HAL
open science

Improvement of the energy method for strongly non resonant dispersive equations and applications

Luc Molinet, Stéphane Vento

To cite this version:

Luc Molinet, Stéphane Vento. Improvement of the energy method for strongly non resonant dispersive equations and applications. Analysis & PDE, 2015, 8 (6), pp.1455-1495. hal-01064252

HAL Id: hal-01064252

https://hal.science/hal-01064252v1

Submitted on 15 Sep 2014

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

IMPROVEMENT OF THE ENERGY METHOD FOR STRONGLY NON RESONANT DISPERSIVE EQUATIONS AND APPLICATIONS

LUC MOLINET AND STÉPHANE VENTO

Abstract. In this paper we propose a new approach to prove the local well-posedness of the Cauchy problem associated with strongly non resonant dispersive equations. As an example we obtain unconditional well-posedness of the Cauchy problem below H^1 for a large class of one-dimensional dispersive equations with a dispersion that is greater or equal to the one of the Benjamin-Ono equation. Since this is done without using a gauge transform, this enables us to prove strong convergence results for solutions of viscous versions of these equations towards the purely dispersive solutions.

1. Introduction

The Cauchy problem associated with dispersive equations with derivative nonlinearity has been extensively studied since the eighties. The first results were obtained by using energy methods that did not make use of the dispersive effects (see for instance [14] and [1]). These methods were restricted to regular initial data ($s > d/2$ were $d \geq 1$ is the spatial dimension) and only ensured the continuity of the solution-map. At the end of the eighties, Kenig, Ponce and Vega proved new dispersive estimates that enable them to lower the regularity requirement on the initial data (see for instance [15], [16], [26]). They even succeed to obtain local well-posedness for a large class of dispersive equations by a fixed point argument in a suitable Banach space related to linear dispersive estimates. Then in the early nineties, Bourgain introduced the now so-called Bourgain's spaces where one can solve by a fixed point argument a wide class of dispersive equations with very rough initial data ([4], [5]). It is worth noticing that, since the nonlinearity of these equations is in general algebraic, the fixed point argument ensures the real analyticity of the solution-map. Now, in the early 2000's, Molinet, Saut and Tzvetkov [24] noticed that a large class of "weakly" dispersive equations, including in particular the Benjamin-Ono equation, cannot be solved by a fixed point argument for initial data in any Sobolev spaces H^s. This obstruction is due to bad interactions between high frequencies and very low frequencies. Since then, roughly speaking, two approaches have been developed to lower the regularity requirement for such equations. The first one is the so called gauge method. This consists in introducing a nonlinear gauge

transform of the solution that solved an equation with less bad interactions than the original one. This method was proved to be very efficient to obtain the lowest regularity index for solving canonical equations (see [28], [12], [6], [23] for the BO equation and [11] for dispersive generalized BO equation) but has the disadvantage to behave very bad with respect to perturbation of the equation. The second one consists in improving the dispersive estimates by localizing it on space frequency depending time intervals and then mixing it with classical energy estimates. This type of method was first introduced by Koch and Tzvetkov [19] (see also [17] for some improvements) in the framework of Strichartz's spaces and then by Koch and Tataru [18] (see also [13]) in the framework of Bourgain's spaces. It is less efficient to get the best regularity index but it is surely more flexible with respect to perturbation of the equation.

In this paper we propose a new approach to derive local and global well-posedness results for dispersive equations that do not exhibit too strong resonances. This approach combines classical energy estimates with Bourgain's type estimates on an interval of time that does not depend on the space frequency. Here, we will apply this method to prove unconditional local well-posedness results on both $\mathbb{R}$ and $\mathbb{T}$ without the use of a gauge transform for a large class of one-dimensional quadratic dispersive equations with a dispersion between the one of the Benjamin-Ono equation and the KdV equation. This class contains in particular the equations with pure power dispersion that read

$$u_t + \partial_x D_x^\alpha u + u u_x = 0 \tag{1.1}$$

with $\alpha \in [1,2]$.
The principle of the method is particularly simple in the regular case $s > 1/2$. We start with the classical space frequency localized energy estimate

$$\|P_N u\|^2_{L^\infty_T H^s} \lesssim \|P_N u_0\|^2_{H^s} + \sup_{t\in]0,T[} \langle N\rangle^{2s} \Big| \int_0^t \int_{\mathbb{R}} \partial_x P_N(u^2) P_N u \Big| \tag{1.2}$$

obtained by projecting the equation on frequencies of order N and taking the inner product with $J_x^s u$. Note that the second term in the RHS of (1.2) is easily controlled (after summing in N) by $\|u\|^3_{L^\infty_T H^s}$ for $s > 3/2$. This is the main point in the standard energy method that lead to LWP in H^s, $s > 3/2$. In order to take into account the dispersive effects of the equation, we will decompose the three factors in the integral term into dyadic pieces for the modulation variables and use the Bourgain's spaces $X^{s,b}$ in a non conventional way. Actually, it is known that standard bilinear estimates in $X^{s,b}$-spaces with $b = 1/2+$ fail, for equation (1.1), for any $s \in \mathbb{R}$ as soon as $\alpha < 2$. On the other hand, as noticed in [20], it is easy to deduce from the equation that a solution $u \in L^\infty(0,T;H^s)$ to (1.3) has to belong to the space $X_T^{s-1,1}$. This means that, if we accept to lose a few spatial derivatives on the solution, then we may gain some regularity in the modulation variable. This is particularly profitable when the equation enjoys a strongly non resonance

relation as (2.5). Actually, this formally allows to estimate the second term in (1.2) at the desired level. However, this term involves a multiplication by $1_{]0,t[}$ and it is well-known that such multiplication is not bounded in $X^{s-1,1}$. To overcome this difficulty we decompose this function into two parts. A high frequency part that will be very small in L^1_T and a low-frequency part that will have good properties with respect to multiplication with high modulation functions in $X^{s-1,1}$. This decomposition will depend on the space frequency localization of the three functions that appear in the trilinear term.

1.1. **Presentation of the results.** In this paper we consider the dispersive equation

$$u_t + L_{\alpha+1}u + uu_x = 0 \tag{1.3}$$

where $x \in \mathbb{R}$ or $\mathbb{T}$, $u = u(t,x)$ is a real-valued function and the linear operator $L_{\alpha+1}$satisfies the following hypothesis.

Hypothesis 1. *We assume that $L_{\alpha+1}$ is the Fourier multiplier operator by $ip_{\alpha+1}$ where $p_{\alpha+1}$ is a real-valued odd function satisfying :*
For any $(\xi_1,\xi_2) \in \mathbb{R}^2$ with $|\xi_1| \gg 1$ and any $0 < \lambda \ll 1$ it holds

$$\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \sim |\xi|_{\min}|\xi|^{\alpha}_{\max}\,, \tag{1.4}$$

where

$$\Omega(\xi_1,\xi_2) := p_{\alpha+1}(\xi_1+\xi_2) - p_{\alpha+1}(\xi_1) - p_{\alpha+1}(\xi_2)\,,$$

$|\xi|_{\min} := \min(|\xi_1|, |\xi_2|, |\xi_1+\xi_2|)$ *and* $|\xi|_{\max} := \max(|\xi_1|, |\xi_2|, |\xi_1+\xi_2|)$.

Remark 1.1. *We will see in Lemma 2.1 below that, for $\alpha > 0$, a very simple criterion on p ensures* (1.4)*. With this criterion in hand, it is not too hard to check that the following linear operators satisfy Hypothesis 1 :*

(1) The purely dispersive operators $L := \partial_x D_x^{\alpha}$ with $\alpha > 0$.

(2) The linear Intermediate Long Wave operator $L := \partial_x D_x \coth(D_x)$. Note that here $\alpha = 1$.

(3) Some perturbations of the Benjamin-Ono equation as the Smith operator $L := \partial_x(D_x^2+1)^{1/2}$ (see [27]*). Here again $\alpha = 1$.*

Theorem 1.1. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$, $L_{\alpha+1}$ satisfying Hypothesis 1 with $1 \le \alpha \le 2$ and let $s \ge 1 - \frac{\alpha}{2}$ with $(s,\alpha) \neq (\frac{1}{2}, 1)$. Then the Cauchy problem associated with* (1.3) *is unconditionally locally well-posed in $H^s(\mathbb{K})$ with a maximal time of existence $T \gtrsim (1 + \|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$.*

Remark 1.2. *In the regular case (Cauchy problem in H^s with $s > 1/2$), we actually need* (1.4) *only for $|\xi_1| \wedge |\xi_2| \gg 1$.*

Remark 1.3. *Our method also work in the case $\alpha > 2$. In this case we get the unconditional well-posedness in $H^s(\mathbb{K})$ for $s \ge 0$.*

Remark 1.4. *In the appendix we indicate the small modifications that enable to obtain the local-well posedness in the limit case* $(s,\alpha) = (1/2,1)$. *However, in this limit case we are not able to prove the unconditional uniqueness in* $L^\infty_T H^{1/2}$.

Remark 1.5. *For* $L_{\alpha+1} := \partial_x^3$, *we recover the unconditional LWP result in* $L^2(\mathbb{T})$ *obtained in* [2] *for the KdV equation. However, our Lipschitz bound on the solution-map holds at the level* $H^{-1/2}$ *whereas in* [2] *it holds at the level* H^{-1}. *Note that the* $L^2(\mathbb{R})$ *case was treated in* [29].

Let us assume now that the symbol $p_{\alpha+1}$ satisfies moreover

$$|p_{\alpha+1}(\xi)| \lesssim |\xi| \text{ for } |\xi| \leq 1 \text{ and } |p_{\alpha+1}(\xi)| \sim |\xi|^{\alpha+1} \text{ for } |\xi| \geq 1. \tag{1.5}$$

Then it is not too hard to check that equation (1.3) enjoys the following conservation laws:

$$\frac{d}{dt}\int_{\mathbb{K}} u^2 dx = 0,$$
$$\frac{d}{dt}\int_{\mathbb{K}} (|\Lambda^{\alpha/2}u|^2 + \frac{1}{3}u^3)dx = 0,$$

where $\Lambda^{\alpha/2}$ is the space Fourier multiplier defined by

$$\widehat{\Lambda^{\alpha/2}v}(\xi) = \left|\frac{p_{\alpha+1}(\xi)}{\xi}\right|^{1/2} \widehat{v}(\xi).$$

Combined with the embedding $H^{\alpha/2} \hookrightarrow L^3$, we get an a priori bound of the $H^{\alpha/2}$-norm of the solution. We may then iterate Theorem 1.1 and obtain the following corollary.

Corollary 1.1. *Let* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{T}$, $L_{\alpha+1}$ *satisfying Hypothesis 1 and (1.5) with* $1 < \alpha \leq 2$. *Then the Cauchy problem associated with* (1.3) *is unconditionally globally well-posed in* $H^{\alpha/2}(\mathbb{K})$.

Remark 1.6. *The linear operators given in Remark 1.1 also satisfy assumption (1.5).*

It is well-known that gauge transform often do not well behave with respect to perturbation of the equation. On the other hand it is well-known that, taking into account some damping or dissipative effects, dissipative versions of (1.3) can be derived (see for instance [25], [7]). One quite direct application of the fact that we do not need a gauge transform to solve (1.3), is that we can easily treat the dissipative limit of dissipative versions of (1.3). Such dissipative limit was for example studied for the Benjamin-Ono equation on the real line in [9] and [21].

Let us introduce the following dissipative version

$$u_t + L_{\alpha+1}u + \varepsilon A_\beta u + uu_x = 0 \tag{1.6}$$

where $\varepsilon > 0$ is a small parameter, $\beta \geq 0$ and A_β is a linear operator satisfying the following hypothesis :

Hypothesis 2. *We assume that A_β is the Fourier multiplier operator by q_β where q_β is a real-valued even function, bounded on bounded intervals, satisfying : For all $0 < \lambda \ll 1$ and $\xi \gg 1$,*

$$\lambda^\beta q_\beta(\lambda^{-1}\xi) \sim |\xi|^\beta .$$

Remark 1.7. *The homogeneous operators D_x^β and the non homogeneous operators J_x^β satisfy Hypothesis 2.*

Theorem 1.2. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$, $1 \le \alpha \le 2$, $0 \le \beta \le 1+\alpha$ and $s \ge 1 - \frac{\alpha}{2}$.*

(1) Then the Cauchy problem associated with (1.6) is locally well-posed in $H^s(\mathbb{K})$.

(2) For $u_0 \in H^s(\mathbb{K})$, let u be solution to (1.3) emanating from u_0. We call $T \gtrsim (1 + \|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$ the maximal time of existence of u in H^s. Then for $\varepsilon > 0$ small enough, the maximal time of existence T_ε of the solution u_ε to (1.6) emanating from u_0 satisfies $T_\varepsilon \ge T$. Moreover, $u_\varepsilon \to u$ in $C([0,T[; H^s)$ as $\varepsilon \to 0$.

Remark 1.8. *The constraint $\beta \le 1+\alpha$ is clearly an artefact of the method we used. We think that it could be dropped by replacing, in some estimates, the dispersive Bourgain's spaces by dispersive-dissipative Bourgain's spaces that were first introduced in [22]. But since the dissipative operators involved in wave motions are commonly of order less or equal to 2 we do not pursue this issue.*

The rest of the paper is organized as follows: in Section 2, we introduce the notations, define the function spaces and state some preliminary lemmas. In Section 3 we develop our method in the simplest case $s > 1/2$, while the non regular case is treated in Section 4. Section 5 is devoted to the proof of Theorem 1.2. We conclude the paper with an appendix explaining how to deal with the special case $(s, \alpha) = (1/2, 1)$.

2. Notations, function spaces and preliminary lemmas

2.1. **Notation.** For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \le cb$. We also denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, α_+, respectively α_-, will denote a number slightly greater, respectively lesser, than α.

For $u = u(x,t) \in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}u = \widehat{u}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = (u)^{\wedge_x}$, respectively $\mathcal{F}_t u = (u)^{\wedge_t}$, will denote its Fourier transform in space, respectively in time. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J_x^s and D_x^s, by

$$J_x^s u = \mathcal{F}_x^{-1}\big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_x u\big) \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}\big(|\xi|^s \mathcal{F}_x u\big).$$

Throughout the paper, we fix a smooth cutoff function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \le \eta \le 1, \quad \eta_{|_{[-1,1]}} = 1 \quad \text{and} \quad \text{supp}(\eta) \subset [-2,2].$$

We set $\phi(\xi) := \eta(\xi) - \eta(2\xi)$. For $l \in \mathbb{Z}$, we define

$$\phi_{2^l}(\xi) := \phi(2^{-l}\xi),$$

and, for $l \in \mathbb{N}^*$,

$$\psi_{2^l}(\xi,\tau) = \phi_{2^l}(\tau - p_{\alpha+1}(\xi)),$$

where $ip_{\alpha+1}$ is the Fourier symbol of $L_{\alpha+1}$. By convention, we also denote

$$\psi_0(\xi,\tau) := \eta(2(\tau - p_{\alpha+1}(\xi))).$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic with N, L, K or $M > 0$, *i.e.*, these variables range over numbers of the form $\{2^k : k \in \mathbb{Z}\}$. Then, we have that

$$\sum_N \phi_N(\xi) = 1, \quad \text{supp}\,(\phi_N) \subset \{\frac{N}{2} \leq |\xi| \leq 2N\},\ N \geq 1, \quad \text{and} \quad \text{supp}\,(\phi_0) \subset \{|\xi| \leq 1\}.$$

Let us define the Littlewood-Paley multipliers by

$$P_N u = \mathcal{F}_x^{-1}\big(\phi_N \mathcal{F}_x u\big), \quad Q_L u = \mathcal{F}^{-1}\big(\psi_L \mathcal{F} u\big),$$

$P_{\geq N} := \sum_{K \geq N} P_K$, $P_{\leq N} := \sum_{K \leq N} P_K$, $Q_{\geq L} := \sum_{K \geq L} Q_K$ and $Q_{\leq L} := \sum_{K \leq L} Q_K$. For brevity we also write $u_N = P_N u$, $u_{\leq N} = P_{\leq N} u$, ...

Let χ be a (possibly complex-valued) bounded function on $\mathbb{R}^2$ and define the pseudo-product operator $\Pi = \Pi_\chi$ on $\mathcal{S}(\mathbb{R})^2$ by

$$\mathcal{F}(\Pi(f,g))(\xi) = \int_{\mathbb{R}} \widehat{f}(\xi_1)\widehat{g}(\xi - \xi_1)\chi(\xi,\xi_1)d\xi_1.$$

Throughout the paper, we write $\Pi = \Pi_\chi$ where χ may be different at each occurrence of Π. This bilinear operator behaves like a product in the sense that it satisfies the following properties

$$\Pi(f,g) = fg \text{ if } \chi \equiv 1,$$

$$\int_{\mathbb{R}} \Pi(f,g)h = \int_{\mathbb{R}} f\Pi(g,h) = \int_{\mathbb{R}} \Pi(f,h)g \tag{2.1}$$

for any $f, g, h \in \mathcal{S}(\mathbb{R})$. Moreover, we easily check from Bernstein inequality that if $f_i \in L^2(\mathbb{R})$ has a Fourier transform localized in an annulus $\{|\xi| \sim N_i\}$, $i = 1, 2, 3$, then

$$\left|\int_{\mathbb{R}} \Pi(f_1, f_2)f_3\right| \lesssim N_{min}^{1/2} \prod_{i=1}^{3} \|f_i\|_{L^2}, \tag{2.2}$$

where the implicit constant only depends on $\|\chi\|_{L^\infty(\mathbb{R}^2)}$ and $N_{min} = \min\{N_1, N_2, N_3\}$. With this notation in hand, we will be able to systematically estimate terms of the form

$$\int_{\mathbb{R}} P_N(u^2)\partial_x P_N u$$

to put the derivative on the lowest frequency factor.

2.2. **Function spaces.** For $1 \leq p \leq \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$, and for $s \in \mathbb{R}$, the real-valued Sobolev spaces $H^s(\mathbb{R})$ denote the spaces of all real-valued functions with the usual norms

$$\|\phi\|_{H^s} = \|J_x^s \phi\|_{L^2} \ .$$

If B is one of the spaces defined above, $1 \leq p \leq \infty$, we will define the space-time spaces $L_t^p B$ and $\widetilde{L_t^p} B$ equipped with the norms

$$\|f\|_{L_t^p B} = \Big(\int_{\mathbb{R}} \|f(\cdot, t)\|_B^p dt \Big)^{\frac{1}{p}},$$

with obvious modifications for $p = \infty$, and

$$\|f\|_{\widetilde{L_t^p} B} = \Big(\sum_{N>0} \|P_N f\|_{L_t^p B}^2 \Big)^{\frac{1}{2}} \ .$$

For s, $b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ related to the linear part of (1.3) as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X^{s,b}} := \left(\int_{\mathbb{R}^2} \langle \tau - p_{\alpha+1}(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\widehat{v}(\xi, \tau)|^2 d\xi d\tau \right)^{\frac{1}{2}}, \tag{2.3}$$

where $\langle x \rangle := 1 + |x|$ and $ip_{\alpha+1}$ is Fourier symbol of $L_{\alpha+1}$. Recall that

$$\|v\|_{X^{s,b}} = \|U_\alpha(-t)v\|_{H_{t,x}^{s,b}}$$

where $U_\alpha(t) = \exp(tL_{\alpha+1})$ is the generator of the free evolution associated with (1.3).
Finally, we will use restriction in time versions of these spaces. Let $T > 0$ be a positive time and Y be a normed space of space-time functions. The restriction space Y_T will be the space of functions $v : \mathbb{R} \times]0, T[\to \mathbb{R}$ satisfying

$$\|v\|_{Y_T} := \inf\{\|\tilde{v}\|_Y \mid \tilde{v} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ \tilde{v}|_{\mathbb{R} \times]0,T[} = v\} < \infty \ .$$

2.3. **Preliminary lemmas.**

Lemma 2.1. *Let $p : \mathbb{R} \to \mathbb{R}$ be an odd function belonging to $C^1(\mathbb{R}) \cap C^2(\mathbb{R}^*)$ such that for all $|\xi| \gg 1$,*

$$|p'(\xi)| \sim |\xi|^\alpha \quad \text{and} \quad |p''(\xi)| \sim |\xi|^{\alpha-1}, \tag{2.4}$$

for some $\alpha > 0$. Then the Fourier multiplier $L_{\alpha+1}$ by $i\,p$ satisfies Hypothesis 1.

Proof. By symmetry we can assume $|\xi_2| \leq |\xi_1|$. We separate different cases:
1. $|\xi_2| \ll |\xi_1|$. Since $|\xi_1| \gg 1$, we can assume that (2.4) holds for any $\xi \geq |\xi_1|$ and thus there exists $\theta \in [\xi_1, \xi_1 + \xi_2]$ such that

$$\begin{aligned} \lambda^{\alpha+1} \Big| p(\lambda^{-1}(\xi_1 + \xi_2)) - p(\lambda^{-1}\xi_1) \Big| &= \lambda^\alpha |\xi_2| |p'(\lambda^{-1}\theta)| \\ &\sim \lambda^\alpha |\xi_2| |\lambda^{-1}\theta|^\alpha \\ &\sim |\xi_2| |\xi_1|^\alpha \end{aligned}$$

for all $0 \leq \lambda \ll 1$. On the other hand, if $\lambda^{-1}|\xi_2| \leq |\xi_1|$ then

$$\lambda^{\alpha+1}|p(\lambda^{-1}\xi_2)| \leq \lambda^{\alpha}|\xi_2| \max_{\xi\in[0,|\xi_1|]} |p'(\xi)| \ll |\xi_2||\xi_1|^{\alpha}$$

and if $\lambda^{-1}|\xi_2| \geq |\xi_1|$ then

$$\begin{aligned}\lambda^{\alpha+1}|p(\lambda^{-1}\xi_2)| &= \lambda^{\alpha+1}|p(\xi_1) + p(\lambda^{-1}\xi_2) - p(\xi_1)| \\ &\leq \lambda^{\alpha+1}\Big(|\xi_1| \max_{\xi\in[0,|\xi_1|]} |p'(\xi)| + \lambda^{-1}|\xi_2||\lambda^{-1}\xi_2|^{\alpha}\Big) \\ &\leq |\xi_2|^{\alpha+1} + \lambda^{\alpha}|\xi_2| \max_{\xi\in[0,|\xi_1|]} |p'(\xi)| \ll |\xi_2||\xi_1|^{\alpha}\end{aligned}$$

Gathering these two estimates leads for $0 < \lambda \ll 1$ to

$$\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \sim |\xi_2||\xi_1|^{\alpha}\,.$$

2. $|\xi_2| \gtrsim |\xi_1|$. In this case we can assume that (2.4) holds for any $\xi \geq |\xi_2|$.
2.1. $\xi_1.\xi_2 \geq 0$. Then we have $0 < \xi_2 \leq \xi_1 < \xi_1 + \xi_2$. We notice that

$$\begin{aligned}\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| =& \lambda^{\alpha}\int_{\lambda\xi_2}^{\xi_2} (p'(\lambda^{-1}(\xi_1+\theta)) - p'(\lambda^{-1}\theta))\,d\theta \\ &+ \lambda^{\alpha+1}\Big(p(\lambda^{-1}\xi_1+\xi_2) - p(\lambda^{-1}\xi_1)\Big) - \lambda^{\alpha+1}p(\xi_2)\,.\end{aligned}$$

with

$$|p(\lambda^{-1}\xi_1+\xi_2) - p(\lambda^{-1}\xi_1)| \lesssim \xi_2\lambda^{-\alpha}\xi_1^{\alpha} \ll \lambda^{-\alpha-1}\xi_2\xi_1^{\alpha}$$

and

$$p'(\lambda^{-1}(\xi_1+\theta)) - p'(\lambda^{-1}\theta) = \lambda^{-1}\int_0^{\xi_1} p''(\lambda^{-1}(\theta+\mu))\,d\mu.$$

But for $\xi \geq \xi_2$, p'' does not change sign since $|p''(\xi)| \sim |\xi|^{\alpha-1}$ and p'' is continuous outside 0. Therefore,

$$\begin{aligned}\lambda^{-1}\int_0^{\xi_1} p''(\lambda^{-1}(\theta+\mu))\,d\mu &\sim \lambda^{-1}\int_0^{\xi_1} (\lambda^{-1}(\theta+\mu))^{\alpha-1}\,d\mu \\ &\sim \lambda^{-\alpha}\Big((\xi_1+\theta)^{\alpha} - \theta^{\alpha}\Big) \\ &\sim \lambda^{-\alpha}\xi_1^{\alpha}\end{aligned}$$

Gathering these estimates we obtain

$$\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \sim \xi_2\xi_1^{\alpha}\,.$$

2.2. $\xi_1.\xi_2 < 0$. We can assume that $\xi_1 > 0$. Then we have $0 < \xi_1 + \xi_2 < -\xi_2 \leq \xi_1$. For $\xi_1 + \xi_2 \ll -\xi_2$, recalling that p is an odd function, we can argue exactly as in the case **1.** but with $\xi_1 + \xi_2$, $-\xi_2$ and ξ_1 playing the role of respectively ξ_2, ξ_1 and $\xi_1 + \xi_2$. Finally, for $\xi_1 + \xi_2 \gtrsim -\xi_2$, we argue exactly as in the case **2.1** with the same exchange of roles than above. □

Lemma 2.2. *Assume that $p_{\alpha+1}$ satisfies* (1.4) *with $\lambda = 1$. Let $L_1, L_2, L_3 > 0$, $0 < N_1 \leq N_2 \leq N_3$ be dyadic numbers and $u, v, w \in \mathcal{S}'(\mathbb{R}^2)$. Then*

$$\int_{\mathbb{R}^2} \Pi(Q_{L_1} P_{N_1} u,\, Q_{L_2} P_{N_2} v)\, Q_{L_3} P_{N_3} w = 0$$

whenever the following relation is not satisfied :

$$\max(L_1, L_2, L_3) \sim \max(N_1 N_2^{\alpha}, L_{med})\ , \tag{2.5}$$

where $L_{med} = \max(\{L_1, L_2, L_3\} - \{L_{\max}\})$.

Proof. This is a direct consequence of the hypothesis (1.4) on the resonance function $\Omega(\xi_1, \xi_2)$ since

$$\Omega(\xi_1, \xi_2) = \sigma(\tau_1 + \tau_2, \xi_1 + \xi_2) - \sigma(\tau_1, \xi_1) - \sigma(\tau_2, \xi_2)$$

with $\sigma(\tau, \xi) := \tau - p_{\alpha+1}(\xi)$. □

Lemma 2.3. *Let $L > 0$, $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. The operator $Q_{\leq L}$ is bounded in $L^p_t H^s$ uniformly in $L > 0$.*

Proof. Let $R_{\leq L}$ be the Fourier multiplier by $\eta_L(\tau)$ where η_L is defined in Section 2.1. The trick is to notice that $Q_{\leq L} u = U_\alpha(t)(R_{\leq L} U_\alpha(-t) u)$. Therefore, using the unitarity of $U_\alpha(\cdot)$ in $H^s(\mathbb{R})$ we infer that

$$\begin{aligned} \|Q_{\leq L} u\|_{L^p_t H^s} &= \|U_\alpha(t)(R_{\leq L} U_\alpha(-t) u)\|_{L^p_t H^s} = \|R_{\leq L} U_\alpha(-t) u\|_{L^p_t H^s} \\ &\lesssim \|U_\alpha(-t) u\|_{L^p_t H^s} = \|u\|_{L^p_t H^s}\ . \end{aligned}$$

□

For any $T > 0$, we consider 1_T the characteristic function of $[0, T]$ and use the decomposition

$$1_T = 1^{low}_{T,R} + 1^{high}_{T,R}, \quad \widehat{1^{low}_{T,R}}(\tau) = \eta(\tau/R)\widehat{1_T}(\tau) \tag{2.6}$$

for some $R > 0$.
The properties of this decomposition we will need are listed in the following lemmas.

Lemma 2.4. *For any $R > 0$ and $T > 0$ it holds*

$$\|1^{high}_{T,R}\|_{L^1} \lesssim T \wedge R^{-1}. \tag{2.7}$$

and

$$\|1^{low}_{T,R}\|_{L^\infty} \lesssim 1. \tag{2.8}$$

Proof. A direct computation provide

$$\begin{aligned}\|1_{T,R}^{high}\|_{L^1} &= \int_{\mathbb{R}} \left| \int_{\mathbb{R}} (1_T(t) - 1_T(t-s/R))\mathcal{F}^{-1}\eta(s)ds \right| dt \\ &\leq \int_{\mathbb{R}} \int_{([0,T]\setminus[s/R,T+s/R])\cup([s/R,T+s/R]\setminus[0,T])} |\mathcal{F}^{-1}\eta(s)|dtds \\ &\lesssim \int_{\mathbb{R}} (T \wedge |s|/R)|\mathcal{F}^{-1}\eta(s)|ds \\ &\lesssim T \wedge R^{-1}.\end{aligned}$$

Finally, the proof of (2.8) is obvious. □

Lemma 2.5. *Let $u \in L^2(\mathbb{R}^2)$. Then for any $T > 0$, $R > 0$ and $L \gg R$ it holds*

$$\|Q_L(1_{T,R}^{low}u)\|_{L^2} \lesssim \|Q_{\sim L}u\|_{L^2}$$

Proof. By Plancherel we get

$$\begin{aligned}I_L &= \|Q_L(1_{T,R}^{low}u)\|_{L^2} \\ &= \|\varphi_L(\tau - \omega(\xi))\widehat{1_{T,R}^{low}} *_\tau \widehat{u}(\tau,\xi)\|_{L^2} \\ &= \left\| \sum_{L_1} \varphi_L(\tau - \omega(\xi)) \int_{\mathbb{R}} \varphi_{L_1}(\tau' - \omega(\xi))\widehat{u}(\tau',\xi)\eta((\tau-\tau')/R)\frac{e^{-iT(\tau-\tau')}-1}{\tau-\tau'}d\tau' \right\|_{L^2}.\end{aligned}$$

In the region where $L_1 \ll L$ or $L_1 \gg L$, we have $|\tau - \tau'| \sim L \vee L_1 \gg R$, thus I_L vanishes. On the other hand, for $L \sim L_1$, we get

$$I_L \lesssim \sum_{L\sim L_1} \|Q_L(1_{T,R}^{low}Q_{L_1}u)\|_{L^2} \lesssim \|Q_{\sim L}u\|_{L^2}.$$

□

3. Unconditional well-posedness in the regular case $s > 1/2$

In this section we develop our method in the regular case $s > 1/2$. This will emphasize the simplicity of this approach to prove unconditional well-posedness for equation (1.3) posed on $\mathbb{R}$ or $\mathbb{T}$.

Let $\lambda > 0$ and $L_{\alpha+1}^\lambda$ be the Fourier multiplier by $i\lambda^{\alpha+1}p_{\alpha+1}(\lambda^{-1}\cdot)$. We notice that if u is solution to (1.3) on $]0,T[$ then $u_\lambda(t,x) = \lambda^\alpha u(\lambda^{\alpha+1}t, \lambda x)$ is solution to (1.3) on $]0,\lambda^{-(\alpha+1)}T[$ with $L_{\alpha+1}$ replaced by $L_{\alpha+1}^\lambda$. Therefore, up to this change of unknown and equation, we can always assume that the operator $L_{\alpha+1}$ verifies (1.4) with $0 < \lambda \leq 1$.

3.1. A priori estimates.

For $s \in \mathbb{R}$ we define the function space M^s as $M^s := L_t^\infty H^s \cap X^{s-1,1}$, endowed with the natural norm

$$\|u\|_{M^s} = \|u\|_{L_t^\infty H^s} + \|u\|_{X^{s-1,1}} .$$

For $u_0 \in H^s(\mathbb{R})$, $s > 1/2$, we will construct a solution to (1.3) in M_T^s, whereas the difference of two solutions emanating from initial data belonging to $H^s(\mathbb{R})$ will take place in M_T^{s-1}.

Lemma 3.1. *Let $0 < T < 2$, $s > 1/2$ and $u \in L_T^\infty H^s$ be a solution to* (1.3)*. Then $u \in M_T^s$ and it holds*

$$\|u\|_{M_T^s} \lesssim \|u\|_{L_T^\infty H^s} + \|u\|_{L_T^\infty H^s} \|u\|_{L_T^\infty H^{\frac{1}{2}+}} \ . \tag{3.1}$$

Moreover, for any couple $(v, w) \in (L_T^\infty H^s)^2$ of solutions to (1.3)*, it holds*

$$\|u - v\|_{M_T^{s-1}} \lesssim \|u - v\|_{L_T^\infty H^{s-1}} + \|u + v\|_{L_T^\infty H^s} \|u - v\|_{L_T^\infty H^{s-1}} \ . \tag{3.2}$$

Proof. We have to extend the function u from $(0, T)$ to $\mathbb{R}$. For this we follow [20] and introduce the extension operator ρ_T defined by

$$\rho_T u(t) := \eta(t) u(T\mu(t/T)) \ , \tag{3.3}$$

where η is the smooth cut-off function defined in Section 2.1 and $\mu(t) = \max(1 - |t - 1|, 0)$. ρ_T is a bounded operator from $X_T^{\theta,b}$ into $X^{\theta,b}$ and from $L^p(0, T; X)$ into $L^p(\mathbb{R}; X)$ for any $b \in]-\infty, 1]$, $s \in \mathbb{R}$, $p \in [1, \infty]$ and any Banach space X. Moreover, these bounds are uniform for $0 < T < 1$.
By using this extension operator, it is clear that we only have to estimate the $X_T^{s-1,1}$-norm of u to prove (3.1). But by the Duhamel formula of (1.3) and standard linear estimates in Bourgain's spaces, we have

$$\begin{aligned} \|u\|_{X_T^{s-1,1}} &\lesssim \|u_0\|_{H^{s-1}} + \|\partial_x(u^2)\|_{X_T^{s-1,0}} \\ &\lesssim \|u_0\|_{H^{s-1}} + \|u^2\|_{L_T^2 H^s} \\ &\lesssim \|u_0\|_{H^{s-1}} + \|u\|_{L_T^\infty H^{\frac{1}{2}+}} \|u\|_{L_T^\infty H^s} \ , \end{aligned}$$

by standard product estimates in Sobolev spaces.
In the same way we have

$$\begin{aligned} \|u - v\|_{X_T^{s-2,1}} &\lesssim \|u_0\|_{H^{s-2}} + \|(u + v)(u - v)\|_{L_T^2 H^{s-1}} \\ &\lesssim \|u_0\|_{H^{s-2}} + \|u + v\|_{L_T^\infty H^s} \|u - v\|_{L_T^\infty H^{s-1}} \ , \end{aligned}$$

which proves (3.2) □

Lemma 3.2. *Assume $u_i \in M^0$, $i = 1, 2, 3$ are functions with spatial Fourier support in $\{|\xi| \sim N_i\}$ with $N_i > 0$ dyadic satisfying $N_1 \leq N_2 \leq N_3$. For any $t > 0$ we set*

$$I_t(u_1, u_2, u_3) = \int_0^t \int_{\mathbb{R}} \Pi(u_1, u_2) u_3.$$

If $N_1 \leq 2^9$ it holds

$$|I_t(u_1, u_2, u_3)| \lesssim N_1^{1/2} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_{tx}^2} \|u_3\|_{L_{tx}^2} . \tag{3.4}$$

In the case $N_1 > 2^9$ it holds

$$\begin{aligned}|I_t(u_1,u_2,u_3)| \lesssim & N_1^{-1/2}N_3^{1-\alpha}\|u_1\|_{L^\infty_t L^2_x}(\|u_2\|_{L^2_{tx}}\|u_3\|_{X^{-1,1}} + \|u_2\|_{X^{-1,1}}\|u_3\|_{L^2_{tx}}) \\ & + N_1^{1/2}N_3^{-\alpha}\|u_1\|_{X^{-1,1}}\|u_2\|_{L^2_{tx}}\|u_3\|_{L^\infty_t L^2_x} \\ & + N_1^{-1}N_3^{-1/8}\|u_1\|_{L^\infty_t L^2_x}\|u_2\|_{L^\infty_t L^2_x}\|u_3\|_{L^\infty_t L^2_x}.\end{aligned}$$

Proof. Estimate (3.4) easily follows from (2.2) together with Hölder inequality, thus it suffices to estimate $|I_t|$ for $N_1 > 2^9$. Note that I_t vanishes unless $N_2 \sim N_3$. Setting $R = N_1^{3/2}N_3^{1/8}$, we split I_t as

$$\begin{aligned} I_t(u_1,u_2,u_3) &= I_\infty(1^{high}_{t,R}u_1,u_2,u_3) + I_\infty(1^{low}_{t,R}u_1,u_2,u_3) \\ &:= I_t^{high} + I_t^{low}, \end{aligned} \tag{3.5}$$

where $I_\infty(u,v,w) = \int_{\mathbb{R}^2}\Pi(u,v)w$. The contribution of I_t^{high} is estimated thanks to Lemma 2.4 as well as (2.2) and Hölder inequality by

$$\begin{aligned} I_t^{high} &\lesssim N_1^{1/2}\|1^{high}_{t,R}\|_{L^1}\|u_1\|_{L^\infty_t L^2_x}\|u_2\|_{L^\infty_t L^2_x}\|u_3\|_{L^\infty_t L^2_x} \\ &\lesssim N_1^{-1}N_3^{-1/8}\|u_1\|_{L^\infty_t L^2_x}\|u_2\|_{L^\infty_t L^2_x}\|u_3\|_{L^\infty_t L^2_x} \end{aligned} \tag{3.6}$$

To evaluate the contribution I_t^{low} we use that, according to Lemma 2.2, we have the following decomposition:

$$\begin{aligned} I_\infty(1^{low}_{t,R}\,u_1,u_2,u_3) =& I_\infty(Q_{>2^{-4}N_1N_3^\alpha}(1^{low}_{t,R}u_1),u_2,u_3) \\ &+ I_\infty(Q_{\le 2^{-4}N_1N_2^\alpha}(1^{low}_{t,R}u_1),Q_{>2^{-4}N_1N_3^\alpha}u_2,u_3) \\ &+ I_\infty(Q_{\le 2^{-4}N_1N_2^\alpha}(1^{low}_{t,R}u_1),Q_{\le 2^{-4}N_1N_3^\alpha}u_2,Q_{\sim N_1N_3^\alpha}u_3)\ . \end{aligned} \tag{3.7}$$

It is worth noticing that since $N_1 \ge 2^9$, $R \ll 2^{-4}N_1N_3^\alpha$. Therefore the contribution $I_t^{1,low}$ of the first term of the above RHS to I_t^{low} is easily estimated thanks to Lemma 2.5 by

$$\begin{aligned} I_t^{1,low} \lesssim & N_1^{1/2}(N_1N_3^\alpha)^{-1}\|u_1\|_{X^{0,1}}\|u_2\|_{L^2_{tx}}\|u_3\|_{L^\infty_t L^2_x} \\ \lesssim & N_1^{1/2}N_3^{-\alpha}\|u_1\|_{X^{-1,1}}\|u_2\|_{L^2_{tx}}\|u_3\|_{L^\infty_t L^2_x}\ . \end{aligned} \tag{3.8}$$

Thanks to Lemmas 2.3 and 2.5, the contribution $I_t^{2,low}$ of the second term can be handle in the following way

$$\begin{aligned} I_t^{2,low} \lesssim & N_1^{1/2}(N_1N_3^\alpha)^{-1}\|u_1\|_{L^\infty_t L^2_x}\|u_2\|_{X^{0,1}}\|u_3\|_{L^\infty_t L^2_x} \\ \lesssim & N_1^{-1/2}N_3^{1-\alpha}\|u_1\|_{L^\infty_t L^2_x}\|u_2\|_{X^{-1,1}}\|u_3\|_{L^2_{tx}}\ . \end{aligned} \tag{3.9}$$

Finally the contribution of the third term is estimated in the same way. □

Remark 3.1. *From (2.1) we see that estimates in Lemma 3.2 also hold for any other rearrangements of N_1, N_2, N_3.*

We are now in position to derive our "improved" energy estimate on smooth solutions to (1.3).

Proposition 3.1. *Let $0 < T < 2$ and $u \in L^\infty_T H^s$ with $s > 1/2$ be a solution to* (1.3)*. Then it holds*

$$\|u\|_{L^\infty_T H^s} \lesssim \|u_0\|_{H^s} + (1 + \|u\|^2_{L^\infty_T H^{\frac{1}{2}+}})\|u\|_{L^\infty_T H^{\frac{1}{2}+}} \|u\|_{L^\infty_T H^s} \ . \tag{3.10}$$

Proof. Applying the operator P_N with $N > 0$ dyadic to equation (1.3), taking the H^s scalar product with $P_N u$ and integrating on $]0, t[$ we obtain

$$\|P_N u\|^2_{L^\infty_T H^s} \lesssim \|P_N u_0\|^2_{H^s} + \sup_{t\in]0,T[} \langle N\rangle^{2s} \Big| \int_0^t \int_{\mathbb{R}} P_N(u^2)\partial_x P_N u \Big| \tag{3.11}$$

Thus it remains to estimate

$$I := \sum_{N>0} \langle N\rangle^{2s} \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} P_N(u^2)\partial_x P_N u \Big| \ . \tag{3.12}$$

We take an extension $\tilde{u}$ of u supported in time in $]-2, 2[$ such that $\|\tilde{u}\|_{M^s} \lesssim \|u\|_{M^s_T}$. To simplify the notation we drop the tilde in the sequel.
By localization considerations, we get

$$P_N(u^2) = P_N(u_{\gtrsim N} u_{\gtrsim N}) + 2P_N(u_{\ll N} u). \tag{3.13}$$

Moreover, using a Taylor expansion of ϕ_N, we easily get

$$P_N(u_{\ll N} u) = u_{\ll N} P_N u + N^{-1}\Pi(\partial_x u_{\ll N}, u), \tag{3.14}$$

where $\Pi = \Pi_\chi$ with $\chi(\xi, \xi_1) = -i \int_0^1 \phi'(N^{-1}(\xi - \theta\xi_1))d\theta \in L^\infty$. Inserting (3.13)-(3.14) into (3.12) and integrating by parts we deduce

$$\begin{aligned} I \lesssim & \sum_{N>0} \sum_{N_1 \ll N} N_1 \langle N\rangle^{2(s-1)} \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_1}(u_{N_1}, u_N) u_N \Big| \\ & + \sum_{N>0} \sum_{N_1 \ll N} N_1 \langle N\rangle^{2(s-1)} \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_2}(u_{N_1}, u_{\sim N}) u_N \Big| \\ & + \sum_{N>0} \sum_{N_1 \gtrsim N} N \langle N\rangle^{2(s-1)} \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_3}(u_{N_1}, u_{\sim N_1}) u_N \Big| \end{aligned}$$

where χ_i, $1 \le i \le 3$ are bounded uniformly in N, N_1 and defined by

$$\chi_1(\xi, \xi_1) = \frac{\xi_1}{N_1} 1_{\operatorname{supp}\phi_{N_1}}(\xi_1), \tag{3.15}$$

$$\chi_2(\xi, \xi_1) = \chi(\xi, \xi_1) \frac{\xi_1}{N_1} \frac{\xi}{N} \frac{1_{\operatorname{supp}\phi_N}(\xi) 1_{\operatorname{supp}\phi_{N_1}}(\xi_1)}{\phi_{\sim N}(\xi - \xi_1)} \tag{3.16}$$

$$\chi_3(\xi, \xi_1) = \frac{\xi}{N} \phi_N(\xi). \tag{3.17}$$

Recalling now the definition of I_t (see Lemma 3.2), it follows that

$$I \lesssim \sum_{N>0} \sum_{N_1 \gtrsim N} N \langle N_1\rangle^{2s} \sup_{t\in]0,T[} |I_t(u_N, u_{\sim N_1}, u_{N_1})|. \tag{3.18}$$

The contribution of the sum over $N \leq 2^9$ is easily estimated thanks to (3.4) and Cauchy-Schwarz by

$$\sum_{N\leq 2^9}\sum_{N_1\gtrsim N} N\langle N_1\rangle^{2s}\,\|u_N\|_{L^\infty_t L^2_x}\|u_{N_1}\|^2_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x}\|u\|^2_{L^\infty_t H^s} \tag{3.19}$$

Finally the contribution of the sum over $N > 2^9$ is controlled with the second part of Lemma 3.2 by

$$\begin{aligned}\sum_{N>2^9}\sum_{N_1\gtrsim N} NN_1^{2s}\Big[&N^{-1/2}N_1^{1-\alpha}\|u_N\|_{L^\infty_t L^2_x}\|u_{N_1}\|_{L^2_{tx}}\|u_{N_1}\|_{X^{-1,1}}\\ &+N^{1/2}N_1^{-\alpha}\|u_N\|_{X^{-1,1}}\|u_{N_1}\|^2_{L^\infty_t L^2_x}\\ &+N^{-1}N_1^{-1/8}\|u_N\|_{L^\infty_t L^2_x}\|u_{N_1}\|^2_{L^\infty_t L^2_x}\Big]\\ \lesssim\ &\|u\|_{M^{\frac12+}_T}\|u\|_{M^s_T}\|u\|_{L^\infty_T H^s}.\end{aligned} \tag{3.20}$$

Gathering all the above estimates leads to

$$\|u\|^2_{L^\infty_T H^s} \lesssim \|u_0\|^2_{H^s} + \|u\|_{M^{\frac12+}_T}\|u\|_{M^s_T}\|u\|_{L^\infty_T H^s} \tag{3.21}$$

which, together with (3.1) completes the proof of the proposition. □

Let us now establish an a priori estimate at the regularity level $s-1$ on the difference of two solutions.

Proposition 3.2. *Let $0 < T < 2$ and $u, v \in L^\infty_T H^s$ with $s > 1/2$ be two solutions to* (1.3)*. Then it holds*

$$\|u-v\|_{L^\infty_T H^{s-1}} \lesssim \|u_0 - v_0\|_{H^{s-1}} + \|u+v\|_{M^s_T}\|u-v\|_{M^{s-1}_T}\ . \tag{3.22}$$

Proof. The difference $w = u - v$ satisfies

$$w_t + D^\alpha w_x = \partial_x(zw), \tag{3.23}$$

where $z = u + v$. Proceeding as in the proof of the preceding proposition, we infer that for $N > 0$,

$$\|P_N w\|^2_{L^\infty_T H^{s-1}} \lesssim \|P_N w_0\|^2_{H^{s-1}} + \sup_{t\in]0,T[}\langle N\rangle^{2(s-1)}\Big|\int_0^t\int_{\mathbb{R}} P_N(zw)\partial_x P_N w\Big| \tag{3.24}$$

We take extensions $\tilde z$ and $\tilde w$ of z and w supported in time in $]-2,2[$ such that $\|\tilde z\|_{M^s} \lesssim \|u\|_{M^s_T}$ and $\|\tilde w\|_{M^{s-1}} \lesssim \|u\|_{M^{s-1}_T}$. To simplify the notation we drop the tilde in the sequel.
Setting

$$J := \sum_{N>0}\langle N\rangle^{2(s-1)}\sup_{t\in]0,T[}\Big|\int_0^t\int_{\mathbb{R}} P_N(zw)\partial_x P_N w\Big| \tag{3.25}$$

it follows from (3.14) and classical dyadic decomposition that for all $N > 0$,

$$
\begin{aligned}
P_N(zw) &= P_N(z_{\ll N} w) + P_N(z_{\sim N} w_{\lesssim N}) + \sum_{N_1 \gg N} P_N(z_{N_1} w_{\sim N_1}) \\
&= z_{\ll N} w_N + N^{-1} \Pi_\chi(\partial_x z_{\ll N}, w) + P_N(z_{\sim N} w_{\lesssim N}) + \sum_{N_1 \gg N} P_N(z_{N_1} w_{\sim N_1}).
\end{aligned} \tag{3.26}
$$

Inserting this into (3.25) and integrating by parts we infer

$$
\begin{aligned}
J \lesssim & \sum_{N>0} \sum_{N_1 \ll N} N_1 \langle N \rangle^{2(s-1)} \sup_{t \in]0,T[} \left| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_1}(z_{N_1}, w_N) w_N \right| \\
& + \sum_{N>0} \sum_{N_1 \ll N} N_1 \langle N \rangle^{2(s-1)} \sup_{t \in]0,T[} \left| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_2}(z_{N_1}, w_{\sim N}) w_N \right| \\
& + \sum_{N>0} \sum_{N_1 \lesssim N} N \langle N \rangle^{2(s-1)} \sup_{t \in]0,T[} \left| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_3}(z_{\sim N}, w_{N_1}) w_N \right| \\
& + \sum_{N>0} \sum_{N_1 \gg N} N \langle N \rangle^{2(s-1)} \sup_{t \in]0,T[} \left| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_3}(z_{N_1}, w_{\sim N_1}) w_N \right|
\end{aligned}
$$

where χ_i, $1 \leq i \leq 3$ are defined in (3.15)-(3.16)-(3.17). Therefore it suffices to estimate

$$
\begin{aligned}
J \lesssim & \sum_{N>0} \sum_{N_1 \gtrsim N} N \langle N_1 \rangle^{2(s-1)} \sup_{t \in]0,T[} |I_t(z_N, w_{\sim N_1}, w_{N_1})| \\
& + \sum_{N>0} \sum_{N_1 \gtrsim N} N_1 \langle N_1 \rangle^{2(s-1)} \sup_{t \in]0,T[} |I_t(z_{\sim N_1}, w_N, w_{N_1})| \\
& + \sum_{N>0} \sum_{N_1 \gtrsim N} N \langle N \rangle^{2(s-1)} \sup_{t \in]0,T[} |I_t(z_{N_1}, w_{N_1}, w_N)| \\
:= & \; J_1 + J_2 + J_3.
\end{aligned} \tag{3.27}
$$

The contribution of the sum over $N \leq 2^9$ in (3.27) is easily estimated thanks to (3.4) by

$$
\begin{aligned}
\sum_{N \leq 2^9} \sum_{N_1 \gtrsim N} & N^{1/2} \Big(N \, \|z_N\|_{L^\infty_t L^2_x} \|w_{N_1}\|^2_{L^2_t H^{s-1}} \\
& + N_1 \langle N_1 \rangle^{-1} \|z_{N_1}\|_{L^2_t H^s} \|w_N\|_{L^\infty_t L^2_x} \|w_{N_1}\|_{L^2_t H^{s-1}} \\
& + N \langle N_1 \rangle^{1-2s} \|z_{N_1}\|_{L^2_t H^s} \|w_{N_1}\|_{L^2_t H^{s-1}} \|w_N\|_{L^\infty_t L^2_x} \Big) \\
\lesssim & \; \|z\|_{L^\infty_t L^2_x} \|w\|^2_{L^\infty_t H^{s-1}} + \|w\|_{L^\infty_t H^{-\frac{1}{2}}_x} \|z\|_{L^\infty_t H^s} \|w\|_{L^\infty_t H^{s-1}}
\end{aligned} \tag{3.28}
$$

For the contribution of the sum over $N > 2^9$, it is worth noticing that since $s > 1/2$, the term J_3 is controlled by J_2. The contribution of J_1 is estimated

thanks to Lemma 3.2 by

$$\begin{aligned}\sum_{N>2^9}\sum_{N_1\gtrsim N} NN_1^{2(s-1)}\Big[&N^{-1/2}N_1^{1-\alpha}\|z_N\|_{L^\infty_t L^2_x}\|w_{N_1}\|_{L^2_{tx}}\|w_{N_1}\|_{X^{-1,1}}\\ &+N^{1/2}N_1^{-\alpha}\|z_N\|_{X^{-1,1}}\|w_{N_1}\|^2_{L^\infty_t L^2_x}\\ &+N^{-1}N_1^{-1/8}\|z_N\|_{L^\infty_t L^2_x}\|w_{N_1}\|^2_{L^\infty_t L^2_x}\Big]\\ &\lesssim \|z\|_{M^{1/2+}}\|w\|_{M^{s-1}}\|w\|_{L^\infty_t H^{s-1}}.\end{aligned} \tag{3.29}$$

Finally, we bound in the same way J_2 by

$$\begin{aligned}\sum_{N>2^9}\sum_{N_1\gtrsim N} N_1^{2s-1}\Big[&N^{-1/2}N_1^{1-\alpha}\|w_N\|_{L^\infty_t L^2_x}(\|z_{N_1}\|_{L^2_{tx}}\|w_{N_1}\|_{X^{-1,1}}+\|z_{N_1}\|_{X^{-1,1}}\|w_{N_1}\|_{L^2_{tx}})\\ &+N^{1/2}N_1^{-\alpha}\|w_N\|_{X^{-1,1}}\|z_{N_1}\|_{L^\infty_t L^2_x}\|w_{N_1}\|_{L^\infty_t L^2_x}\\ &+N^{-1}N_1^{-1/8}\|w_N\|_{L^\infty_t L^2_x}\|z_{N_1}\|_{L^\infty_t L^2_x}\|w_{N_1}\|_{L^\infty_t L^2_x}\Big]\\ &\lesssim \|z\|_{M^s}\|w\|_{M^{-1/2+}}\|w\|_{L^\infty_t H^{s-1}}+\|z\|_{M^s}\|w\|_{M^{s-1}}\|w\|_{L^\infty_t H^{-1/2+}}.\end{aligned} \tag{3.30}$$

Gathering estimates (3.27)-(3.30) we obtain

$$J\lesssim (\|z\|_{M^{\frac12+}_T}\|w\|_{M^{s-1}}+\|z\|_{M^s_T}\|w\|_{M^{-\frac12+}})\|w\|_{L^\infty_T H^{s-1}}+\|z\|_{M^s_T}\|w\|_{M^{s-1}}\|w\|_{L^\infty_T H^{-\frac12+}} \tag{3.31}$$

which leads to (3.22) and completes the proof of the proposition. □

3.2. **Unconditional well-posedness.** It is well known (see for instance [1]) that (1.3) is locally well-posed in H^s for $s>3/2$ with a minimum time of existence which depends on $\|u_0\|_{H^{3/2}+}$. As in the beginning of this section, we will use that $u_\lambda(t,x):=\lambda^\alpha u(\lambda^{\alpha+1}t,\lambda x)$ is solution to (1.3) with $L_{\alpha+1}$ replaced by $L^\lambda_{\alpha+1}$ that is the Fourier multiplier by $i\lambda^{\alpha+1}p_{\alpha+1}(\lambda^{-1}\cdot)$. Let u be a smooth solution to (1.3) emanating from a smooth initial data u_0, it follows from (1.4) that the estimate (3.10) also holds for u_λ with $0<\lambda\le 1$. Since $\|u_\lambda(0)\|_{H^{\frac12+}}\lesssim\lambda^{\alpha-1/2}\|u_0\|_{H^{\frac12+}}$, a classical continuity argument ensures that for $\lambda\sim(1+\|u_0\|_{H^{\frac12+}})^{-\frac{1}{\alpha-1/2}}$, u_λ exists at least on $[0,1]$ with $\|u_\lambda\|_{L^\infty(0,1;H^{\frac12+})}\lesssim\lambda^{\alpha-1/2}\|u_0\|_{H^{\frac12+}}$. Going back to u we obtain that it exists at least on $[0,T]$, with $T=T(\|u_0\|_{H^{\frac12+}}):=(1+\|u_0\|_{H^{\frac12+}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$ and

$$\|u\|_{L^\infty(0,T;H^s)}\lesssim\|u_0\|_{H^s}\text{ for }s>1/2\,.$$

Now, let $u_0\in H^s(\mathbb{R})$ with $s>1/2$. From the above remark, we infer that we can pass to the limit on a sequence of solutions emanating from smooth approximations of u_0 to obtain the existence of a solution $u\in L^\infty_T H^s$, with initial data u_0, of (1.3). Moreover, (3.22)-(3.2) ensure that this solution is the only one in this class. Now the continuity of u with values in $H^s(\mathbb{R})$ as

well as the continuity of the flow-map in $H^s(\mathbb{R})$ will follow from the Bona-Smith argument (see [3]). For any $\varphi \in H^s(\mathbb{R})$, any dyadic integer $N \geq 1$ and any $r \geq 0$, straightforward calculations in Fourier space lead to

(3.32)
$$\|P_{\leq N}\varphi\|_{H_x^{s+r}} \lesssim N^r \|\varphi\|_{H_x^s} \quad \text{and} \quad \|\varphi - P_{\leq N}\varphi\|_{H_x^{s-r}} \lesssim o(N^{-r})\|\varphi\|_{H_x^s} \ .$$

Let $u_0 \in H^s$ with $s > 1/2$. We denote by u^N the solution of (1.3) emanating from $P_{\leq N}u_0$ and for $1 \leq N_1 \leq N_2$, we set

$$w := u^{N_1} - u^{N_2} \ .$$

It follows from the estimates of the previous subsection applied to w that for $T = T(\|u_0\|_{H^{\frac{1}{2}+}})$ and any $-\frac{1}{2}+ < r \leq s$ it holds

$$\|w\|_{M_T^r} \lesssim \|w(0)\|_{H^r} \lesssim N_1^{r-s}\varepsilon(N_1) \tag{3.33}$$

with $\varepsilon(y) \to 0$ as $y \to +\infty$. Moreover, for any $r \geq 0$ we have

$$\|u^{N_i}\|_{M_T^{s+r}} \lesssim \|u_0^{N_i}\|_{H^{s+r}} \lesssim N_i^r \|u_0\|_{H^s} \tag{3.34}$$

Next, we observe that w solves the equation

$$w_t + L_{\alpha+1}w = \frac{1}{2}\partial_x(w^2) + \partial_x(u^{N_1}w) \tag{3.35}$$

Proposition 3.3. *Let $0 < T < 2$ and $w \in L_T^\infty H^s$ with $s > 1/2$ be a solution to* (3.35)*. Then it holds*

$$\begin{aligned} \|w\|^2_{L_T^\infty H^s} \lesssim\ & \|w_0\|^2_{H^s} + \|w\|^3_{M_T^s} \\ & + \|u^{N_1}\|_{M_T^s}\|w\|^2_{M_T^s} + \|u^{N_1}\|_{M_T^{s+1}}\|w\|_{M_T^s}\|w\|_{M_T^{s-1}} \ . \end{aligned} \tag{3.36}$$

Proof. Actually it is a consequence of estimates derived in the proof of Propositions 3.1 and 3.2. We separate the contributions of $\partial_x(w^2)$ and $\partial_x(u^{N_1}w)$. Let $t \in]0,T[$. First (3.21) leads to

$$\sum_N N^{2s}\Big|\int_0^t \int_{\mathbb{R}} P_N\partial_x(w^2)P_N w\Big| \lesssim \|w\|^3_{M_T^s} \ .$$

Second, applying (3.31) at the level s with z replaced by u^{N_1} we obtain

$$\sum_N N^{2s}\Big|\int_0^t \int_{\mathbb{R}} P_N\partial_x(u^{N_1}w)P_N w\Big| \lesssim \|u^{N_1}\|_{M_T^s}\|w\|^2_{M_T^s} + \|u^{N_1}\|_{M_T^{s+1}}\|w\|_{M_T^s}\|w\|_{M_T^{-\frac{1}{2}+}}$$

which leads to (3.36) since $s > 1/2$. □

(3.33)-(3.34) together with (3.36) lead to

$$\begin{aligned} \|w\|^2_{L_T^\infty H^s} &\lesssim \|w_0\|^2_{H^s} + \varepsilon(N_1) + N_1\, N_1^{-1}\varepsilon(N_1) \\ &\lesssim \varepsilon(N_1) \ . \end{aligned} \tag{3.37}$$

This shows that $\{u^N\}$ is a Cauchy sequence in $C([0,T];H^s)$ and thus $\{u^N\}$ converges in $C([0,T];H^s)$ to a solution of (1.3) emanating from u_0. Therefore, the uniqueness result ensures that $u \in C([0,T];H^s)$.

3.3. **Continuity of the flow map.** Let $s > 1/2$ and $\{u_{0,n}\} \subset H^s(\mathbb{R})$ be such that $u_{0,n} \to u_0$ in $H^s(\mathbb{R})$. We want to prove that the emanating solution u_n tends to u in $C([0,T];H^s)$. By the triangle inequality,

$$\|u-u_n\|_{L^\infty_T H^s} \leq \|u-u^N\|_{L^\infty_T H^s} + \|u^N-u_n^N\|_{L^\infty_T H^s} + \|u_n^N-u_n\|_{L^\infty_T H^s} \quad .$$

Using the above estimates on the solution to (3.35) we first infer that

$$\|u-u^N\|_{L^\infty_T H^s} + \|u_n-u_n^N\|_{L^\infty_T H^s} = \varepsilon(N), \tag{3.38}$$

where $\varepsilon(y) \to 0$ as $y \to \infty$. Next, we get the bound

$$\begin{aligned} \|u^N-u_n^N\|_{L^\infty_T H^s} &\lesssim \|u^N(0)-u_n^N(0)\|_{H^s} + o(1) \\ &= \|P_{\leq N}(u(0)-u^n(0))\|_{H^s} + \varepsilon(N) \\ &\lesssim \|u_0-u_{0,n}\|_{H^s} + \varepsilon(N) \end{aligned} \tag{3.39}$$

where again $\varepsilon(y) \to 0$ as $y \to \infty$. Collecting (3.38) and (3.39) ends the proof of the continuity of the flow map. Thus the proof of Theorem 1.1 is now completed in the case $s > 1/2$.

4. Estimates in the non regular case

In this section, we provide the needed estimates at level $s \geq 1-\alpha/2$ for $1 < \alpha \leq 2$. We introduce the space

$$F^{s,b} = F^{s,\alpha,b} = X^{s-\frac{\alpha+1}{2},b+1/2} + X^{s-\frac{1+\alpha}{8},b+\frac{1}{8}} \ , \tag{4.1}$$

endowed with the usual norm and we define

$$Y^s = Y^{s,\alpha} = L^\infty_t H^s \cap F^{s,\alpha,1/2} = L^\infty_t H^s \cap (X^{s-\frac{\alpha+1}{2},1} + X^{s-\frac{1+\alpha}{8},\frac{5}{8}}) \quad .$$

For $u_0 \in H^s(\mathbb{R})$ we will construct a solution to (1.3) that belongs to Y^s_T for some $T = T(\|u_0\|_{H^{1-\frac{\alpha}{2}}}) > 0$. As in the regular case, by a dilation argument, we may assume that $L_{\alpha+1}$ satisfies (1.4) for $0 < \lambda \leq 1$.

Remark 4.1. *Actually except in the case $(s,\alpha) = (0,2)$ we could simply take $Y^{s,\alpha} := L^\infty_t H^s \cap X^{s-\frac{\alpha+1}{2},1}$. But to include the case $(s,\alpha) = (0,2)$ in the general case we prefer to introduce the sum space $F^{s,\alpha,1/2}$ (see (4.1)) in all the cases.*

Lemma 4.1. *Let $0 < T < 2$, $1 < \alpha \leq 2$, $s \geq 1-\alpha/2$ and $u \in L^\infty_T H^s$ be a solution to (1.3). Then u belongs to $Y^{s,\alpha}_T$. Moreover, if $(s,\alpha) \neq (0,2)$ it holds*

$$\|u\|_{Y^{s,\alpha}_T} \lesssim \|u\|_{L^\infty_T H^s}(1+\|u\|_{L^\infty_T H^{1-\frac{\alpha}{2}}}) \tag{4.2}$$

and if $(s,\alpha) = (0,2)$,

$$\|u\|_{Y^{0,2}_T} \lesssim \|u\|_{L^\infty_T L^2_x}(1+\|u\|^2_{L^\infty_T L^2_x}) \ . \tag{4.3}$$

Proof. As in Lemma 3.1 we will work with the extension $\tilde{u} = \rho_T u$ of u (see (3.3)). Recall that $\operatorname{supp} \widetilde{u} \subset [-2,2] \times \mathbb{R}$ and that

$$\|\widetilde{u}\|_{L^\infty_t H^s} \lesssim \|u\|_{L^\infty_T H^s} \quad \text{and} \quad \|\widetilde{u}\|_{X^{\theta,b}} \lesssim \|u\|_{X^{\theta,b}_T} \quad ,$$

for any $(\theta, b) \in \mathbb{R} \times]-\infty, 1]$. It thus remains to control the $F^{s,\alpha,1/2}_T$-norm of u. In the case $(s,\alpha) \neq (0,2)$ we actually simply control the $X^{s-\frac{\alpha+1}{2},1}_T$-norm of u. Using the integral formulation, standard linear estimates in Bourgain's spaces and standard product estimates in Sobolev spaces we infer that

$$\begin{aligned}
\|u\|_{X^{s-\frac{1+\alpha}{2},1}_T} &\lesssim \|u_0\|_{H^{s-\frac{1+\alpha}{2}}} + \|\partial_x(u^2)\|_{X^{s-\frac{1+\alpha}{2},0}_T} \\
&\lesssim \|u_0\|_{H^{s-\frac{1+\alpha}{2}}} + \|u^2\|_{L^2_T H^{s+\frac{1-\alpha}{2}}} \\
&\lesssim \|u_0\|_{H^{s-\frac{1+\alpha}{2}}} + \|u\|_{L^\infty_T H^{1-\frac{\alpha}{2}}} \|u\|_{L^\infty_T H^s} \ ,
\end{aligned}$$

since for $1 < \alpha \leq 2$ and $s \geq 1 - \frac{\alpha}{2}$ with $(s,\alpha) \neq (0,2)$, it holds $s + 1 - \frac{\alpha}{2} > 0$ and $s + 1 - \frac{\alpha}{2} - (s + \frac{1-\alpha}{2}) = 1/2$.
Let us now tackle the case $(s,\alpha) = (0,2)$. First we notice that since $L^1(\mathbb{R}) \hookrightarrow H^{-\frac{3}{2}-}(\mathbb{R})$, we have

$$\|u\|_{X^{-\frac{7}{4},1}_T} \lesssim \|u_0\|_{H^{-\frac{7}{4}}} + \|u^2\|_{L^2_t H^{-\frac{3}{4}}} \lesssim \|u\|_{L^\infty_T L^2_x}(1 + \|u\|_{L^\infty_T L^2_x}) \ . \tag{4.4}$$

To bound the $F^{0,2,\frac{1}{2}}$-norm of u, we decompose u^2 as

$$u^2 = P_{\leq 2} u^2 + \sum_{N>2} \Big(P_N(P_{\ll N} u \, u_{\sim N}) + \sum_{N_1' \sim N_1 \gtrsim N} P_N(u_{N_1} u_{N_1'}) \Big) \ . \tag{4.5}$$

The contribution of the first term in the right hand side is easily controlled by $\|u\|^2_{L^\infty_T L^2_x}$. The contribution of the (LH)- interactions is easily estimated by

$$\begin{aligned}
\Big\| \sum_{N>2} \partial_x P_N(P_{\ll N} u \, u_{\sim N}) \Big\|_{F^{0,2,-\frac{1}{2}}_T} &\lesssim \Big\| \sum_{N>2} P_N \partial_x (P_{\ll N} u \, u_{\sim N}) \Big\|_{X^{-\frac{3}{2},0}_T} \\
&\lesssim \Big(\sum_{N>2} \|P_N(P_{\ll N} u \, u_{\sim N})\|^2_{L^2_T L^1_x} \Big)^{1/2} \\
&\lesssim \Big(\sum_{N \geq 1} \|u_N\|^2_{L^2_T L^2_x} \|P_{\ll N} u\|^2_{L^\infty_T L^2_x} \Big)^{1/2} \\
&\lesssim \|u\|_{L^\infty_T L^2} \|u\|_{L^\infty_T L^2_x} \quad . \qquad (4.6)
\end{aligned}$$

To estimate the (HH)-interactions, we take advantage of the $X^{-\frac{3}{8},-\frac{3}{8}}$-part of $F^{0,2,-\frac{1}{2}}$. For $N>2$ we have

$$\sum_{N_1'\sim N_1\gtrsim N}\|\partial_x P_N(P_{N_1}uP_{N_1'}u)\|_{F_T^{0,2,-\frac{1}{2}}} \lesssim \sum_{N_1'\sim N_1\gtrsim N} N\left\|\sum_{(L,L_1,L_2)\text{ satisfying (2.5)}}\partial_x P_N Q_L(Q_{L_1}\widetilde{u}_{N_1}Q_{L_2}\widetilde{u}_{N_1'})\right\|_{X^{-\frac{3}{8},-\frac{3}{8}}}. \tag{4.7}$$

For the contribution of the sum over $L\gtrsim NN_1^2$ in (4.7) we obtain

$$\begin{aligned}\sum_{N_1\sim N_1'\gtrsim N}&\|\partial_x P_N Q_{\gtrsim NN_1^2}(\widetilde{u}_{N_1}\widetilde{u}_{N_1'})\|_{X^{-\frac{3}{8},-\frac{3}{8}}}\\ &\lesssim \sum_{N_1\sim N_1'\gtrsim N} N^{\frac{5}{8}}N^{1/2}(NN_1^2)^{-3/8}\|\widetilde{u}_{N_1}\|_{L^2_{tx}}\|\widetilde{u}_{N_1'}\|_{L^\infty_t L^2_x}\\ &\lesssim \|\widetilde{u}\|_{L^\infty_t L^2_x}\sum_{N_1\gtrsim N}\Big(\frac{N}{N_1}\Big)^{3/4}\|\widetilde{u}_{N_1}\|_{L^2_{tx}}\\ &\lesssim \gamma_N\,\|\widetilde{u}\|^2_{L^\infty_t L^2_x}\,,.\end{aligned} \tag{4.8}$$

with $\|(\gamma_{2^j})\|_{l^2(\mathbb{N})}\leq 1$. The contribution of the region ($L\ll NN_1^2$ and $L_1\gtrsim NN_1^2$) in (4.7) is controlled by

$$\begin{aligned}\sum_{N_1\sim N_1'\gtrsim N}&\|\partial_x P_N Q_{\ll NN_1^2}(Q_{\gtrsim NN_1^2}\widetilde{u}_{N_1}\widetilde{u}_{N_1'})\|_{X^{-\frac{3}{8},-\frac{3}{8}}}\\ &\lesssim \sum_{N_1\sim N_1'\gtrsim N} N^{\frac{5}{8}}N^{1/2}(NN_1^2)^{-1}N_1^{\frac{7}{4}}\|\widetilde{u}_{N_1}\|_{X^{-\frac{7}{4},1}}\|\widetilde{u}_{N_1'}\|_{L^\infty_t L^2_x}\\ &\lesssim N^{-1/8}\,\|\widetilde{u}\|_{L^\infty_t L^2_x}\|\widetilde{u}\|_{X^{-\frac{7}{4},1}}\,.\end{aligned} \tag{4.9}$$

Finally, the contribution of the last region ($L,L_1\ll NN_1^2$ and $L_2\sim NN_1^2$) in (4.7) is controlled in the same way. Gathering (4.4) and (4.7)-(4.9), we obtain the desired result for the case $(s,\alpha)=(0,2)$. □

In the sequel we will need the following straightforward estimates.

Lemma 4.2. *Let $\alpha\geq 0$ and $w\in F^{0,\frac{1}{2}}$. For $1\leq B\lesssim N^{\alpha+1}$ it holds*

$$\|Q_{\gtrsim B}w_N\|_{L^2}\lesssim B^{-1}N^{\frac{1+\alpha}{2}}\|Q_{\gtrsim B}w_N\|_{F^{0,\frac{1}{2}}} \tag{4.10}$$

and, for $B\gtrsim\langle N\rangle^{\alpha+1}$, it holds

$$\|Q_{\gtrsim B}w_N\|_{L^2}\lesssim B^{-5/8}\langle N\rangle^{\frac{1+\alpha}{8}}\|Q_{\gtrsim B}w_N\|_{F^{0,\frac{1}{2}}}\,. \tag{4.11}$$

Proof. Noticing that $F^{0,\frac{1}{2}} = F^{0,\alpha,\frac{1}{2}} = X^{-\frac{1+\alpha}{2},1} + X^{-\frac{1+\alpha}{8},\frac{5}{8}}$, it is direct to check that

$$\begin{aligned}\|Q_{\gtrsim B} w_N\|_{L^2} &\lesssim \max(B^{-1}\langle N\rangle^{\frac{1+\alpha}{2}}, B^{-5/8}\langle N\rangle^{\frac{1+\alpha}{8}}\Big)\|Q_{\gtrsim B} w_N\|_{F^{0,\frac{1}{2}}} \\ &\lesssim B^{-5/8}\langle N\rangle^{\frac{1+\alpha}{8}} \max\Big((\langle N\rangle^{1+\alpha}/B)^{\frac{3}{8}}, 1\Big)\|Q_{\gtrsim B} w_N\|_{F^{0,\frac{1}{2}}}\end{aligned}$$

which leads to the desired result. □

Now we rewrite Lemma 3.2 in the context of the $F^{s,b}$ spaces.

Lemma 4.3. *Assume $u_i \in Y^0$, $i = 1, 2, 3$ are functions with spatial Fourier support in $\{|\xi| \sim N_i\}$ with $N_i > 0$ dyadic satisfying $N_1 \le N_2 \le N_3$.*
If $N_3 > 2^9$ and $N_1 \gtrsim N_3^{\frac{2}{3}(1-\alpha)} \wedge 2^9$, it holds for $(p, q) \in \{(2, \infty), (\infty, 2)\}$

$$\begin{aligned}|I_t(u_1, u_2, u_3)| \lesssim \sum_{l\ge -4} & 2^{-l} N_1^{-1/2} N_3^{\frac{1-\alpha}{2}} \|u_1\|_{L_t^p L_x^2} \|Q_{2^l N_1 N_3^\alpha} u_2\|_{F^{0,\frac{1}{2}}} \|u_3\|_{L_t^q L_x^2} \\ & + N_1^{-1/2} N_3^{\frac{1-\alpha}{2}} \|u_1\|_{L_t^p L_x^2} \|u_2\|_{L_t^q L_x^2} \|Q_{\sim N_1 N_3^\alpha} u_3\|_{F^{0,\frac{1}{2}}} \\ & + N_1^{-1/8} \langle N_1\rangle^{\frac{1+\alpha}{8}} N_3^{-5\alpha/8} \|u_1\|_{F^{0,\frac{1}{2}}} \|u_2\|_{L_{tx}^2} \|u_3\|_{L_t^\infty L_x^2} \\ & + N_1^{-1/4} N_3^{\frac{1}{8}-\frac{\alpha}{2}} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|u_3\|_{L_t^\infty L_x^2}.\end{aligned}$$

Proof. For $R = N_1^{3/4} N_3^{\frac{\alpha}{2}-\frac{1}{8}}$ we decompose I_t as in (3.5) and obtain from (3.6) that

$$|I_t^{high}| \lesssim N_1^{-1/4} N_3^{\frac{1}{8}-\frac{\alpha}{2}} \prod_{i=1}^{3} \|u_i\|_{L_t^\infty L_x^2}.$$

To evaluate I_t^{low} we use decomposition (3.7) and notice that

$$R = N_1^{3/4} N_3^{\frac{\alpha}{2}-\frac{1}{8}} \le N_1 N_2^{\frac{2\alpha}{3}-\frac{7}{24}} \ll N_1 N_3^\alpha \text{ and } N_1 N_3^\alpha \gtrsim N_3^{\frac{2+\alpha}{3}} \gg 1.$$

Therefore the contribution $I_t^{1,low}$ of the first term of the RHS of (3.7) to I_t^{low} is easily estimated thanks to Lemmas 2.5 and 4.2 by

$$|I_t^{1,low}| \lesssim N_1^{1/2} (N_1 N_3^\alpha)^{-5/8} \langle N_1\rangle^{\frac{\alpha+1}{8}} \|u_1\|_{F^{0,\frac{1}{2}}} \|u_2\|_{L_{tx}^2} \|u_3\|_{L_t^\infty L_x^2},$$

which is acceptable. Thanks to Lemmas 2.3, 2.5 and 4.2, the contribution $I_t^{2,low}$ of the second term can be handle in the following way

$$\begin{aligned}|I_t^{2,low}| &\lesssim \sum_{l\ge -4} N_1^{1/2} (2^l N_1 N_3^\alpha)^{-1} N_3^{\frac{\alpha+1}{2}} \|u_1\|_{L_t^p L_x^2} \|Q_{2^l N_1 N_3^\alpha} u_2\|_{F^{0,\frac{1}{2}}} \|u_3\|_{L_t^q L_x^2} \\ &\lesssim \sum_{l\ge -4} 2^{-l} N_1^{-1/2} N_3^{\frac{1-\alpha}{2}} \|u_1\|_{L_t^p L_x^2} \|Q_{2^l N_1 N_3^\alpha} u_2\|_{F^{0,\frac{1}{2}}} \|u_3\|_{L_t^q L_x^2}. \qquad (4.12)\end{aligned}$$

In the same way, we get that the contribution $I_t^{3,low}$ of the third term to I_t^{low} is bounded by

$$|I_t^{3,low}| \lesssim N_1^{1/2}(N_1N_3^\alpha)^{-1}N_3^{\frac{\alpha+1}{2}}\|u_1\|_{L_t^pL_x^2}\|u_2\|_{L_t^qL_x^2}\|Q_{\sim N_1N_3^\alpha}u_3\|_{F^{0,\frac12}}$$

$$\lesssim N_1^{-1/2}N_3^{\frac{1-\alpha}{2}}\|u_1\|_{L_t^pL_x^2}\|u_2\|_{L_t^qL_x^2}\|Q_{\sim N_1N_3^\alpha}u_3\|_{F^{0,\frac12}}\ . \tag{4.13}$$

Gathering all these estimates, we obtain the desired bound. □

Proposition 4.1. *Let $0<T<2$, $1<\alpha\le 2$, $s\ge 1-\alpha/2$ and $u\in L_T^\infty H^s$ be a solution to* (1.3)*. Then u belongs to $\widetilde{L_T^\infty}H^s$ and it holds*

$$\|u\|_{\widetilde{L_T^\infty}H^s}\lesssim \|u_0\|_{H^s}+\|u\|_{L_T^\infty H^{1-\frac\alpha2}}\|u\|_{Y_T^s}+\|u\|_{L_T^\infty H^s}\|u\|_{Y_T^{1-\frac\alpha2}}\ . \tag{4.14}$$

Proof. Applying the operator P_N with $N>0$ dyadic to equation (1.3), taking the H^s scalar product with P_Nu and integrating on $]0,t[$ we obtain

$$\|P_Nu\|^2_{L_T^\infty H^s}\lesssim \|P_Nu_0\|^2_{H^s}+\sup_{t\in]0,T[}\langle N\rangle^{2s}\Big|\int_0^t\int_{\mathbb{R}}P_N(u^2)\partial_xP_Nu\Big|. \tag{4.15}$$

We take an extension $\tilde u$ of u supported in time in $]-4,4[$ such that $\|\tilde u\|_{Y^s}\lesssim \|u\|_{Y_T^s}$. To simplify the notation we drop the tilde in the sequel.
We infer from (3.18) that it suffices to estimate

$$I=\sum_{N>0}\sum_{N_1\gtrsim N}N\langle N_1\rangle^{2s}\sup_{t\in]0,T[}|I_t(u_N,u_{\sim N_1},u_{N_1})|.$$

The low frequencies part $N\le 2^9$ is estimated exactly as in (3.19) by

$$\|u\|_{L_t^\infty L_x^2}\|u\|^2_{L_t^\infty H^s}.$$

On the other hand, the contribution of the sum over $N>2^9$ is controlled thanks to Lemma 4.3 by

$$\begin{aligned}\sum_{N>2^9}\sum_{N_1\gtrsim N}\Big[&\Big(\frac{N}{N_1}\Big)^{\frac{\alpha-1}{2}}\|u_N\|_{L_t^2H^{1-\frac\alpha2}}\|u_{N_1}\|_{L_t^\infty H^s}\|u_{N_1}\|_{F^{s,\frac12}}\\&+\Big(\frac{N}{N_1}\Big)^{5\alpha/8}\|u_N\|_{F^{1-\frac\alpha2,\frac12}}\|u_{N_1}\|_{L_t^2H^s}\|u_{N_1}\|_{L_t^\infty H^s}\\&+N^{\frac\alpha2-\frac14}N_1^{\frac18-\frac\alpha2}\|u_N\|_{L_t^\infty H^{1-\frac\alpha2}}\|u_{N_1}\|^2_{L_t^\infty H^s}\Big]\\&\lesssim \|u\|_{Y^{1-\frac\alpha2}}\|u\|^2_{L_t^\infty H^s}+\|u\|_{L_t^\infty H^{1-\frac\alpha2}}\|u\|_{L_t^\infty H^s}\|u\|_{Y^s},\end{aligned} \tag{4.16}$$

where we use discrete Young's inequality in N_1 and then Cauchy-Schwarz in N to bound the first two terms.
Gathering the above estimates we eventually obtain

$$I\lesssim \|u\|_{Y_T^{1-\frac\alpha2}}\|u\|^2_{L_T^\infty H^s}+\|u\|_{L_T^\infty H^{1-\frac\alpha2}}\|u\|_{L_T^\infty H^s}\|u\|_{Y_T^s}\ , \tag{4.17}$$

which completes the proof of the proposition.

□

4.1. **Estimates on the difference of two solutions.** First we introduce the function spaces where we will estimate the difference of two solutions of (1.3). Contrary to the regular case, we will have to work in a function space that put a weight on the very low frequencies. For $\theta \in \mathbb{R}$ we denote by $\overline{H}^\theta$ the completion of $\mathcal{S}(\mathbb{R})$ for the norm

$$\|\varphi\|_{\overline{H}^\theta} := \|\langle |\xi|^{-1/2}\rangle \langle \xi\rangle^\theta \hat{\varphi}\|_{L^2}$$

Then we define the space $\widetilde{L^\infty_t}\overline{H}^\theta$ by

$$\|w\|_{\widetilde{L^\infty_t}\overline{H}^\theta} := \Big(\sum_{N-dyadic>0} \|w_N\|^2_{L^\infty_t \overline{H}^\theta} \Big)^{1/2} . \tag{4.18}$$

We then define the function spaces $\tilde{Y}^\theta$ and Z^θ, $\theta \in \mathbb{R}$, by respectively

$$\tilde{Y}^\theta = \widetilde{L^\infty_t} H^\theta \cap F^{\theta,1/2} \text{ and } Z^\theta = \widetilde{L^\infty_t}\overline{H}^\theta \cap F^{\theta,1/2} ,$$

with $F^{\theta,b}$ defined in (4.1).

If $u, v \in L^\infty_T H^s$ are two solutions of (1.3) with $s \geq 1-\alpha/2$, then according to Lemma 4.1 and Proposition 4.1 we know that u and v belong to $Y^s_T \cap \widetilde{L^\infty_T} H^s$. Moreover, using again the extension operator ρ_T it is easy to check that

$$Y^s_T \cap \widetilde{L^\infty_T} H^s \hookrightarrow \tilde{Y}^s_T \tag{4.19}$$

with an embedding constant that does not depend on $0 < T \leq 2$. Hence, u and v belong to $\tilde{Y}^s_T$. Assuming that $u_0 - v_0 \in \overline{H}^s$, we claim that the difference $u - v$ belongs to Z^s_T. Indeed, according to the above definitions of $\tilde{Y}^s$ and Z^s , it suffices to check that $P_1(u-v)$ belongs to $\widetilde{L^\infty_T}\overline{H}^s$. But it is straightforward, since by the Duhamel formula for any dyadic integer $0 < N < 1$ it holds

$$\|P_N(u-v)\|_{L^\infty_T \overline{H}^s} \lesssim \|u_0 - v_0\|_{\overline{H}^s} + N(\|u\|^2_{L^\infty_T L^2_x} + \|v\|^2_{L^\infty_T L^2_x}) .$$

We are thus allowed to estimate the difference $w = u - v$ in the space $Z^{s-\frac{3}{2}+\frac{\alpha}{2}}_T$.

Proposition 4.2. *Let $0 < T < 1$, $1 < \alpha \leq 2$, $s \geq 1-\alpha/2$ and $u, v \in L^\infty_T H^s$ be two solutions to (1.3) on $]0,T[$. Then we have*

$$\|u-v\|_{Z^{s-\frac{3}{2}+\frac{\alpha}{2}}_T} \lesssim \|u-v\|_{L^\infty_T \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u+v\|_{\tilde{Y}^s_T} \|u-v\|_{Z^{-1/2}_T} + \|u+v\|_{\tilde{Y}^{1-\frac{\alpha}{2}}} \|u-v\|_{Z^{s-\frac{3}{2}+\frac{\alpha}{2}}_T} . \tag{4.20}$$

Proof. Recall that $w = u - v$ satisfies (3.23) with $z = u + v$. We extend w from $(0,T)$ to $\mathbb{R}$ by using the extension operator ρ_T defined in (3.3). On account of the uniform bounds on ρ_T (see the paragraph just after (3.3)), it remains to estimate the $F^{s-\frac{3}{2}+\frac{\alpha}{2},\alpha,\frac{1}{2}}_T$-norm of w. From classical linear

estimates in the framework of Bourgain's spaces, the Duhamel formulation associated with (3.23) leads to

$$\|w\|_{F_T^{s-\frac{3}{2}+\frac{\alpha}{2},1/2}} \lesssim \|w_0\|_{H^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|\partial_x(zw)\|_{F_T^{s-\frac{3}{2}+\frac{\alpha}{2},-1/2}}. \tag{4.21}$$

Let $\widetilde{z}$ and $\widetilde{w}$ be time extensions of z and w satisfying $\|\widetilde{z}\|_{\tilde{Y}^s} \lesssim \|z\|_{\tilde{Y}_T^s}$ and $\|\widetilde{w}\|_{Z^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}$. To simplify the notation we drop the tilde in the sequel. From (4.21) we see that it suffices to estimate

$$\|\partial_x(zw)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},\alpha,-1/2}} \lesssim \left(\sum_N \|P_N\partial_x(zw)\|^2_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-1/2}}\right)^{1/2}.$$

We first estimate the (low-high) contribution $P_N(P_{\lesssim N}zP_Nw)$:

$$\begin{aligned}
\|\partial_x P_N(P_{\lesssim N}zP_Nw)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},\alpha,-1/2}} &\lesssim \sum_{N_1\lesssim N} N\|P_N(P_{N_1}zP_Nw)\|_{X^{s-2,0}}\\
&\lesssim \sum_{N_1\lesssim N} N_1^{1/2}N\langle N\rangle^{s-2}\|P_{N_1}z\|_{L_t^\infty L_x^2}\|P_Nw\|_{L_t^2L_x^2}\\
&\lesssim \|P_Nw\|_{L_t^2H^{s-\frac{3}{2}+\frac{\alpha}{2}}}\sum_{N_1\lesssim N}\Big(\frac{N_1}{\langle N\rangle}\Big)^{\frac{\alpha-1}{2}}\|P_{N_1}z\|_{L_t^\infty H^{1-\frac{\alpha}{2}}}\\
&\lesssim \|z\|_{L_t^\infty H^{1-\frac{\alpha}{2}}}\|P_Nw\|_{L_t^\infty H^{s-\frac{3}{2}+\frac{\alpha}{2}}}.
\end{aligned}$$

Similarly, the (high-low) interactions are estimated as follows:

$$\begin{aligned}
\|\partial_x P_N(P_NzP_{\lesssim N}w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-1/2}} &\lesssim N\|P_N(P_NzP_{\lesssim N}w)\|_{X^{s-2,0}}\\
&\lesssim \|P_Nz\|_{L_t^2H^s}\sum_{N_1\lesssim N}\Big(\frac{N_1}{\langle N\rangle}\Big)^{1/2}\|P_{N_1}w\|_{L_t^\infty H^{-1/2}}\\
&\lesssim \|P_Nz\|_{L_t^2H^s}\|w\|_{L_t^\infty H^{-\frac{1}{2}}}.
\end{aligned}$$

Now we deal with the high-high interactions term

$$\begin{aligned}
&\|\partial_x P_N(P_{\gg N}zP_{\gg N}w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}}\\
&\qquad\lesssim \sum_{N_1\gg N} N\left\|\sum_{(L,L_1,L_2)\text{ satisfying (2.5)}}\partial_xP_NQ_L(Q_{L_1}z_{N_1}Q_{L_2}w_{N_1})\right\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}},
\end{aligned}$$

We may assume that $N_1\gg 1$ since otherwise, it holds $N\ll N_1\lesssim 1$ and we have

$$\|P_{\lesssim 1}\partial_x(P_{\lesssim 1}zP_{\lesssim 1}w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-1/2}} \lesssim \|P_{\lesssim 1}z\|_{L_t^\infty L^2}\|P_{\lesssim 1}w\|_{L_t^\infty H^{-\frac{1}{2}}}.$$

For $N_1\gg 1$, we will take advantage of the fact that $X^{s-\frac{13}{8}+\frac{3\alpha}{8},-\frac{3}{8}}\hookrightarrow F^{s-\frac{3}{2}+\frac{\alpha}{2},-1/2}$. The contribution of the sum over $L\gtrsim NN_1^\alpha$ can be thus

controlled by

$$
\begin{aligned}
&\sum_{N_1\gg N}\|\partial_x P_N Q_{\gtrsim NN_1^\alpha}(z_{N_1}w_{N_1})\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-1/2}}\\
&\lesssim \sum_{N_1\gg N} N\|P_N Q_{\gtrsim NN_1^\alpha}(z_{N_1}w_{N_1})\|_{X^{s-\frac{13}{8}+\frac{3\alpha}{8},-3/8}}\\
&\lesssim \sum_{N_1\gg N}\sum_{L\gtrsim NN_1^\alpha} N\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}}L^{-3/8}\|P_N Q_L(z_{N_1}w_{N_1})\|_{L^2}\\
&\lesssim \sum_{N_1\gg N} N^{3/2}\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}}(NN_1^\alpha)^{-3/8}N_1^{\frac{1}{2}-s}\|z_{N_1}\|_{L^2_tH^s}\|w_{N_1}\|_{L^\infty_tH^{-1/2}}\\
&\lesssim \sum_{N_1\gg N}(N/N_1)^{1/2-\alpha/8}\Big(\frac{\langle N\rangle}{\langle N_1\rangle}\Big)^{s-1+\frac{\alpha}{2}}\|z_{N_1}\|_{L^2_tH^s}\|w_{N_1}\|_{L^\infty_tH^{-1/2}}\\
&\lesssim \delta_N\|z\|_{L^2_tH^s}\|w\|_{L^\infty_tH^{-1/2}},
\end{aligned}
$$

where $\|(\delta_{2^j})_j\|_{l^2(\mathbb{Z})}\lesssim 1$. The contribution of the region ($L\ll NN_1^\alpha$ and $L_1\gtrsim NN_1^\alpha$) is estimated thanks to (4.10) by

$$
\begin{aligned}
&\sum_{N_1\gg N}\|\partial_x P_N Q_{\ll NN_1^\alpha}(Q_{\gtrsim NN_1^\alpha}z_{N_1}w_{N_1})\|_{X^{s-\frac{13}{8}+\frac{3\alpha}{8},-3/8}}\\
&\lesssim \sum_{N_1\gg N} N\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}}\|P_N(Q_{\gtrsim NN_1^\alpha}z_{N_1}w_{N_1})\|_{L^2}\\
&\lesssim \sum_{N_1\gg N} N^{3/2}\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}}(NN_1^\alpha)^{-1}N_1^{1-s+\frac{\alpha}{2}}\|Q_{\gtrsim NN_1^\alpha}z_{N_1}\|_{F^{s,\frac{1}{2}}}\|w_{N_1}\|_{L^\infty_tH^{-1/2}}\\
&\lesssim \sum_{N_1\gg N}\Big(\frac{N}{\langle N\rangle}\Big)^{1/2}\langle N\rangle^{-\frac{1+\alpha}{8}}\Big(\frac{\langle N\rangle}{\langle N_1\rangle}\Big)^{s-1+\frac{\alpha}{2}}\|Q_{\gtrsim NN_1^\alpha}z_{N_1}\|_{F^{s,\frac{1}{2}}}\|w_{N_1}\|_{L^\infty_tH^{-1/2}}\\
&\lesssim \delta_N\|z\|_{Y^s}\|w\|_{\widetilde{L^\infty_t}H^{-1/2}}
\end{aligned}
$$

where $\|(\delta_{2^j})_j\|_{l^2(\mathbb{Z})}\lesssim 1$. Finally the contribution of the last region can be bounded thanks to (4.10) by

$$
\begin{aligned}
&\sum_{N_1\gg N}\|\partial_x P_N Q_{\ll NN_1^\alpha}(Q_{\ll NN_1^\alpha}zN_1\,Q_{\sim NN_1^\alpha}w_{N_1})\|_{X^{s-\frac{13}{8}+\frac{3\alpha}{8},-\frac{3}{8}}}\\
&\lesssim \sum_{N_1\gg N} N\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}}\|P_N Q_{\ll NN_1^\alpha}(Q_{\ll NN_1^\alpha}z_{N_1}\,Q_{\sim NN_1^\alpha}w_{N_1})\|_{L^2}\\
&\lesssim \sum_{N_1\gg N} N^{3/2}\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}}N_1^{-s}(NN_1^\alpha)^{-1}N_1^{1+\alpha/2}\|Q_{\ll NN_1^\alpha}z_{N_1}\|_{L^\infty_tH^s}\|Q_{\sim NN_1^\alpha}w_{N_1}\|_{F^{-1/2,1/2}}\\
&\lesssim \sum_{N_1\gg N}\Big(\frac{N}{\langle N\rangle}\Big)^{1/2}\langle N\rangle^{-\frac{1+\alpha}{8}}\Big(\frac{\langle N\rangle}{\langle N_1\rangle}\Big)^{s-1+\frac{\alpha}{2}}\|z_{N_1}\|_{L^\infty_tH^s}\|w_{N_1}\|_{F^{-1/2,1/2}}\\
&\lesssim \delta_N\|z\|_{\widetilde{L^\infty_t}H^s}\|w\|_{Z^{-1/2}}
\end{aligned}
$$

which is acceptable. This concludes the proof of Proposition 4.2. □

Proposition 4.3. *Let $1 \leq \alpha \leq 2$, $0 < T < 2$ and $u, v \in L^\infty_T H^s$ with $s \geq 1 - \alpha/2$ be two solutions to* (1.3)*. Then it holds*[1]

$$\|u - v\|_{\widetilde{L^\infty_T}\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|u_0 - v_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u + v\|_{Y^s_T}\|u - v\|_{Z^{s-\frac{3}{2}+\frac{\alpha}{2}}_T} . \tag{4.22}$$

Proof. Recall that the difference $w = u - v$ satisfies (3.23) with $z = u + v$. Applying the operator P_N with $N > 0$ dyadic to equation (3.23), taking the H^s scalar product with $P_N w$ and integrating on $]0, t[$ we obtain

$$\|w_N\|^2_{L^\infty_T\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|P_N w_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3}{2}+\frac{\alpha}{2})} \sup_{t\in[0,T]}\Big|\int_0^t\int_{\mathbb{R}} P_N(zw)\partial_x w_N\Big|.$$

Therefore we have to estimate

$$J := \sum_{N>0}\langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in[0,T]}\Big|\int_0^t\int_{\mathbb{R}} P_N(zw)\partial_x w_N\Big|.$$

We take extensions $\tilde{z}$ and $\tilde{w}$ of z and w supported in time in $]-4, 4[$ such that $\|\tilde{z}\|_{Y^s} \lesssim \|u\|_{Y^s_T}$ and $\|\tilde{w}\|_{Z^s} \lesssim \|w\|_{Z^s_T}$. To simplify the notation we drop the tilde in the sequel.
Proceeding as in (3.27) we get

$$\begin{aligned} J \lesssim & \sum_{N>0}\sum_{N_1\gtrsim N} N\langle N_1^{-1}\rangle\langle N_1\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in]0,T[}|I_t(z_N, w_{\sim N_1}, w_{N_1})| \\ & + \sum_{N>0}\sum_{N_1\gtrsim N} N_1\langle N_1^{-1}\rangle\langle N_1\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in]0,T[}|I_t(z_{\sim N_1}, w_N, w_{N_1})| \\ & + \sum_{N>0}\sum_{N_1\gtrsim N} N\langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in]0,T[}|I_t(z_{N_1}, w_{N_1}, w_N)| \\ := & J_1 + J_2 + J_3. \end{aligned} \tag{4.23}$$

Estimates for J_1. The contribution of the sum over $N \leq 2^9$ in J_1 is estimated thanks to (3.4) by

$$\begin{aligned} & \sum_{N\leq 2^9}\sum_{N_1\gtrsim N} N^{3/2}\|z_N\|_{L^\infty_t L^2_x}\|w_{N_1}\|^2_{L^\infty_t\overline{H}^{s-\frac{3-\alpha}{2}}} \\ & \lesssim \|z\|_{L^\infty_t L^2_x}\|w\|^2_{\widetilde{L^\infty_T}\overline{H}^{s-\frac{3-\alpha}{2}}}. \end{aligned}$$

1. We include the case $\alpha = 1$ here since it does not lead to additional difficulties and will be useful in the appendix to prove LWP for $(\alpha, s) = (1, 1/2)$.

The contribution $N > 2^9$ in J_1 can be controlled with Lemma 4.3 by

$$\sum_{N>2^9}\sum_{N_1\gtrsim N}\Big[\sum_{l\geq -4}2^{-l}\left(\frac{N}{N_1}\right)^{\frac{\alpha-1}{2}}\|z_N\|_{L^2_tH^{1-\frac{\alpha}{2}}}\|Q_{2^lNN_1^\alpha}w_{N_1}\|_{F^{s-\frac{3-\alpha}{2},\frac{1}{2}}}\|w_{N_1}\|_{L^\infty_tH^{s-\frac{3-\alpha}{2}}}$$
$$+\left(\frac{N}{N_1}\right)^{5\alpha/8}\|z_N\|_{F^{1-\frac{\alpha}{2},\frac{1}{2}}}\|w_{N_1}\|_{L^2_tH^{s-\frac{3-\alpha}{2}}}\|w_{N_1}\|_{L^\infty_tH^{s-\frac{3-\alpha}{2}}}$$
$$+N^{\frac{\alpha}{2}-\frac{1}{4}}N_1^{\frac{1}{8}-\frac{\alpha}{2}}\|z_N\|_{L^\infty_tH^{1-\frac{\alpha}{2}}}\|w_{N_1}\|^2_{L^\infty_tH^{s-\frac{3-\alpha}{2}}}\Big]$$
$$\lesssim\|z\|_{Y^{1-\frac{\alpha}{2}}}\|w\|_{\widetilde{L^\infty_t}\overline{H}^{s-\frac{3-\alpha}{2}}}\|w\|_{Z^{s-\frac{3-\alpha}{2}}}$$

where we used for the first term Cauchy-Schwarz in (N,N_1) and then sum in l. Note that for $\alpha>1$ we could replace the $\widetilde{L^\infty_t}H^{s-\frac{3}{2}+\frac{\alpha}{2}}$-norm by a standard $L^\infty_tH^{s-\frac{3}{2}+\frac{\alpha}{2}}$ by invoking the discrete Young inequality.

Estimates for J_2. We separate different contributions. First, the contribution of the sum over $N_1\leq 2^9$ is directly estimated by $\|z\|_{L^\infty_TL^2}\|w\|^2_{L^\infty_TH^{-\frac{1}{2}}}$.

The contribution of the sum over $N\leq N_1^{\frac{2}{3}(1-\alpha)}$ and $N_1>2^9$ is then easily estimated by

$$\sum_{N_1>2^9}\sum_{N\leq N_1^{\frac{2}{3}(1-\alpha)}}NN_1^{\frac{\alpha-1}{2}}\|z_{N_1}\|_{L^2_TH^s}\|w_N\|_{L^\infty_T\overline{H}^{-1/2}}\|w_{N_1}\|_{L^2_TH^{s-\frac{3}{2}+\frac{\alpha}{2}}}$$
$$\lesssim\sum_{N_1>2^9}N_1^{\frac{1-\alpha}{6}}\|z_{N_1}\|_{L^2_TH^s}\|w\|_{L^\infty_T\overline{H}^{-1/2}}\|w_{N_1}\|_{L^2_TH^{s-\frac{3}{2}+\frac{\alpha}{2}}}$$
$$\lesssim\|z\|_{L^\infty_TH^s}\|w\|_{L^\infty_T\overline{H}^{-1/2}}\|w\|_{L^\infty_TH^{s-\frac{3}{2}+\frac{\alpha}{2}}}\,. \tag{4.24}$$

Finally the contribution of the sum over $N_1>2^9$ and $N\gg N_1^{\frac{2}{3}(1-\alpha)}$ is bounded thanks to Lemma 4.3 by

$$\sum_{N_1>2^9}\sum_{N\gg N_1^{\frac{2}{3}(1-\alpha)}}\Big[\sum_{l\geq-4}\|w_N\|_{L^\infty_t\overline{H}^{-1/2}}\|Q_{2^lNN_1^\alpha}w_{N_1}\|_{F^{s-\frac{3-\alpha}{2},\frac{1}{2}}}\|z_{N_1}\|_{L^2_tH^s}$$
$$+\|w_N\|_{L^\infty_t\overline{H}^{-1/2}}\|w_{N_1}\|_{L^2_tH^{s-\frac{3-\alpha}{2}}}\|Q_{\sim NN_1^\alpha}z_{N_1}\|_{F^{s,\frac{1}{2}}}$$
$$+N^{-1/8}\langle N\rangle^{\frac{5+\alpha}{8}}N_1^{-\frac{\alpha}{8}-\frac{1}{2}}\|w_N\|_{F^{-\frac{1}{2},\frac{1}{2}}}\|w_{N_1}\|_{L^\infty_tH^{s-\frac{3-\alpha}{2}}}\|z_{N_1}\|_{L^2_tH^s}$$
$$+N^{1/4}N_1^{-3/8}\|w_N\|_{L^\infty_t\overline{H}^{-1/2}}\|w_{N_1}\|_{L^\infty_tH^{s-\frac{3-\alpha}{2}}}\|z_{N_1}\|_{L^\infty_tH^s}$$
$$\lesssim\|z\|_{Y^s}(\|w\|_{\widetilde{L^\infty_t}\overline{H}^{-1/2}}\|w\|_{Z^{s-\frac{3-\alpha}{2}}}+\|w\|_{Z^{-1/2}}\|w\|_{\widetilde{L^\infty_t}\overline{H}^{-1/2}})$$

where again we used Cauchy-Schwarz in (N,N_1) and then sum over l.

Estimates for J_3. We first notice that for $N\lesssim N_1$ and $N_1>2^9$, since $1+2(s-\frac{3-\alpha}{2})\geq 0$, it holds

$$N\langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3-\alpha}{2})}\lesssim N_1\langle N_1^{-1}\rangle\langle N_1\rangle^{2(s-\frac{3-\alpha}{2})}.$$

Therefore the contribution of this region to J_3 is controlled by J_2. Finally the contribution of $N \lesssim N_1 \leq 2^9$ is easily bounded by $\|z\|_{L^\infty_t L^2_x} \|w\|^2_{L^\infty_t \overline{H}^{-1/2}}$.

Gathering all the estimates, we eventually obtain

(4.25)
$$J \lesssim \|z\|_{Y^s} \|w\|_{L^\infty_T \overline{H}^{-1/2}} \|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|z\|_{Y_T^{1-\frac{\alpha}{2}}} \|w\|_{\widetilde{L^\infty_T} H^{s-\frac{3}{2}+\frac{\alpha}{2}}} \|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}$$

which completes the proof of (4.22). □

4.2. **Unconditional well-posedness.** We argue as in Section 3.2. We notice that $1 - \frac{\alpha}{2} \geq 0 > s_c = \frac{1}{2} - \alpha$ that is the critical Sobolev exponent associated with (1.3) for dilation symmetry. Estimates (4.2), (4.3), (4.14), dilations and continuity arguments ensure that the time of existence of a smooth solution is bounded from below by $T = T(\|u_0\|_{H^{1-\frac{\alpha}{2}}}) \sim (1 + \|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$. Passing to the limit on a sequence of smooth solutions we construct a solution $u \in \tilde{Y}^s_T$ to (1.3) emanating from $u_0 \in H^s(\mathbb{R})$. On the other hand, Lemma 4.1, Proposition 4.1 and (4.19) ensure that any $L^\infty_T H^s$-solution to (1.3) on $]0, T[$ belongs to $\widetilde{Y}^s_T$. Therefore, according to (4.20) and (4.22), u is the only solution emanating from u_0 that belongs to $L^\infty_{loc} H^s$. Now the continuity of u with values in $H^s(\mathbb{R})$ as well as the continuity of the flow-map in $H^s(\mathbb{R}^2)$ will follow from the Bona-Smith argument. Let $u_0 \in H^s$ with $s \geq 1 - \frac{\alpha}{2}$. We denote by u^N the solution of (1.3) emanating from $P_{\leq N} u_0$ and we set for $1 \leq N_1 \leq N_2$, we set

$$w := u^{N_1} - u^{N_2}\,.$$

Let us notice that $P_{\leq 1} w_0 = P_{\leq 1}(u^{N_1} - u^{N_2}) = 0$ and thus $\|w_0\|_{\overline{H}^s} \sim \|w_0\|_{H^s}$. It thus follows from (4.20) -(4.22) and dilation arguments that for $T \sim (1 + \|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$ and any $-\frac{1}{2} \leq r \leq s$ it holds

$$\|w\|_{Z^r_T} \lesssim \|w(0)\|_{H^r} \lesssim N_1^{r-s} \varepsilon(N_1) \tag{4.26}$$

with $\varepsilon(y) \to 0$ as $y \to +\infty$. Moreover, on account of Lemma 4.1 and Proposition 4.1, for any $r \geq 0$ we have

$$\|u^{N_i}\|_{Y_T^{s+r}} \lesssim \|u_0^{N_i}\|_{H^{s+r}} \lesssim N_i^r \|u_0\|_{H^s}\,. \tag{4.27}$$

Next, observing that w solves the equation

$$w_t + L_{\alpha+1} w = \frac{1}{2} \partial_x (w^2) + \partial_x (u^{N_1} w)\,, \tag{4.28}$$

we derive the following estimate on w.

Proposition 4.4. *Let $1 < \alpha \leq 2$, $0 < T < 2$ and $w \in L^\infty_T H^s$ with $s \geq 1 - \frac{\alpha}{2}$ be a solution to (4.28). Then it holds*

$$\begin{aligned} \|w\|^2_{L^\infty_T H^s} \lesssim\ & \|w_0\|^2_{H^s} + \|w\|^3_{Z^s_T} \\ & + \|u^{N_1}\|_{Y^s_T} \|w\|^2_{Z^s_T} + \|u^{N_1}\|_{Y_T^{s+\frac{3}{2}-\frac{\alpha}{2}}} \|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} \|w\|_{Z^s_T}\,. \end{aligned} \tag{4.29}$$

Proof. We separate the contribution of $\partial_x(w^2)$ and $\partial_x(u^{N_1}w)$. First (4.17) leads to

$$\sum_N N^{2s}\Big|\int_0^t\int_{\mathbb{R}} P_N\partial_x(w^2)P_N w\Big| \lesssim \|w\|^3_{Y^s_T}\ .$$

Second, applying (4.25) at the level s with z replaced by u^{N_1} we obtain

$$\sum_N N^{2s}\Big|\int_0^t\int_{\mathbb{R}} P_N\partial_x(u^{N_1}w)P_N w\Big| \lesssim \|u^{N_1}\|_{Y_T^{s+\frac{3}{2}-\frac{\alpha}{2}}}\|w\|_{Z_T^{-\frac{1}{2}}}\|w\|_{Z^s_T}+\|u^{N_1}\|_{Y_T^{1-\frac{\alpha}{2}}}\|w\|^2_{Z^s_T}$$

which leads to (4.29) since $s-\frac{3}{2}+\frac{\alpha}{2}\geq -1/2$ for $s\geq 1-\frac{\alpha}{2}$ and $Z^s_T\hookrightarrow Y^s_T$. $\square$

Estimates (4.26)-(4.27) together with (4.29) lead to

$$\begin{aligned}\|w\|^2_{L^\infty_T H^s} &\lesssim \|w_0\|^2_{H^s}+\varepsilon(N_1)+N_1^{-\frac{3}{2}+\frac{\alpha}{2}}\,N_1^{-(-\frac{3}{2}+\frac{\alpha}{2})}\varepsilon(N_1)\\ &\lesssim \varepsilon(N_1)\ .\end{aligned}$$

This shows that $\{u^N\}$ is a Cauchy sequence in $C([0,T];H^s)$ and thus $\{u^N\}$ converges in $C([0,T];H^s)$ to a solution of (1.3) emanating from u_0. Therefore, the uniqueness result ensures that $u\in C([0,T];H^s)$. The proof of the continuity of the solution-map is exactly the same as in Subsection 3.3 as thus will be omitted .

4.3. **The periodic case.** We notice that all our estimates still hold in the periodic case and are uniform with respect to the period $L\geq 1$ as soon as only frequencies with modulus greater or equal to $\frac{2\pi}{L}$ are involved. We thus have only to care about the contribution of the null frequencies. In the regular case, it is not too hard to check that all the estimates still hold when we also consider the contribution of the null frequencies. This is because we only use the resonance hypothesis (1.4) for high input frequencies (see Remark 1.2). In the non regular case, this is no longer true. Anyway, it is easy to check that (1.3) preserves the mean-value and it is well-known that the map $u\mapsto v(t,x):=u(t,x-t⨍u_0)-⨍u_0$ maps a solution of (1.3) with mean-value $⨍u_0$ to a solution of (1.3) with mean value zero. Therefore, up to this change of unknown, we may always assume that our solutions have mean-value zero and thus all the estimates still hold in the periodic setting. The proof of Theorem 1.1 is now completed.

5. Dissipative limits

First, we notice that if u is solution to $(1.6)_\varepsilon$ then u_λ defined by $u_\lambda(t,x)=\lambda^\alpha u(\lambda^{1+\alpha}t,\lambda x)$ is solution to

$$\partial_t u_\lambda + L^\lambda_{\alpha+1}u_\lambda+\varepsilon\lambda^{\alpha+1-\beta}A^\lambda_\beta\, u_\lambda+\frac{1}{2}\partial_x(u_\lambda)^2=0 \tag{5.1}$$

with

$$\widehat{L^\lambda_{\alpha+1}v}(\xi)=i\lambda^{\alpha+1}p_{\alpha+1}(\lambda^{-1}\xi)\hat{v}(\xi)\,.$$

and

$$\widehat{A^\lambda_\beta v}(\xi)=\lambda^\beta q_\beta(\lambda^{-1}\xi)\hat{v}(\xi),\ \forall\xi\in\mathbb{R}\,.$$

Therefore, as in the preceding section, up to this change of unknown, of parameter ε and of operators we may assume that u satisfies (1.6) with $L_{\alpha+1}$ and A_β that verify Hypotheses 1 and 2 for all $0<\lambda\leq 1$.
Second, we notice that Hypothesis 2 now ensures that for $0<\lambda\leq 1$ and $N\gg 1$ dyadic,

$$(A_\beta^\lambda P_N v, P_N v)_{L^2} \gtrsim N^{\beta/2}\|P_N v\|_{L^2}^2 \tag{5.2}$$

and

$$\|A_\beta^\lambda P_N v\|_{L^2} \lesssim N^\beta \|P_N v\|_{L^2}\,. \tag{5.3}$$

The main point is now to prove that the Cauchy problem (1.6) is locally well-posed in H^s uniformly in $\varepsilon>0$.

Proposition 5.1. *Let $1\leq\alpha\leq 2$, $0\leq\beta\leq 1+\alpha$ and $s\geq 1-\frac{\alpha}{2}$.*
For any $\varphi\in H^s(\mathbb{R})$ there exists $T\sim(1+\|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$ and a solution $u_\varepsilon\in C([0,T];H^s)$ to $(1.6)_\varepsilon$ that is unique in some function space[2] *embedded in $L^\infty_T(0,T;H^s)$. Moreover, there exists $C>0$ such that for any $\varepsilon\in]0,1[$,*

$$\sup_{t\in[0,T]}\|u_\varepsilon(t)\|_{H^s}\leq C\|\varphi\|_{H^s}\,. \tag{5.4}$$

Finally, for any $R>0$, the family of solution-maps $S_\varepsilon:\varphi\mapsto u_\varepsilon$, $\varepsilon\in]0,1[$, from $B(0,R)_{H^s}$ into $C([0,T(R)];H^s(\mathbb{R}))$ is equi-continuous, i.e. for any sequence $\{\varphi_n\}\subset B(0,R)_{H^s}$ converging to φ in $H^s(\mathbb{R})$ it holds

$$\lim_{n\to+\infty}\sup_{\varepsilon\in]0,1[}\|S_\varepsilon\varphi-S_\varepsilon\varphi_n\|_{L^\infty(0,T(R);H^s(\mathbb{R}))}=0\,. \tag{5.5}$$

Proof. We treat the cases $(\alpha,s)\neq(1,1/2)$. This last case can be treated in the same way by using the estimates derived in the appendix. First we notice that for $(1.6)_\varepsilon$, in view of (5.2), the energy estimate (4.14) becomes

$$\|u\|_{\widetilde{L^\infty_T}H^s}+\sqrt{\varepsilon}\|u\|_{L^2_TH^{s+\frac{\beta}{2}}}\lesssim\|u_0\|_{H^s}+\|u\|_{L^\infty_TH^{1-\frac{\alpha}{2}}}\|u\|_{Y^s_T}+\|u\|_{L^\infty_TH^s}\|u\|_{Y_T^{1-\frac{\alpha}{2}}}\,. \tag{5.6}$$

On the other hand, viewing $\varepsilon A_\beta u$ as a forced term, (4.2)-(4.3) together with (5.3) lead to

$$\|u\|_{Y^s_T}\lesssim\|u\|_{L^\infty_TH^s}(1+\|u\|^2_{L^\infty_TH^{1-\frac{\alpha}{2}}})+\varepsilon\|u\|_{L^2_TH^{s-\frac{1+\alpha}{2}+\beta}}\,. \tag{5.7}$$

To derive an a priori bound from the above estimates, as in the previous section, we have to use the dilation argument that is described in the beginning of this section. So the dilation function u_λ, defined by $u_\lambda(t,x)=\lambda^\alpha u(\lambda^{1+\alpha}t,\lambda x)$, satisfies (5.1) and we set

$$\|v\|_{N^s}:=\|v\|_{L^\infty_TH^s}+\sqrt{\varepsilon\lambda^{\alpha+1-\beta}}\|v\|_{L^2_TH^{s+\frac{\beta}{2}}}\,.$$

2. For $(\alpha,s)\neq(1,1/2)$, this space is simply the space $L^\infty_TH^s\cap L^2_TH^{s+\frac{\beta}{2}}$

Since $\beta \leq \alpha+1$, this ensures that for $\lambda \lesssim (1+\|\varphi\|_{H^s})^{-\frac{2(\alpha+1)}{2\alpha-1}}$ and $0 < T \leq 2$, it holds

$$\|u_\lambda\|_{N_T^s} \quad \lesssim \quad \|\varphi_\lambda\|_{H^s} + (1+\|u_\lambda\|^2_{N_T^{1-\frac{\alpha}{2}}})\|u_\lambda\|_{N_T^{1-\frac{\alpha}{2}}}\|u_\lambda\|_{N_T^s} \ .$$

with $\|\varphi_\lambda\|_{H^s} \lesssim \lambda^{\alpha-\frac{1}{2}}\|\varphi\|_{H^s} \ll 1$. This leads to the uniform bound (5.4) for smooth solutions to $(1.6)_\varepsilon$ by a classical continuity argument. Then passing to the limit on sequence of smooth solutions we obtain the existence of a solution $u_\varepsilon \in L^\infty_T H^s \cap L^2_T H^{s+\frac{\beta}{2}}$ to $(1.6)_\varepsilon$ with $T \gtrsim (1+\|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$ and $\varphi \in H^s$ as initial data. Obviously, this solution satisfies (5.4).

Now, proceeding in the same way for the difference of two solutions, it is not too hard to check that (4.20) becomes

$$\begin{aligned}\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim & \|u-v\|_{\widetilde{L^\infty_T}\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u-v\|_{L^2_T H^{s-\frac{3}{2}+\frac{\alpha}{2}+\beta}} \\ & + \|u+v\|_{\widetilde{Y}^s_T}\|u-v\|_{Z_T^{-1/2}} + \|u+v\|_{\widetilde{Y}^{1-\frac{\alpha}{2}}}\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} \ .\end{aligned}$$

whereas (4.22) becomes

$$\begin{aligned}\|u_\varepsilon - v_\varepsilon\|_{\widetilde{L^\infty_T}\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \sqrt{\varepsilon}\|u_\varepsilon - v_\varepsilon\|_{L^2_T\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}+\frac{\beta}{2}}} \\ \lesssim \|u_0 - v_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u_\varepsilon + v_\varepsilon\|_{\widetilde{Y}^s_T}\|u_\varepsilon - v_\varepsilon\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} \ .\end{aligned}$$

By the same dilation arguments as above this leads to

$$\|u-v\|_{Z^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \sqrt{\varepsilon}\|u-v\|_{L^2_T H^{s-\frac{3}{2}+\frac{\alpha}{2}+\frac{\beta}{2}}} \lesssim \|u_0 - v_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} \tag{5.8}$$

and proves the uniqueness in the class $L^\infty_{loc}H^s \cap L^2_{loc}H^{s+\beta/2}$. Finally the continuity of the solution and the equi-continuity of the solution-map in $C(0,T;H^s)$ follows from Bona-Smith arguments as in the previous section. □

It is clear that the above proposition implies part (1) of Theorem 1.2. Now, part (2) will follow from general arguments (see for instance [10]). Let us denote by S_ε and S the nonlinear group associated with respectively $(1.6)_\varepsilon$ and (1.3). Let $\varphi \in H^s(\mathbb{R})$, $s \geq 1-\frac{\alpha}{2}$ and let $T = T(\|\varphi\|_{H^{1-\frac{\alpha}{2}}}) > 0$ be given by Proposition 5.1. For any $N > 0$ we can rewrite $S_\varepsilon(\varphi) - S(\varphi)$ as

$$\begin{aligned}S_\varepsilon(\varphi) - S(\varphi) \quad = \quad & \Big(S_\varepsilon(\varphi) - S_\varepsilon(P_{\leq N}\varphi)\Big) + \Big(S_\varepsilon(P_{\leq N}\varphi) - S(P_{\leq N}\varphi)\Big) \\ & + \Big(S(P_{\leq N}\varphi) - S(\varphi)\Big) = I_{\varepsilon,N} + J_{\varepsilon,N} + K_N \ .\end{aligned}$$

By continuity with respect to initial data in $H^s(\mathbb{R})$ of the solution map associated with (1.3), we have $\lim_{N\to\infty}\|K_N\|_{L^\infty(0,T;H^s)} = 0$. Moreover, (5.5) ensures that

$$\lim_{N\to\infty}\sup_{\varepsilon\in]0,1[}\|I_{\varepsilon,N}\|_{L^\infty(0,T;H^s)} = 0 \ .$$

It thus remains to check that for any fixed $N>0$, $\lim_{\varepsilon\to 0}\|J_{\varepsilon,N}\|_{L^\infty(0,T;H^s_x)}=0$. Since $P_{\leq N}\varphi\in H^\infty(\mathbb{R})$, it is worth noticing that $S_\varepsilon(P_{\leq N}\varphi)$ and $S(P_{\leq N}\varphi)$ belong to $C^\infty(\mathbb{R};H^\infty(\mathbb{R}))$. Moreover, according to Theorem 1.2 and Proposition 5.1, for all $\theta\in\mathbb{R}$ and $\varepsilon\in]0,1[$,

$$\|S_\varepsilon(P_{\leq N}\varphi)\|_{L^\infty_T H^\theta_x}+\|S(P_{\leq N}\varphi)\|_{L^\infty_T H^\theta_x}\leq C(N,\theta,\|\varphi\|_{L^2_x})\ .$$

Now, setting $v_\varepsilon:=S_\varepsilon(P_{\leq N}\varphi)$ and $v:=S(P_{\leq N}\varphi)$, we observe that $w_\varepsilon:=v_\varepsilon-v$ satisfies

$$\partial_t w_\varepsilon+L_{\alpha+1}w_\varepsilon=-\frac{1}{2}\partial_x\Big(w_\varepsilon(v+v_\varepsilon)\Big)-\varepsilon A_\beta v_\varepsilon$$

with initial data $w_\varepsilon(0)=0$. For $s\geq 0$, taking the H^s-scalar product of this last equation with w_ε and integrating by parts we get

$$\frac{d}{dt}\|w_\varepsilon\|_{H^s}\lesssim\Big(1+\|\partial_x(v+v_\varepsilon)\|_{L^\infty_x}\Big)\|w_\varepsilon\|^2_{H^s}+\|[J^s\partial_x,(v+v_\varepsilon)]w_\varepsilon\|_{L^2_x}\|w_\varepsilon\|_{H^s}+\varepsilon^2\|D^\beta_x v_\varepsilon\|^2_{H^s}\ .$$

Applying the mean-value theorem on the Fourier transform of the commutator term, it is not too hard to check that

$$\|[J^s_x\partial_x,f]g\|_{L^2_x}\lesssim\|f_x\|_{H^{s+1}}\|g\|_{H^s_x}\ , \tag{5.9}$$

that leads to

$$\frac{d}{dt}\|w_\varepsilon(t)\|^2_{H^s}\lesssim C(N,s+2,\|\varphi\|_{L^2_x})\|w_\varepsilon(t)\|^2_{H^s_x}+\varepsilon^2C(N,s+\beta,\|\varphi\|_{L^2_x})^2\ .$$

Integrating this differential inequality on $[0,T]$, this ensures that $\lim_{\varepsilon\to 0}\|w_\varepsilon\|_{L^\infty(0,T;H^s)}=0$ and proves that

$$u_\varepsilon\longrightarrow u\ \text{ in } C([0,T];H^s) \tag{5.10}$$

with $T\sim(1+\|u_0\|_{H^{1-\frac{\alpha}{2}}})^{-\frac{2(\alpha+1)}{2\alpha-1}}$. Now, the fact that, φ being fixed, the time of existence T_ε of $S_\varepsilon(\varphi)$ in H^s is greater or equal for $\varepsilon>0$ small enough to the time of existence T_0 of $S(\varphi)$ follows by a classical contradiction argument. Indeed, assuming that this is not true, there exists $\varepsilon_n\searrow 0$ such that $\lim T_{\varepsilon_n}=T^*<T_0$. We set

$$\delta=(1+\|S(\varphi)\|_{L^\infty(0,T^*;H^{1-\frac{\alpha}{2}})})^{-\frac{2(\alpha+1)}{2\alpha-1}}$$

which is well-defined since $T^*<T$. Applying (5.10) about T^*/δ times we eventually obtain that for n large enough

$$\|S_{\varepsilon_n}(\varphi)(T^*-\frac{\delta}{100})\|_{H^{1-\frac{\alpha}{2}}}\leq 2\|S(\varphi)\|_{L^\infty(0,T^*;H^{1-\frac{\alpha}{2}})}\ .$$

But then the uniform bound from below on the existence time ensures that $T_{\varepsilon_n}\geq T^*+\delta/2$ that contradicts $\lim T_{\varepsilon_n}=T^*$. This ensures that $T_\varepsilon\geq T_0$ for $\varepsilon>0$ small enough and, for $0<T^*<T_0$, applying (5.10) about T^*/δ times we get (5.10) with $T=T^*$. This completes the proof of Theorem 1.2.

6. Appendix: The case $\alpha = 1$ and $s = 1/2$.

This case is important since $H^{1/2}$ is the energy space for the Benjamin-Ono equation and also the Intermediate Long Waves equation. Unfortunately, we are not able to prove the unconditional well-posedness in this case. However, we are able to prove the well-posedness without using a gauge transform. This is useful to treat perturbations of these equations as we explained in the preceding section. In this section we indicate the modifications of the proofs in this case. In the sequel we set

$$\widetilde{M}^{1/2} := \widetilde{L^\infty_t} H^{1/2} \cap X^{-1/2,1} \ .$$

Lemma 6.1. *Let $\alpha = 1$, $0 < T < 2$, and $u \in \widetilde{M}^{1/2}_T$ be a solution to (1.3). Then it holds*

$$\|u\|_{\widetilde{M}^{1/2}_T} \lesssim \|u\|_{\widetilde{L^\infty_T} H^{1/2}} + \|u\|^2_{\widetilde{M}^{1/2}_T} \ . \tag{6.1}$$

Proof. Working with the extension $\tilde{u} = \rho_T u$ (see (3.3)), still denoted u, if suffices to estimate the $X^{-1/2,1}$-norm of u. First we notice that the low frequency part can be easily controlled by

$$\|P_{\leq 2^9} u\|_{X^{-1/2,1}_T} \lesssim \|u\|^2_{L^\infty_T L^2_x} \ .$$

Now for $N \geq 2^9$, we have

$$\begin{aligned}
\|u_N\|_{X^{-1/2,1}_T} &\lesssim \|P_N u_0\|_{H^{-1/2}} + N^{1/2} \Big\| \sum_{N_2' \sim N_2 \geq N} u_{N_2} u_{N_2'} \Big\|_{L^2_T L^2_x} \\
&\quad + N^{1/2} \Big\| \sum_{N_2 \ll N} P_N(\widetilde{u}_{\sim N}\, u_{N_2}) \Big\|_{L^2_T L^2_x} \\
&= \|P_N u_0\|_{H^{-1/2}} + I_N + II_N \ .
\end{aligned}$$

Clearly, it holds

$$\begin{aligned}
I_N &\lesssim N^{1/2} \sum_{N_2' \sim N_2 \geq N} \|u_{N_2}\|_{L^2_t H^{1/2}} \|u_{N_2'}\|_{L^\infty_t H^{1/2}} \\
&\lesssim \delta_N \|u\|^2_{L^\infty_t H^{1/2}} \ ,
\end{aligned}$$

with $\|(\delta_N)\|_{l^2} \lesssim 1$. On the other hand,

$$\begin{aligned}
II_N &\lesssim N^{1/2} \Big\| \sum_{N_2 \ll N} Q_{\sim N N_2} P_N(u_{\sim N}\, u_{N_2}) \Big\|_{L^2_{tx}} + N^{1/2} \Big\| \sum_{N_2 \ll N} Q_{\not\sim N N_2} P_N(u_{\sim N}\, u_{N_2}) \Big\|_{L^2_{tx}} \\
&\lesssim II^1_N + II^2_N \ .
\end{aligned}$$

By almost orthogonality, we have

$$\begin{aligned}
I\!I_N^1 &\lesssim N^{1/2}\Big(\sum_{N_2\ll N}\Big\|Q_{\sim NN_2}P_N(u_{\sim N}\,u_{N_2})\Big\|^2_{L^2_{tx}}\Big)^{1/2}\\
&\lesssim N^{1/2}\Big(\sum_{N_2\ll N}\|u_{\sim N}\|^2_{L^2_tH^{1/2}}\|u_{N_2}\|^2_{L^\infty_tL^2_x}\Big)^{1/2}\\
&\lesssim \|u_{\sim N}\|_{L^2_tH^{1/2}}\|u\|_{\widetilde{L^\infty_t}H^{1/2}}\\
&\lesssim \delta_N\|u\|_{L^\infty_tH^{1/2}}\|u\|_{\widetilde{L^\infty_t}H^{1/2}}\ ,
\end{aligned}$$

with $\|(\delta_N)\|_{l^2}\lesssim 1$. It remains to control $I\!I_N^2$. Since the Fourier projectors ensure that $\langle\tau-p_2(\xi)\rangle\not\sim NN_2$, the resonance relation (1.4) leads to $|\tau_1-p_2(\xi_1)|\vee|\tau-\tau_1-p_2(\xi-\xi_1)|)\geq NN_2$ for $I\!I_N^2$. We separate the contributions of $Q_{\geq NN_2}\widetilde{u}_{\sim N}$ and $Q_{\geq NN_2}\widetilde{u}_{N_2}$. For the first contribution we have

$$\begin{aligned}
I\!I_N^2 &\lesssim N^{1/2}\sum_{N_2\ll N}(NN_2)^{-1/4}N^{1/4}\|Q_{\geq NN_2}u_{\sim N}\|_{X^{1/4,1/4}}\|u_{N_2}\|_{L^\infty_tH^{1/2}}\\
&\lesssim \|u_{\sim N}\|_{X^{1/4,1/4}}\|u\|_{L^\infty_tH^{1/2}}\\
&\lesssim \delta_N\|u\|^{1/4}_{X^{-1/2,1}}\|u\|^{3/4}_{L^\infty_tH^{1/2}}\|u\|_{L^\infty_tH^{1/2}}\ ,
\end{aligned}$$

with $\|(\delta_N)\|_{l^2}\lesssim 1$ and where we used interpolation at the last step. For the second contribution we write

$$\begin{aligned}
I\!I_N^2 &\lesssim N^{1/2}\sum_{N_2\ll N}\|Q_{<NN_2}u_{\sim N}\|_{L^\infty_tL^4_x}\|Q_{\geq NN_2}u_{N_2}\|_{L^2_tL^4_x}\\
&\lesssim N^{1/2}\sum_{N_2\ll N}N^{-1/4}\|Q_{<NN_2}u_{\sim N}\|_{L^\infty_tH^{1/2}}\|Q_{\geq NN_2}u_{N_2}\|_{L^2_tH^{1/4}}\\
&\lesssim N^{1/2}\sum_{N_2\ll N}N^{-1/4}(NN_2)^{-1/4}\|u_{\sim N}\|_{L^\infty_tH^{1/2}}\|u_{N_2}\|_{X^{1/4,1/4}}\\
&\lesssim \delta_N\|u\|_{\widetilde{L^\infty_t}H^{1/2}}\|u\|^{1/4}_{X^{-1/2,1}}\||u\|^{3/4}_{L^\infty_tH^{1/2}}\ ,
\end{aligned}$$

with $\|(\delta_N)\|_{l^2}\lesssim 1$. Gathering the above estimates, (5.2) follows. □

Lemma 6.2. *Let $\alpha=1$, $0<T<2$ and $u\in\widetilde{M}^{1/2}_T$ be a solution to* (1.3). *Then it holds*

$$\|u\|_{\widetilde{L^\infty_T}H^{1/2}}\lesssim\|u\|_{\widetilde{L^\infty_t}H^{1/2}}+\|u\|_{\widetilde{L^\infty_t}H^{1/2}}\|u\|_{\widetilde{M}^{1/2}_T}\ . \tag{6.2}$$

Proof. We follow the proof of Proposition 4.1. Note that $\widetilde{M}^{1/2}\hookrightarrow\widetilde{Y}^{1/2}$. According to (4.15) it suffices to control

$$I=\sum_{N>0}\sum_{N_1\gtrsim N}N\langle N_1\rangle\sup_{t\in]0,T[}|I_t(u_N,u_{\sim N_1},u_{N_1})|.$$

It is easy to check that the only term of the left-hand side of (4.16) that causes trouble in the case $\alpha=1$ is the first one. This term corresponds to the contribution of $Q_{2^lNN_1^\alpha}u_{N_1}$ and $Q_{\sim NN_1^\alpha}u_{N_1}$. For $\alpha=1$ we control

these contributions by applying Cauchy-Schwarz in (N, N_1). For instance, the contribution of $Q_{2^l N N_1^\alpha} u_{N_1}$ is estimated thanks to Lemma 4.3 by

$$\sum_{N>2^9} \sum_{N_1 \gtrsim N} N\langle N_1\rangle \sum_{l\geq -4} 2^{-l} N^{-1/2} \|u_N\|_{L^2_{tx}} \|Q_{2^l N N_1^\alpha} u_{N_1}\|_{F^{0,1/2}} \|u_{\sim N_1}\|_{L^\infty_t L^2_x}$$
$$\lesssim \sum_{l\geq -4} 2^{-l} \Big(\sum_{N_1 \gtrsim N \geq 2^9} \|u_N\|^2_{L^2_t H^{1/2}} \|u_{\sim N_1}\|^2_{L^\infty_t H^{1/2}} \Big)^{1/2} \Big(\sum_{N_1 \gtrsim N \geq 2^9} \|Q_{2^l N N_1^\alpha} u_{N_1}\|^2_{F^{1/2,1/2}} \Big)^{1/2}$$
$$\lesssim \|u\|_{L^2_t H^{1/2}} \|u\|_{\widetilde{L^\infty_t} H^{1/2}} \|u\|_{F^{1/2,1/2}} \ .$$

□

Lemma 6.3. *Let $0 < T < 1$ and $u, v \in \widetilde{M}^{1/2}_T$ be two solutions to (1.3) on $]0, T[$. Then we have*

(6.3)
$$\|u-v\|_{Z_T^{-\frac{1}{2}}} \lesssim \|u-v\|_{L^\infty_T \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u+v\|_{\widetilde{M}^{1/2}_T} \|u-v\|_{Z_T^{-1/2}} + \|u+v\|_{\widetilde{M}^{1/2}_T} \|u-v\|_{Z_T^{-\frac{1}{2}}} \ .$$

and

$$\|u-v\|_{\widetilde{L^\infty_T} \overline{H}^{-\frac{1}{2}}} \lesssim \|u_0 - v_0\|_{\overline{H}^{-\frac{1}{2}}} + \|u+v\|_{\widetilde{M}^{1/2}_T} \|u-v\|_{Z_T^{-\frac{1}{2}}} \ . \tag{6.4}$$

Remark 6.1. *Actually we could avoid to put a weight on the low frequencies of the difference $u - v$. However, working in $Z_T^{-1/2}$ allow us to use directly some results of Section 4.*

Proof. First we notice that (6.4) is already proven in Proposition 4.3. since $\widetilde{M}^{1/2}_T \hookrightarrow \widetilde{Y}^{1/2}_T \hookrightarrow Y^{1/2}_T$. It remains to prove (6.3). We follow the proof of Proposition 4.1. It is not to hard to check that the only contribution that causes troubles in the right-hand side member of (4.21), in the case $\alpha = 1$, is the contribution of the low-high interaction term : $P_N(P_{\lesssim N} z\, w_N)$. We proceed as in Lemma 6.1 . We take extensions $\widetilde{z}$ and $\widetilde{w}$, supported in $]-4, 4[$ of z and w such that $\|\widetilde{z}\|_{\widetilde{M}^{1/2}} \lesssim \|z\|_{\widetilde{M}^{1/2}_T}$ and $\|\widetilde{w}\|_{Z^{-1/2}} \lesssim \|w\|_{Z_T^{-1/2}}$. For simplicity we drop the tilde. We first notice that the contribution of $P_{\lesssim 1} z$ is easily estimated by

$$\|\partial_x P_N(P_{\lesssim 1} z\, w_{\sim N})\|_{F^{-1/2,1,-1/2}} \lesssim \langle N\rangle^{-1/2} \|P_N(P_{\lesssim 1} z\, w_{\sim N})\|_{L^2_{tx}} \lesssim \|z\|_{L^\infty_t L^2_x} \|w_{\sim N}\|_{L^2_t H^{-1/2}}$$

which is acceptable. Now we decompose the remaining contribution as

$$\begin{aligned}
\|\partial_x P_N(P_{\gg 1} P_{\lesssim N} z w_{\sim N})\|_{F^{-1/2,1,-1/2}} &\lesssim N \| \sum_{1 \ll N_1 \lesssim N} P_N(P_{N_1} z w_{\sim N})\|_{X^{-3/2,0}} \\
&\lesssim \langle N\rangle^{-1/2} \| \sum_{1 \ll N_1 \lesssim N} Q_{\sim N N_1} P_N(P_{N_1} z\, w_{\sim N})\|_{L^2_{tx}} \\
&+ \langle N\rangle^{-1/2} \| \sum_{1 \ll N_1 \lesssim N} Q_{\not\sim N N_1} P_N(P_{N_1} z\, w_{\sim N})\|_{L^2_{tx}} \\
&= J_{1,N} + J_{2,N} \ .
\end{aligned}$$

By almost-orthogonality it holds

$$\begin{aligned} J_{1,N} &\lesssim \langle N\rangle^{-1/2}\Big(\sum_{1\ll N_1\lesssim N}\|Q_{\sim NN_1}P_N(P_{N_1}z\,w_{\sim N})\|^2_{L^2_{tx}}\Big)^{1/2}\\ &\lesssim \langle N\rangle^{-1/2}\Big(\sum_{1\ll N_1\lesssim N}\|P_{N_1}z\|^2_{L^2_tH^{1/2}}\|\,w_{\sim N}\|^2_{L^\infty_{tx}}\Big)^{1/2}\\ &\lesssim \|w_{\sim N}\|_{L^\infty_tH^{-1/2}}\|z\|_{L^2_tH^{1/2}}, \end{aligned}$$

which is acceptable. To treat J_2, we notice that the Fourier projectors ensure that $\langle\tau-p_2(\xi)\rangle\not\sim NN_1$, the resonance relation (1.4) leads to $|\tau_1-p_2(\xi_1)|\vee|\tau-\tau_1-p_2(\xi-\xi_1)|)\geq NN_1$ for $J_{2,N}$. We separate the contributions of $Q_{\geq NN_1}\widetilde{z}_{N_1}$ and $Q_{\geq NN_1}\widetilde{w}_{\sim N}$. For the first contribution we write

$$\begin{aligned} J_{2,N} &\lesssim \langle N\rangle^{-1/2}\sum_{1\ll N_1\lesssim N}N_1^{1/2}\|Q_{\geq NN_1}P_{N_1}z\|_{L^2_{tx}}\|w_{\sim N}\|_{L^\infty_tL^2_x}\\ &\lesssim \langle N\rangle^{-1/2}\sum_{1\ll N_1\lesssim N}(NN_1)^{-1/4}N_1^{1/4}\|Q_{\geq NN_1}P_{N_1}z\|_{X^{1/4,1/4}}\|w_{\sim N}\|_{L^\infty_tL^2_x}\\ &\lesssim \|z\|^{-1/4}_{X^{-1/2,1}}\|z\|^{3/4}_{L^\infty_tH^{1/2}}\|w_{\sim N}\|_{L^\infty_tH^{-1/2}}, \end{aligned}$$

which is acceptable. For the second contribution, according to (4.10) we have

$$\begin{aligned} J_2 &\lesssim \langle N\rangle^{-1/2}\sum_{1\ll N_1\lesssim N}\|z_{N_1}\|_{L^\infty_tH^{1/2}}\|Q_{\geq NN_1}w_{\sim N}\|_{L^2_{tx}}\\ &\lesssim \langle N\rangle^{-1/2}\sum_{1\ll N_1\lesssim N}(NN_1)^{-1}N^{3/2}\|z_{N_1}\|_{L^\infty_tH^{1/2}}\|w_{\sim N}\|_{F^{-1/2,1/2}}\\ &\lesssim \|w_{\sim N}\|_{F^{-1/2,1/2}}\|z\|_{L^\infty_tH^{1/2}} \end{aligned}$$

which is acceptable. Gathering the above estimates we obtain (6.3). □

Gathering Lemmas 6.1-6.3 and proceeding as in Subsection 4.2 we obtain the local-well-posedness in $H^{1/2}$ of (1.3) for $\alpha=1$. Note that the uniqueness holds in the space $\widetilde{M}^{1/2}_T$.

References

1. L. Abdelouhab, J. L. Bona, M. Felland, and J-C. Saut, *Nonlocal models for nonlinear, dispersive waves,* Phys. D, **40** (1989), no. 3, 360–392.
2. A. Babin, A. Ilyin and E.S. Titi, Edriss, *On the regularization mechanism for the periodic Korteweg-de Vries equation,*) Comm. Pure Appl. Math. **64** (2011), 591–648.
3. J.L. Bona, R. Smith, *The initial-value problem for the Korteweg-de Vries equation*, Philos. Trans. Roy. Soc. London Ser. A 278 (1975), 1287, 555–601.
4. J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and application to nonlinear evolution equations, I. Schrödinger equations II. The KdV equation,* GAFA, **3** (1993), 209–262.
5. J. Bourgain, *On the Cauchy problem for the Kadomtsev-Petviashvili equation,* GAFA, **3** (1993), 315–341.

6. N. Burq, F. Planchon, *On well-posedness for the Benjamin-Ono equation,* Math. Annal., **440** (2008), 497–542.
7. P.M. Edwin, B. Roberts, *The Benjamin-Ono-Burgers equation: an application in solar physics,* Wave Motion 8 (2) (1986) 151–158.
8. J. Ginibre, *Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace (d'après Bourgain),* Astérisque, **237** (1996), 163–187.
9. Z. Guo, L. Peng, B. Wang and Y. Wang *Uniform well-posedness and inviscid limit for the Benjamin-Ono-Burgers equation* , Advances in Mathematics **228** (2011), 647-677.
10. Z. Guo and B. Wang, *Global well-posedness and inviscid limit for the Korteweg-de Vries-Burgers equation,* J. Diff. Eq., **246** (2009), 3864–3901.
11. S. Herr, A. D. Ionescu, C. E. Kenig and H. Koch, *A para-differential renormalization technique for nonlinear dispersive equations,* Comm. Partial Differential Equations **35** (2010), 1827–1875.
12. A. D. Ionescu and C. E. Kenig, *Global well-posedness of the Benjamin-Ono equation in low-regularity spaces,* J. Amer. Math. Soc., **20** (2007), 753–798.
13. A. D. Ionescu, C. Kenig, D. Tataru, *Global well-posedness of the initial value problem for the KP-I equation in the energy space*, Invent. Math. **1**73 2 (2008), 265–304.
14. T. Kato, *On the Cauchy problem for the (generalized) Korteweg-de Vries equation*, Stud. Appl. Math. **8** (1983), 93–126.
15. C. E. Kenig, G. Ponce, and L. Vega, newblock *Well-posedness of the initial value problem for the Korteweg-de Vries equation*, J. Amer. Math. Soc. **4** (1991), 323–347
16. C. E. Kenig, G. Ponce and L. Vega, *Well-posedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle,* Comm. Pure Appl. Math. **46** (1993), 527–620.
17. C. Kenig and K. Koenig, *On the local well-posedness of the Benjamin-Ono and modified Benjamin-Ono equations*, Math. Res. Letters **10** (2003) 879-895.
18. H. Koch and D.Tataru, *A priori bounds for the 1D cubic NLS in negative Sobolev spaces* Int. Math. Res. Not. (2007), no. 16.
19. H. Koch and N. Tzvetkov, *Local well-posedness of the Benjamin-Ono equation in* $H^s(\mathbb{R})$, I.M.R.N. **26** (2003) 1449-1464.
20. N. Masmoudi and K. Nakanishi, *From the Klein-Gordon-Zakharov system to the nonlinear Schrödinger equation* , J. Hyperbolic Differ. Equ. **2** (2005), 975–1008.
21. L. Molinet, *A note on the inviscid limit of the Benjamin-Ono-Burgers equation in the energy space,* Proc. Amer. Math. Soc. **14** (2013), 2793–2798.
22. L. Molinet and F. Ribaud, *On the low regularity of the Korteweg-de Vries-Burgers equation,* I.M.R.N. **37** (2002), 1979-2005.
23. L. Molinet and D. Pilod, *The Cauchy problem for the Benjamin-Ono equation in* L^2 *revisited,* Anal. PDE, **5** (2012), No. 2, 365–395.
24. L. Molinet, J-C. Saut and N. Tzvetkov, *Ill-posedness issues for the Benjamin-Ono and related equations,* SIAM J. Math. Anal., **33** (2001), no. 4, 982–988.
25. E. Ott and N. Sudan, *Damping of solitary waves*, Phys. Fluids, **13**(6) (1970), pp. 1432–1434.
26. G. Ponce, *On the global well-posedness of the Benjamin-Ono equation,* Diff. Int. Eq. **4** (1991), 527–542.
27. R. Smith, *Nonlinear Kelvin and continental-shelf waves*, J. Fluid Mech. **57** (1972), 379-391.
28. T. Tao, *Global well-posedness of the Benjamin-Ono equation in* $H^1(\mathbb{R})$*,* J. Hyp. Diff. Eq. **1** (2004), 27–49.

29. Y. Zhou, *Uniqueness of weak solution of the KdV equation*, I.M.R.N. (1997), no. 6, 271–283.

Laboratoire de Mathématiques et Physique Théorique, Universté Francois Rabelais Tours,, CNRS UMR 7350- Fédération Denis Poisson, Parc Grandmont, 37200 Tours, France

Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS (UMR 7539), 99, avenue Jean-Baptiste Clément, F-93 430 Villetaneuse, France