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# Traces of weighted Sobolev spaces. Old and new

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## Abstract

We give short simple proofs of Uspenskii's results characterizing Besov spaces as trace spaces of weighted Sobolev spaces. We generalize Uspenskii's results and prove the optimality of these generalizations. We next show how classical results on the functional calculus in the Besov spaces can be obtained as straightforward consequences of the theory of weighted Sobolev spaces.

## Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Preliminaries</b>	<b>6</b>
<b>3</b>	<b>Direct trace theorem: proof of Theorem 1.3</b>	<b>10</b>
<b>4</b>	<b>Inverse trace theorem: proof of Theorem 1.4</b>	<b>12</b>
<b>5</b>	<b>Further results</b>	<b>14</b>
<b>6</b>	<b>From weighted spaces to functional calculus</b>	<b>28</b>

## 1 Introduction

### 1.1 Traces

Let us start by recalling three well-known facts from trace theory. First fact: every function  $U \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$  has a trace  $f \in L^1(\mathbb{R}^n)$ . Second fact: if  $f \in L^1(\mathbb{R}^n)$ , then there is some  $U = U(f) \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$  the trace of which is  $f$ . Third fact: one cannot pick  $U$  such that the mapping  $f \mapsto U$  is linear continuous.

The first two facts are due to Gagliardo [9], the third one to Peetre [13].

Uspenskii [26] discovered that the expected generalizations of the first and of the third fact to two (or more) derivatives are wrong. More specifically, Uspenskii proved the following results.

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First fact: if  $U \in W^{2,1}(\mathbb{R}^n \times (0, \infty))$ , then  $U$  has a trace  $f$  in the Besov space  $B_{1,1}^1(\mathbb{R}^n)$ , which is strictly contained in  $W^{1,1}(\mathbb{R}^n)$ . Second fact: if  $f \in B_{1,1}^1(\mathbb{R}^n)$ , then there is some  $U = U(f) \in W^{2,1}(\mathbb{R}^n \times (0, \infty))$  the trace of which is  $f$ . Third fact: one can pick  $U$  such that the mapping  $f \mapsto U$  is linear continuous.

As one may expect, the above results contain some routine information: e.g. when  $U \in W^{2,1}$  we have  $f \in L^1$ , and when  $f \in B_{1,1}^1$  we may pick  $U \in W^{1,1}$ . Additionally, by straightforward arguments we may assume that  $f$  and  $U$  are smooth and that  $f$  is compactly supported. Therefore, the heart of the proof consists in proving the maximal order estimates for smooth functions, e.g. the fact that the second order derivatives of  $U$  are controlled by a suitable semi-norm of  $f$  in  $B_{1,1}^1$ . With this in mind, Uspenskii's results can be essentially rephrased as follows.

**1.1 Theorem.** Let  $s > 0$ ,  $1 \leq p < \infty$  and let  $l$  be an integer such that  $l > s$ . Let  $F \in C^\infty(\mathbb{R}^n \times [0, \infty))$ . Set  $f(x) = F(x, 0)$ ,  $x \in \mathbb{R}^n$ . Then

$$|f|_{B_{p,p}^s}^p \lesssim \sum_{|\alpha|=l} \int_0^\infty \varepsilon^{p(l-s)-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^p d\varepsilon. \quad (1.1)$$

**1.2 Theorem.** Let  $s > 0$  and  $1 \leq p < \infty$ . Let  $f \in C_c^\infty(\mathbb{R}^n)$ . Then  $f$  has an extension  $U \in C^\infty(\mathbb{R}^n \times [0, \infty))$  such that:

$$\int_0^\infty \varepsilon^{p(|\alpha|-s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^p d\varepsilon \lesssim |f|_{B_{p,p}^s}^p, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| > s. \quad (1.2)$$

Moreover, we may choose  $U$  depending linearly on  $f$ .

One recovers the results on the trace of  $W^{2,1}$  by taking in Theorems 1.1 and 1.2  $s = 1$ ,  $p = 1$  and  $|\alpha| = l = 2$ .

In the above, the suitable semi-norms to be considered on  $B_{p,p}^s$  will be described in the body of the paper.

Uspenskii's proof is rather elementary, but long and tricky. The above results are quoted in Maz'ya [11, Section 10.1.1, Theorem 1, p. 512], with a proof of Theorem 1.1 and a partial proof of Theorem 1.2. The first goal of this paper is to present a very short proof of the above results, based only on standard ingredients. Our arguments apply to more general Besov spaces and range of partial derivatives  $\partial^\alpha$ , and yield Theorems 1.3 and 1.4 below.

Before stating these results, let us introduce some notation. A multi-index  $\alpha \in \mathbb{N}^{n+1}$  is split as  $\alpha = (\beta, \gamma)$ , with  $\beta \in \mathbb{N}^n$  and  $\gamma \in \mathbb{N}$ .

Given an integer  $l$ , we set

$$\mathcal{M}_l = \{(\beta, 0) \in \mathbb{N}^{n+1}; |\beta| = l\} \cup \{(0, l)\}. \quad (1.3)$$

Given a real  $s$ , we set

$$\mathcal{P}_s = \{(\beta, \gamma) \in \mathbb{N}^{n+1}; \gamma > 0\} \cup \{(\beta, 0) \in \mathbb{N}^{n+1}; |\beta| > s\}. \quad (1.4)$$

We may now state our first results.

**1.3 Theorem.** Let  $s > 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ , and let  $l$  be an integer such that  $l > s$ . Let  $F \in C^\infty(\mathbb{R}^n \times [0, \infty))$ . Set  $f(x) = F(x, 0)$ ,  $x \in \mathbb{R}^n$ . Then

$$|f|_{B_{p,q}^s}^q \lesssim \sum_{\alpha \in \mathcal{M}_l} \int_0^\infty \varepsilon^{q(l-s)-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon. \quad (1.5)$$

**1.4 Theorem.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in C_c^\infty(\mathbb{R}^n)$ . Then  $f$  has an extension  $U \in C^\infty(\mathbb{R}^n \times [0, \infty))$  such that:

$$\int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q, \quad \forall \alpha \in \mathcal{P}_s. \quad (1.6)$$

Moreover, we may choose  $U$  depending linearly on  $f$ .

The semi-norms we consider in Theorems 1.3 and 1.4 will be specified in the body of the paper.

**1.5 Remark.** Clearly, Theorem 1.3 improves Theorem 1.1 as soon as  $l \geq 2$ .

On the other hand, Theorem 1.4 improves Theorem 1.2 even when  $s > 0$ , since we allow  $|\alpha| \leq s$  provided we have  $\gamma > 0$ . Thus, for example, when  $f \in B_{1,1}^1$ , then we may pick  $U$  such that not only  $U \in W^{2,1}$ , but we also have

$$\int_0^\infty \varepsilon^{-1} \left\| \frac{\partial U}{\partial \varepsilon}(\cdot, \varepsilon) \right\|_{L^1(\mathbb{R}^n)} d\varepsilon < \infty.$$

**1.6 Remark.** An explicit example of function  $f$  which is in  $W^{1,1}$  but not in  $B_{1,1}^1$ , and thus is not the trace of any  $W^{2,1}$  function, can be found in [6, Remark A.1, p. 1238].

**1.7 Remark.** Theorem 1.3 is inspired by Maz'ya's remarkably simple proof of Theorem 1.1 [11, Section 10.1.1, Theorem 1, p. 512–513], and the proof we present in Section 3 merely extends the ideas in [11].

In contrast, Theorem 1.2 is more difficult to prove, at least when  $l \geq 2$ . When  $l \geq 2$  the most delicate part in Uspenskii's proof of Theorem 1.2 consists in controlling the cross terms  $\partial^\alpha U$ , with  $\alpha = (\beta, \gamma)$  such that  $\beta \neq 0$  and  $\gamma \neq 0$ ; and this is the part of the proof missing in [11]. Most of the variants of Theorem 1.2 that we could find in the literature do not involve the cross terms, and are thus easier to establish than Theorem 1.2. Such variants were e.g. obtained by Triebel [22] and Bui [7] following a pioneering work of Taibleson on Hölder spaces [19], [20]. See also the historical references in Triebel's monographs [21, p. 192–196], [23, p. 184], [24, p. 52–54].

The reader may wonder why, when  $\gamma = 0$ , we impose in Theorem 1.4 the condition  $|\beta| > s$ . The reason is given by the following simple result.

**1.8 Proposition.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in C_c^\infty(\mathbb{R}^n)$ ,  $f \neq 0$ . Let  $F \in C^\infty(\mathbb{R}^n \times [0, \infty))$  satisfy  $F(\cdot, 0) = f$ . Then

$$\int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = \infty, \quad \forall \alpha \notin \mathcal{P}_s. \quad (1.7)$$

We now return to Theorem 1.2. Uspenskii proved that, in Theorem 1.2, we may let  $U$  be “the” harmonic extension of  $f$ .<sup>1</sup> This still the case in the setting of Theorem 1.4, but in general not for the full set  $\mathcal{P}_s$  of multi-indices.

**1.9 Theorem.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in C_c^\infty(\mathbb{R}^n)$ . Then the harmonic extension  $V$  of  $f$  satisfies:

$$\int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha V(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| > s. \quad (1.8)$$

The condition  $|\alpha| > s$  is optimal, as shown by the following

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<sup>1</sup>That is,  $U(x, \varepsilon) = f * P_\varepsilon(x)$ , where  $P$  is the Poisson kernel.

**1.10 Proposition.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in C_c^\infty(\mathbb{R}^n)$ ,  $f \neq 0$ . Let  $V$  be the harmonic extension of  $f$ . Then

$$\int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha V(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = \infty, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| \leq s. \quad (1.9)$$

When  $s > 0$ , by combining Theorem 1.3 with Theorem 1.9 we obtain the first part of the next result, already noticed by Uspenskii when  $p = q$ .

In order to state Theorem 1.11 below, let us define, for  $F \in C^\infty(\mathbb{R}^n \times [0, \infty))$ , the “energy”

$$E(F) = E_{l,s,p,q}(F) = \sum_{|\alpha|=l} \int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon, \quad (1.10)$$

where  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$ , and  $l \in \mathbb{N}$ .

**1.11 Theorem.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $l > s$  be a non negative integer.

1. If  $s > 0$ , then for every  $f \in C_c^\infty(\mathbb{R}^n)$  the harmonic extension  $V$  of  $f$  satisfies the following “almost Dirichlet principle”

$$E(V) \lesssim E(F), \text{ for every smooth extension } F \text{ of } f. \quad (1.11)$$

2. If  $s \leq 0$ , then we have the semi-norm equivalence

$$E(V) \sim |f|_{B_{p,q}^s}^q. \quad (1.12)$$

**1.12 Remark.** When  $s < 0$ ,  $l = 0$ , and  $p = q \in (1, \infty)$ , item 2 in Theorem 1.11 was obtained by Marcus and Véron [10].<sup>2</sup> The approach in [10], based on interpolation, excludes the case where  $p = 1$ .

When  $s \leq 0$ , the conclusions of Theorem 1.3 and item 1 in Theorem 1.11 do not hold, as shown by the next two results.

**1.13 Proposition.** Let  $f \in C_c^\infty(\mathbb{R}^n)$ .

1. Let  $s < 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Then there exists a sequence  $(F_j) \subset C_c^\infty(\mathbb{R}^n \times [0, \infty))$  of extensions of  $f$  such that

$$\lim_{j \rightarrow \infty} \int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha F_j(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = 0, \quad \forall \alpha \in \mathbb{N}^{n+1}. \quad (1.13)$$

2. Let  $1 \leq p \leq \infty$  and  $1 < q < \infty$ . Then we may choose  $F_j$  such that

$$\lim_{j \rightarrow \infty} \int_0^\infty \varepsilon^{q|\alpha|-1} \|\partial^\alpha F_j(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = 0, \quad \forall \alpha \in \mathbb{N}^{n+1} \setminus \{(0, 0)\}. \quad (1.14)$$

The above proposition shows that (1.11) does not hold when  $s < 0$  or when  $s = 0$  and  $q > 1$ . The case where  $s = 0$  and  $q = 1$  is more delicate. Indeed, if  $F \in C_c^\infty(\mathbb{R}^n \times [0, \infty))$  is an extension of  $f \in C_c^\infty(\mathbb{R}^n)$ , then

$$\int_0^\infty \left\| \frac{\partial}{\partial \varepsilon} F(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)} d\varepsilon \geq \left\| \int_0^\infty \frac{\partial}{\partial \varepsilon} F(\cdot, \varepsilon) d\varepsilon \right\|_{L^p(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)},$$

and thus (1.14) with  $q = 1$  does not hold when  $f \neq 0$  and  $\alpha = (0, 1)$ .

In this case, we establish the following substitute of Proposition 1.13.

<sup>2</sup>In a ball, but this is not relevant for the result.

**1.14 Proposition.** Let  $1 \leq p \leq \infty$  and let  $l > 0$  be an integer. Then:

1. Every  $f \in C_c^\infty(\mathbb{R}^n)$  has an extension  $F \in C^\infty(\mathbb{R}^n \times [0, \infty))$  such that

$$\int_0^\infty \varepsilon^{l-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} d\varepsilon \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| = l. \quad (1.15)$$

2. There exists a sequence  $(f^k) \subset C_c^\infty(\mathbb{R}^n)$  such that  $\|f^k\|_{L^p(\mathbb{R}^n)} = 1$  and  $|f^k|_{B_{p,1}^0} \rightarrow \infty$ .

The first item of Proposition 1.14 can be seen as an  $L^p$  version of Gagliardo's inverse trace result “ $\text{tr } W^{1,1} \supset L^1$ ”.

To summarize: on the one hand, Theorems 1.3 and 1.4 and Proposition 1.8 exhaust the (non) estimates that can be achieved using extensions. On the other hand, Theorem 1.9 and Proposition 1.10 describe all the (non) estimates satisfied by the harmonic extension. Finally, Theorem 1.11 and Propositions 1.13 and 1.14 give necessary and sufficient conditions for the validity of the “almost Dirichlet principle”.

We next return to Peetre's result on the non existence of a linear continuous map

$$L^1(\mathbb{R}^n) \ni f \mapsto U(f) \in W^{1,1}(\mathbb{R}^n \times (0, \infty))$$

such that  $\text{tr } U(f) = f$  [13]. This result has the following consequence. Let  $\zeta \in C_c^\infty(\mathbb{R})$  be such that  $\zeta(0) = 1$ . Consider the linear map

$$L^1(\mathbb{R}^n) \ni f \mapsto U(f)(x, \varepsilon) = \zeta(\varepsilon)V(x, \varepsilon), \text{ with } V \text{ the harmonic extension of } f.$$

Since clearly  $\text{tr } U(f) = f$  and  $f \mapsto U(f) \in L^1(\mathbb{R}^n \times (0, \infty))$  is linear continuous, we find (by Peetre's result and a straightforward closed graph argument) that for an arbitrary  $f \in L^1(\mathbb{R}^n)$  we need not have  $\nabla U \in L^1$ , and thus that we need not have  $\nabla V \in L^1$ . This leads to the question answered in our next result.

**1.15 Theorem.** Let  $f \in L^1(\mathbb{R}^n)$ . Let  $V$  be the harmonic extension of  $f$ . Then

$$\nabla V \in L^1(\mathbb{R}^n \times (0, \infty)) \iff f \in \dot{B}_{1,1}^0.$$

Moreover, we have the norm equivalence

$$\int_0^1 \int_{\mathbb{R}^n} |V(x, \varepsilon)| dx d\varepsilon + \int_0^\infty \int_{\mathbb{R}^n} |\nabla V(x, \varepsilon)| dx d\varepsilon \sim \|f\|_{L^1(\mathbb{R}^n)} + |f|_{B_{1,1}^0}. \quad (1.16)$$

Here,  $\dot{B}_{1,1}^0$  is a homogeneous Besov space that will be described in the body of the paper, as well as its corresponding semi-norm  $| \cdot |_{B_{1,1}^0}$ .

**1.16 Remark.** Condition  $f \in \dot{B}_{1,1}^0$  is quite restrictive, and a map in  $C_c^\infty$  may not belong to  $\dot{B}_{1,1}^0$  (see Proposition 5.8). The reason is that even if  $f$  is smooth,  $\nabla V(\cdot, \varepsilon)$  may not decay sufficiently fast as  $\varepsilon \rightarrow \infty$ . Thus the following version of Theorem 1.15 accounts better of the smoothness of  $f$ .

**1.17 Theorem.** Let  $f \in L^1(\mathbb{R}^n)$ . Let  $V$  be the harmonic extension of  $f$ . Then we have the norm equivalence

$$\int_0^1 \int_{\mathbb{R}^n} |V(x, \varepsilon)| dx d\varepsilon + \int_0^1 \int_{\mathbb{R}^n} |\nabla V(x, \varepsilon)| dx d\varepsilon \sim \|f\|_{B_{1,1}^0}. \quad (1.17)$$

More generally, when  $s \leq 0$  local versions of our results (i.e., not involving  $\int_0^\infty d\varepsilon$ , but only  $\int_0^1 d\varepsilon$ ) are needed for characterizing the inhomogeneous spaces  $B_{p,q}^s$ .

**1.18 Remark.** A general remark concerning the above results, and in particular Uspenskii's Theorems 1.1 and 1.2. The most difficult part consists in estimating  $U$  (or  $V$ ) in terms of  $f$ . As we will see in the proofs, estimates are much easier when, instead of considering the harmonic extension  $V = f * P_\varepsilon$ , we consider another extension  $U = f * \rho_\varepsilon$ , where  $\rho$  is a better suited kernel. Actually, for such  $U$  estimates can be obtained almost for free and using very little technology; see Section 4. The difficulties arise from the bad properties of the Poisson kernel  $P$ . We will explain how to cope with this using methods developed in the late 60's in the theory of the Hardy spaces (and detailed in Stein's monograph [17, Chapter III]).

**1.19 Remark.** Estimates which do not involve cross terms adapt to the case where  $q = \infty$  [22], [7]. When  $q = \infty$ , the case of cross terms has not been investigated in the present work. Some of our arguments relying on Hardy's inequalities with exponent  $q$  (which are not valid when  $q = \infty$ ), extending our results to this limiting case would require a new ingredient.

Our paper is organized as follows. The basic facts on Besov spaces relevant to the statements and proofs are recalled in Section 2. In Section 3, we prove Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.4. Additional results implying in particular the optimality of Theorems 1.3 and 1.4 are discussed in Section 5. Also in Section 5, we prove Theorem 1.15. In Section 6, we explain how to recover standard results on the superposition operators in Besov spaces by combining the theory of weighted Sobolev spaces with modest technology.

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## 2 Preliminaries

### Hardy inequalities and Nikolskiĭ's estimates

We work in  $\Omega := \mathbb{R}^n \times \mathbb{R}_+$ .

A point in  $\Omega$  is of the form  $(x, \varepsilon)$ ,  $x \in \mathbb{R}^n$ ,  $\varepsilon \geq 0$ .

As already mentioned in the introduction, a multi-index  $\alpha \in \mathbb{N}^{n+1}$  is split as  $\alpha = (\beta, \gamma)$ , with  $\beta \in \mathbb{N}^n$  and  $\gamma \in \mathbb{N}$ .

As standard when working in usual function spaces, we deal only with smooth maps. The case of general maps is obtained from the special case of smooth maps using standard techniques. Ideally, we would like to deal only with compactly supported functions, but when we work in  $\Omega$  it will be more convenient to consider the slightly larger class of smooth maps  $U \in C^\infty(\Omega; \mathbb{C})$  satisfying the following decay condition at infinity:

$$\frac{\partial^\gamma}{\partial \varepsilon^\gamma} \frac{\partial^{|\beta|}}{\partial x^\beta} U(x, \varepsilon) = - \int_\varepsilon^\infty \frac{\partial^{\gamma+1}}{\partial \varepsilon^{\gamma+1}} \frac{\partial^{|\beta|}}{\partial x^\beta} U(x, t) dt, \quad \forall \gamma \in \mathbb{N}, \quad \forall \beta \in \mathbb{N}^n, \quad \forall (x, \varepsilon) \in \Omega. \quad (2.1)$$

Note that  $C_c^\infty(\Omega)$  maps satisfy (2.1). So do functions of the form  $U(x, \varepsilon) = f * \rho_\varepsilon(x)$ , where  $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$  and  $\rho \in \mathcal{S}'(\mathbb{R}^n)$ . Another example of such map is "the" harmonic extension  $U(x, \varepsilon) = f * P_\varepsilon(x)$  (with  $P$  the Poisson kernel) of  $f \in C_c^\infty(\mathbb{R}^n; \mathbb{C})$ .

In addition to straightforward identities and estimates, our arguments rely on two simple well-known results: Hardy's inequalities and Nikolskii's estimates, that we recall here.

**2.1 Proposition.** Let  $q \in [1, \infty)$ ,  $r \in (0, \infty)$  and let  $g$  be a nonnegative measurable function. Then we have "Hardy's inequality at 0"

$$\left( \int_0^\infty t^{-r-1} \left( \int_0^t g(u) du \right)^q dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty u^{-r+q-1} (g(u))^q du \right)^{1/q} \quad (2.2)$$

and "Hardy's inequality at  $\infty$ "

$$\left( \int_0^\infty t^{r-1} \left( \int_t^\infty g(u) du \right)^q dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty u^{r+q-1} (g(u))^q du \right)^{1/q}. \quad (2.3)$$

**2.2 Proposition.** Let  $0 < r_1 < r_2 < \infty$  and  $1 \leq p \leq \infty$  be fixed. Then for every  $\alpha \in \mathbb{N}^n$ ,  $u \in L^p(\mathbb{R}^n)$  and  $R > 0$  we have the "direct Nikolskii's estimates"

$$\text{supp } \hat{u} \subset B(0, R) \implies \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)} \lesssim R^{|\alpha|} \|u\|_{L^p(\mathbb{R}^n)} \quad (2.4)$$

and the "reverse Nikolskii's estimates"

$$\text{supp } \hat{u} \subset B(0, r_2 R) \setminus B(0, r_1 R) \implies \|u\|_{L^p(\mathbb{R}^n)} \lesssim R^{-k} \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^p(\mathbb{R}^n)}. \quad (2.5)$$

See e.g. [18, Chapter 5, Lemma 3.14] for the first result, and [8, Lemma 2.1.1] for the second one.

## Some basic facts about the Besov spaces

The Besov spaces  $B_{p,q}^s(\mathbb{R}^n)$  can be defined for  $0 < p \leq \infty$  and  $0 < q \leq \infty$ , but we discuss here only the range  $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$ , which relevant for our results on traces.

We first focus on inhomogeneous Besov spaces. Fix a sequence of functions  $(\varphi^j)_{j \geq 0} \in \mathcal{S}(\mathbb{R}^n)$  such that:

1.  $\text{supp } \varphi^0 \subset B(0, 2)$  and  $\text{supp } \varphi^j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$  for all  $j \geq 1$ .
2. For all multi-index  $\alpha \in \mathbb{N}^n$ , there exists  $c_\alpha > 0$  such that  $|D^\alpha \varphi^j(x)| \leq c_\alpha 2^{-j|\alpha|}$ , for all  $x \in \mathbb{R}^n$  and all  $j \geq 0$ .
3. For all  $x \in \mathbb{R}^n$ , it holds  $\sum_{j \geq 0} \varphi^j(x) = 1$ .

An example of such a sequence  $(\varphi^j)_{j \geq 0}$  is given by  $\varphi^0 = \varphi$  and

$$\varphi^j = \varphi_{2^{-j}} - \varphi_{2^{1-j}}, \quad \forall j \geq 1, \text{ where } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ satisfies } \text{supp } \hat{\varphi} \subset B(0, 2) \text{ and } \hat{\varphi} \equiv 1 \text{ in } \overline{B}(0, 3/2). \quad (2.6)$$

Let  $f \in \mathcal{S}'(\mathbb{R}^n)$ . For all  $j \geq 0$ , let  $f_j := f * \varphi^j$ . One has

$$f = \sum_{j \geq 0} f_j, \text{ where the series converges in } \mathcal{S}'. \quad (2.7)$$

**2.3 Definition** (Definition of inhomogeneous Besov spaces). Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Define  $B_{p,q}^s = B_{p,q}^s(\mathbb{R}^n)$  as the space of tempered distributions  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$\|f\|_{B_{p,q}^s} = \|f\|_{B_{p,q}^s(\mathbb{R}^n)} := \left( \sum_{j \geq 0} 2^{sjq} \|f_j\|_{L^p}^q \right)^{1/q} < \infty.$$



Recall [23, Section 2.3.2, Proposition 1, p. 46] that  $B_{p,q}^s(\mathbb{R}^n)$  is a Banach space which does not depend on the choice of the sequence  $(\varphi^j)_{j \geq 0}$ , in the sense that two different choices for the sequence  $(\varphi^j)_{j \geq 0}$  give rise to equivalent norms. Once the  $\varphi^j$ 's are fixed, we refer to the equality  $f = \sum_{j \geq 0} f_j$  in  $\mathcal{S}'$  as the (inhomogeneous) Littlewood-Paley decomposition of  $f$ .

Let us now turn to the definition of homogeneous Besov spaces. Let  $(\varphi^j)_{j \in \mathbb{Z}}$  be a sequence of functions satisfying:

1.  $\text{supp } \varphi^j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1})$  for all  $j \in \mathbb{Z}$ .
2. For all multi-index  $\alpha \in \mathbb{N}^n$ , there exists  $c_\alpha > 0$  such that  $|D^\alpha \varphi^j(x)| \leq c_\alpha 2^{-j|\alpha|}$ , for all  $x \in \mathbb{R}^n$  and all  $j \in \mathbb{Z}$ .
3. For all  $x \in \mathbb{R}^n$ , it holds  $\sum_{j \in \mathbb{Z}} \varphi^j(x) = 1$ .

An example of such a sequence  $(\varphi^j)_{j \in \mathbb{Z}}$  is given by

$$\varphi^j = \varphi_{2^{-j}} - \varphi_{2^{1-j}}, \quad \forall j \in \mathbb{Z}, \quad \text{where } \varphi \in \mathcal{S}(\mathbb{R}^n) \text{ satisfies } \int_{\mathbb{R}^n} \varphi = 1. \quad (2.8)$$

Define again  $f_j := f * \varphi^j$  for all  $j \in \mathbb{Z}$ . Note that, in general, when  $f \in \mathcal{S}'(\mathbb{R}^n)$ , the sum  $\sum_{j \in \mathbb{Z}} f_j$  is not  $f$ . E.g. when  $f \equiv 1$  we have  $f_j \equiv 0$ ,  $\forall j \in \mathbb{Z}$ . However, we have

$$f = \sum_{j \in \mathbb{Z}} f_j \text{ in } \mathcal{S}'(\mathbb{R}^n) \text{ if } f \in L^p(\mathbb{R}^n) \text{ for some } 1 \leq p < \infty. \quad (2.9)$$

Although the equality  $f = \sum_j f_j$  does not always hold, the series  $\sum_{j \in \mathbb{Z}} f_j$  will be referred to as “the homogeneous Littlewood-Paley decomposition” of  $f$ , and we write “ $f = \sum_{j \in \mathbb{Z}} f_j$ ”.

In the special case where the  $\varphi^j$ 's are as in (2.8), the homogeneous Littlewood-Paley decomposition reads

$$f = \sum_{j \in \mathbb{Z}} f_j, \quad \text{where } f_j = f_j(f, \varphi) = f * (\varphi_{2^{-j}} - \varphi_{2^{1-j}}), \quad \forall j \in \mathbb{Z}. \quad (2.10)$$

**2.4 Definition** (Definition of homogeneous Besov spaces). Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Define  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  as the space of  $f \in \mathcal{S}'(\mathbb{R}^n)$  such that

$$|f|_{\dot{B}_{p,q}^s} = |f|_{\dot{B}_{p,q}^s(\mathbb{R}^n)} := \left( \sum_{j \in \mathbb{Z}} 2^{sjq} \|f_j\|_{L^p}^q \right)^{1/q} < \infty. \quad (2.11)$$

Recall the following result [23, Section 5.1.5, Theorem, p. 240].

**2.5 Lemma.** The space  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  does not depend on the choice of the sequence  $(\varphi^j)_{j \in \mathbb{Z}}$ .

Among the various characterizations of Besov spaces, we will need the one by finite order differences. Let  $f : \mathbb{R}^n \rightarrow \mathbb{C}$ . For all integers  $M \geq 0$  and all  $x, h \in \mathbb{R}^n$ , set

$$\Delta_h^M f(x) = \sum_{l=0}^M \binom{M}{l} (-1)^{M-l} f(x + lh). \quad (2.12)$$

An immediate consequence of the definition of  $|f|_{\dot{B}_{p,q}^s}$  is

**2.6 Lemma.** Let  $0 < r_1 < r_2$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . If  $f \in \mathcal{S}'(\mathbb{R}^n)$  is such that  $\text{supp } \hat{f} \subset B(0, r_2 R) \setminus B(0, r_1 R)$ , then we have

$$|f|_{\dot{B}_{p,q}^s(\mathbb{R}^n)}^q \sim R^{sq} \|f\|_{L^p(\mathbb{R}^n)}^q. \quad (2.13)$$

*Proof.* Let  $I$  denote the integer part, and set

$$k := I(\log_2(r_1 R)), \quad \ell = I(\log_2(r_2 R)).$$

Note that

$$\ell - k < \log_2 r_2 - \log_2 r_1 + 1, \quad (2.14)$$

and that

$$f_j = 0 \text{ if } j < k \text{ or } j > \ell + 1. \quad (2.15)$$

We next note that the  $\varphi^j$ 's given by (2.6) satisfy

$$\|\varphi^j\|_{L^1(\mathbb{R}^n)} \lesssim 1. \quad (2.16)$$

By combining (2.15) and (2.16), we find that

$$\begin{aligned} |f|_{B_{p,q}^s}^q &= \sum_k^{\ell+1} 2^{sjq} \|f_j\|_{L^p(\mathbb{R}^n)}^q = \sum_k^{\ell+1} 2^{sjq} \|f * \varphi^j\|_{L^p(\mathbb{R}^n)}^q \leq \sum_k^{\ell+1} 2^{sjq} \|f\|_{L^p(\mathbb{R}^n)}^q \|\varphi^j\|_{L^p(\mathbb{R}^n)}^q \\ &\lesssim R^{sq} \|f\|_{L^p(\mathbb{R}^n)}^q. \end{aligned} \quad (2.17)$$

On the other hand, by (2.14) and (2.15) we have

$$R^{sq} \|f\|_{L^p(\mathbb{R}^n)}^q = R^{sq} \left\| \sum_k^{\ell+1} f_j \right\|_{L^p(\mathbb{R}^n)}^q \lesssim R^{sq} \sum_k^{\ell+1} \|f_j\|_{L^p(\mathbb{R}^n)}^q \lesssim \sum_k^{\ell+1} 2^{sjq} \|f_j\|_{L^p(\mathbb{R}^n)}^q = |f|_{B_{p,q}^s}^q. \quad (2.18)$$

We complete the proof of the lemma by combining (2.17) and (2.18).  $\square$

Besov spaces can be characterized by means of the differences  $\Delta_h^M$  [23, Section 5.2.3], [14, Theorem, p.41], [25, Section 1.11.9, Theorem 1.118, p. 74]. Let us recall the following results.

**2.7 Proposition.** Let  $s > 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $M > s$  be an integer. Then:

1. In the space  $\dot{B}_{p,q}^s(\mathbb{R}^n)$  we have the equivalence of semi-norms

$$|f|_{B_{p,q}^s(\mathbb{R}^n)} \sim \left( \int_{\mathbb{R}^n} |h|^{-sq} \left\| \Delta_h^M f \right\|_{L^p}^q \frac{dh}{|h|^n} \right)^{1/q}. \quad (2.19)$$

2. The full  $B_{p,q}^s$  norm satisfies

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \left( \int_{|h| \leq 1} |h|^{-sq} \left\| \Delta_h^M f \right\|_{L^p}^q \frac{dh}{|h|^n} \right)^{1/q}$$

and

$$\|f\|_{B_{p,q}^s(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + |f|_{B_{p,q}^s(\mathbb{R}^n)}. \quad (2.20)$$

### 3 Direct trace theorem: proof of Theorem 1.3

Here, we fix  $l > s$  and consider the semi-norm

$$|f|_{B_{p,q}^s}^q = |f|_{B_{p,q}^s(\mathbb{R}^n)}^q = \int_{\mathbb{R}^n} \|\Delta_h^l f\|_{L^p}^q \frac{dh}{|h|^{n+sq}}; \quad (3.1)$$

considering an  $l$ -dependent semi-norm considerably simplifies the proof.

As mentioned in the introduction, we adapt here Maz'ya's elegant proof [11, p. 512–513]. We rely on the following elementary lemma, whose proof is postponed.

**3.1 Lemma.** Let  $l > 0$  be an integer and  $F \in C^\infty(\Omega; \mathbb{C})$ . We set

$$|D_l F(x, \varepsilon)| = \sum_{\alpha \in \mathcal{M}_l} |\partial^\alpha F(x, \varepsilon)|.$$

Let  $x, h \in \mathbb{R}^n$ . Set  $f(x) = F(x, 0)$  and  $r = |h|$ . Then

$$|\Delta_h^l f(x)| \lesssim r^l \sum_{j=1}^l \int_0^l t^{l-1} |D_l F(x + th, jr)| dt + r^l \sum_{j=0}^l \int_0^l t^{l-1} |D_l F(x + jh, tr)| dt. \quad (3.2)$$

Recall that  $\mathcal{M}_l$  consists of all the multi-indices  $\alpha = (\beta, \gamma)$  of length  $l$  such that either  $\beta = 0$ , or  $\gamma = 0$ .

Assuming Lemma 3.1 proved for the moment, we proceed to the

*Proof of Theorem 1.3.* Set  $K(\varepsilon) = \|D_l F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}$ . Integrating (3.2) in  $x$ , we obtain (with  $r = |h|$ )

$$\|\Delta_h^l f\|_{L^p(\mathbb{R}^n)} \lesssim r^l \sum_{j=1}^l K(jr) + r^l \int_0^l t^{l-1} K(tr) dt \lesssim r^l \sum_{j=1}^l K(jr) + \int_0^{lr} t^{l-1} K(t) dt. \quad (3.3)$$

In view of (3.1) and (3.3), in order to establish (1.5) it suffices to prove that

$$\int_{\mathbb{R}^n} |h|^{lq-n-sq} [K(j|h|)]^q dh \lesssim \int_0^\infty \varepsilon^{q(l-s)-1} [K(\varepsilon)]^q d\varepsilon, \forall j > 0, \quad (3.4)$$

and

$$\int_{\mathbb{R}^n} \left( \int_0^{l|h|} t^{l-1} K(t) dt \right)^q \frac{dh}{|h|^{n+sq}} \lesssim \int_0^\infty \varepsilon^{q(l-s)-1} [K(\varepsilon)]^q d\varepsilon. \quad (3.5)$$

Passing to spherical coordinates, we see that the two quantities in (3.4) are proportional, and thus (3.4) holds. Also in spherical coordinates, (3.5) amounts to

$$\int_0^\infty \frac{1}{\varepsilon^{sq+1}} \left( \int_0^{l\varepsilon} t^{l-1} K(t) dt \right)^q \lesssim \int_0^\infty \varepsilon^{q(l-s)-1} [K(\varepsilon)]^q d\varepsilon. \quad (3.6)$$

In turn, (3.6) follows from Hardy's inequality at 0 (2.2) applied with  $r = sq$  and  $g(\varepsilon) = \varepsilon^{l-1} K(\varepsilon)$ .

The proof of Theorem 1.3 (except Lemma 3.1) is complete.  $\square$

**3.2 Remark.** In establishing (3.4) and (3.5), we did not rely on the fact that  $l > s$ . However, if the right-hand side of (3.1) is finite for some  $l \leq s$ , then  $f$  is a polynomial of degree  $\leq l - 1$ ; see Proposition 5.1 below. Thus, if say  $F \in C_c^\infty$  and the right-hand side of (1.5) is finite for some  $l \leq s$ , then  $f = 0$ .<sup>3</sup>

<sup>3</sup>This can also be obtained from Proposition 1.8.

In order to complete the proof of Theorem 1.3, it remains to prove Lemma 3.1. This lemma is a clear consequence of the identity (3.7) and of the estimate (3.8) below, that we state as two lemmas.

**3.3 Lemma.** For  $x, h \in \mathbb{R}^n$  and  $l \in \mathbb{N}^*$  we have, with  $r = |h|$ :

$$\Delta_h^l f(x) = \sum_{j=0}^l \binom{l}{j} (-1)^j \Delta_{re_n}^l F(x + jh, 0) + \sum_{j=1}^l \binom{l}{j} (-1)^{j+1} \Delta_h^l F(x, jr). \quad (3.7)$$

*Proof.* We start from the identity

$$f(x) = \sum_{j=0}^l \binom{l}{j} (-1)^j F(x, jt) + \sum_{j=1}^l \binom{l}{j} (-1)^{j+1} F(x, jr), \quad \forall x \in \mathbb{R}^n.$$

As a consequence,

$$\begin{aligned} \Delta_h^l f(x) &= \sum_{k=0}^l \binom{l}{k} (-1)^{l-k} f(x + kh) \\ &= \sum_{k=0}^l \binom{l}{k} (-1)^{l-k} \left[ \sum_{j=0}^l \binom{l}{j} (-1)^j F(x + kh, jr) + \sum_{j=1}^l \binom{l}{j} (-1)^{j+1} F(x + kh, jr) \right] \\ &= \sum_{k=0}^l \binom{l}{k} (-1)^k \Delta_{re_n}^l F(x + kh, 0) + \sum_{j=1}^l \binom{l}{j} (-1)^{j+1} \Delta_h^l F(x, jr). \end{aligned}$$

In the second term of the last equality, we exchanged the sums over  $l$  and  $j$ . □

**3.4 Lemma.** Let  $l > 0$  be an integer. Let  $y = (x, \varepsilon) \in \Omega$  and let  $h \in \mathbb{R}^{n+1}$  be such that  $[y, y + lh] \subset \Omega$ . Write  $h = (h', h_{n+1}) \in \mathbb{R}^n \times \mathbb{R}$ , and assume that either  $h' = 0$  or  $h_{n+1} = 0$ . Then, with  $r = |h|$ , we have

$$|\Delta_h^l F(y)| \lesssim r^l \int_0^l t^{l-1} |D_l F(y + th)| dt. \quad (3.8)$$

*Proof.* Set  $G(t) = F(y + th)$ ,  $t \in [0, l]$ . Then clearly

$$\Delta_h^l F(y) = \Delta_1^l G(0) \text{ and } |G^{(l)}(t)| \leq r^l |D_l F(y + th)|.$$

Therefore, it suffices to prove that

$$|\Delta_1^l G(0)| \lesssim \int_0^l t^{l-1} |G^{(l)}(t)| dt. \quad (3.9)$$

In turn, estimate (3.9) is obtained as follows. Let  $H_1 = \mathbb{1}_{[-1,0]}$  and, for  $j \geq 2$ , set  $H_j = H_0 * H_0 * \dots * H_0$  ( $j$  times). By a straightforward induction on  $j$ , the distributional derivative  $H_j^{(j-1)}$  is bounded, and  $H_j(t) = 0$  when  $t \geq 0$  or when  $t \leq -j$ . This leads to the inequality

$$|H_j(-t)| \lesssim t^{j-1}, \quad \forall j \geq 1, \quad \forall t \geq 0. \quad (3.10)$$

On the other hand, again by a straightforward induction on  $l$ , we have

$$\Delta_1^l G(0) = G^{(l)} * H_l(0) = \int_0^l G^{(l)}(t) H_l(-t) dt. \quad (3.11)$$

We obtain (3.9) by combining (3.10) and (3.11). □

**3.5 Remark.** For further use, let us note that if  $f \in C_c^\infty(\mathbb{R}^n)$  then the identity (3.11) applied to the function

$$t \mapsto G(t) = f(x + t\varepsilon\omega),$$

with  $\omega \in S^{n-1}$  and  $x$  in a compact  $K \subset \mathbb{R}^n$ , leads to

$$\lim_{\varepsilon \rightarrow 0} \frac{|\Delta_{\varepsilon\omega}^l f(x)|}{\varepsilon^l} = |D^l f(x)(\omega, \dots, \omega)| \text{ uniformly in } \omega \in S^{n-1} \text{ and in } x \in K. \quad (3.12)$$

## 4 Inverse trace theorem: proof of Theorem 1.4

*Proof of Theorem 1.4.* We consider a radial mollifier

$$\rho(x) = g(|x|) \in \mathcal{S}(\mathbb{R}^n), \text{ with } \text{supp } \hat{\rho} \subset B(0, 1) \text{ and } \hat{\rho} = 1 \text{ in } B(0, 1/2).^4 \quad (4.1)$$

Given  $f \in C_c^\infty(\mathbb{R}^n)$ , we let  $U(x, \varepsilon) = f * \rho_\varepsilon(x)$ . We will prove that  $U$  satisfies (1.6) whenever  $\alpha \in \mathcal{P}_s$ .

*Step 1.* Reduction to the case where  $\gamma > 0$ .

This reduction is based on the next lemma.

**4.1 Lemma.** Let  $W \in C^\infty(\Omega)$  satisfy

$$W(x, \varepsilon) = - \int_\varepsilon^\infty \frac{\partial}{\partial t} W(x, t) dt, \quad \forall (x, \varepsilon) \in \Omega. \quad (4.2)$$

Let  $a > -1$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Then

$$\int_0^\infty \varepsilon^a \|W(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim \int_0^\infty \varepsilon^{a+q} \left\| \frac{\partial}{\partial \varepsilon} W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon.$$

Lemma 4.1 is a straightforward consequence of Hardy's inequality at  $\infty$ , and its proof is postponed.

Assume that we have proved (1.6) for every  $\alpha = (\beta, \gamma) \in \mathbb{N}^{n+1}$  such that  $\gamma > 0$ . Let  $\alpha = (\beta, \gamma) \in \mathcal{P}_s$  be such that  $\gamma = 0$  (and thus  $|\beta| > s$ ). Since  $U$  satisfies (2.1), we may apply the above lemma to

$$W = \frac{\partial^{|\beta|}}{\partial x^\beta} U \text{ and with}$$

$$a = q(|\beta| - s) - 1 > -1.$$

Since (1.6) holds for  $(\beta, 1)$ , Lemma 4.1 implies that (1.6) holds for  $\alpha = (\beta, 0)$ .

In conclusion, it suffices to prove (1.6) when  $\gamma > 0$ .

*Step 2.* Proof of (1.6) when  $\gamma > 0$ .

This is the heart of the proof, and will be obtained as a consequence of the estimate (4.8) below.

We start by noting that the partial Fourier transform  $\mathcal{F}_x$  in  $x$  of  $\frac{\partial^\gamma U}{\partial \varepsilon^\gamma}$  satisfies

$$\mathcal{F}_x \left( \frac{\partial^\gamma U}{\partial \varepsilon^\gamma} \right) (\xi) = \frac{\partial^\gamma}{\partial \varepsilon^\gamma} (\hat{\rho}(\varepsilon\xi) \hat{f}(\xi)) = |\xi|^\gamma \varphi^{(\gamma)}(\varepsilon|\xi|) \hat{f}(\xi) = \frac{1}{\varepsilon^\gamma} \hat{\eta}(\varepsilon\xi) \hat{f}(\xi). \quad (4.3)$$

<sup>4</sup>Unlike the other assumptions listed in (4.1), the fact that  $\rho$  is radial is not crucial for the arguments (but just leads to shorter formulas). In particular,  $\rho$  needs not be even. This contrasts to several tricks in [26] and [11], where it is essential to have even mollifiers.

Here,

$$\varphi \in \mathcal{S}(\mathbb{R}), \eta = \eta(\gamma) \in \mathcal{S}(\mathbb{R}^n) \text{ satisfy } \widehat{\rho}(\xi) = \varphi(|\xi|) \text{ and } \widehat{\eta}(\xi) = |\xi|^\gamma \varphi^{(\gamma)}(|\xi|). \quad (4.4)$$

By (4.3) and the assumption on  $\rho$ , we have

$$\text{supp } \mathcal{F}_x \left( \frac{\partial^\gamma U}{\partial \varepsilon^\gamma} \right) \subset B(0, 1/\varepsilon) \setminus B(0, 1/(2\varepsilon)). \quad (4.5)$$

By combining (4.3), (4.4) and (4.5) with the fact that

$$\text{supp } \widehat{f}_j \subset B(0, 2^{j+1}) \setminus B(0, 2^{j-1}),$$

we find that

$$\frac{\partial^\gamma U}{\partial \varepsilon^\gamma}(x, \varepsilon) = \frac{1}{\varepsilon^\gamma} \eta_\varepsilon * f(x) = \frac{1}{\varepsilon^\gamma} \sum_{1/(4\varepsilon) < 2^j < 2/\varepsilon} \eta_\varepsilon * f_j(x). \quad (4.6)$$

Using (4.6), we obtain that

$$\frac{\partial^{|\alpha|} U}{\partial \varepsilon^\gamma \partial x^\beta}(x, \varepsilon) = \frac{1}{\varepsilon^\gamma} \sum_{1/(4\varepsilon) < 2^j < 2/\varepsilon} \eta_\varepsilon * \partial^\beta f_j(x), \quad \forall \gamma > 0, \quad \forall \beta \in \mathbb{N}^n. \quad (4.7)$$

Finally, by combining (4.7) with the fact that  $\eta \in L^1$  (by (4.4)) and with the Nikolskii's estimates (2.4), we obtain

$$\left\| \frac{\partial^{|\alpha|} U}{\partial \varepsilon^\gamma \partial x^\beta} \right\|_{L^p(\mathbb{R}^n)} \lesssim \frac{1}{\varepsilon^\gamma} \sum_{1/(4\varepsilon) < 2^j < 2/\varepsilon} \left\| \partial^\beta f_j \right\|_{L^p(\mathbb{R}^n)} \lesssim \frac{1}{\varepsilon^{|\alpha|}} \sum_{1/(4\varepsilon) < 2^j < 2/\varepsilon} \|f_j\|_{L^p(\mathbb{R}^n)}, \quad \forall \gamma > 0, \quad \forall \beta \in \mathbb{N}^n. \quad (4.8)$$

Estimate (1.6) with  $\gamma > 0$  is an easy consequence of (4.8). Indeed, noting that for every  $\varepsilon$  there are at most three  $j$ 's such that  $1/4\varepsilon < 2^j < 2/\varepsilon$ , we find that

$$\begin{aligned} \int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon &\lesssim \int_0^\infty \varepsilon^{-sq-1} \left( \sum_{1/(4\varepsilon) < 2^j < 2/\varepsilon} \|f_j\|_{L^p(\mathbb{R}^n)} \right)^q d\varepsilon \\ &\lesssim \int_0^\infty \varepsilon^{-sq-1} \sum_{1/(4\varepsilon) < 2^j < 2/\varepsilon} \|f_j\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \\ &= \sum_{j \in \mathbb{Z}} \|f_j\|_{L^p(\mathbb{R}^n)}^q \int_{2^{-j-2}}^{2^{1-j}} \varepsilon^{-sq-1} d\varepsilon \sim \sum_{j \in \mathbb{Z}} 2^{jsq} \|f_j\|_{L^p(\mathbb{R}^n)}^q = |f|_{B_{p,q}^s}^q. \end{aligned}$$

Granted Lemma 4.1, the proof of Theorem 1.4 is complete.  $\square$

*Proof of Lemma 4.1.* In view of (4.2), we have

$$\|W(\cdot, \varepsilon)\|_{L^p} = \left\| \int_\varepsilon^\infty \frac{\partial}{\partial t} W(\cdot, t) dt \right\|_{L^p} \leq \int_\varepsilon^\infty \left\| \frac{\partial}{\partial t} W(\cdot, t) \right\|_{L^p} dt. \quad (4.9)$$

It then suffices to combine (4.9) with Hardy's inequality at  $\infty$

$$\int_0^\infty \varepsilon^a \left( \int_\varepsilon^\infty f(t) dt \right)^q d\varepsilon \lesssim \int_0^\infty \varepsilon^{a+q} |f(\varepsilon)|^q d\varepsilon. \quad \square$$

## 5 Further results

In this section, we prove Proposition 1.8 (and its cousin Proposition 5.1), Theorem 1.9 and its complement Proposition 1.10, item 2 in Theorem 1.11 and its negative counterparts Propositions 1.13 and 1.14 (that we establish in more general forms). We also prove Theorem 1.15; the proof of Theorem 1.17 is very similar to the one of Theorem 1.15 and is left to the reader. In a related direction, we establish Proposition 5.8. Finally, we explain why the non homogeneous (i.e., for the full norms) counterparts of Theorems 1.3 and 1.4 are trivial consequences of Theorems 1.3 and 1.4.

*Proof of Proposition 1.8.* Let  $\alpha \notin \mathcal{P}_s$ . Thus  $\alpha = (\beta, 0)$ , with  $|\beta| \leq s$ . We argue by contradiction and assume that

$$\int_0^\infty \varepsilon^a \left\| \frac{\partial^{|\beta|} F}{\partial x^\beta}(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon < \infty, \text{ where } a = q(|\beta| - s) - 1 \leq -1. \quad (5.1)$$

By (5.1) and the fact that  $a \leq -1$ , we find that there exists a sequence  $\varepsilon_j \rightarrow 0$  such that

$$\lim_{j \rightarrow \infty} \left\| \frac{\partial^{|\beta|} F}{\partial x^\beta}(\cdot, \varepsilon_j) \right\|_{L^p(\mathbb{R}^n)} = 0.$$

This implies that  $\frac{\partial^{|\beta|} f}{\partial x^\beta} = 0$ , which contradicts the assumption  $f \neq 0$ .  $\square$

In the same spirit, we have the following elementary result.

**5.1 Proposition.** Let  $l > 0$  be an integer and let  $s \geq l$ . Let  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in L^1_{loc}(\mathbb{R}^n; \mathbb{C})$  be such that

$$\int_{\mathbb{R}^n} \|\Delta_h^l f\|_{L^p(\mathbb{R}^n)}^q \frac{dh}{|h|^{n+sq}} < \infty. \quad (5.2)$$

Then  $f$  is a polynomial of degree  $\leq l - 1$ .

*Proof.* In spherical coordinates, (5.2) reads

$$\int_0^\infty \varepsilon^a \int_{\mathbb{S}^{n-1}} \left( \frac{\|\Delta_{\varepsilon\omega}^l f\|_{L^p(\mathbb{R}^n)}}{\varepsilon^l} d\omega \right)^q d\varepsilon < \infty, \text{ with } a = -1 - q(s-l) \leq -1. \quad (5.3)$$

In view of (5.3) and of the fact that  $a \leq -1$ , there exists a sequence  $\varepsilon_j \rightarrow 0$  such that

$$\lim_{j \rightarrow \infty} \int_{\mathbb{S}^{n-1}} \frac{\|\Delta_{\varepsilon_j \omega}^l f\|_{L^p(\mathbb{R}^n)}}{\varepsilon_j^l} d\omega = 0. \quad (5.4)$$

Using (5.4), we will obtain the desired conclusion assuming temporarily in addition that  $f$  is smooth. In view of (3.12), for every compact  $K \subset \mathbb{R}^n$  we have

$$\lim_{j \rightarrow \infty} \int_{\mathbb{S}^{n-1}} \frac{\|\Delta_{\varepsilon_j \omega}^l f\|_{L^p(K)}}{\varepsilon_j^l} d\omega = \int_{\mathbb{S}^{n-1}} \|D^l f(\cdot)(\omega, \dots, \omega)\|_{L^p(K)} d\omega. \quad (5.5)$$

By combining (5.4) and (5.5), we find that  $D^l f(\cdot)(\omega, \dots, \omega) = 0$ ,  $\forall \omega \in \mathbb{S}^{n-1}$ , and thus  $D^l f = 0$ .<sup>5</sup> Therefore,  $f$  is a polynomial of degree  $\leq l - 1$ .

<sup>5</sup>If  $T$  is a symmetric  $l$ -linear form in  $\mathbb{R}^n$  and  $T(\omega, \dots, \omega) = 0$ ,  $\forall \omega \in \mathbb{S}^{n-1}$ , then  $T = 0$ .

We now consider an arbitrary  $f$ . Let  $\varphi \in C_c^\infty(\mathbb{R}^n)$ , and set  $g = f * \varphi$ . Since

$$\left\| \Delta_h^l g \right\|_{L^p(\mathbb{R}^n)} = \left\| (\Delta_h^l f) * \varphi \right\|_{L^p(\mathbb{R}^n)} \lesssim \left\| \Delta_h^l f \right\|_{L^p(\mathbb{R}^n)},$$

we find that the smooth function  $g$  satisfies (5.2). In view of the above, we have  $\partial^\alpha g = 0$  whenever  $|\alpha| = l$ . If we now let  $\varphi = \rho_\varepsilon$ , with  $\rho$  a standard mollifier and  $\varepsilon \rightarrow 0$ , we find that  $\partial^\alpha f = 0$  whenever  $|\alpha| = l$ , and thus  $f$  is a polynomial of degree  $\leq l - 1$ .  $\square$

*Proof of Theorem 1.9.* Let  $f \in C_c^\infty(\mathbb{R}^n)$ , and let  $U(x, \varepsilon) = f * \rho_\varepsilon(x)$ , with  $\rho$  as in the proof of Theorem 1.4. Let  $P$  be the Poisson kernel, and let  $V(x, \varepsilon) = f * P_\varepsilon(x)$  be “the” harmonic extension of  $f$ . Theorem 1.9 is a consequence of the general estimate

$$\int_0^\infty \varepsilon^a \|\partial_x^\beta V(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim \int_0^\infty \varepsilon^a \|\partial_x^\beta U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon, \quad (5.6)$$

valid for every  $a > -1$ ,  $\beta \in \mathbb{N}^n$ ,  $1 \leq p \leq \infty$ , and  $1 \leq q < \infty$ .

In turn, (5.6) follows from the next lemma, widely used in the theory of (real) Hardy spaces and the proof of which is postponed.

**5.2 Lemma.** Let  $\rho$  be as in (4.1). Let  $M > 0$ . Then there exists a sequence  $(\eta^j)_{j \in \mathbb{Z}} \subset L^1(\mathbb{R}^n)$  such that

$$\left\| \eta^j \right\|_{L^1} \lesssim \begin{cases} 2^{-j}, & \text{if } j \geq 0 \\ 2^{Mj}, & \text{if } j < 0 \end{cases} \quad \text{and } P = \sum_{j \in \mathbb{Z}} \eta^j * \rho_{2^j}. \quad (5.7)$$

Taking Lemma 5.2 for granted, we proceed to the proof of (5.6), and explain why (5.6) implies Theorem 1.9.

*Step 1.* Estimate (5.6) implies Theorem 1.9.

This step relies on a trick of Uspenskii. Assume that (5.6) is known to hold. We will then establish the estimate (1.8) for every  $\alpha = (\beta, \gamma)$  such that  $|\alpha| > s$ . When  $\gamma = 0$ , the conclusion of (1.8) is obtained by combining (1.6) with (5.6). So let us assume that  $\gamma > 0$ . Arguing as in Step 1 in the proof of Theorem 1.4, if (1.8) holds for  $(\beta, \gamma + 1)$ , then it also holds for  $(\beta, \gamma)$ .<sup>6</sup> Therefore, possibly after replacing  $\gamma$  by  $\gamma + 1$ , we may always assume that  $\gamma$  is even, say  $\gamma = 2k$ , with  $k \geq 1$ . Now comes Uspenskii’s trick. Since  $V$  is harmonic, we have (by a straightforward induction on  $k$ ) the identity

$$\partial^\alpha V = \frac{\partial^{2k}}{\partial \varepsilon^{2k}} \partial_x^\beta V = (-1)^k (\Delta_x)^k \partial_x^\beta V.$$

Thus

$$\partial^\alpha V = (-1)^k \sum_{1 \leq j_1, \dots, j_k \leq n} \frac{\partial^2}{\partial x_{j_1}^2} \cdots \frac{\partial^2}{\partial x_{j_k}^2} \partial_x^\beta V.$$

Consequently, we have

$$\partial^\alpha V = \sum_{|\delta| = |\alpha|} c_{\alpha, \delta} \partial_x^\delta V \quad (5.8)$$

for some suitable coefficients  $c_{\alpha, \delta}$ . Since by (5.6) and (1.6), (1.8) holds for  $\partial_x^\delta$  when  $|\delta| = |\alpha|$ , we obtain from (5.8) that (1.8) holds for every  $\alpha$ .

<sup>6</sup> This relies on Lemma 4.1 combined with the fact that  $q(|\alpha| - s) - 1 > -1$ , as well as (2.1).



*Step 2. Proof of (5.6).*

Set  $G_\varepsilon = \|\partial_x^\beta U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}$  and  $b_j = \|\eta^j\|_{L^1(\mathbb{R}^n)}$ . By (5.7), we have the identity

$$V(\cdot, \varepsilon) = \sum_{j \in \mathbb{Z}} \left( \eta^j \right)_\varepsilon * \rho_{2^j \varepsilon} * f,$$

and thus

$$\|\partial_x^\beta V(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} \leq \sum_{j \in \mathbb{Z}} \left\| \left( \eta^j \right)_\varepsilon \right\|_{L^1(\mathbb{R}^n)} \|\partial_x^\beta (f * \rho_{2^j \varepsilon})\|_{L^p(\mathbb{R}^n)} = \sum_{j \in \mathbb{Z}} b_j G(2^j \varepsilon). \quad (5.9)$$

In view of (5.9), in order to prove (5.6) it suffices to establish the estimate

$$\int_0^\infty \varepsilon^\alpha \left( \sum_{j \in \mathbb{Z}} b_j G(2^j \varepsilon) \right)^q d\varepsilon \lesssim \int_0^\infty \varepsilon^\alpha [G(\varepsilon)]^q d\varepsilon. \quad (5.10)$$

Let  $N$  denote the right-hand side of (5.10). Then

$$\left( \int_0^\infty \varepsilon^\alpha [G(2^j \varepsilon)]^q d\varepsilon \right)^{1/q} = 2^{-j(\alpha+1)/q} N^{1/q}. \quad (5.11)$$

Minkowski's inequality (applied to the measure  $\varepsilon^\alpha d\varepsilon$ ) combined with (5.11) implies that

$$\int_0^\infty \varepsilon^\alpha \left( \sum_{j \in \mathbb{Z}} b_j G(2^j \varepsilon) \right)^q d\varepsilon \leq \left( \sum_{j \in \mathbb{Z}} b_j 2^{-j(\alpha+1)/q} \right)^q N \lesssim N.$$

The last inequality follows from the fact that  $\sum b_j 2^{-j(\alpha+1)/q} < \infty$ , by the first part of (5.7).

Granted Lemma 5.2, the proof of Theorem 1.9 is complete.  $\square$

*Proof of Lemma 5.2.* We rely on two decompositions that can be found in Stein [17]. First decomposition: given a function  $\varphi \in \mathcal{S}(\mathbb{R}^n)$ , we may write

$$\varphi = \sum_{j \geq 0} \lambda^{j, \varphi} * \rho_{2^{-j}}. \quad (5.12)$$

Here  $(\lambda^{j, \varphi})_j \subset \mathcal{S}(\mathbb{R}^n)$  is a sequence that decays rapidly as  $j \rightarrow \infty$ , in the following sense: if  $\varphi$  belongs to a bounded subset  $\mathcal{B} \subset \mathcal{S}(\mathbb{R}^n)$ , then for every  $M > 0$  there exists a constant  $C$  such that

$$\|\lambda^{j, \varphi}\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{2^M j}, \quad \forall j \geq 0, \quad \forall \varphi \in \mathcal{B}; \quad (5.13)$$

see [17, Lemma 2, p. 93].

Second decomposition: the Poisson kernel  $P$  can be decomposed as

$$P = \sum_{k \geq 0} \frac{1}{2^k} \left( \varphi^k \right)_{2^k}, \quad (5.14)$$

where  $(\varphi^k)_k \subset \mathcal{S}(\mathbb{R}^n)$  is a bounded sequence; see [17, eq (18), p. 98].

By combining (5.12) with (5.14), we obtain the decomposition

$$\begin{aligned} P &= \sum_{k \geq 0} \frac{1}{2^k} \left( \sum_{j \geq 0} \lambda^{j, \varphi^k} * \rho_{2^{-j}} \right)_{2^k} = \sum_{k \geq 0} \frac{1}{2^k} \sum_{j \geq 0} \left( \lambda^{j, \varphi^k} \right)_{2^k} * \rho_{2^{k-j}} \\ &= \sum_{l \in \mathbb{Z}} \eta^l * \rho_{2^l}, \quad \text{with } \eta^l = \sum_{j \geq \max\{0, -l\}} \frac{1}{2^{j+l}} \left( \lambda^{j, \varphi^{j+l}} \right)_{2^{j+l}}. \end{aligned} \quad (5.15)$$

By (5.15), we thus have  $P = \sum_{l \in \mathbb{Z}} \eta^l * \rho_{2^l}$ , with

$$\|\eta^l\|_{L^1(\mathbb{R}^n)} \leq \sum_{j \geq \max\{0, -l\}} \frac{1}{2^{j+l}} \|\lambda^{j, \varphi^{j+l}}\|_{L^1(\mathbb{R}^n)}. \quad (5.16)$$

We obtain the first part of (5.7) by combining (5.13) with (5.16).  $\square$

**5.3 Remark.** For further use, we note that the decomposition (5.12) holds not only for a mollifier  $\rho$  as in the proof of Theorem 1.4, but also for any  $\rho \in \mathcal{S}(\mathbb{R}^n)$  such that  $\int_{\mathbb{R}^n} \rho = 1$ . In addition, the estimate (5.13) can be improved to

$$\left\| x^\delta \partial^\beta \lambda^{j,\varphi}(x) \right\|_{L^p(\mathbb{R}^n)} \leq \frac{C}{2^{Mj}}, \quad \forall j \geq 0, \quad \forall 1 \leq p \leq \infty, \quad \forall \beta, \delta \in \mathbb{N}^n, \quad \forall \varphi \in \mathcal{B}; \quad (5.17)$$

see [17, Lemma 2, p. 98].

Now let us pause and compare Theorem 1.4 to Theorem 1.9. In view of their respective conclusions, it is natural to examine the properties of

$$W(\cdot, \varepsilon) := f * \varphi_\varepsilon(x), \quad \text{where } \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (5.18)$$

We have the following

**5.4 Theorem.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in C_c^\infty(\mathbb{R}^n)$  and let  $W$  be as in (5.18). Then

$$\int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha W(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| > s. \quad (5.19)$$

*Proof.* Let  $\alpha = (\beta, \gamma) \in \mathbb{N}^n \times \mathbb{N}$  be such that  $|\alpha| = l > s$ . We will establish (5.19) for such  $\alpha$ .

*Step 1.* Reduction to the case where  $\alpha = (\beta, 0)$ .

Assume for the moment that (5.4) holds under the additional assumption that  $\gamma = 0$ . Starting from the identity

$$\mathcal{F}_x W(\cdot, \varepsilon)(\xi) = \widehat{f}(\xi) \widehat{\varphi}(\varepsilon \xi),$$

we find that, with appropriate  $\lambda^\mu, \varphi^\mu \in \mathcal{S}(\mathbb{R}^n)$ , we have

$$\mathcal{F}_x (\partial^\alpha W(\cdot, \varepsilon))(\xi) = \widehat{f}(\xi) \sum_{\substack{\mu \in \mathbb{N}^n \\ |\mu|=l}} \xi^\mu \widehat{\lambda^\mu}(\varepsilon \xi)$$

and

$$\partial^\alpha W(\cdot, \varepsilon) = \sum_{\substack{\mu \in \mathbb{N}^n \\ |\mu|=l}} \partial_x^\mu [f * (\varphi^\mu)]_\varepsilon = \sum_{\substack{\mu \in \mathbb{N}^n \\ |\mu|=l}} \partial_x^\mu W^\mu(\cdot, \varepsilon), \quad (5.20)$$

where

$$W^\mu(\cdot, \varepsilon) = f * (\varphi^\mu)_\varepsilon.$$

In view of our assumption that (5.19) holds (for  $\varphi^\mu$  and  $\mu$ ), we obtain from (5.20) that (5.19) holds for  $\varphi$  and  $\alpha$ .

*Step 2.* Proof of (5.19) when  $\alpha = (\beta, 0)$ .

This is an easy consequence of (5.12). Indeed, (5.12) implies that (with  $\rho$  as in (4.1) and  $U(\cdot, \varepsilon) := f * \rho_\varepsilon$ ) we have

$$\partial_x^\beta W(\cdot, \varepsilon) = \partial_x^\beta \left( \sum_{j \geq 0} f * (\lambda^{j,\varphi})_\varepsilon * \rho_{2^{-j}\varepsilon} \right) = \sum_{j \geq 0} (\lambda^{j,\varphi})_\varepsilon * \partial_x^\beta U(\cdot, 2^{-j}\varepsilon).$$

Using (5.13), for every  $M > 0$  we have

$$\left\| \partial_x^\beta W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)} \leq \sum_{j \geq 0} \left\| \lambda^{j,\varphi} \right\|_{L^1(\mathbb{R}^n)} \left\| \partial_x^\beta U(\cdot, 2^{-j}\varepsilon) \right\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j \geq 0} 2^{-Mj} \left\| \partial_x^\beta U(\cdot, 2^{-j}\varepsilon) \right\|_{L^p(\mathbb{R}^n)}. \quad (5.21)$$

By (5.21) and Theorem 1.4, for every  $N < M$  we have

$$\begin{aligned} \int_0^\infty \varepsilon^{q(l-s)-1} \left\| \partial_x^\beta W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon &\lesssim \int_0^\infty \varepsilon^{q(l-s)-1} \left( \sum_{j \geq 0} 2^{-Mj} \left\| \partial_x^\beta U(\cdot, 2^{-j}\varepsilon) \right\|_{L^p(\mathbb{R}^n)} \right)^q d\varepsilon \\ &\lesssim \int_0^\infty \varepsilon^{q(l-s)-1} \sum_{j \geq 0} 2^{-Nj} \left\| \partial_x^\beta U(\cdot, 2^{-j}\varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \\ &= \int_0^\infty \varepsilon^{q(l-s)-1} \left\| \partial_x^\beta U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \sum_{j \geq 0} 2^{-j(N-q(l-s))} \lesssim |f|_{B_{p,q}^s}^q, \end{aligned}$$

provided we choose  $M > N > q(l-s)$ . □

We next continue with the

*Proof of Proposition 1.10.* Let  $\alpha = (\beta, \gamma) \in \mathbb{N}^{n+1}$  be such that  $|\alpha| \leq s$ . Assume, by contradiction, that the integral in (1.9) is finite. Arguing as in the proof of Proposition 1.8, there exists a sequence  $\varepsilon_j \rightarrow 0$  such that

$$\lim_{j \rightarrow \infty} \partial^\alpha U(\cdot, \varepsilon_j) = 0 \text{ in } L^p(\mathbb{R}^n).$$

This implies that  $\partial^\alpha U(\cdot, 0) = 0$ . By taking the partial Fourier transform in  $x$ , we obtain that

$$0 = \mathcal{F}_x(\partial^\alpha U)(\xi, 0) = (\iota\xi)^\beta (-|\xi|)^\gamma \widehat{f}(\xi),$$

and thus  $f = 0$ , which is a contradiction. □

*Proof of Theorem 1.15.* Assume that  $f \in L^1(\mathbb{R}^n) \cap B_{1,1}^0$ . Let  $U$  be as in Theorem 1.4. Then, by Theorem 1.4, we have

$$\int_{\mathbb{R}^n \times (0, \infty)} \varepsilon^{|\alpha|-1} |\partial^\alpha U(x, \varepsilon)| dx d\varepsilon \lesssim |f|_{B_{1,1}^0}, \quad \forall \alpha \neq 0. \quad (5.22)$$

On the other hand, (5.6) implies in particular that

$$\int_{\mathbb{R}^n \times (0, \infty)} \varepsilon \left| \frac{\partial^2}{\partial x_j^2} V(x, \varepsilon) \right| dx d\varepsilon \lesssim \int_{\mathbb{R}^n \times (0, \infty)} \varepsilon \left| \frac{\partial^2}{\partial x_j^2} U(x, \varepsilon) \right| dx d\varepsilon, \quad \forall j \in \llbracket 1, n \rrbracket. \quad (5.23)$$

Using successively Hardy's inequality at  $\infty$ , Uspenskii's trick, (5.22) with  $|\alpha| = 2$  and (5.23), we find that

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, \infty)} \left| \frac{\partial}{\partial \varepsilon} V(x, \varepsilon) \right| dx d\varepsilon &\lesssim \int_{\mathbb{R}^n \times (0, \infty)} \varepsilon \left| \frac{\partial^2}{\partial \varepsilon^2} V(x, \varepsilon) \right| dx d\varepsilon \lesssim \sum_{j=1}^n \int_{\mathbb{R}^n \times (0, \infty)} \varepsilon \left| \frac{\partial^2}{\partial x_j^2} V(x, \varepsilon) \right| dx d\varepsilon \\ &\lesssim \sum_{j=1}^n \int_{\mathbb{R}^n \times (0, \infty)} \varepsilon \left| \frac{\partial^2}{\partial x_j^2} U(x, \varepsilon) \right| dx d\varepsilon \lesssim |f|_{B_{1,1}^0}. \end{aligned} \quad (5.24)$$

Since we also clearly have

$$\|V\|_{L^1(\mathbb{R}^n \times (0,1))} \leq \|f\|_{L^1(\mathbb{R}^n)}, \quad (5.25)$$

estimates (5.22)-(5.25), as well as (5.6), imply " $\lesssim$ " in (1.16).

<sup>7</sup>In our case,  $f$  is not assumed to be smooth, but (5.23) and (5.24) are obtained from the corresponding estimates for smooth  $f$  by a standard smoothing argument.

Conversely, assume that

$$\int_0^1 \int_{\mathbb{R}^n} |V(x, \varepsilon)| dx d\varepsilon + \int_{\mathbb{R}^n \times (0, \infty)} |\nabla V(x, \varepsilon)| dx d\varepsilon < \infty.$$

Then, by standard trace theory, we have  $f \in L^1(\mathbb{R}^n)$  and

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \int_0^1 \int_{\mathbb{R}^n} |V(x, \varepsilon)| dx d\varepsilon + \int_{\mathbb{R}^n \times (0, \infty)} |\nabla V(x, \varepsilon)| dx d\varepsilon.$$

Thus the heart of the proof consists in proving the estimate

$$|f|_{B_{1,1}^0} \lesssim \int_{\mathbb{R}^n \times (0, \infty)} |\nabla V(x, \varepsilon)| dx d\varepsilon. \quad (5.26)$$

In turn, (5.26) will follow by combining

$$|f|_{B_{1,1}^0} \lesssim \int_{\mathbb{R}^n \times (0, \infty)} |\nabla U(x, \varepsilon)| dx d\varepsilon \quad (5.27)$$

with

$$\int_{\mathbb{R}^n \times (0, \infty)} |\nabla U(x, \varepsilon)| dx d\varepsilon \lesssim \int_{\mathbb{R}^n \times (0, \infty)} |\nabla V(x, \varepsilon)| dx d\varepsilon. \quad (5.28)$$

Estimates (5.27) and (5.28) will be obtained below. For the time being, let us note that (5.27) can be seen as variant of Theorem 1.11 with  $s = 0$  and  $l = 1$ , and that (5.28) is a reverse of (1.11).

*Step 1. Proof of (5.27).*

More generally, we will prove the following cousin of Theorem 1.3.

**5.5 Lemma.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $f \in \mathcal{S}'(\mathbb{R}^n)$  and set  $U(x, \varepsilon) = f * \rho_\varepsilon(x)$ , with  $\rho$  as in the proof of Theorem 1.4. Then

$$|f|_{B_{p,q}^s}^q \lesssim \int_0^\infty \varepsilon^{q(1-s)-1} \left\| \frac{\partial}{\partial \varepsilon} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon. \quad (5.29)$$

Clearly, (5.29) with  $p = q = 1$  and  $s = 0$  implies (5.27).

*Proof of Lemma 5.5.* Since the choice of the mollifier  $\varphi$  leading to the Littlewood-Paley decomposition is irrelevant (Lemma 2.5), we consider the Littlewood-Paley decomposition associated to  $\rho$ , that is, we let

$$f_j = U(\cdot, 2^{-j}) - U(\cdot, 2^{1-j}), \quad \forall j \in \mathbb{Z}.$$

Hölder's inequality leads to

$$\begin{aligned} |f|_{B_{p,q}^s}^q &= \sum_{j \in \mathbb{Z}} 2^{sjq} \|U(\cdot, 2^{-j}) - U(\cdot, 2^{1-j})\|_{L^p(\mathbb{R}^n)}^q \leq \sum_{j \in \mathbb{Z}} 2^{sjq} \left\| \int_{2^{-j}}^{2^{1-j}} \frac{\partial}{\partial \varepsilon} U(\cdot, \varepsilon) d\varepsilon \right\|_{L^p(\mathbb{R}^n)}^q \\ &\leq \sum_{j \in \mathbb{Z}} 2^{sjq} 2^{-j(q-1)} \int_{2^{-j}}^{2^{1-j}} \left\| \frac{\partial}{\partial \varepsilon} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \sim \int_0^\infty \varepsilon^{q(1-s)-1} \left\| \frac{\partial}{\partial \varepsilon} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon. \quad \square \end{aligned}$$

**5.6 Remark.** When  $s < 1$ , Lemma 5.5 combined with Theorem 1.4 leads to the equivalence

$$|f|_{B_{p,q}^s}^q \sim \int_0^\infty \varepsilon^{q(1-s)-1} \left\| \frac{\partial}{\partial \varepsilon} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon. \quad (5.30)$$

This is similar to results of Triebel [22] and Bui [7], who obtained this equivalence when  $U$  is replaced by the harmonic extension  $V$ . [22] and [7] also contain analog equivalences involving higher order derivatives  $\frac{\partial^l}{\partial \varepsilon^l} V$ .

*Step 2 (in the proof of Theorem 1.15).* Proof of (5.28).

Estimates of the type (5.28) are standard in the theory of Hardy spaces, and we will explain how (5.28) can be obtained using the techniques detailed in Stein's monograph [17, Chapter III, p. 92–94, p. 99].<sup>8</sup> To start with, there exists a rapidly decreasing function  $\eta : [1, \infty) \rightarrow \mathbb{C}$  such that the function  $\Phi \in L^1(\mathbb{R}^n)$  defined by

$$\Phi(x) = \int_1^\infty \eta(t) P_t(x) dt, \quad \forall x \in \mathbb{R}^n,$$

belongs to  $\mathcal{S}(\mathbb{R}^n)$  and has integral 1 [17, Section III.1.7, p. 99]. Let

$$W(x, \varepsilon) = f * \Phi_\varepsilon(x) = \int_1^\infty \eta(t) f * P_{t\varepsilon}(x) dt = \int_1^\infty \eta(t) V(x, t\varepsilon) dt, \quad \varepsilon > 0, x \in \mathbb{R}^n.$$

Then

$$\left| \frac{\partial}{\partial \varepsilon} W(x, \varepsilon) \right| \leq \int_1^\infty t |\eta(t)| \left| \left( \frac{\partial}{\partial \varepsilon} V \right) (x, t\varepsilon) \right| dt,$$

and thus

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, \infty)} \left| \frac{\partial}{\partial \varepsilon} W(x, \varepsilon) \right| dx d\varepsilon &\leq \int_{\mathbb{R}^n \times (0, \infty)} \int_1^\infty t |\eta(t)| \left| \left( \frac{\partial}{\partial \varepsilon} V \right) (x, t\varepsilon) \right| dx d\varepsilon dt \\ &= C \int_{\mathbb{R}^n \times (0, \infty)} \left| \frac{\partial}{\partial \varepsilon} V(x, \varepsilon) \right| dx d\varepsilon, \end{aligned} \quad (5.31)$$

with

$$C = \int_1^\infty |\eta(t)| dt < \infty.$$

Similarly, we have

$$\int_{\mathbb{R}^n \times (0, \infty)} |\nabla_x W(x, \varepsilon)| dx d\varepsilon \leq \int_1^\infty \frac{|\eta(t)|}{t} dt \int_{\mathbb{R}^n \times (0, \infty)} |\nabla_x V(x, \varepsilon)| dx d\varepsilon. \quad (5.32)$$

Thus  $\nabla V$  controls  $\nabla W$  (via (5.31) and (5.32)). It remains to prove that  $\nabla W$  controls  $\nabla U$ . This is obtained with the help of Remark 5.3. We write

$$\rho = \sum_{k \geq 0} \lambda^k * \Phi_{2^{-k}}, \quad \text{and thus } \rho_\varepsilon = \sum_{k \geq 0} \left( \lambda^k \right)_\varepsilon * \Phi_{2^{-k}\varepsilon}, \quad \forall \varepsilon > 0, \quad (5.33)$$

where the functions  $\lambda^k$  satisfy (5.17).

<sup>8</sup>The same technique was used by Stein [16, Section VI.3] and by Adams, Aronszajn and Smith [2] in order to construct universal extension operators for Sobolev spaces of functions in domains. See also the reference Stein [10] in [16].

We note that (5.33) implies that

$$U(x, \varepsilon) = \sum_{k \geq 0} \left( \lambda^k \right)_\varepsilon * W(\cdot, 2^{-k} \varepsilon)(x). \quad (5.34)$$

In turn, (5.34) combined with (5.17) leads to the following estimate:

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, \infty)} |\nabla_x U(x, \varepsilon)| dx d\varepsilon &\leq \sum_{k \geq 0} \|\lambda^k\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n \times (0, \infty)} |\nabla_x W(x, 2^{-k} \varepsilon)| dx d\varepsilon \\ &\lesssim \sum_{k \geq 0} \frac{1}{2^{2k}} \int_{\mathbb{R}^n \times (0, \infty)} |\nabla_x W(x, 2^{-k} \varepsilon)| dx d\varepsilon \\ &\lesssim \int_{\mathbb{R}^n \times (0, \infty)} |\nabla_x W(x, \varepsilon)| dx d\varepsilon. \end{aligned} \quad (5.35)$$

In order to estimate  $\partial U / \partial \varepsilon$ , we start from the identity

$$\frac{\partial}{\partial \varepsilon} (\lambda_\varepsilon) = -\operatorname{div}_x [(\lambda x)_\varepsilon] = -\sum_{j=1}^n \frac{\partial}{\partial x_j} \left( \frac{1}{\varepsilon^n} \lambda \left( \frac{x}{\varepsilon} \right) \frac{x_j}{\varepsilon} \right) \text{ for all } \lambda \in \mathcal{S}(\mathbb{R}^n), \quad (5.36)$$

which implies that

$$\frac{\partial}{\partial \varepsilon} \left[ \left( \lambda^k \right)_\varepsilon * W(\cdot, 2^{-k} \varepsilon)(x) \right] = 2^{-k} \left( \lambda^k \right)_\varepsilon * \left( \frac{\partial}{\partial \varepsilon} W \right)(x, 2^{-k} \varepsilon) - \sum_{j=1}^n \left( \lambda^k x_j \right)_\varepsilon * \left( \frac{\partial}{\partial x_j} W \right)(\cdot, 2^{-k} \varepsilon)(x). \quad (5.37)$$

By combining the identities (5.34) and (5.37), we find that

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, \infty)} \left| \frac{\partial}{\partial \varepsilon} U(x, \varepsilon) \right| dx d\varepsilon &\leq \sum_{k \geq 0} 2^{-k} \|\lambda^k\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n \times (0, \infty)} \left| \left( \frac{\partial}{\partial \varepsilon} W \right)(x, 2^{-k} \varepsilon) \right| dx d\varepsilon \\ &\quad + \sum_{k \geq 0} \|x \mapsto \lambda^k(x)x\|_{L^1(\mathbb{R}^n)} \int_{\mathbb{R}^n \times (0, \infty)} \left| \left( \frac{\partial}{\partial x_j} W \right)(x, 2^{-k} \varepsilon) \right| dx d\varepsilon. \end{aligned}$$

We now use the estimate (5.17) and obtain

$$\begin{aligned} \int_{\mathbb{R}^n \times (0, \infty)} \left| \frac{\partial}{\partial \varepsilon} U(x, \varepsilon) \right| dx d\varepsilon &\lesssim \sum_{k \geq 0} 2^{-2k} \int_{\mathbb{R}^n \times (0, \infty)} \left| \left( \frac{\partial}{\partial \varepsilon} W \right)(x, 2^{-k} \varepsilon) \right| dx d\varepsilon \\ &\quad + \sum_{k \geq 0} 2^{-2k} \int_{\mathbb{R}^n \times (0, \infty)} \left| \left( \frac{\partial}{\partial x_j} W \right)(x, 2^{-k} \varepsilon) \right| dx d\varepsilon \\ &\lesssim \int_{\mathbb{R}^n \times (0, \infty)} |\nabla W(x, \varepsilon)| dx d\varepsilon. \end{aligned} \quad (5.38)$$

Estimate (5.28) follows from (5.31), (5.32), (5.35) and (5.38).

The proof of Theorem 1.15 is complete.  $\square$

*Proof of Theorem 1.11, item 2.* Let  $l > s$ , with  $s \leq 0$ , and let  $E = E_l$  be the “energy” defined by formula (1.10). The estimate

$$E(V) \lesssim |f|_{B_{p,q}^s}^q, \forall f \in C_c^\infty(\mathbb{R}^n), \quad (5.39)$$

follows from Theorem 1.9.

On the other hand, we claim that

$$|f|_{B_{p,q}^s}^q \lesssim \int_0^\infty \varepsilon^{q(l-s)-1} \left\| \frac{\partial^l}{\partial \varepsilon^l} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \leq E_l(U), \forall l \in \mathbb{N}, \forall s < 1, \quad (5.40)$$

where  $U$  is as in the proof of Theorem 1.4. Indeed, when  $l = 1$  estimate (5.40) was established in Lemma 5.5. The case where  $l \geq 2$  follows by induction from the case where  $l = 1$ , using Hardy's inequality at 0 (2.2).<sup>9</sup> Finally, the direct Nikolskii's inequality (2.4) implies that

$$E_1(U) = \int_0^\infty \varepsilon^{q(1-s)-1} \left\| \frac{\partial}{\partial \varepsilon} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim \int_0^\infty \varepsilon^{-qs-1} \|U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = E_0(U),$$

and thus (5.40) holds also when  $l = 0$ .

We complete the proof of the theorem by combining (5.39) and (5.40) with the next lemma.  $\square$

In the next statement, we consider some  $f \in C_c^\infty(\mathbb{R}^n)$ . We let  $U$  be as in the proof of Theorem 1.4, and we let  $V$  be the harmonic extension of  $f$ .

**5.7 Lemma.** Let  $s \in \mathbb{R}$ ,  $1 \leq p \leq \infty$ ,  $1 \leq q < \infty$ , and  $l \in \mathbb{N}$ . Then

$$E_l(U) \lesssim E_l(V). \quad (5.41)$$

*Proof.* We argue as in Step 2 in the proof of Theorem 1.15, which leads to the estimate (5.28). As in formula (5.33), we write

$$\rho_\varepsilon = \sum_{k \geq 0} (\lambda^k)_\varepsilon * \left( \int_1^\infty \eta(t) P_{2^{-k}t\varepsilon} dt \right), \quad (5.42)$$

where  $\eta : [1, \infty) \rightarrow \mathbb{C}$  is a rapidly decreasing function and where the  $\lambda^k$ 's are rapidly decreasing in the sense of (5.17). We note that (5.42) implies that

$$U(x, \varepsilon) = \sum_{k \geq 0} \int_1^\infty \eta(t) (\lambda^k)_\varepsilon * V(\cdot, 2^{-k}t\varepsilon)(x) dt. \quad (5.43)$$

If we combine (5.43) with the identity (5.36) and with the Leibniz's rule, we obtain, for every  $\alpha = (\beta, \gamma) \in \mathbb{N}^{n+1}$ , an estimate of the form

$$\begin{aligned} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} &\lesssim \sum_{\delta=0}^\gamma \sum_{|\xi|=|\beta|+\gamma-\delta} \sum_{k \geq 0} \int_1^\infty |\eta(t)| t^\delta 2^{-\delta k} \left\| (\eta^{k, \gamma, \delta})_\varepsilon * \frac{\partial^\delta}{\partial \varepsilon^\delta} \partial_x^\xi V(\cdot, 2^{-k}t\varepsilon) \right\|_{L^p(\mathbb{R}^n)} dt \\ &\lesssim \sum_{\delta=0}^\gamma \sum_{|\xi|=|\beta|+\gamma-\delta} \sum_{k \geq 0} \int_1^\infty |\eta(t)| t^\delta 2^{-\delta k} \left\| \eta^{k, \gamma, \delta, \xi} \right\|_{L^1(\mathbb{R}^n)} \times \\ &\quad \times \left\| \frac{\partial^\delta}{\partial \varepsilon^\delta} \partial_x^\xi V(\cdot, 2^{-k}t\varepsilon) \right\|_{L^p(\mathbb{R}^n)} dt. \end{aligned} \quad (5.44)$$

In addition, we have (using Remark 5.3 and the identity (5.33)) the estimates

$$\left\| \eta^{k, \gamma, \delta, \xi} \right\|_{L^1(\mathbb{R}^n)} \leq \frac{C}{2^{Mk}}, \quad \forall M, \forall \gamma, \forall \delta, \forall \xi. \quad (5.45)$$

By combining (5.44) with (5.45) we derive, for every every  $M > 0$ , for every integer  $l$  and for every  $\alpha \in \mathbb{N}^{n+1}$  such that  $|\alpha| = l$ , the estimate

$$\|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{|\mu|=l} \sum_{k \geq 0} \frac{1}{2^{Mk}} \int_1^\infty |\eta(t)| t^l \left\| \partial^\mu V(\cdot, 2^{-k}t\varepsilon) \right\|_{L^p(\mathbb{R}^n)} dt. \quad (5.46)$$

<sup>9</sup>It is only at this stage that we use the assumption  $s < 1$ .

Applying Hölder's inequality with exponents  $q$  and  $q/(q-1)$  to (5.46), we find (using the fact that  $\eta$  is rapidly decreasing) that

$$\|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q \lesssim \sum_{|\mu|=l} \sum_{k \geq 0} \frac{1}{2^{Mk}} \int_1^\infty |\eta(t)| t^l \|\partial^\mu V(\cdot, 2^{-k} t \varepsilon)\|_{L^p(\mathbb{R}^n)}^q dt. \quad (5.47)$$

Using (5.47) we find (using the change of variables  $2^{-k} t \varepsilon = u$ ) that

$$\begin{aligned} E_l(U) &= \sum_{|\alpha|=l} \int_0^\infty \varepsilon^{q(l-s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim \sum_{|\mu|=l} \sum_{k \geq 0} \frac{1}{2^{Mk}} \int_1^\infty \int_0^\infty \varepsilon^{q(l-s)-1} \times \\ &\quad \times |\eta(t)| t^l \|\partial^\mu V(\cdot, 2^{-k} t \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon dt \leq E_l(V) \sum_{k \geq 0} \frac{1}{2^{[M-q(l-s)]k}} \int_1^\infty |\eta(t)| t^{l-q(l-s)} dt \\ &\lesssim E_l(V), \end{aligned} \quad (5.48)$$

provided we take  $M$  sufficiently large.  $\square$

We next justify Remark 1.16, asserting that homogeneous norms are not always suited to the Besov spaces with non positive exponent  $s$ . This is explained by the next example.

**5.8 Proposition.** Let  $s \leq 0$  and  $1 \leq q < \infty$ . Let  $f \in \mathcal{S}(\mathbb{R}^n)$  be such that  $\int_{\mathbb{R}^n} f \neq 0$ . Then  $|f|_{B_{1,q}^s} = \infty$ .<sup>10</sup>

*Proof.* We consider the homogeneous Littlewood-Paley decomposition (2.10) associated to the sequence given by (2.8). Let  $\xi$  be such that  $\widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi) \neq 0$ . Then, as  $j \rightarrow -\infty$ , we have

$$\widehat{f}_j(2^j \xi) = \widehat{f}(2^j \xi) (\widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)) \rightarrow (\widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)) \widehat{f}(0) = (\widehat{\varphi}(\xi) - \widehat{\varphi}(2\xi)) \int_{\mathbb{R}^n} f \neq 0,$$

so that

$$\liminf_{j \rightarrow -\infty} \|f_j\|_{L^1} \geq \liminf_{j \rightarrow -\infty} \|\widehat{f}_j\|_{L^\infty} \geq \liminf_{j \rightarrow -\infty} \left| \widehat{f}_j(2^j \xi) \right| > 0.$$

We find that  $\sum_{j \leq 0} \|f_j\|_{L^1}^q = \infty$ , and thus  $|f|_{B_{1,q}^s} = \infty$ .  $\square$

The above proposition and the comparison between the statements of Theorems 1.15 and 1.17 suggest that it is natural to strengthen the conclusions of Propositions 1.13 and 1.14 as follows.

**5.9 Proposition.** Let  $f \in C_c^\infty(\mathbb{R}^n)$ .

1. Let  $s < 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Then there exists a sequence  $(F_j) \subset C_c^\infty(\mathbb{R}^n \times [0, 1])$  of extensions of  $f$  such that

$$\lim_{j \rightarrow \infty} \int_0^1 \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha F_j(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = 0, \quad \forall \alpha \in \mathbb{N}^{n+1}. \quad (5.49)$$

2. Let  $1 \leq p \leq \infty$  and  $1 < q < \infty$ . Then we may choose  $F_j$  such that

$$\lim_{j \rightarrow \infty} \int_0^1 \varepsilon^{q|\alpha|-1} \|\partial^\alpha F_j(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = 0, \quad \forall \alpha \in \mathbb{N}^{n+1} \setminus \{(0, 0)\}. \quad (5.50)$$

**5.10 Proposition.** Let  $1 \leq p \leq \infty$  and let  $l > 0$  be an integer. Then:

<sup>10</sup>Though, of course,  $f$  belongs to the space  $B_{1,q}^s$ .



1. Every  $f \in C_c^\infty(\mathbb{R}^n)$  has an extension  $F \in C^\infty(\mathbb{R}^n \times [0, 1))$  such that

$$\int_0^1 \varepsilon^{l-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} d\varepsilon \lesssim \|f\|_{L^p(\mathbb{R}^n)}, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| = l. \quad (5.51)$$

2. There exists a sequence  $(f^k) \subset C_c^\infty(\mathbb{R}^n)$  such that  $\|f^k\|_{L^p(\mathbb{R}^n)} = 1$  and  $\|f^k\|_{B_{p,1}^0} \rightarrow \infty$ .

*Proof of Proposition 5.9.*

*Step 1.* The case where  $s < 0$ .

Let  $g \in C_c^\infty(\mathbb{R})$  be such that  $g(0) = 1$ . We let

$$F_j(x, \varepsilon) = f(x)g(j\varepsilon), \quad \forall j \geq 1, \quad \forall x \in \mathbb{R}^n, \quad \forall \varepsilon \geq 0.$$

Clearly,  $F \in C_c^\infty(\Omega)$  is an extension of  $f$ .

If  $\alpha = (\beta, \gamma)$ , then

$$\begin{aligned} \int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\alpha F_j(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon &= j^{q\gamma} \int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \|\partial^\beta f\|_{L^p(\mathbb{R}^n)}^q \left|g^{(\gamma)}(j\varepsilon)\right|^q d\varepsilon \\ &= j^{q(s-|\beta|)} \|\partial^\beta f\|_{L^p(\mathbb{R}^n)}^q \int_0^\infty \varepsilon^{q(|\alpha|-s)-1} \left|g^{(\gamma)}(\varepsilon)\right|^q d\varepsilon \rightarrow 0 \text{ as } j \rightarrow \infty. \end{aligned}$$

*Step 2.* The case where  $s = 0$ ,  $q > 1$  and  $\alpha \neq \{(0, 0)\}$ .

Fix a function  $\psi \in C^\infty(\mathbb{R}; [0, 1])$  such that  $\psi(t) = \begin{cases} 1, & \text{if } t \leq -2 \\ 0, & \text{if } t \geq -1 \end{cases}$ . For  $j \geq 2$ , let

$$g^j(\varepsilon) = \begin{cases} \psi(\ln \varepsilon / \ln j), & \text{if } \varepsilon > 0 \\ 1, & \text{if } \varepsilon \leq 0 \end{cases}.$$

Then clearly  $g^j \in C^\infty(\mathbb{R})$ . We set  $F_j(x, \varepsilon) = f(x)g^j(\varepsilon)$ , which is a smooth extension of  $f$ . Let  $\alpha = (\beta, \gamma) \in \mathbb{N}^{n+1} \setminus \{(0, 0)\}$ . Then we have

$$\int_0^\infty \varepsilon^{q|\alpha|-1} \|\partial^\alpha F_j(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon = \|\partial^\beta f\|_{L^p(\mathbb{R}^n)}^q \int_0^\infty \varepsilon^{q|\alpha|-1} \left|(g^j)^{(\gamma)}(\varepsilon)\right|^q d\varepsilon,$$

and thus, setting  $a = |\alpha|$ , we have to prove that

$$\int_0^\infty \varepsilon^{qa-1} \left|(g^j)^{(\gamma)}(\varepsilon)\right|^q d\varepsilon \rightarrow 0 \text{ as } j \rightarrow \infty, \text{ provided either } a \geq \gamma > 0 \text{ or } a > \gamma = 0. \quad (5.52)$$

We establish (5.52) only when  $a \geq \gamma > 0$ ; the case where  $a > \gamma = 0$  is similar and is left to the reader. By a straightforward induction on  $\gamma \geq 1$ , we have

$$\left|(g^j)^{(\gamma)}(\varepsilon)\right| \lesssim \frac{1}{\varepsilon^\gamma} \sum_{k=1}^\gamma \frac{1}{\ln^k j} \left|\psi^{(k)}\left(\frac{\ln \varepsilon}{\ln j}\right)\right|. \quad (5.53)$$

Inserting (5.53) into the integral in (5.52) and performing the change of variables  $t = \frac{\ln \varepsilon}{\ln j}$ , we are led to

$$\begin{aligned} \int_0^\infty \varepsilon^{qa-1} \left|(g^j)^{(\gamma)}(\varepsilon)\right|^q d\varepsilon &\lesssim \sum_{k=1}^\gamma \frac{1}{\ln^{kq} j} \int_0^\infty \varepsilon^{q(a-\gamma)} \left|\psi^{(k)}\left(\frac{\ln \varepsilon}{\ln j}\right)\right|^q \frac{d\varepsilon}{\varepsilon} = \sum_{k=1}^\gamma \frac{1}{\ln^{kq-1} j} \int_{\mathbb{R}} j^{q(a-\gamma)t} \left|\psi^{(k)}(t)\right|^q dt \\ &= \sum_{k=1}^\gamma \frac{1}{\ln^{kq-1} j} \int_{-2}^{-1} j^{q(a-\gamma)t} \left|\psi^{(k)}(t)\right|^q dt \lesssim \sum_{k=1}^\gamma \frac{1}{\ln^{kq-1} j} \rightarrow 0 \text{ as } j \rightarrow \infty. \quad \square \end{aligned}$$

*Proof of Proposition 5.10.*

*Step 1.* Proof of item 1.

We simplify Gagliardo's idea in proving “tr  $W^{1,1} \supset L^1$ ”. We fix a function  $\psi \in C^\infty(\mathbb{R})$  such that  $\psi(t) = \begin{cases} 1, & \text{if } t \leq 1/2 \\ 0, & \text{if } t \geq 1 \end{cases}$ . For every  $f \in C_c^\infty(\mathbb{R}^n)$ , we let

$$F(x, \varepsilon) = f(x) \psi\left(\frac{\varepsilon}{\delta}\right), \quad \forall x \in \mathbb{R}^n, \forall \varepsilon \geq 0,$$

with  $\delta > 0$  a constant to be fixed later. Clearly,  $F$  is an extension of  $f$ .

If  $\alpha = (\beta, \gamma) \in \mathbb{N}^{n+1} \setminus \{(0, 0)\}$ , then we have

$$\int_0^\infty \varepsilon^{|\alpha|-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)} d\varepsilon \lesssim \frac{1}{\delta^\gamma} \|\partial^\beta f\|_{L^p(\mathbb{R}^n)} \int_0^\delta \varepsilon^{|\alpha|-1} d\varepsilon \lesssim \delta^{|\beta|} \|\partial^\beta f\|_{L^p(\mathbb{R}^n)}.$$

We find that (5.51) holds for small  $\delta$ .

*Step 2.* Proof of item 2 when  $p = 1$ .

Let  $\alpha \in (1, 2)$  and

$$f(x) = C \begin{cases} 1/[|x|^n \ln^\alpha |x|], & \text{if } |x| < 1/2 \\ 0, & \text{if } |x| \geq 1/2 \end{cases}.$$

It is easy to see that

$$f \in L^1(\mathbb{R}^n), \quad \|f\|_{L^1(\mathbb{R}^n)} = 1 \text{ (for an appropriate } C > 0) \text{ and } f \notin L \log L(B(0, \delta)), \quad \forall \delta \in (0, 1/2). \quad (5.54)$$

The non embedding (5.54) implies that, whenever  $h \in L_{loc}^\infty(\mathbb{R}^n)$ , we cannot have  $f - h \in \mathcal{H}^1(\mathbb{R}^n)$  (where  $\mathcal{H}^1(\mathbb{R}^n)$  is the Hardy space). Indeed, argue by contradiction and assume that we do have  $f - h \in \mathcal{H}^1(\mathbb{R}^n)$ . Then, for sufficiently small  $\delta$ , we have  $f - h > 0$  in  $B(0, 2\delta)$  and thus we have  $f - h \in L \log L(B(0, \delta))$  [17, III.5.3, p. 128]; see also [27], [15]. This contradicts (5.54).

We are now ready to construct our “bad” sequence  $(f^k)$ . Let  $(f^k) \subset C_c^\infty(\mathbb{R}^n)$  be such that  $\|f_k\|_{L^1(\mathbb{R}^n)} \equiv 1$  and  $f^k \rightarrow f$  in  $L^1$ . We claim that  $\|f^k\|_{B_{1,1}^0} \rightarrow \infty$  as  $k \rightarrow \infty$ . Indeed, argue by contradiction and assume that, possibly up to a subsequence, the inhomogeneous Littlewood-Paley decompositions  $f^k = \sum_{j \geq 0} (f^k)_j$  of the  $f^k$ 's satisfy

$$\sum_{j \geq 0} \left\| (f^k)_j \right\|_{L^1(\mathbb{R}^n)} \leq C < \infty, \quad \forall k. \quad (5.55)$$

Here, the Littlewood-Paley decomposition relies on a sequence  $(\varphi^j)_{j \geq 0}$  as in (2.6).

Since  $f^k \rightarrow f$  in  $L^1$ , we find (by Young's inequality) that  $(f^k)_j \rightarrow f_j$  in  $L^1$  as  $k \rightarrow \infty$ , and thus (5.55) leads to

$$\sum_{j \geq 0} \|f_j\|_{L^1(\mathbb{R}^n)} < \infty, \quad (5.56)$$

where  $f = \sum_{j \geq 0} f_j$  is the Littlewood-Paley decomposition of  $f$ .

We complete the proof by proving that (5.56) cannot hold. Indeed, we have  $f_0 \in C^\infty$ , and  $f - f_0 = \sum_{j \geq 1} f_j$  in  $L^1$ . We obtain a contradiction by proving that  $\sum_{j \geq 1} f_j \in \mathcal{H}^1(\mathbb{R}^n)$ . This conclusion will be obtained by combining (5.56) with

$$\|f_j\|_{\mathcal{H}^1(\mathbb{R}^n)} \lesssim \|f_j\|_{L^1(\mathbb{R}^n)} \quad (5.57)$$

and with the fact that  $\mathcal{H}^1(\mathbb{R}^n)$  is a Banach space [17, III.5.1, p. 127]. In turn, (5.57) is obtained as follows. Let  $R_l$  be the  $l^{\text{th}}$  Riesz transform in  $\mathbb{R}^n$ , i.e.,

$$\mathcal{F}(R_l f)(\xi) = \frac{\xi_l}{|\xi|} \widehat{f}(\xi), \quad \forall 1 \leq l \leq n, \quad \forall f \in L^1(\mathbb{R}^n), \quad \forall \xi \in \mathbb{R}^n.$$

Fix any  $\eta^l \in \mathcal{S}(\mathbb{R}^n)$  is such that

$$\mathcal{F}\eta^l \in C_c^\infty(\mathbb{R}^n) \text{ and } \mathcal{F}\eta^l(\xi) = \frac{\xi^l}{|\xi|} \text{ in } B(0, 4) \setminus B(0, 1/4). \quad (5.58)$$

For such  $\eta^l$ , (2.6) and (5.58) lead to

$$R_l f_j = \left(\eta^l\right)_{2^{-j}} * f_j, \quad \forall 1 \leq l \leq n, \quad \forall j \in \mathbb{Z},$$

and thus we have (using the characterization of  $\mathcal{H}^1(\mathbb{R}^n)$  via the Riesz transforms [17, III.4.3, p. 123-124])

$$\|f_j\|_{\mathcal{H}^1(\mathbb{R}^n)} \sim \|f_j\|_{L^1(\mathbb{R}^n)} + \sum_{l=1}^n \|R_l(f_j)\|_{L^1(\mathbb{R}^n)} \leq \|f_j\|_{L^1(\mathbb{R}^n)} \left(1 + \sum_{l=1}^n \left\| \left(\eta^l\right)_{2^{-j}} \right\|_{L^1(\mathbb{R}^n)}\right) \lesssim \|f_j\|_{L^1(\mathbb{R}^n)},$$

i.e., (5.57) holds.

*Step 3. Proof of item 2 when  $p = \infty$ .*

We take  $f = \mathbb{1}_{B(0,1)}$  and we let  $(f^k) \subset C_c^\infty(\mathbb{R}^n)$  be such that  $\|f^k\|_{L^\infty(\mathbb{R}^n)} \equiv 1$  and  $f^k \rightarrow f$  in  $L^1$ . We prove that  $\|f^k\|_{B_{\infty,1}^0} \rightarrow \infty$  as  $k \rightarrow \infty$ . For otherwise, arguing by contradiction as in Step 2, we have  $\|f\|_{B_{\infty,1}^0} < \infty$ , and thus  $f = \sum_{j \geq 0} f_j$  in  $L^\infty$ . But this cannot happen, since each  $f_j$  is smooth, while  $f$  is essentially discontinuous.

*Step 4. Proof of item 2 when  $1 < p < \infty$ .*

Let us introduce, only in this proof, the following notation:

$$\langle f \rangle = \sum_{j \geq 1} \|f_j\|_{L^p(\mathbb{R}^n)}, \text{ where } f = \sum_{j \geq 0} f_j \text{ is the Littlewood-Paley decomposition of } f.$$

Let  $\psi \in \mathcal{S}(\mathbb{R}^n)$  be such that

$$\|\psi\|_{L^p(\mathbb{R}^n)} = 1 \text{ and } \text{supp } \widehat{\psi} \subset B(0, 2) \setminus B(0, 1). \quad (5.59)$$

We note that

$$\psi_\varepsilon(x) \rightarrow 0 \text{ as } \varepsilon \rightarrow 0, \quad \forall x \in \mathbb{R}^n \setminus \{0\}, \quad (5.60)$$

and that Lemma 2.6 combined with (5.59) implies that

$$\langle \psi_\varepsilon \rangle \sim \|\psi_\varepsilon\|_{L^p(\mathbb{R}^n)} = \varepsilon^{-n(1-1/p)} \text{ for sufficiently small } \varepsilon > 0. \quad (5.61)$$

We claim that there exist sequences  $\varepsilon^l \rightarrow 0$  and  $(b^l)$  such that

$$\left\| \sum_{l=1}^k b^l \psi_{\varepsilon^l} \right\|_{L^p(\mathbb{R}^n)} \rightarrow L \in (0, \infty) \text{ as } k \rightarrow \infty \quad (5.62)$$

and

$$\left\langle \sum_{l=1}^k b^l \psi_{\varepsilon^l} \right\rangle \rightarrow \infty \text{ as } k \rightarrow \infty. \quad (5.63)$$

Assuming (5.62) and (5.63) proved for the moment, we complete Step 4 as follows.

We may assume that  $L = 1$ . By suitably approximating (with  $k$  fixed) every  $b^l \psi_{\varepsilon^l}$ , with  $1 \leq l \leq k$ , we may find (using (5.62), (5.63) and a straightforward Fatou type argument) functions  $\eta_l^k$ , with  $k \geq 1$  and  $1 \leq l \leq k$ , such that

$$\eta_l^k \in C_c^\infty(\mathbb{R}^n), \quad \left\| \sum_{l=1}^k \eta_l^k \right\|_{L^p(\mathbb{R}^n)} \equiv 1 \text{ and } \left\langle \sum_{l=1}^k \eta_l^k \right\rangle \rightarrow \infty \text{ as } k \rightarrow \infty.$$

This completes Step 4. It thus remains to prove the next step.

*Step 5.* Proof of (5.62) and of (5.63).

Our construction of  $b^l$  and  $\varepsilon^l$  is based on two observations. Consider a fixed sum

$$f = \sum_{l=1}^k b^l \psi_{\varepsilon^l}; \quad (5.64)$$

we do not make any size assumption on  $b^l$ , and we assume the  $\varepsilon^l$ 's sufficiently small in order to be in position to apply (5.61).

The first observation is that, when  $f$  is as in (5.64) and  $\varepsilon$  is sufficiently small (smallness depending on  $f$ ), the functions  $f_j(f)$  and  $f_j(\psi_\varepsilon)$  (appearing in the Littlewood-Paley decomposition (2.7) associated to  $\varphi$  as in (2.6)) satisfy either  $f_j(f) = 0$  or  $f_j(\psi_\varepsilon) = 0$ ,  $\forall j \in \mathbb{Z}$ , and thus we have

$$\langle f + b\psi_\varepsilon \rangle = \langle f \rangle + |b| \langle \psi_\varepsilon \rangle, \quad \forall \varepsilon > 0 \text{ sufficiently small}, \forall b. \quad (5.65)$$

Let now  $\varepsilon > 0$  and  $C > 0$  be arbitrary, and define  $b = b(C, \varepsilon)$  through the equality

$$\|b\psi_\varepsilon\|_{L^p(\mathbb{R}^n)}^p = C, \text{ that is, } b(C, \varepsilon) = C^{1/p} \varepsilon^{n(1-1/p)}. \quad (5.66)$$

The second observation is that, with  $b$  given by (5.66), we have

$$\lim_{\varepsilon \rightarrow 0} \|f + b\psi_\varepsilon\|_{L^p(\mathbb{R}^n)}^p = \|f\|_{L^p(\mathbb{R}^n)}^p + C. \quad (5.67)$$

This follows from the Brezis-Lieb lemma [5] used in conjunction with the first equality in (5.66) and with the fact that (by (5.60) and the second equality in (5.66)) we have  $b\psi_\varepsilon \rightarrow 0$  a.e. as  $\varepsilon \rightarrow 0$ .

Let  $(C_l)$  be a sequence of positive numbers to be fixed later. By combining the two above observations with (5.61) and with the second equality in (5.66), we easily construct by induction on  $l$  sequences  $b^l$  and  $\varepsilon_l$  such that

$$\left\| \sum_{l=1}^k b^l \psi_{\varepsilon^l} \right\|_{L^p(\mathbb{R}^n)}^p \sim \sum_{l=1}^k C_l \text{ as } k \rightarrow \infty \quad (5.68)$$

and

$$\left\langle \sum_{l=1}^k b^l \psi_{\varepsilon^l} \right\rangle \sim \sum_{l=1}^k C_l^{1/p} \text{ as } k \rightarrow \infty. \quad (5.69)$$

We obtain (5.62) and (5.63) if we take e.g.  $C_l = l^{-a}$ , where  $a$  is a constant such that  $1 < a < p$ .  $\square$

We next turn to the non homogeneous counterparts of Theorems 1.3 and 1.4.

**5.11 Theorem.** Let  $s > 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ , and let  $l$  be an integer such that  $l > s$ . Let  $F \in C^\infty(\mathbb{R}^n \times [0, \infty))$ . Set  $f(x) = F(x, 0)$ ,  $x \in \mathbb{R}^n$ . Then

$$\|f\|_{B_{p,q}^s}^q \lesssim \|F\|_{L^p(\mathbb{R}^n \times (0,1))}^q + \sum_{\alpha \in \mathcal{M}_l} \int_0^\infty \varepsilon^{q(l-s)-1} \|\partial^\alpha F(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon. \quad (5.70)$$

*Proof.* Let  $M$  denote the right-hand side of (5.70). In view of Theorem 1.3, it suffices to establish the estimate

$$\|f\|_{L^p(\mathbb{R}^n)}^q \lesssim M. \quad (5.71)$$

We start by noting that Hardy's inequality at 0 (2.2) applied with

$$r = qs, \quad g(\varepsilon) = \varepsilon^{l-1} \left\| \frac{\partial^l}{\partial \varepsilon^l} F(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)},$$

leads to

$$\int_0^\infty \varepsilon^{-qs-1} \left( \int_0^\varepsilon t^{l-1} \left\| \frac{\partial^l}{\partial \varepsilon^l} F(\cdot, t) \right\|_{L^p(\mathbb{R}^n)} dt \right)^q d\varepsilon \lesssim M. \quad (5.72)$$

Using (5.72) and a straightforward mean value argument, we obtain the existence of some  $\varepsilon \in (0, 1/l)$  such that

$$\sum_{j=1}^l \|F(\cdot, j\varepsilon)\|_{L^p(\mathbb{R}^n)}^q + \left( \int_0^{l\varepsilon} t^{l-1} \left\| \frac{\partial^l}{\partial \varepsilon^l} F(\cdot, t) \right\|_{L^p(\mathbb{R}^n)} dt \right)^q \lesssim M. \quad (5.73)$$

We next note the inequality

$$|f(x)| \lesssim |\Delta_{\varepsilon e_{n+1}}^l F(x, 0)| + \sum_{j=1}^l |F(x, j\varepsilon)|,$$

which combined with the proof of (3.8) leads to

$$|f(x)| \lesssim \sum_{j=1}^l |F(x, j\varepsilon)| + \int_0^{l\varepsilon} t^{l-1} \left| \frac{\partial^l}{\partial \varepsilon^l} F(x, t) \right| dt. \quad (5.74)$$

Integrating (5.74), we find that

$$\|f\|_{L^p(\mathbb{R}^n)} \lesssim \sum_{j=1}^l \|F(\cdot, j\varepsilon)\|_{L^p(\mathbb{R}^n)} + \int_0^{l\varepsilon} t^{l-1} \left\| \frac{\partial^l}{\partial \varepsilon^l} F(\cdot, t) \right\|_{L^p(\mathbb{R}^n)} dt. \quad (5.75)$$

We obtain (5.71) by combining (5.73) with (5.75).  $\square$

The non homogeneous counterpart of Theorem 1.4 simply asserts that, when  $s > 0$ , the extension  $U$  satisfies in addition

$$\|U\|_{L^p(\mathbb{R}^n \times (0,1))} \lesssim \|f\|_{B_{p,q}^s}.$$

This is clear, and the proof will be omitted.

## 6 From weighted spaces to functional calculus

In this section, we recall some standard results related to Besov algebras or to the continuity properties of superposition operators in Besov spaces, and explain how these results can be obtained as straightforward consequences of Theorems 1.3 and 1.4. Since our main purpose is to illustrate the effectiveness of Theorems 1.3 and 1.4, we will always assume that  $q < \infty$ . However, most of the results we prove below still hold for  $q = \infty$ .

We start with some algebra and embedding properties (Propositions 6.1 to 6.3).

**6.1 Proposition.** Let  $s > 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ , and let  $0 < \theta < 1$ . Then  $B_{p,q}^s \cap L^\infty \subset B_{p/\theta, q/\theta}^{\theta s}$ .

*Proof.* Let  $f \in C_c^\infty(\mathbb{R}^n)$  and  $U(x, \varepsilon) := f * \rho_\varepsilon(x)$ . Here,  $\rho$  is as in (4.1), and thus (1.6) holds. Then, clearly,

$$\|\partial^\alpha U(\cdot, \varepsilon)\|_{L^\infty(\mathbb{R}^n)} \lesssim \frac{1}{\varepsilon^{|\alpha|}} \|f\|_{L^\infty(\mathbb{R}^n)}, \quad \forall \alpha \in \mathbb{N}^{n+1}, \quad (6.1)$$

so that

$$\begin{aligned} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^{p/\theta}(\mathbb{R}^n)} &\leq \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^\theta \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^\infty(\mathbb{R}^n)}^{1-\theta} \\ &\lesssim \frac{1}{\varepsilon^{(1-\theta)|\alpha|}} \|f\|_{L^\infty(\mathbb{R}^n)}^{1-\theta} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^\theta. \end{aligned} \quad (6.2)$$

Let  $l > s$  and let  $\alpha \in \mathcal{M}_l$ . By (6.2), one has

$$\int_0^\infty \varepsilon^{\frac{q}{\theta}(l-\theta s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^{p/\theta}(\mathbb{R}^n)}^{q/\theta} d\varepsilon \lesssim \|f\|_{L^\infty(\mathbb{R}^n)}^{q\frac{1-\theta}{\theta}} \int_0^\infty \varepsilon^{q(l-s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon. \quad (6.3)$$

By (6.3), Theorem 1.3 and (1.6), we therefore have

$$|f|_{B_{p/\theta, q/\theta}^{\theta s}}^{q/\theta} \lesssim |f|_{B_{p, q}^s}^q \|f\|_{L^\infty(\mathbb{R}^n)}^{q\frac{1-\theta}{\theta}}, \quad (6.4)$$

and thus

$$|f|_{B_{p/\theta, q/\theta}^{\theta s}} \lesssim |f|_{B_{p, q}^s} \|f\|_{L^\infty(\mathbb{R}^n)}^{1-\theta}. \quad (6.5)$$

On the other hand, we clearly have

$$\|f\|_{L^{p/\theta}(\mathbb{R}^n)} \leq \|f\|_{L^p(\mathbb{R}^n)}^\theta \|f\|_{L^\infty(\mathbb{R}^n)}^{1-\theta}. \quad (6.6)$$

We complete the proof by combining (6.5) and (6.6) with (2.20).  $\square$

An easy consequence of Proposition 6.1 is

**6.2 Proposition.** Let  $s > 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Then  $B_{p, q}^s \cap L^\infty$  is an algebra.

*Proof.* Let  $f, g \in C_c^\infty(\mathbb{R}^n)$ . We will only estimate  $|fg|_{B_{p, q}^s}$ , since the estimate of  $\|fg\|_{L^p(\mathbb{R}^n)}$  is trivial and  $fg$  clearly belongs to  $L^\infty(\mathbb{R}^n)$ . By combining (6.4) with Theorem 1.4,  $f$  has an extension  $U \in C^\infty(\mathbb{R}^n \times [0, \infty))$  such that

$$\int_0^\infty \varepsilon^{\frac{q}{\theta}(l-\theta s)-1} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^{p/\theta}(\mathbb{R}^n)}^{q/\theta} d\varepsilon \lesssim |f|_{B_{p, q}^s}^q \|f\|_{L^\infty(\mathbb{R}^n)}^{q\frac{1-\theta}{\theta}}, \quad \forall \theta \in (0, 1), \forall l > \theta s, \forall \alpha \in \mathcal{P}_{\theta s}. \quad (6.7)$$

Similarly,  $g$  has an extension  $V$  satisfying the corresponding analog of (6.7). Let  $l > s$ . In view of Theorem 1.3, in order to estimate  $|fg|_{B_{p, q}^s}$  it suffices to control

$$\int_0^\infty \varepsilon^{q(l-s)-1} \|\partial^\alpha(UV)(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon, \quad \forall \alpha \in \mathcal{M}_l.$$

In turn, by Leibniz's rule, it suffices to control

$$\int_0^\infty \varepsilon^{q(l-s)-1} \|\partial^\beta U(\cdot, \varepsilon) \partial^\gamma V(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon, \quad \forall \beta \in \mathcal{M}_{l_1}, \forall \gamma \in \mathcal{M}_{l_2} \text{ for all } l_1, l_2 \in \llbracket 0, l \rrbracket \text{ with } l_1 + l_2 = l.$$

The cases where  $l_1 = 0$  or  $l_2 = 0$  are clear (by using (6.1) with  $\alpha = 0$ ), so that we may assume that  $0 < l_1, l_2 < l$ . Let  $\theta \in (0, 1)$  be such that  $l_1 > \theta s$  and  $l_2 > (1-\theta)s$ .<sup>11</sup> By (6.7), we have

$$\int_0^\infty \varepsilon^{\frac{q}{\theta}(l_1-\theta s)} \|\partial^\beta U(\cdot, \varepsilon)\|_{L^{p/\theta}(\mathbb{R}^n)}^{q/\theta} \frac{d\varepsilon}{\varepsilon} \lesssim |f|_{B_{p, q}^s}^q \|f\|_{L^\infty(\mathbb{R}^n)}^{q\frac{1-\theta}{\theta}} \quad (6.8)$$

and

$$\int_0^\infty \varepsilon^{\frac{q}{1-\theta}(l_2-(1-\theta)s)} \|\partial^\gamma V(\cdot, \varepsilon)\|_{L^{p/(1-\theta)}(\mathbb{R}^n)}^{q/(1-\theta)} \frac{d\varepsilon}{\varepsilon} \lesssim |g|_{B_{p, q}^s}^q \|g\|_{L^\infty(\mathbb{R}^n)}^{q\frac{\theta}{1-\theta}}. \quad (6.9)$$

<sup>11</sup>Such a  $\theta$  exists since  $l_1 + l_2 = l > s$ .

By combining (6.8), (6.9) and Hölder inequality, we find that

$$\begin{aligned}
\int_0^\infty \varepsilon^{q(l-s)} \left\| \partial^\beta U(\cdot, \varepsilon) \partial^\gamma V(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} &\leq \int_0^\infty \varepsilon^{q(l-s)} \left\| \partial^\beta U(\cdot, \varepsilon) \right\|_{L^{p/\theta}(\mathbb{R}^n)}^q \left\| \partial^\gamma V(\cdot, \varepsilon) \right\|_{L^{p/(1-\theta)}(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} \\
&\leq \left( \int_0^\infty \varepsilon^{\frac{q}{\theta}(l_1-\theta s)} \left\| \partial^\beta U(\cdot, \varepsilon) \right\|_{L^{p/\theta}(\mathbb{R}^n)}^{q/\theta} \frac{d\varepsilon}{\varepsilon} \right)^\theta \times \\
&\quad \times \left( \int_0^\infty \varepsilon^{\frac{q}{1-\theta}(l_2-(1-\theta)s)} \left\| \partial^\gamma V(\cdot, \varepsilon) \right\|_{L^{p/(1-\theta)}(\mathbb{R}^n)}^{q/(1-\theta)} \frac{d\varepsilon}{\varepsilon} \right)^{1-\theta} \\
&\lesssim |f|_{B_{p,q}^s}^{\theta q} \|f\|_{L^\infty(\mathbb{R}^n)}^{q(1-\theta)} |g|_{B_{p,q}^s}^{(1-\theta)q} \|g\|_{L^\infty(\mathbb{R}^n)}^{\theta q}.
\end{aligned} \tag{6.10}$$

We complete the proof of the proposition using (6.10) and Theorem 1.3.  $\square$

**6.3 Proposition.** Let  $s_1, s_2 > 0$ ,  $1 \leq p_1, p_2 \leq \infty$  and  $1 \leq q_1, q_2 < \infty$ . Let  $0 < \theta < 1$  and define  $s$ ,  $p$  and  $q$  by

$$s = \theta s_1 + (1-\theta)s_2, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.$$

Then  $B_{p_1, q_1}^{s_1} \cap B_{p_2, q_2}^{s_2} \subset B_{p, q}^s$ .

*Proof.* Let  $f \in B_{p_1, q_1}^{s_1} \cap B_{p_2, q_2}^{s_2}$ . It is clear that  $f \in L^p$ . Moreover, let  $U = \rho_\varepsilon * f$ , with  $\rho$  as in (4.1). Recall that  $U$  satisfies

$$\int_0^\infty \varepsilon^{q_1(l-s_1)} \left\| \partial^\alpha U(\cdot, \varepsilon) \right\|_{L^{p_1}(\mathbb{R}^n)}^{q_1} \frac{d\varepsilon}{\varepsilon} \lesssim |f|_{B_{p_1, q_1}^{s_1}}^{q_1} \quad \text{for all } l > s_1 \text{ and for all } \alpha \in \mathcal{P}_{s_1},$$

and a similar estimate involving  $s_2, p_2$  and  $q_2$ . Let  $l > \max(s_1, s_2)$  and  $\alpha \in \mathcal{M}_l$ . By Hölder's inequality, we have

$$\begin{aligned}
\int_0^\infty \varepsilon^{q(l-s)} \left\| \partial^\alpha U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} &\leq \int_0^\infty \varepsilon^{q\theta(l-s_1)} \left\| \partial^\alpha U(\cdot, \varepsilon) \right\|_{L^{p_1}(\mathbb{R}^n)}^{\theta q} \varepsilon^{q(1-\theta)(l-s_2)} \left\| \partial^\alpha U(\cdot, \varepsilon) \right\|_{L^{p_2}(\mathbb{R}^n)}^{(1-\theta)q} \frac{d\varepsilon}{\varepsilon} \\
&\leq \left( \int_0^\infty \varepsilon^{q_1(l-s_1)} \left\| \partial^\alpha U(\cdot, \varepsilon) \right\|_{L^{p_1}(\mathbb{R}^n)}^{q_1} \frac{d\varepsilon}{\varepsilon} \right)^{\theta q/q_1} \times \\
&\quad \times \left( \int_0^\infty \varepsilon^{q_2(l-s_2)} \left\| \partial^\alpha U(\cdot, \varepsilon) \right\|_{L^{p_2}(\mathbb{R}^n)}^{q_2} \frac{d\varepsilon}{\varepsilon} \right)^{(1-\theta)q/q_2} \\
&\lesssim |f|_{B_{p_1, q_1}^{s_1}}^{\theta q} |f|_{B_{p_2, q_2}^{s_2}}^{(1-\theta)q},
\end{aligned}$$

which entails that

$$|f|_{B_{p,q}^s} \lesssim |f|_{B_{p_1, q_1}^{s_1}}^\theta |f|_{B_{p_2, q_2}^{s_2}}^{1-\theta}. \quad \square$$

We now turn to mapping properties of superposition operators (Proposition 6.4 to Theorem 6.7).

**6.4 Proposition.** Let  $0 < s < 1$  and let  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\Phi'$  is bounded on  $\mathbb{R}$  and  $\Phi(0) = 0$ . Then  $f \mapsto \Phi \circ f$  maps  $B_{p,q}^s$  into  $B_{p,q}^s$ .

*Proof.* Let  $f \in B_{p,q}^s$ . It is plain to see that  $\Phi(f) \in L^p(\mathbb{R}^n)$ , and we therefore have to estimate  $|\Phi(f)|_{B_{p,q}^s}$ . Let  $U$  be an extension of  $f$  such that

$$\int_0^\infty \varepsilon^{q(1-s)} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} \lesssim |f|_{B_{p,q}^s}^q \quad \text{for all } \alpha \text{ such that } |\alpha| = 1.$$

When  $|\alpha| = 1$ , the chain rule and the boundedness of  $\Phi'$  yield at once

$$\int_0^\infty \varepsilon^{q(1-s)} \|\partial^\alpha [\Phi \circ U](\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} \lesssim \int_0^\infty \varepsilon^{q(1-s)} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} \lesssim |f|_{B_{p,q}^s}^q,$$

and Theorem 1.3 provides the conclusion.  $\square$

The next result is required in the proof of Corollary 6.6.

**6.5 Proposition.** Let  $s \geq 1$ . Let  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  have  $I(s) + 1$  bounded derivatives<sup>12</sup> and satisfy  $\Phi(0) = 0$ . Let  $\sigma$  be a number such that

$$0 < \sigma < \min \left\{ 1, \frac{s}{I(s) + 1} \right\}.$$

Then  $f \mapsto \Phi \circ f$  maps  $B_{p,q}^s \cap B_{sp/\sigma, sq/\sigma}^\sigma$  into  $B_{p,q}^s$ .

*Proof.* Let  $f \in B_{p,q}^s \cap B_{sp/\sigma, sq/\sigma}^\sigma$ . As in the proof to the previous proposition, it suffices to estimate  $|\Phi(f)|_{B_{p,q}^s}$ . Let  $U$  be an extension of  $f$  such that

$$\int_0^\infty \varepsilon^{q(l-s)} \|\partial^\alpha U(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} \lesssim |f|_{B_{p,q}^s}^q \quad \text{for all } l > s \text{ and for all } \alpha \in \mathcal{D}_s \text{ with } |\alpha| = l.$$

Fix  $l := I(s) + 1$  and consider some  $\alpha$  in  $\mathcal{M}_l$ . By Theorem 1.3, we have to estimate

$$\int_0^\infty \varepsilon^{q(l-s)} \|\partial^\alpha [\Phi \circ U](\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon}. \quad (6.11)$$

In turn, the chain rule and the fact that  $\Phi^{(j)}$  is bounded for all  $j \in \llbracket 0, l \rrbracket$  reduce the control of the quantity in (6.11) to the control of the integrals

$$\int_0^\infty \varepsilon^{q(l-s)} \left\| \prod_{i=1}^k \partial^{\alpha_i} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon},$$

where  $k \in \llbracket 1, l \rrbracket$ ,  $\sum_{i=1}^k l_i = l$  and  $\alpha_i \in \mathcal{M}_{l_i}$ . For all  $i \in \llbracket 1, k \rrbracket$ , pick up  $s_i \in (\sigma, s]$  such that  $s_i < l_i$  and  $\sum_{i=1}^k s_i = s$ .<sup>13</sup> We define  $\theta_i \in (0, 1)$  by  $s_i = \theta_i s + (1 - \theta_i)\sigma$ . Define also  $p_i, q_i$  by

$$\frac{1}{p_i} := \frac{\theta_i}{p} + \frac{(1 - \theta_i)\sigma}{ps}$$

and

$$\frac{1}{q_i} := \frac{\theta_i}{q} + \frac{(1 - \theta_i)\sigma}{qs}.$$

We next note that

$$\sum_{i=1}^k \theta_i + \frac{\sigma}{s} \sum_{i=1}^k (1 - \theta_i) = 1,$$

<sup>12</sup>Here,  $I(s)$  denotes the integer part of  $s$ .

<sup>13</sup>We note that such  $s_i$ 's do exist, since by assumption on  $\sigma$  we have  $\sigma k \leq \sigma l < s$ .



and therefore we have

$$\sum_{i=1}^k \frac{1}{p_i} = \frac{1}{p} \quad \text{and} \quad \sum_{i=1}^k \frac{1}{q_i} = \frac{1}{q}.$$

Therefore, by Hölder's inequality, we obtain

$$\begin{aligned} \int_0^\infty \varepsilon^{q(l-s)} \left\| \prod_{i=1}^k \partial^{\alpha_i} U(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} &\leq \int_0^\infty \varepsilon^{q(l-s)} \prod_{i=1}^k \left\| \partial^{\alpha_i} U(\cdot, \varepsilon) \right\|_{L^{p_i}(\mathbb{R}^n)}^q \frac{d\varepsilon}{\varepsilon} \\ &\leq \prod_{i=1}^k \left( \int_0^\infty \varepsilon^{q_i(l_i-s_i)} \left\| \partial^{\alpha_i} U(\cdot, \varepsilon) \right\|_{L^{p_i}(\mathbb{R}^n)}^{q_i} \frac{d\varepsilon}{\varepsilon} \right)^{q/q_i}. \end{aligned}$$

By Theorem 1.4 and Proposition 6.3,

$$\int_0^\infty \varepsilon^{q_i(l_i-s_i)} \left\| \partial^{\alpha_i} U(\cdot, \varepsilon) \right\|_{L^{p_i}(\mathbb{R}^n)}^{q_i} \frac{d\varepsilon}{\varepsilon} \leq |f|_{B_{p_i, q_i}^{s_i}}^{q_i} \leq |f|_{B_{p, q}^{s_i}}^{\theta_i q_i} |f|_{B_{\frac{p}{\theta}, \frac{q}{\theta}}^{(1-\theta_i)q_i}}^{(1-\theta_i)q_i},$$

which ends the proof.  $\square$

An immediate consequence of Proposition 6.5 is

**6.6 Corollary.** Let  $s > 0$ ,  $1 \leq p \leq \infty$  and  $1 \leq q < \infty$ . Let  $\Phi$  be as in Proposition 6.5. Then

1. The application  $f \mapsto \Phi \circ f$  maps  $B_{p, q}^s \cap L^\infty$  into itself.
2. Assume in addition that that  $sp \geq n$ . Then  $f \mapsto \Phi \circ f$  maps  $B_{p, q}^s$  into itself.

*Proof.* When  $s < 1$ , the conclusions are given by Proposition 6.4.

Assume next that  $s \geq 1$  and let  $l := I(s) + 1$ . We prove item 2, the proof of item 1 being simpler.

If  $sp > n$ , then  $B_{p, q}^s \hookrightarrow L^\infty$  [23, Section 2.7.1, Remark 2, pp. 130-131], and since  $B_{p, q}^s \cap L^\infty \hookrightarrow B_{p/\theta, q/\theta}^{\theta s}$  for all  $\theta \in (0, 1)$  by Proposition 6.1, the conclusion is given by Proposition 6.5.<sup>14</sup>

If  $sp = n$ , then, by [14, Theorem 1, p. 82] and the fact that Besov spaces are increasing in  $q$ , we have the embedding  $B_{p, q}^s \hookrightarrow B_{p/\theta, q/\theta}^{\theta s}$  whenever  $0 < \theta < 1$ . We conclude again by applying Proposition 6.5.<sup>15</sup>  $\square$

We end this section by discussing a more difficult result.

**6.7 Theorem.** Let  $0 < s < 2$ ,  $1 < p < \infty$  and  $1 \leq q < \infty$ . Let  $\Phi \in C_c^2$  satisfy  $\Phi(0) = 0$ . If  $f$  is a non negative function in  $B_{p, q}^s$ , then  $\Phi \circ f$  belongs to  $B_{p, q}^s$ .

The above result (for a more general  $\Phi$ ) was obtained by Bourdaud and Meyer [4, Corollaire, p. 359] using a remarkable inequality due to Maz'ya (and presented in [1, Theorem 3]<sup>16</sup>) combined with a nonlinear interpolation result (due to Peetre [12]). As we will see below, the theory of weighted Sobolev spaces can serve as a substitute for the interpolation theory.

*Proof.* By standard arguments, it suffices to estimate  $\|\Phi \circ f\|_{B_{p, q}^s}$  when  $f \in C_c^\infty(\mathbb{R}^n)$  is non negative. The estimate of  $\|\Phi \circ f\|_{L^p(\mathbb{R}^n)}$  being obvious, we proceed to estimating  $|\Phi \circ f|_{B_{p, q}^s}$ . For this purpose, we extend  $f$  by setting  $W(\cdot, \varepsilon) := f * \varphi_\varepsilon$ , where  $\varphi \in C_c^\infty(\mathbb{R}^n)$  has the following properties:

$$\int_{\mathbb{R}^n} \varphi = 1, \quad \varphi \geq 0, \quad \text{supp } \varphi \subset (0, 1)^n. \quad (6.12)$$

<sup>14</sup>In order to be in position to apply Proposition 6.5, we have to choose  $\theta$  such that  $l\theta < 1$ .

<sup>15</sup>Once again, we take  $\theta$  such that  $l\theta < 1$ .

<sup>16</sup>See also references [11] and [12] in [1].

In view of Theorem 5.4, we have

$$\int_0^\infty \varepsilon^{q(2-s)-1} \|\partial^\alpha W(\cdot, \varepsilon)\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q, \quad \forall \alpha \in \mathbb{N}^{n+1} \text{ such that } |\alpha| = 2. \quad (6.13)$$

Note that  $\Phi \circ W$  is an extension of  $\Phi \circ f$ . By Theorem 1.3, it suffices to establish, for all  $1 \leq j, k \leq n$ , the estimates

$$\int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{jk}^2 [\Phi \circ W](\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q \quad (6.14)$$

and

$$\int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{\varepsilon\varepsilon}^2 [\Phi \circ W](\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q. \quad (6.15)$$

Let us first check (6.14). A simple computation yields

$$\begin{aligned} \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{jk}^2 [\Phi \circ W](\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon &\lesssim \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \Phi'(W) \partial_{jk}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \\ &\quad + \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \Phi''(W) \partial_j W(\cdot, \varepsilon) \partial_k W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon := A + B. \end{aligned}$$

In view of (6.13), it is plain that

$$A \lesssim \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{jk}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \lesssim |f|_{B_{p,q}^s}^q.$$

In order to estimate  $B$ , we first recall that for a given  $\Psi \in C_c(\mathbb{R}^n)$ , the following estimate holds for all nonnegative function  $g \in W^{2,p}(\mathbb{R}^n)$  [3, p. 438]:

$$\int_{\mathbb{R}^n} |\Psi(g)|^p |\partial_j g|^{2p} \lesssim \int_{\mathbb{R}^n} |\partial_{jj}^2 g|^p. \quad (6.16)$$

The functions  $f$  and  $\varphi$  being non negative, we may apply (6.16) to  $W(\cdot, \varepsilon)$ . Using successively Cauchy-Schwarz, (6.16) and (6.13), we find that

$$\begin{aligned} B &\leq \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \Phi''(W) |\partial_j W(\cdot, \varepsilon)|^2 \right\|_{L^p(\mathbb{R}^n)}^{q/2} \left\| \Phi''(W) |\partial_k W(\cdot, \varepsilon)|^2 \right\|_{L^p(\mathbb{R}^n)}^{q/2} d\varepsilon \\ &\lesssim \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{jj}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^{q/2} \left\| \partial_{kk}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^{q/2} d\varepsilon \\ &\leq \left( \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{jj}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \right)^{1/2} \times \\ &\quad \times \left( \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{kk}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \right)^{1/2} \lesssim |f|_{B_{p,q}^s}^q. \end{aligned}$$

Let us now turn to (6.15). Arguing as before,

$$\begin{aligned} \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{\varepsilon\varepsilon}^2 [\Phi \circ W](\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon &\lesssim \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \Phi'(W) \partial_{\varepsilon\varepsilon}^2 W(\cdot, \varepsilon) \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon \\ &\quad + \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \Phi''(W) |\partial_\varepsilon W(\cdot, \varepsilon)|^2 \right\|_{L^p(\mathbb{R}^n)}^q d\varepsilon := C + D. \end{aligned}$$

The term  $C$  is estimated as  $A$  above. As far as  $D$  is concerned, observe first that, since  $W(\cdot, \varepsilon) = f * \varphi_\varepsilon$ , we have

$$\partial_\varepsilon W(\cdot, \varepsilon) = -f * \operatorname{div}_x((\varphi x)_\varepsilon) = -\sum_{j=1}^n \partial_j [f * (\varphi x_j)_\varepsilon] := -\sum_{j=1}^n \partial_j V_j(\cdot, \varepsilon).$$

We next invoke (6.12) and obtain that

$$0 \leq \varphi(x)x_j \leq \varphi(x), \quad \forall x \in \mathbb{R}^n, \quad \forall j \in \llbracket 1, n \rrbracket. \quad (6.17)$$

Next, by (6.17) and the non negativity of  $f$ , we have

$$0 \leq V_j \leq W. \quad (6.18)$$

Consider now a function  $\Psi \in C_c(\mathbb{R})$  such that

$$|\Phi''(t)| \leq |\Psi(\tau)|, \quad \forall t, \tau \in \mathbb{R} \text{ such that } 0 \leq \tau \leq t. \quad (6.19)$$

For such  $\Psi$  we have (using (6.18) and (6.19))

$$|\Phi''(W)| \leq |\Psi(V_j)|, \quad j \in \llbracket 1, n \rrbracket.$$

By combining (6.16) and (6.18) with Theorem 5.4, we obtain the estimate

$$\begin{aligned} D &\lesssim \sum_j \int_0^\infty \varepsilon^{q(2-s)-1} \left\| |\Psi(V_j(\cdot, \varepsilon))| |\partial_j V_j(\cdot, \varepsilon)|^2 \right\|_{L^q(\mathbb{R}^n)}^p d\varepsilon \\ &\lesssim \sum_j \int_0^\infty \varepsilon^{q(2-s)-1} \left\| \partial_{jj} V_j(\cdot, \varepsilon) \right\|_{L^q(\mathbb{R}^n)}^p d\varepsilon \lesssim |f|_{B_{p,q}^s}^q. \end{aligned} \quad (6.20)$$

**6.8 Remark.** As observed in [4], it is possible to weaken the assumption  $\Phi \in C_c^2$  to

$$\Phi \in C^2(\mathbb{R}) \text{ and } |\Phi^{(j)}(t)| \lesssim t^{1-j} \text{ for } t > 0, \quad j = 0, 1, 2; \quad (6.21)$$

see also [14, Section 5.4.2, Proposition, p. 361]. The main reason is that (6.16) still holds when the condition  $\Psi \in C_c$  is weakened to

$$\Psi \in C(\mathbb{R}) \text{ and } |\Psi(t)| \lesssim t^{-1} \text{ for } t > 0. \quad (6.22)$$

Our proof extends to  $\Phi$ 's satisfying (6.21). Indeed, the last assumption in (6.12) implies that (6.18) can be strengthened to

$$C W \leq V_j \leq W \text{ for some } C > 0. \quad (6.23)$$

Therefore, the first line in estimate (6.20) holds with

$$\Psi(t) = \max\{|\Phi''(\tau)|; C t \leq \tau \leq t\}, \quad \forall t \geq 0.$$

Clearly, if  $\Phi$  satisfies (6.21), then  $\Psi$  satisfies (6.22), and thus estimate (6.20) holds. This justifies our remark.

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<sup>17</sup>If  $\Phi$  is supported in  $[-a, a]$ , then it suffices to let  $\Psi$  such that  $\Psi(x) = \max|\Phi''|$ ,  $\forall x \in [-a, a]$ .

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