh BO \cdot \cosh OF \cdot \sin \widehat{BOD}, \end{aligned}$$

where the last equality follows from the fact that $\widehat{AOF} + \widehat{BOF} + \widehat{BOD} = \pi$.
This implies

$$\begin{aligned} \sin \widehat{BOD} &= \frac{\sinh AO \cdot \cosh BO \cdot \sinh OF \cdot \sin \widehat{AOF} + \sinh BO \cdot \cosh AO \cdot \sinh OF \cdot \sin \widehat{BOF}}{\sinh AO \cdot \sinh BO \cdot \cosh OF} \\ &= \tanh OF \cdot \left(\frac{\sin \widehat{BOF}}{\tanh AO} + \frac{\sin \widehat{AOF}}{\tanh BO} \right). \end{aligned}$$

□

For simplicity, we denote by $p = \widehat{BOF}, q = \widehat{AOF}$ and $r = \widehat{BOD}$. Then $p + q + r = \pi$. We write Lemma 3.4 as

$$\frac{\sin r}{\tanh OF} = \frac{\sin p}{\tanh AO} + \frac{\sin q}{\tanh BO}.$$

By repeating the proof of Lemma 3.4, we have

$$\frac{\sin p}{\tanh OD} = \frac{\sin q}{\tanh BO} + \frac{\sin r}{\tanh CO}$$

and

$$\frac{\sin q}{\tanh OE} = \frac{\sin r}{\tanh CO} + \frac{\sin p}{\tanh AO}.$$

Setting $P = \dfrac{\sin p}{\tanh AO}, Q = \dfrac{\sin q}{\tanh BO}$ and $R = \dfrac{\sin r}{\tanh CO}$, it follows from the above three equations that

$$\gamma R = P + Q, \tag{9}$$

$$\alpha P = Q + R, \tag{10}$$

$$\beta Q = R + p. \tag{11}$$

Using (9) and (10), we have

$$R = \frac{P+Q}{\gamma} = \alpha P - Q.$$

As a result,

$$\frac{P}{Q} = \frac{\gamma + 1}{\alpha\gamma - 1}.$$

On the other hand, it follows from (10) and (11) that

$$\alpha P = Q + R = Q + \beta Q - P.$$

Then we have

$$\frac{P}{Q} = \frac{\beta + 1}{\alpha + 1}.$$

Thus,

$$\frac{\gamma + 1}{\alpha\gamma - 1} = \frac{\beta + 1}{\alpha + 1},$$

which implies

$$\alpha\beta\gamma = \alpha + \beta + \gamma + 2.$$

Remark 3.5. In [1], Euler also writes Equation (2) as

$$\frac{1}{\alpha + 1} + \frac{1}{\beta + 1} + \frac{1}{\gamma + 1} = 1,$$

and this leads, in the hyperbolic case, to the relation:

$$\frac{\tanh Oa}{\tanh Aa} + \frac{\tanh Ob}{\tanh Bb} + \frac{\tanh Oc}{\tanh Cc} = 1.$$

3.2. **The converse.**

Construction 3.6. From the six arcs AO, BO, CO, OD, OE, OF satisfying the relation (4), we can construct a unique triangle ABC in which three line segments AD, BE, CF are drawn from each vertex to the opposite side, meeting at a point O and leading to the given arcs.

The construction is the same as Euler's in the case of a spherical triangle. We are given three segments AOD, BOE, COF intersecting at a common point O, and we wish to find the angles $p = \widehat{AOF}, q = \widehat{BOF}, r = \widehat{BOD}$ (Figure 2), so that the three points A, B, C are vertices of a triangle and D, E, F are on the lines joining the opposite sides.

Setting

$$G=\frac{\tanh OA}{\alpha+1}, H=\frac{\tanh OB}{\beta+1}, I=\frac{\tanh OC}{\gamma+1}$$

and using the fact that the angles satisfy the further equation

$$p+q+r=\pi,$$

we get (by writing the formula for the sum of two supplementary angles):

$$\Delta=\frac{\sqrt{(G+H+I)(G+H-I)(I+G-H)(H+I-G)}}{2GHI}$$

and

$$\Delta=\frac{2M^2}{GHI}.$$

A calculation gives then the following formula for the angles:

$$\sin p=\frac{2M^2}{HI}, \sin q=\frac{2M^2}{IG}, \sin r=\frac{2M^2}{GH}.$$

From these angles, we can construct the triangle.

3.3. **Another proof of Theorem 3.3.** We present another proof of Theorem 3.3, based on the hyperboloid model of the hyperbolic plane.

We denote by $\mathbb{R}^{2,1}$ the three-dimensional Minkowski space, that is, the real vector space of dimension three equipped with the following pseudo-inner product:

$$<\mathbf{x},\mathbf{y}>=-x_0y_0+x_1y_1+x_2y_2.$$

We consider the space

$$\mathbb{H}:=\{\mathbf{x}\in\mathbb{R}^{2+1}\mid x_0>0,<\mathbf{x},\mathbf{x}>=-1\}$$

This is one of the two connected components of the "unit sphere" in this space, that is, the sphere of radius $\sqrt{-1}$. We shall call this component the *pseudo-sphere*.

At every point $\mathbf{x}$ of the pseudo-sphere, we equip the tangent space $\mathrm{T}_{\mathbf{x}}\mathbb{H}$ at x with the pseudo-inner product induced from that of $\mathbb{R}^{2+1}$. It is well known that this induced pseudo-inner product is a scalar product, and the pseudo-sphere equipped with the length metric induced from these inner products on tangent spaces is isometric to the hyperbolic plane. This is a model of the hyperbolic plane, called the Minkowski model. See [20] for some details.

Let $\mathbf{x},\mathbf{y}$ be two points on $\mathbb{H}$. It is well known and not hard to show that their distance $d(\mathbf{x},\mathbf{y})$ is given by

$$\cosh d(\mathbf{x},\mathbf{y})=-<\mathbf{x},\mathbf{y}>.$$

Up to an isometry of $\mathbb{H}$, we may assume that $\mathbf{x}=(1,0,0)$ and $\mathbf{y}=a\mathbf{x}+b\mathbf{n}$, where $\mathbf{n}=(0,1,0)$. The equation $<\mathbf{y},\mathbf{y}>=-1$ implies $a^2-b^2=1$. We may also assume that $b\geq 0$. See Figure 3. Then we have

$$\cosh d(\mathbf{x},\mathbf{y})=-<\mathbf{x},\mathbf{y}>=-a<\mathbf{x},\mathbf{x}>-b<\mathbf{x},\mathbf{n}>=a.$$

It follows that

$$\mathbf{y}=\cosh\left(d(\mathbf{x},\mathbf{y})\right)\mathbf{x}+\sinh\left(d(\mathbf{x},\mathbf{y})\right)\mathbf{n} \tag{12}$$

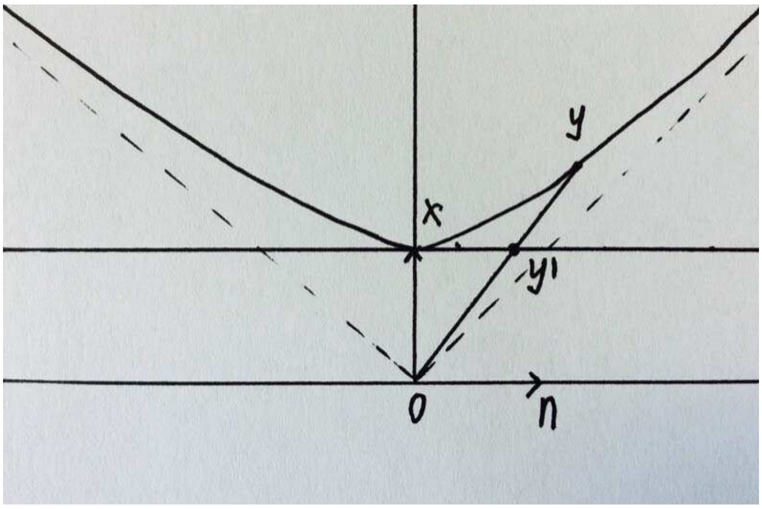

FIGURE 3. The hyperboloid model of the hyperbolic plane.

Another proof of Theorem 3.3. Consider a triangle ABC in $\mathbb{H}$, with D, E, F on the lines BC, CA, AB, respectively. Suppose that the lines AD, BE and CF intersect at O. Up to an isometry, we may suppose that the point O is $(1, 0, 0)$.

Let Σ be the plane tangent to $\mathbb{H}$ at $O = (1, 0, 0)$. We shall use the Euclidean metric on this plane. For any point $\mathbf{x} \in \mathbb{H}$, the line drawn from $\mathbf{0} = (0, 0, 0)$ through $\mathbf{x}$ intersects Σ at a unique point, which we denote by $\mathbf{x}'$. Consider the points A', B', C', D', E', F' obtained from the intersections of the lines OA, OB, OC, OD, OE, OF with Σ, respectively. By (12),

$$\tan \widehat{O\mathbf{0}A} = \frac{OA'}{1} = \frac{\sinh OA}{\cosh OA}.$$

(Here OA' denotes the Euclidean distance between O and A', and OA denotes the hyperbolic distance between O and A.) As a result, $OA' = \tanh OA$. Similarly, $OB' = \tanh OB, OC' = \tanh OC$ and $OD' = \tanh OD, OE' = \tanh OE, OF' = \tanh OF$.

Since

$$\frac{\tanh OA}{\tanh OD} = \frac{OA'}{OD'} = \alpha, \frac{\tanh OB}{\tanh OE} = \frac{OB'}{OE'} = \beta, \frac{\tanh OC}{\tanh OF} = \frac{OC'}{OF'} = \gamma,$$

we have reduced the proof to the case of a Euclidean triangle. □

Remark 3.7. The second proof is inspired from an argument that Euler gave in a second proof of his Theorem 3.2. Euler's argument uses a radial projection of the sphere onto a Euclidean plane tangent to the sphere at the point O, which we have transformed into an argument that uses the radial projection of the pseudo-sphere onto a Euclidean plane.

3.4. **Ceva's theorem.** We refer again to Figure 2. The classical theorem of Ceva[4] gives another necessary and sufficient relation for the three lines

[4]Giovanni Ceva (1647-1734) obtained the statement in the Euclidean case, in his *De lineis rectis se invicem secantibus statica constructio*, 1678. According to Hogendijk, Ceva's theorem was already known to the Arabic mathematician Ibn Hūd, cf. [3].

AD, BE, CF to meet in a point, and it has also in Euclidean, spherical and hyperbolic versions. The Ceva identity is different from Euler's. The statement is:

Theorem 3.8. *If the three lines AD, BE, CF meet in a common point, then we have*

- *in Euclidean geometry:*
$$\frac{DE}{DC}\cdot\frac{EC}{EA}\cdot\frac{FA}{FB}=1;$$
- *in spherical geometry:*
$$\frac{\sin DE}{\sin DC}\cdot\frac{\sin EC}{\sin EA}\cdot\frac{\sin FA}{\sin FB}=1.$$
- *in hyperbolic geometry:*
$$\frac{\sinh DE}{\sinh DC}\cdot\frac{\sinh EC}{\sinh EA}\cdot\frac{\sinh FA}{\sinh FB}=1.$$

Proof. We give the proof in th case of hyperbolic geometry. The other proofs are similar. Assume the three lines meet at a point O. By the sine formula, we have
$$\frac{\sinh DB}{\sin \widehat{DOB}}=\frac{\sinh OB}{\sin \widehat{ODB}}$$
and
$$\frac{\sinh DC}{\sin \widehat{DOC}}=\frac{\sinh OC}{\sin \widehat{ODC}}.$$
Dividing both sides of these two equations, we get:
$$\frac{\sinh DB}{\sinh DC}=\frac{\sinh OB}{\sinh OC}\cdot\frac{\sin \widehat{DOB}}{\sin \widehat{DOC}}.$$
In the same way, we have
$$\frac{\sinh EC}{\sinh EA}=\frac{\sinh OC}{\sinh OA}\cdot\frac{\sin \widehat{EOC}}{\sin \widehat{EOA}}$$
and
$$\frac{\sinh FA}{\sinh FB}=\frac{\sinh OA}{\sinh OB}\cdot\frac{\sin \widehat{FOA}}{\sin \widehat{FOB}}.$$
Multiplying both sides of the last three equations and using the relations
$$\sin \widehat{DOB}=\sin \widehat{EOA},\ \sin \widehat{DOC}=\sin \widehat{FOA},\ \sin \widehat{EOC}=\sin \widehat{BOF},$$
we get the desired result. □

Remark 3.9. The classical theorem of Ceva is usually stated with a minus sign at the right hand side (that is, the result is -1 instead of 1), and the length are counted algebraically. In this form, the converse also of the theorem holds. The proof is also easy.

3.5. **A theorem of Lambert.** We present now a result of Lambert,[5] contained in his *Theory of parallel lines* [10], §77. This result says that in an equilateral triangle ABC, if F is the midpoint of BC and D the intersection point of the medians, we have $DF = \frac{1}{3}AF$ (respectively $DF > \frac{1}{3}AF$, $DF < \frac{1}{3}AF$) in Euclidean (respectively spherical, hyperbolic geometry). In fact, we shall obtain a more precise relation between the lengths involved. The hyperbolic case will follow from the following proposition:

Proposition 3.10. *Let ABC be an equilateral triangle in the hyperbolic plane and let D, E, F be the midpoints of BC, AC, AB, respectively (Figure 2). Then the lines AD, BE, CF intersect at a common point O satisfying*

$$\frac{\tanh AO}{\tanh OD} = \frac{\tanh BO}{\tanh OE} = \frac{\tanh CO}{\tanh OF} = 2.$$

In particular,

$$\frac{AD}{OD} = \frac{BE}{OE} = \frac{CF}{OF} < 3.$$

Proof. The fact that the lines AD, BE, CF intersect at a common point O follows from the symmetry of the equilateral triangle.

Let us set

$$\alpha = \frac{\tanh AO}{\tanh OD}, \beta = \frac{\tanh BO}{\tanh OE}, \gamma = \frac{\tanh CO}{\tanh OF}.$$

Then, again by symmetry, $\alpha = \beta = \gamma$. By Euler's Theorem 3.2, we have

$$\alpha^3 = 3\alpha + 2.$$

This implies that $\alpha = \beta = \gamma = 2$.

To see that $\frac{AD}{OD} < 3$ (or, equivalently, $\frac{AO}{OD} < 2$), it suffices to check that $\tanh AD = 2\tanh OD < \tanh(2 \cdot OD)$. This follows from the inequality

$$2\tanh x < \tanh(2x), \forall\ x > 0.$$

□

[5]Johann Heinrich Lambert (1728-1777) was an Alsatian mathematician (born in Mulhouse). He is sometimes considered as the founder of modern cartography, a field which was closely related to spherical geometry. His *Anmerkungen und Zusätze zur Entwerfung der Land- und Himmelscharten* (Remarks and complements for the design of terrestrial and celestial maps, 1772) [5] contains seven new projections of the sphere, some of which are still in use today, for various purposes. Lambert is an important precursor of hyperbolic geometry; he was probably the mathematician who came closest to that geometry, before this geometry was born in the works of Lobachevsky, Bolyai and Gauss. In his *Theorie der Parallellinien*, written in 1766, he developed the bases of a geometry in which all the Euclidean postulates hold except the parallel postulate which is replaced by its negation. His hope was to arrive to a contradiction, which would show that Euclid's parallel postulate is a consequence of the other Euclidean postulates. Instead of leading to a contradiction, Lambert's work turned out to be a set of results in hyperbolic geometry, to which belongs the result that we present here. We refer the reader to [10] for the first translation of this work, together with a mathematical commentary. Lambert was self-taught (he left school at the age of eleven), and he eventually became one of the greatest and most universal minds of the eighteenth century. Euler had a great respect for him, and he helped him joining the Academy of Sciences of Berlin, where Lambert worked during the last ten years of his life. One of Lambert's achievements is that π is irrational. He also conjectured that π is transcendental (a result which was obtained a hundred years later).

The same proof shows that in the Euclidean case, and with the same notation, we have $\frac{AD}{OD} = 3$ and that in the spherical case, we have $\frac{AD}{OD} > 3$. (Notice that $2\sin x > \sin(2x)$ when $0 < x < \pi$.) Thus we recover the results of Lambert.

4. Hyperbolic triangles with the same area

In this section, we will study the following question:

> *Given two distinct points $A, B \in \mathbb{H}^2$, determine the set of points $P \in \mathbb{H}^2$ such that the area of the triangle with vertices P, A, B is equal to some given constant.*

The same question in the case of a spherical triangle was solved by Lexell [8] and Euler [2].[6] The proof for the case of a hyperbolic triangle, which we will give below, is similar in spirit to that of Euler.

Let us note that the analogous locus in the Euclidean case is in Euclid's *Elements* (Propositions 37 and its converse, Proposition 39, of Book I). In this case, the locus consists of a pair of lines parallel to the basis. In spherical and hyperbolic geometries, the locus does not consist of lines (that is, geodesics) but of two hypercycles (equidistant loci to lines). Also note that these hypercycles are not equidistant to the line containing the base of the triangle. The two hypercycles are equidistant to two distinct lines.

This theorem has an interesting history. Both Euler and his student Lexell gave a proof in [2] (published in 1797) and [8] (published in 1784).[7] Jakob Steiner published a proof of the same theorem in 1827 [17], that is, several decades after Euler and Lexell. In 1841, Steiner published a new proof [18]. In the same paper, he says that Liouville, the editor of the journal, before he presented the result at the Academy of Sciences of Paris, looked into the literature and found that Lexell already knew the theorem. Steiner mentions that the theorem was known, "at least in part", by Lexell, and then by Legendre. He does not mention Euler. Steiner adds: "The application of the theorem became easy only after the following complement: *the circle which contains the triangles with the same area passes through the points antipodal to the extremities of the bases*". In fact, this "complement" is contained in both Lexell's and Euler's proof. Legendre gives a proof of the same theorem in his *Éléments de géométrie*, [7] Note X, Problem III. His solution is based on spherical trigonometry, like one of Euler's solutions. In 1855, Lebesgue gave a proof of this theorem [6], which in fact is Euler's proof. At the end of Lebesgue's paper, the editor of the journal adds a comment, saying that one can find a proof of this theorem in the *Éléments de Géométrie* of Catalan (Book VII, Problem VII), but no reference is given to Euler.

In the rest of this this section, we prove the hyperbolic analogue of this theorem. We shall use the unit disc model of the hyperbolic plane.

[6]Despite the difference in the dates of publication, the papers of Euler and Lexell were written the same year.

[7]We already noted that the two memoirs were written in the same year. Euler says that the idea of the result was given to him by Lexell.

Up to an isometry, we may assume that the two vertices A and B are on the real line, with $0 < A = -B < 1$ (that is, A and B are symmetric with respect to the origin). This will simplify the notation and will make our discussion clearer. We denote the hyperbolic distance between A and B by $2x$.

4.1. **Example: A family of triangles with increasing areas.** Let us denote the center of the unit disc by O. We first consider the case where the vertex P lies on the geodesic that goes through O perpendicularly. We denote the hyperbolic distance between O and P by y, and the hyperbolic distance between A and P by c. See Figure 4.

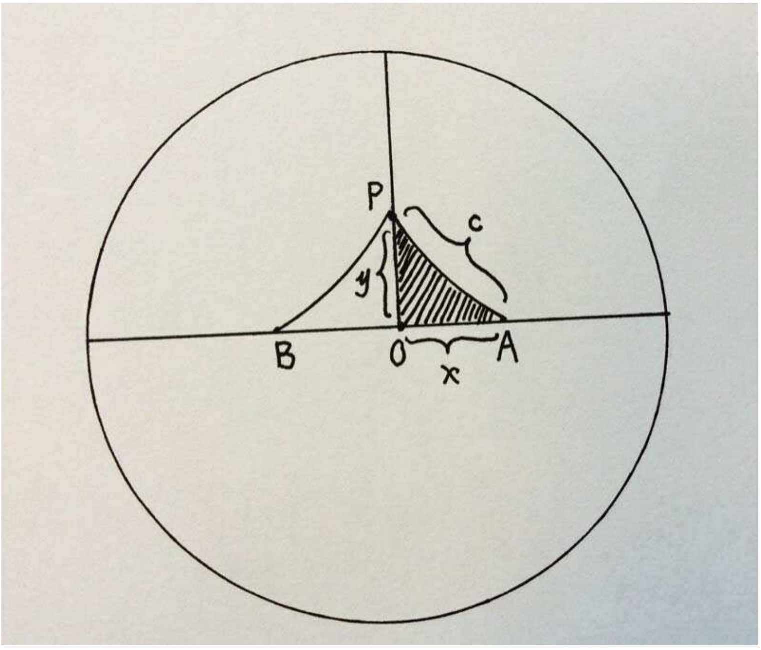

FIGURE 4. The example APB

Since the distances $x = d(O, A) = d(O, B)$ are fixed, we may consider the area of the triangle APB as a function of y. This area is the double of the area of APO. We start with the following:

Proposition 4.1. *The area of APO is an increasing function of y.*

Proof. We use again trigonometry. By the cosine formula for a hyperbolic triangle, we have

$$\cos \widehat{AOP} = \frac{\cosh x \cdot \cosh y - \cosh c}{\sinh x \cdot \sinh y}.$$

Since $\widehat{AOP} = \frac{\pi}{2}$, $\cos \widehat{AOP} = 0$. We have

$$\cosh x = \cosh x \cdot \cosh y.$$

Denote $\widehat{APO} = \alpha, \widehat{PAO} = \beta$. Using again the cosine formula for hyperbolic triangles, we obtain

$$\cos\alpha = \frac{\cosh y\cdot\cosh c - \cosh x}{\sinh y\cdot \sinh c},\ \cos\beta = \frac{\cosh x\cdot\cosh c - \cosh y}{\sinh x\cdot \sinh c}.$$

Using the sine formula for hyperbolic triangles, we have (note that $\sin\widehat{AOP} = 1$)

$$\sin\alpha = \frac{\sinh x}{\sinh c},\ \sin\beta = \frac{\sinh y}{\sinh c}.$$

Applying the above equations, we have

$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cdot\cos\beta + \sin\beta\cdot\cos\alpha\\ &= \frac{\sinh x}{\sinh c}\cdot\frac{\cosh x\cdot\cosh c - \cosh y}{\sinh x\cdot\sinh c} + \frac{\sinh y}{\sinh c}\cdot\frac{\cosh y\cdot\cosh c - \cosh x}{\sinh y\cdot\sinh c}\\ &= \frac{(\cosh c - 1)(\cosh x + \cosh y)}{(\sinh c)^2}\\ &= \frac{(\cosh x\cdot\cosh y - 1)(\cosh x + \cosh y)}{(\cosh x\cdot\cosh y)^2 - 1}\end{aligned}$$

We set $t = \cosh y > 1$ and write the right-hand side of the above equation as

$$\begin{aligned} f(t) &= \frac{(\cosh x\cdot t - 1)(\cosh x + t)}{(\cosh x\cdot t)^2 - 1}\\ &= \frac{\cosh x\cdot t^2 + (\sinh x)^2\cdot t - \cosh x}{(\cosh x)^2\cdot t^2 - 1}\end{aligned}$$

By a calculation, we have

$$f'(t) = -(\sinh x)^2\cdot\frac{\cosh x\cdot t - 1}{\cosh x\cdot t + 1} < 0.$$

This shows that $\sin(\alpha+\beta)$ is a decreasing function of y. Since the area of APO is given by $\frac{\pi}{2} - (\alpha+\beta)$, it is an increasing function of y. □

More generally, let $\Gamma(t),\ t\in[0,\infty)$ be an arbitrary geodesic ray initiating from the real line perpendicularly. Geodesics are parameterized by arc-length. We denote by $F\in(-1,1)$ the foot of $\Gamma(t)$ and by a the (hyperbolic) distance between O and F. Consider the case where $F\in[B,A]$. Then the proof of Proposition 4.1 shows that the area of hyperbolic triangle with vertices $A, B, \Gamma(t)$ is an increasing function of t.

The triangle is naturally separated into two right triangles. (When P coincides with A or B, we consider that one of the triangles is of area 0.) Denoting by $\Delta_1(t)$ and $\Delta_2(t)$ the areas of these two right triangles, we have

$$\Delta_1(t) = \arccos\left(\frac{(\cosh(x-a)\cdot\cosh t - 1)(\cosh(x-a) + \cosh t)}{(\cosh(x-a)\cdot\cosh t)^2 - 1}\right) \tag{13}$$

$$\Delta_2(t) = \arccos\left(\frac{(\cosh(x+a)\cdot\cosh t - 1)(\cosh(x+a) + \cosh t)}{(\cosh(x+a)\cdot\cosh t)^2 - 1}\right) \tag{14}$$

As we have shown, both $\Delta_1(t)$ and $\Delta_2(t)$ are increasing functions of t. By letting $t \to \infty$, we have

$$\lim_{t\to\infty} (\Delta_1(t) + \Delta_2(t)) = \arccos\left(\frac{1}{\cosh(x-a)}\right) + \arccos\left(\frac{1}{\cosh(x+a)}\right).$$

The limit is the area of the ideal triangle with vertices $A, B, \Gamma(\infty)$.

4.2. **The locus of vertices of triangles with given base and area.** As in §4.1, we assume that $A, B \in \mathbb{H}$ are on the real line and symmetric with respect to the imaginary axis. For any $P \in \mathbb{H}$, we denote by $\Delta(P)$ the area of the hyperbolic triangle APB. Let $\mathcal{L}(P)$ be the set of points Z in $\mathbb{H}$ such that the area of the triangle AZB is equal to $\Delta(P)$.

We recall that a *hypercycle* $\mathcal{C}$ in $\mathbb{H}$ is a bi-infinite curve in $\mathbb{H}$ whose points are equidistant from a given geodesic. In the unit disc model of the hyperbolic plane, $\mathcal{C}$ is represented by an arc of circle that intersects the boundary circle at non-right angles. (This angle is right if and only if the hypercycle coincides with the geodesic.) The geodesic to which the points of $\mathcal{C}$ are equidistant intersects the boundary circle in the same point, but with right angles. We shall denote this geodesic by $\mathcal{G}$. There is another hypercycle, on the other side of $\mathcal{G}$, with the same distance, which will be denoted by $\mathcal{C}'$. We need only consider hypercycles that are symmetric with respect to the imaginary axes.

With the above notation, we can state our main result.

Theorem 4.2. *For any $P \in \mathbb{H}$, there is a unique hypercycle $\mathcal{C}$ that passes through A, B such that $\mathcal{C}'$ is one of the two connected components of the locus $\mathcal{L}(P)$ of vertices Z of triangles ABZ having the same area as ABP.*

Theorem 4.2 is illustrated in Figure 5.

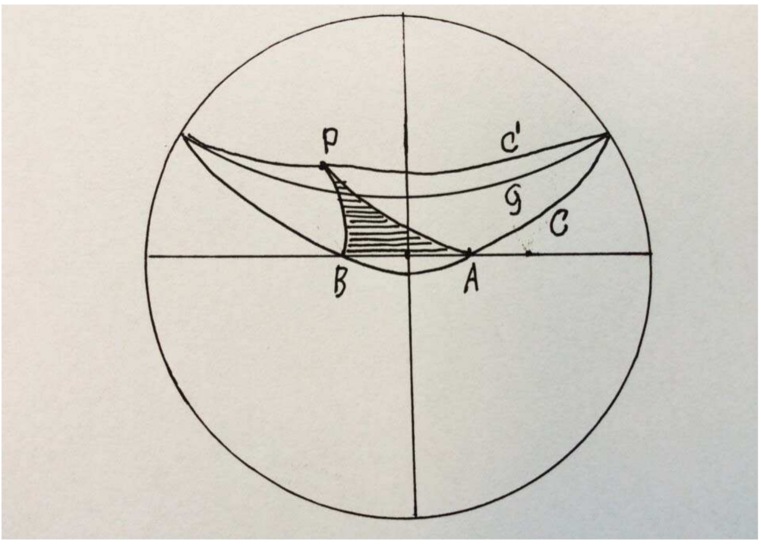

FIGURE 5. The hypercycle $\mathcal{C}$ and $\mathcal{C}'$. When the point P describes the hypercycle $\mathcal{C}'$, the area $\Delta(P)$ is a constant.

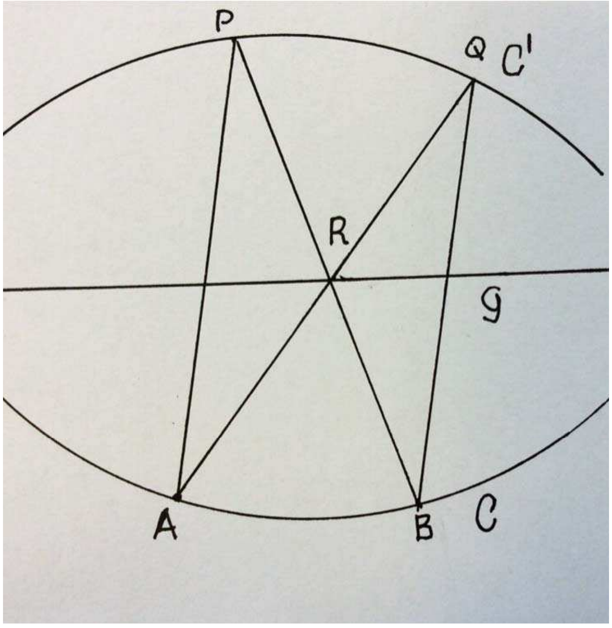

FIGURE 6. The triangle PRB and QRA have the same area.

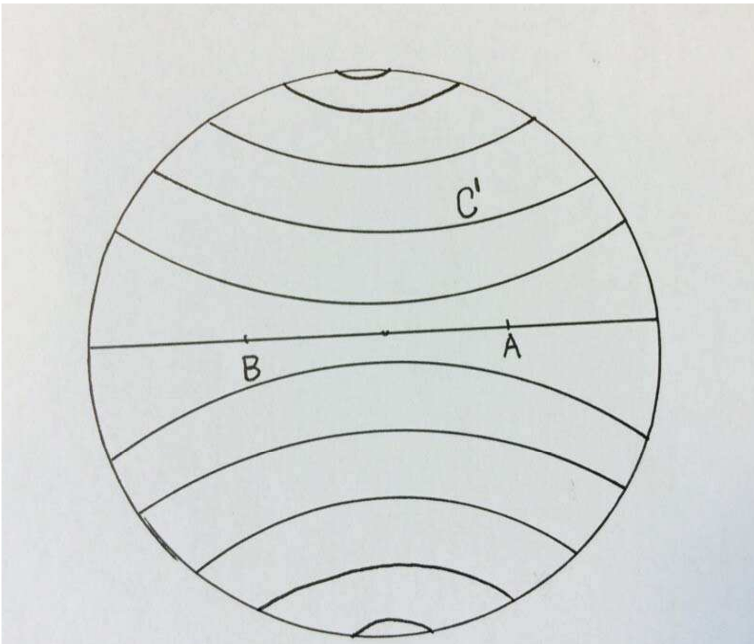

FIGURE 7. The foliation consists of leaves as locus of vertices with the same triangle area.

Proof of Theorem 4.2. Consider any hypercycle $\mathcal{C}$ passing through A, B and intersecting the imaginary axis perpendicularly. As we noticed before, there is a unique geodesic $\mathcal{G}$ equidistant to $\mathcal{C}$. There is another hypercycle $\mathcal{C}'$ symmetric to $\mathcal{C}$ with respect to $\mathcal{G}$.

It is a simple fact that any geodesic arc connecting a point in $\mathcal{C}$ and a point in $\mathcal{C}'$ is cut by $\mathcal{G}$ into two sub-arcs of the same hyperbolic length. In particular, as shown in Figure 6, for any two points P, Q on $\mathcal{C}'$, the geodesic arcs PA (and also QB, QA, PB) are separated by $\mathcal{G}$ into equal segments. It follows that the area of PRB is equal to the area of QRA. Here R denotes the intersection of PA and QB, which necessarily lies on $\mathcal{G}$. It is not hard to see that the triangles PAB and QAB have the same area, that is, $\Delta(P) = \Delta(Q)$. Since the points P, Q are arbitrarily chosen in $\mathcal{C}'$, we conclude that the area $\Delta(P)$ of the triangle PAB, with vertex P varying on $\mathcal{C}'$, is constant.

If the hypercycle $\mathcal{C}$ intersects the imaginary axis at a point which has distance y to the center of the unit disc, then we showed in §4.1 that $\Delta(P)$, for any $P \in \mathcal{C}'$, is equal to

$$2 \arccos \left(\frac{(\cosh x \cdot \cosh y - 1)(\cosh x + \cosh y)}{(\cosh x \cdot \cosh y)^2 - 1} \right).$$

Moreover, we showed that the area is a increasing function of y.

With the above description, we have a fairly clear picture of the locus of vertices with the same triangle area. For if we move the hypercycle $\mathcal{C}$ continuously in that disc, we get a family of hypercycles $\mathcal{C}'$. Such a family forms a foliation filling the unit disc. On each leaf, the area $\Delta(\cdot)$ is constant. On any two distinct leaves which are not symmetric with respect to the real axes, the areas are different. The theorem follows since any point $P \in \mathbb{H}$ lies on a unique leaf of the foliation, and the locus $\mathcal{L}(P)$ consists of two components, which are symmetric with respect to the real axes. □

Remark 4.3. A limit case is when both A and B are on the ideal boundary of the unit disc. In this case, it is easy to see that on any hypercycle $\mathcal{C}$ asymptotic to the geodesic AB, the area $\Delta(\cdot)$ is a constant.

In fact, given such a hypercyle and any point P on it, the perpendicular distance between P and AB is a constant c. the geodesic arc realizing the distance between P and AB divides the triangle PAB into two isometric right triangles, each of which has area

$$\frac{\pi}{2} - \arctan \left(\frac{1}{\sinh c} \right).$$

In this case, the foliation whose leaves are loci of vertices with the same triangle area consists of hypercycles asymptotic to the geodesic AB, see Figure 8. This foliation can be seen as a limit of the foliations constructed in the proof of Theorem 4.2.

4.3. **How to determine the hypercycle.** We conclude by setting up a construction of the hypercycle passing through P.

As illustrated in Figure 9, when P lies on the imaginary axis, we draw the geodesics from P to A and B. To determine the hypercycle through P, we only need to determine the midpoint of the geodesics PA and PB. There is a unique geodesic passing through these two midpoints, and it has two endpoints on the ideal boundary. The hypercycle with the same endpoints is the one we want.

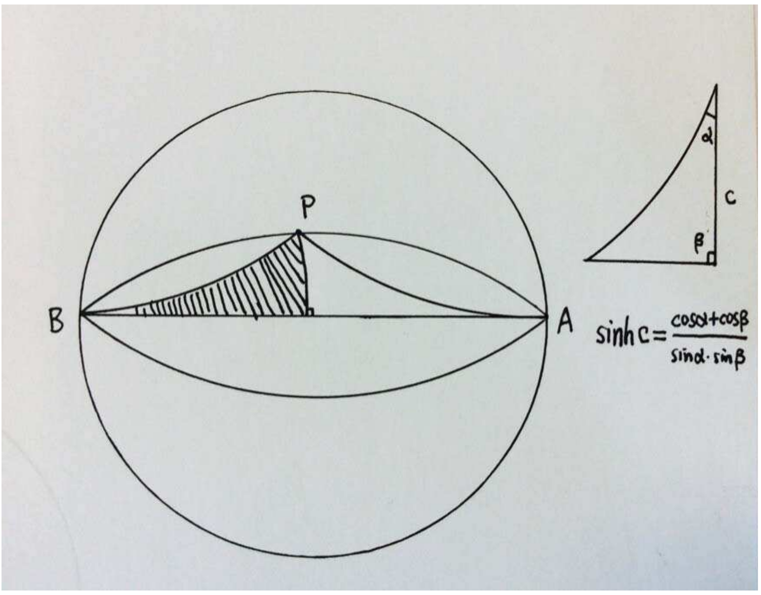

FIGURE 8. The limit case

When P does not lie on the imaginary axis, we take the point P' which is symmetric to P with respect to the imaginary axes. We draw the geodesics connecting P to A and P' to B. To determine the hypercycle through P, we only need to determine the midpoints of the geodesics PA and $P'B$. There is a unique geodesic passing through the two midpoints, with two endpoints on the ideal boundary. The hypercycle with the same endpoints is the one we want.

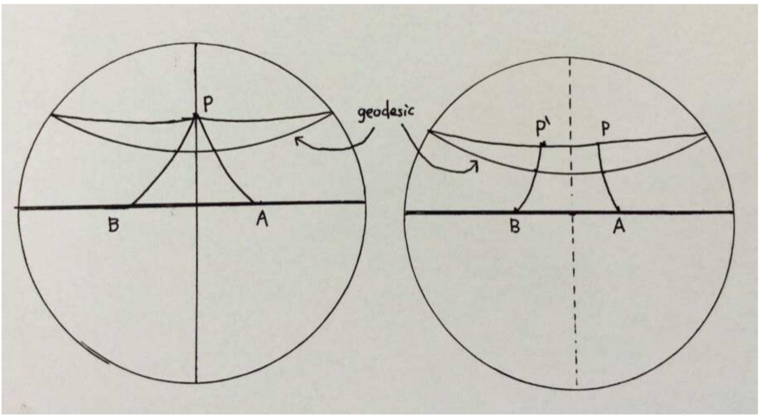

FIGURE 9. To find the hypercycle.

We then have the following complement to Theorem 4.2:

Proposition 4.4. *The midpoints of the variable triangles that are on a given side of the line joining A, B are all on a common line, and the locus of the vertices that we are seeking is a hypercycle with basis that line.*

Problem 4.5 (A'Campo)**.** Work out the three-dimensional analogue of Lexell's theorem, in the non-Euclidean cases.

In conclusion, and independently of the proper interest of the theorems presented, we hope that this paper can motivate the working mathematician to read the original sources.

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