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DIRAC-WEYL-FOCK EQUATION ALONG A CHRONOLOGICAL FIELD

JONOT JEAN LOUIS

ABSTRACT. We will quantify the (p, q) -tensors of the universe using the ∇^γ -quantification process. This method can be generalised for all the states of the physical system. We will give some results on the notion of space-time which permits travel through parts of the universe where a chronological field exists. This foliation leads to a generalisation of the Schrödinger equation and the Dirac-Weyl-Fock equation.

1. THE QUANTIFICATION PRINCIPLE

I) The superposition principle

Each physical system in the universe Ω is associated with a fibre bundle

$$\zeta = (E, \Omega, p, \mathcal{H})$$

which is endowed with a connexion ∇ and a Hilbert fibre. The state of the system is defined by a local section of the fibre $\Gamma(\zeta)$. Any linear superposition of the states $s = \sum_i c_i s_i$, with c_i belonging to $C^\infty(\Omega, \mathbb{C})$ and $\sum_i |c_i|^2 \leq 1$ is an accessible state if $\{s_i(\omega)\}$ form a Hilbert's basis of E_ω for each ω event of the universe Ω .

II) Measuring physical quantities

a) The evolution of a physical system on an open set $U \subset \Omega$, is the datum of a Hermitian section $i\Phi$ of the fibre bundle $E^ \otimes E$ on U .*

b) Let $s(\omega)$ be the state of the system in $\omega \in \Omega$ where we perform the measurement, the only possible results are the observable $i\Phi(\omega)$ eigenvalues $\lambda_\alpha(\omega)$.

c) We note $\hat{\mathcal{P}}_\alpha(\omega) : E_\omega \rightarrow E_\omega$, the projection on the eigenspace associated with the eigenvalue $\lambda_\alpha(\omega)$. The probability of finding the eigenvalue $\lambda_\alpha(\omega)$ during a measurement $i\Phi(\omega)$ is $p_\alpha = \frac{\|s_\alpha(\omega)\|_\omega^2}{\|s(\omega)\|_\omega^2}$, where $s_\alpha(\omega) = \hat{\mathcal{P}}_\alpha(\omega)(s(\omega))$.

d) After a measurement in ω giving $\lambda_\alpha(\omega)$, the new system state is $s_\alpha(\omega)$.

III) The evolution equation

If s is the state of a system, as long as the system is not subject to any observation, s is solution of the evolution equation

$$\Phi(s) = -i\lambda s, \tag{1.1}$$

λ is an eigenfunction of $i\Phi$ defined locally on U , Φ is the evolution system operator.

2. ∇^γ -QUANTIFICATION

In what follows, (E, π, Ω, F) is a real or complex vector bundle of a finite dimension, of fibre bundle F , for which the basis in the universe Ω which is a C^∞ -manifold

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of dimension 4. We note $\Lambda^n \Omega = \wedge^n T^* \Omega$, the fibre bundle of n -differential forms on Ω . The (p, q) -tensor space, p times covariant and q times contravariant, is written

$$T^{p,q} \Omega = (\otimes^p T^* \Omega) \otimes (\otimes^q T \Omega),$$

and the total space E of the fibre bundle is the space of the states, which is endowed with a connection

$$\nabla : \Gamma(E) \rightarrow \Gamma(\Lambda^1 \Omega \otimes E).$$

Definition 1. 1) A section γ of the fibre bundle $L(\Lambda^1 \Omega \otimes E, E)$ is a Dirac section of states E .

2) We define section ∇^γ of $\text{End } E$ by

$$\nabla^\gamma s = \gamma(\nabla s), \quad s \in \Gamma(E).$$

Dirac's endomorphisms are defined by

$$\gamma^d(s) = \gamma(d \otimes s), \quad d \in \Lambda^1 \Omega \text{ et } s \in \Gamma(E).$$

The commutator of a finite family of Dirac's endomorphisms $\{\gamma^{d^\mu}\}$ is

$$[\gamma^{d^1}, \gamma^{d^2}, \dots, \gamma^{d^n}] = \sum_{s \in \text{Perm}\{1,2,\dots,n\}} \varepsilon(s) \gamma^{d^{s(1)}} \circ \gamma^{d^{s(2)}} \circ \dots \circ \gamma^{d^{s(n-1)}} \circ \gamma^{d^{s(n)}},$$

et the anticommutator or Poisson bracket is defined by,

$$\{\gamma^{d^1}, \gamma^{d^2}, \dots, \gamma^{d^n}\} = \sum_{s \in \text{Perm}\{1,2,\dots,n\}} \gamma^{d^{s(1)}} \circ \gamma^{d^{s(2)}} \circ \dots \circ \gamma^{d^{s(n-1)}} \circ \gamma^{d^{s(n)}}.$$

The commutator defines a section of $L(\Lambda^n \Omega \otimes E, E)$. We define the n -rank extension of a section γ , by section $\gamma_n \in \Gamma(L(\Lambda^n(\Omega) \otimes E, E))$ where

$$\gamma_n((d^1 \wedge d^2 \wedge \dots \wedge d^n) \otimes s) = [\gamma^{d^1}, \gamma^{d^2}, \dots, \gamma^{d^n}](s),$$

and we extend γ_n by $C^\infty(\Omega, \mathbb{R})$ -linearity on $\Lambda^n(\Omega) \otimes E$.

Any connection ∇ has an extension $d_n^\nabla : \Gamma(\Lambda^n(\Omega) \otimes E) \rightarrow \Gamma(\Lambda^{n+1}(\Omega) \otimes E)$ with, $d_0^\nabla = \nabla$, $\Lambda^0(\Omega) \otimes E = E$ and $d_n^\nabla(\psi \otimes s) = d\psi \otimes s + (-1)^n \psi \wedge \nabla s$, $\psi \in \Lambda^n(\Omega)$. The Dirac-Einstein equation on the n -rank is written

$$\gamma_n(\nabla_{n-1} s) = \lambda s, \quad \nabla_{n-1} = d_{n-1}^\nabla \circ \dots \circ d_0^\nabla. \quad (2.1)$$

Definition 2. An n -order evolution equation of fibre bundle states E on an open set $U \subset \Omega$, is an equation in the form 2.1. We note $E(\nabla, \gamma, \lambda, n)$ this equation.

For a "local frame" on $E(U)$, the associated matrices γ^d are Dirac's matrices. If the evolution equation of a quantum system is $\Phi = \gamma_n \circ \nabla_{n-1}$ if (E, π, Ω, F) is a real vector bundle and $i\Phi = \gamma_n \circ \nabla_{n-1}$ if (E, π, Ω, F) is a complex vector bundle, the quantification principle becomes the quantification ∇^γ -principle on n -rank.

Example 1. The equation $E(\nabla, \gamma, \lambda, 1)$ is an evolution equation which allows the Dirac equation to be generalised in the curved spaces [1] when we set it out, $\lambda = \frac{mc}{\hbar}$. The fibre bundle E is one of the two fibre bundles $E = \Omega \times \mathbb{C}^4$ or $E = T_{\mathbb{C}} \Omega$ and the Dirac endomorphisms $\gamma^\mu = \gamma(d^\mu)$ in the dual basis $\{d^\mu\}$ of $\{\partial_\mu\}$, solve the anticommutation relation

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I_4, \quad (2.2)$$

with $(g_{\mu\nu})$ which is the matrix of the Lorentzian metric g on Ω in basis $\{\partial_\mu\}$.

Definition 3. A relativistic quantum equation is an equation which links connection ∇ to the Dirac section γ . This type of relation will be recorded as $R(\nabla, \gamma)$.

For the fields, the relativist equation $R(\nabla, \gamma)$ was established in the equation 4.2 and 4.3. The evolution equation $E(\nabla, \gamma, \lambda, 1)$ was proven in 3.2. To generalise this equation across all the states of the universe, it is necessary to determine a tensor that plays an analogue role to the Ricci tensor for connection ∇ defined on the fibre bundle of states E .

3. REPRESENTING FIBRE BUNDLES IN CHARTS

In the following, (x^1, x^2, x^3, x^4) is a chart on an open set U of the universe Ω , on which the fibre bundles $T\Omega$ and E are trivial. Let $\{\partial_\mu = \frac{\partial}{\partial x^\mu}, 1 \leq \mu \leq 4\}$ be the vector fields on U linearly independent associates and $\{d^\nu = dx^\nu, 1 \leq \nu \leq 4\}$ be the dual basis. On $E(U)$ there is a "local frame", written $\{e_\alpha, 1 \leq \alpha \leq n = \dim E\}$.

3.0.1. *Local writing of the Dirac-Einstein equation $E(\nabla, \gamma, \lambda, 1)$.* Let $s = s^\alpha e_\alpha$ then $\nabla s = \nabla(s^\alpha e_\alpha) = ds^\alpha \otimes e_\alpha + s^\alpha \nabla e_\alpha = ds^\alpha \otimes e_\alpha + s^\alpha \Gamma_{\alpha\nu}^\beta d^\nu \otimes e_\beta$. On U the connection is written

$$\nabla e_\alpha = \Gamma_{\alpha\nu}^\beta d^\nu \otimes e_\beta,$$

with C^∞ functions $\Gamma_{\alpha\nu}^\beta$ defined on U with complex or real values, $\Gamma_{\alpha\nu}^\beta$ are the Christoffel symbols associated with connexion ∇ . We have, $ds^\alpha = \partial_\nu s^\alpha d^\nu$ and

$$\begin{aligned} \nabla s &= \partial_\nu s^\alpha d^\nu \otimes e_\alpha + s^\alpha \Gamma_{\alpha\nu}^\beta d^\nu \otimes e_\beta \\ &= (\partial_\nu s^\beta + s^\alpha \Gamma_{\alpha\nu}^\beta) d^\nu \otimes e_\beta. \end{aligned}$$

The Dirac section γ of E can be represented

$$\gamma(d^\nu \otimes e_\beta) = \gamma_\beta^{\nu\sigma} e_\sigma,$$

and if $\gamma^\nu = \gamma(d^\nu)$ then $(\gamma^\nu)_\beta^\alpha = \gamma_\beta^{\nu\alpha}$.

$$\nabla^\gamma s = \gamma((\partial_\nu s^\beta + s^\alpha \Gamma_{\alpha\nu}^\beta) d^\nu \otimes e_\beta) = (\partial_\nu s^\beta + s^\alpha \Gamma_{\alpha\nu}^\beta) \gamma(d^\nu \otimes e_\beta)$$

$$\nabla^\gamma s = \gamma_\beta^{\nu\sigma} (\partial_\nu s^\beta + s^\alpha \Gamma_{\alpha\nu}^\beta) e_\sigma.$$

The Dirac-Einstein equation $E(\nabla, \gamma, \lambda, 1)$ is written locally,

$$\gamma_\beta^{\nu\sigma} (\partial_\nu s^\beta + s^\alpha \Gamma_{\alpha\nu}^\beta) = -i\lambda s^\sigma \text{ for all } \sigma, 1 \leq \sigma \leq n.$$

The Dirac matrices γ^μ , $1 \leq \mu \leq 4$, are square n -order matrices whose coefficients are applications C^∞ on U , with real or complex values. We note that

$$(D_\nu s)^\beta = \partial_\nu s^\beta + s^\alpha \Gamma_{\alpha\nu}^\beta,$$

D_ν is the local section of $\text{End}(E)$ defined by, $D_\nu = L_\nu + \Gamma_\nu$, where Γ_ν is the local section of $\text{End}(E)$ for which the matrix, in the "local frame", is $(\Gamma_\nu)_\alpha^\beta = \Gamma_{\alpha\nu}^\beta$ and L_ν is the Lie derivative along the field $\partial_\nu = \frac{\partial}{\partial x^\nu}$ defined by, $L_\nu s = (\partial_\nu s^\alpha) e_\alpha$.

$$\gamma_\beta^{\nu\sigma} (D_\nu s)^\beta = -i\lambda s^\sigma \text{ for all } \sigma, 1 \leq \sigma \leq n.$$

The previous equation is written

$$\gamma^\nu D_\nu (s) = -i\lambda s,$$

and the evolution equation operator is,

$$\Phi = \gamma^\nu D_\nu. \tag{3.1}$$

We have an evolution equation in the sense of the ∇^γ -quantification if $i\Phi$ is a section that is locally Hermitian.

Definition 4. Φ is the operator of the evolution equation of first order states.

The application to the fields is immediate if we set out $E = T_C\Omega$, the evolution equation of the fields is represented as follows in the "local frame"

$$\gamma^\nu D_\nu X = -i\lambda X,$$

$\lambda \in \text{Spec}_U(i\Phi)$ and $D_\nu = L_\nu + \Gamma_\nu$. The eigenvalues are proportional to the mass of the particles 6.1.

The quantification principle for the fields is as follows:

"If no measurement is made in chart U , the field X evolves according to the equation $\Phi X = -i\lambda X$, where $\lambda \in \text{Spec}_U(i\Phi)$. The field X is a specific field of the evolution operator Φ . If a measurement in $\omega \in U \subset \Omega$ is $a = a(\omega)$, then a is an eigenvalue of $i\Phi(\omega)$. The field takes the value $X_a(\omega)$ where $X_a(\omega) = \hat{P}_a(\omega)(X(\omega))$, $\hat{P}_a(\omega)$ is the projection on the eigenspace associated with the eigenvalue a . The probability of finding the value a during the measurement is

$$p_a = \frac{\|X_a(\omega)\|_\omega^2}{\|X(\omega)\|_\omega^2},$$

and the field evolution equation is $\Phi X = -i\lambda X$ for a $\lambda \in \text{Spec}_U(i\Phi)$, proving $\lambda(\omega) = a$ and a field $X(\omega) = X_a(\omega)$."

We have therefore set in the event ω , new conditions for the system's evolution. We have disrupted the field evolution X by carrying out a measurement in this field.

3.0.2. *Quantifying metrics.* From a connection ∇ on the fields, we can spread the endomorphism ∇_X individually to the tensors with two conditions on ∇_X , ∇_X commuted with the contractions, i.e., $\nabla_X(c(S)) = c(\nabla_X(S))$ and $\nabla_X(S \otimes T) = \nabla_X(S) \otimes T + S \otimes \nabla_X(T)$.

The connection on the $(2, 0)$ -tensors is defined from the initial connection ∇ on the fields by setting out, $\nabla(d^1 \otimes d^2) = \nabla(d^1) \otimes d^2 + d^1 \otimes \nabla(d^2)$ and if we restrict ourselves to the defined sections on a chart U of the universe, in the basis $\{\partial_\mu\}$, the dual connection is,

$$\nabla_{\partial_\beta}^* d^\sigma = -\Gamma_{\beta\tau}^\sigma d^\tau.$$

In particular,

$$\begin{aligned} \nabla_{\partial_\beta}(d^\sigma \otimes d^\tau) &= \nabla(d^\sigma \otimes d^\tau)(\partial_\beta) = (\Gamma_{\rho\gamma\delta}^{\sigma\tau} d^\rho \otimes d^\gamma \otimes d^\delta)(\partial_\beta) \\ \nabla_{\partial_\beta}(d^\sigma \otimes d^\tau) &= \Gamma_{\rho\gamma\delta}^{\sigma\tau} d^\rho(\partial_\beta) d^\gamma \otimes d^\delta = \Gamma_{\beta\gamma\delta}^{\sigma\tau} d^\gamma \otimes d^\delta, \end{aligned}$$

$$\begin{aligned} \nabla_{\partial_\beta}(d^\sigma \otimes d^\tau) &= \nabla_{\partial_\beta}^* d^\sigma \otimes d^\tau + d^\sigma \otimes \nabla_{\partial_\beta}^* d^\tau \\ &= -\Gamma_{\beta\mu}^\sigma d^\mu \otimes d^\tau - \Gamma_{\beta\rho}^\tau d^\sigma \otimes d^\rho \\ &= -(\Gamma_{\beta i}^\sigma + \Gamma_{\beta j}^\tau) d^i \otimes d^j, \end{aligned}$$

and

$$\Gamma_{\beta ij}^{\sigma\tau} = -(\Gamma_{\beta i}^\sigma + \Gamma_{\beta j}^\tau)$$

There is a natural way to describe Dirac's endomorphisms on $(2, 0)$ -tensors, if γ is a Dirac section of field fibre bundles, it's also a section of dual fibre bundle

$$\gamma \in \Gamma(T\Omega \otimes T^*\Omega \otimes T\Omega) \approx \Gamma(T\Omega \otimes T\Omega \otimes T^*\Omega)$$

and,

$$\gamma \in \Gamma(L(\Lambda^1\Omega, \text{End}(T\Omega))) \approx \Gamma(L(\Lambda^1\Omega, \text{End}(\Lambda^1\Omega))).$$

We have

$$(\gamma^d)^*(\zeta) = \zeta \circ \gamma^d, \quad \zeta \text{ et } d \in \Lambda^1\Omega.$$

Then, in order to obtain the Dirac section on the $(2, 0)$ -tensors, we set out

$$[\gamma^d] = (\gamma^d)^* \otimes (\gamma^d)^*.$$

$$[\gamma^\mu](d^\nu \otimes d^\tau) = \gamma_{\rho\sigma}^{\mu\nu\tau} d^\rho \otimes d^\sigma = (\gamma^\mu)^*(d^\nu) \otimes (\gamma^\mu)^*(d^\tau),$$

$$(\gamma^\mu)^*(d^\nu)(\partial_\rho) = d^\nu \circ \gamma^\mu(\partial_\rho) = d^\nu(\gamma_\rho^{\mu\sigma} \partial_\sigma) = \gamma_\rho^{\mu\nu}$$

and

$$(\gamma^\mu)^*(d^\nu) = \gamma_\rho^{\mu\nu} d^\rho,$$

$$[\gamma^\mu](d^\nu \otimes d^\tau) = \gamma_\rho^{\mu\nu} \gamma_\sigma^{\mu\tau} d^\rho \otimes d^\sigma,$$

it follows

$$\gamma_{\rho\sigma}^{\mu\nu\tau} = \gamma_\rho^{\mu\nu} \gamma_\sigma^{\mu\tau}.$$

We replace the previous results in the equation,

$$\gamma_{\rho\sigma}^{\mu\nu\tau} (\partial_\mu a_{\nu\tau} + a_{\alpha\beta} \Gamma_{\mu\nu\tau}^{\alpha\beta}) = \lambda a_{\rho\sigma},$$

we obtain

$$\gamma_\rho^{\mu\nu} \gamma_\sigma^{\mu\tau} (\partial_\mu a_{\nu\tau} - a_{\alpha\beta} \Gamma_{\mu\nu}^\alpha - a_{\alpha\beta} \Gamma_{\mu\tau}^\beta) = \lambda a_{\rho\sigma}.$$

Theorem 1. *If a pseudo-Riemannian metric g follows a Dirac-Einstein evolution equation when in the entire chart U its matrix $G = (g_{ij})$ solves the equation*

$$\gamma_{\rho\sigma}^{\mu\nu\tau} (\partial_\mu g_{\nu\tau} + g_{\alpha\beta} \Gamma_{\mu\nu\tau}^{\alpha\beta}) = \lambda g_{\rho\sigma}, \quad (3.2)$$

where

$$\Gamma_{\mu\nu\tau}^{\alpha\beta} = -(\Gamma_{\mu\nu}^\alpha + \Gamma_{\mu\tau}^\beta) \text{ et } \gamma_{\rho\sigma}^{\mu\nu\tau} = \gamma_\rho^{\mu\nu} \gamma_\sigma^{\mu\tau}.$$

4. RELATIVISTIC QUANTUM EQUATION

We remember that the relativistic quantum equation is that which links connection ∇ to the Dirac section γ defined on the states. We start from the principle that any physical system can be quantified by the quantification procedure mentioned above. Einstein's equation is performed for a connexion ∇ which is the Levi-Civita connection of a metric g . The connection ∇ does not generally allow us to define a metric on the universe Ω , we must therefore find a tensor which plays the role of a metric tensor. The Dirac section γ that lets us build this tensor is the fibre bundle $E = T_C\Omega$.

Definition 5. *The Poisson tensor $\gamma^{\alpha\beta}$ is defined by,*

$$\gamma^{\alpha\beta} = \frac{1}{8} \text{Trace}(\{\gamma^\alpha, \gamma^\beta\}) = \frac{1}{4} \text{Trace}(\gamma^\alpha \gamma^\beta). \quad (4.1)$$

Remark 1. *When $\gamma^{\alpha\beta}$ is regular, $\gamma_{\alpha\beta}$ plays the role of a metric tensor. The Poisson tensor measures the anticommutativity of the Dirac endomorphisms.*

For the curved tensor, the standard notations are,

$$R^\nabla(X, Y)(Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X, Y]} Z,$$

$$R^\nabla(\partial_\alpha, \partial_\beta)(\partial_\sigma) = \nabla_{\partial_\alpha} \nabla_{\partial_\beta} \partial_\sigma - \nabla_{\partial_\beta} \nabla_{\partial_\alpha} \partial_\sigma = R_{\sigma\beta\alpha}^\mu \partial_\mu,$$

and Ricci's tensor is written

$$Ric_{\alpha\beta} = R_{\alpha\nu\beta}^\nu.$$

The scalar curve of ∇ is,

$$R = \gamma^{\alpha\beta} Ric_{\alpha\beta}.$$

Theorem 2. *If the Poisson tensor is regular, Einstein's relativistic equation is*

$$Ric_{\alpha\beta} - \frac{1}{2} R \gamma_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta},$$

with $T_{\alpha\beta} = \gamma_{\alpha\beta} T^{\alpha\beta}$, $T^{\alpha\beta}$ is the energy-momentum equation. If the Poisson tensor is not regular

$$Ric^{\alpha\beta} - \frac{1}{2} R \gamma^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}, Ric^{\alpha\beta} = Ric_{hk} \gamma^{h\alpha} \gamma^{k\beta}. \quad (4.2)$$

The field evolution equation is given by

$$\Phi X = -i\lambda X,$$

and the metric evolution equation

$$\Delta g = \lambda g.$$

Proof. We contract Einstein's relativistic equation twice and we use the fundamental equality 2.2 as if a metric g exists, we have, $g^{\alpha\beta} = \gamma^{\alpha\beta}$. The evolution equations have been proven above. \square

Any (p, q) -tensor changes following an equation $E(\nabla, \gamma, \lambda, 1)$ where ∇ is the sole extension to $T^{p,q}\Omega$ of the initial connection ∇ on the fields and γ is the sole extension to $T^{p,q}\Omega$ for the Direct section γ of the fields. The associated relativistic quantum equation is,

$$Ric^{\alpha\beta} - \frac{1}{2} R \gamma^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta}.$$

The quantification order $n \geq 1$ is given by the evolution equation $E(\nabla, \gamma, \lambda, n)$. We can generalise this quantification procedure to all the Hermitian fibre bundles and set up the evolution equations $E(\nabla, \gamma, \lambda, n)$ in a chart, the only issue is generalising of a Ricci tensor in any kind of fibre bundle. If this construction is possible, we can define a relativistic equation linking connection ∇ to the Dirac section γ of the fibre bundle for each equation of a state of the universe. The choice of the connection ∇ is driving naturally by the Hermitian nature of the fibre bundle of states E .

We would like to generalise Einstein's relativistic quantum equation by introducing the cosmological constant Λ_0 of this equation,

$$Ric^{\alpha\beta} - \frac{1}{2} R \gamma^{\alpha\beta} = \frac{8\pi G}{c^4} T^{\alpha\beta} + \Lambda_0 \gamma^{\alpha\beta}. \quad (4.3)$$

We revert to Einstein's general equation when the Poisson tensor $\gamma^{\alpha\beta}$ is identified with $g^{\alpha\beta}$.

Remark 2. *The Dirac-Einstein equation can be defined from a Koszul connection without adding the Lorentzian metric and the Dirac section is a solution to the relativistic quantum equation defined by that connection.*

5. SPACE-TIME FOLIATION OF PARTS OF THE UNIVERSE

In all that follows, we position ourselves on a part Ω of the universe where no measurement has been made on the metric g . We suppose that Ω is a related Lorentzian manifold, orientated in time, i.e., there is at least one field $T : \Omega \rightarrow T\Omega$ of time $\forall \omega \in \Omega$, $g(\omega)(T(\omega), T(\omega)) < 0$. For the Minkowski space-time (\mathbb{R}^4, η) , $ds^2 = dx^2 + dy^2 + dz^2 - dt^2$ and field $T = \frac{\partial}{\partial t}$ to give the orientation in time.

Let $T_\omega = \mathbb{R}T(\omega)$, the vector subspace of dimension 1, produced by vector $T(\omega) \in T_\omega(\Omega)$. This subspace is not isotropic in the bilinear form $g(\omega)$ on $T_\omega(\Omega)$ and splits $T_\omega(\Omega)$ into a single orthogonal sum $T_\omega(\Omega) = T_\omega \oplus E_\omega$, where $E_\omega = T_\omega^\perp$. The restrictions $g(\omega)|_{E_\omega}$ and $-g(\omega)|_{T_\omega}$ are positively defined. Any field $X : \Omega \rightarrow T(\Omega)$ breaks down into $X = X_E - tT$, where $t : \Omega \rightarrow \mathbb{R}$ is a C^∞ -application, X_E is a field space, i.e., $X_E(\omega) \in E_\omega$ for everything $\omega \in \Omega$. In particular, $|X_E(\omega)|_g^2 = g(\omega)(X_E(\omega), X_E(\omega)) > 0$ if $X_E(\omega) \neq o_{T_\omega(\Omega)}$.

Definition 6. *Field X is orientated towards T -future if $X = X_E - tT$, with $t > 0$ and towards T -past if $t < 0$.*

The space part of the tangent space is $\mathcal{E}\Omega = \cup_{\omega \in \Omega} \{\omega\} \times E_\omega$ and the chronological part is $\mathcal{T}\Omega = \cup_{\omega \in \Omega} \{\omega\} \times T_\omega$.

Lemma 1.
$$\begin{array}{ccc} \mathcal{E}\Omega = \cup_{\omega \in \Omega} \{\omega\} \times E_\omega & & \mathcal{T}\Omega = \cup_{\omega \in \Omega} \{\omega\} \times T_\omega \\ \downarrow \pi_{\mathcal{E}} & \text{and} & \downarrow \pi_{\mathcal{T}} \\ \Omega & & \Omega \end{array}$$
 are fibre bundles.

The chronological fibre bundle is trivial and $T\Omega$ is the Whitney sum of these two fibre bundles, $T\Omega = \mathcal{E}\Omega \oplus \mathcal{T}\Omega$.

Proof. All that is required is to prove that $\begin{array}{c} \mathcal{E}\Omega \\ \downarrow \pi_{\mathcal{E}} \\ \Omega \end{array}$ is a fibre bundle. Let U be

an inter-related open set of Ω on which TU is trivial. There are vector fields A , B , C and D defined on the linearly independent U . We can split each field into its space and chronological parts $A = A_E + aT$, $B = B_E + bT$, $C = C_E + cT$ and $D = D_E + dT$ where a , b , c and d are functions of class C^∞ on U . We look for orthogonal fields in T for the bilinear form g which are linearly independent. We set out $\Gamma = \alpha A + \beta B + \gamma C + \delta D$, the condition $g(\Gamma, T) = 0$ is equivalent to $\alpha a + \beta b + \gamma c + \delta d = 0$. If $X = -bA + aB - dC + cD$, $Y = cA - dB - aC + bD$, $Z = -dA - cB + bC + aD$ then for all $\omega \in U$, $\text{Vect}(X(\omega), Y(\omega), Z(\omega)) = E_\omega = \mathbb{R}T(\omega)^\perp$ since

$$\det \begin{pmatrix} -b & c & -d & a \\ a & -d & -c & b \\ -d & -a & b & c \\ c & b & a & d \end{pmatrix} = (a^2 + b^2 + c^2 + d^2)^2 = 0 \iff a = b = c = d = 0.$$

$$\begin{array}{ccc} \pi_E^{-1}(U) = \cup_{\omega \in U} E_\omega & \xrightarrow{\psi} & U \times \mathbb{R}^3 \\ \downarrow \pi_E & & \downarrow \text{Pr}_1 \\ U & \xrightarrow{\text{Id}_U} & U \end{array}, \text{ with } \psi(v) = (\omega, x, y, z) \text{ and}$$

$v = xX(\omega) + yY(\omega) + zZ(\omega)$ is the isomorphism of the local triviality. \square

We note that on the space fibre bundle

$$\begin{array}{c} \mathcal{E}\Omega = \cup_{\omega \in \Omega} \{\omega\} \times E_\omega \\ \downarrow \pi_{\mathcal{E}} \\ \Omega \end{array}, \text{ on each fibre bundle } E_\omega \text{ we have a positive bilinear}$$

form, written g^R , defined by $g^R(X, Y) = g(X, Y)$ and the Lorentz matrix g , can be used to define a local Riemannian metric g_R as follows, if $|X|^2 = |X_E|^2 - t^2$ then $|X|_R^2 = |X_E|^2 + t^2$ with, $|X|^2 = g(X, X)$ and $|X|_R^2 = g_R(X, X)$.

Definition 7. A field T is stable in $\omega \in \Omega$, if there is a map U of Ω on which if

$$T = t^\nu \partial_\nu$$

then

$$\partial_k (t^j g_{ij}) = \partial_i (t^j g_{kj}), \forall i, k.$$

Afterward, all the chronological fields \mathbf{T} considered are supposed stable on the open where they exist.

Theorem 3. The Pfaff system $\omega \rightarrow \tau(\omega) = E_\omega$ can be fully integrated, for each $\omega \in \Omega$, the interrelated component of the integral associated with this system is a related manifold known as a space leaf in ω which is written \mathcal{E}_ω .

Proof. We use the principle of Einstein summation notation. Let $\varphi : U \rightarrow \mathbb{R}^4$, be a chart of Ω . We set out $\{\partial_i\}$ the associated "local frame", if $X \in \Gamma^\infty(T\Omega)$ then in local coordinates $X = a^i \partial_i \in T_u\Omega$, where $a^i : U \rightarrow \mathbb{R}$ is a C^∞ -application. If $Y = b^i \partial_i$ then $[X, Y] = c^i \partial_i$ with, $c^i = a^j (\partial_j b^i) - b^j (\partial_j a^i)$.

We note $T = t^i \partial_i$, the chronological field. It must be proven that if $g(X, T) = 0$ and $g(Y, T) = 0$ on U , then $g([X, Y], T) = 0$ on U .

We have $g(X, T) = a^i t^j g_{ij} = 0$, as well as $b^i t^j g_{ij} = 0$ with, $g(\partial_i, \partial_j) = g_{ij}$. Then, we calculate

$$\begin{aligned} g([X, Y], T) &= c^i t^j g_{ij} = (a^k (\partial_k b^j) - b^k (\partial_k a^j)) t^j g_{ij} \\ &= a^k t^j g_{ij} (\partial_k b^j) - b^k t^j g_{ij} (\partial_k a^j), \end{aligned}$$

the conditions $a^i t^j g_{ij} = 0$ and $b^i t^j g_{ij} = 0$ imply $\partial_k (a^i t^j g_{ij}) = 0$ and $\partial_k (b^i t^j g_{ij}) = 0$. We deduce

$$\begin{aligned} g([X, Y], T) &= -a^k b^i \partial_k (t^j g_{ij}) + b^k a^i \partial_k (t^j g_{ij}) \\ &= a^i b^k (-\partial_i (t^j g_{kj}) + \partial_k (t^j g_{ij})) = 0. \end{aligned}$$

Therefore, τ can be fully integrated. If \mathcal{E}_ω is the related component of the integral τ containing ω , \mathcal{E}_ω is a sub-manifold of dimension 3 of Ω . Furthermore, $\cup_{\omega \in \Omega} \mathcal{E}_\omega = \Omega$. \square

Conclusion 1. The space-time parts Ω of the universe are related manifolds of dimension 4, endowed with a Lorentzian metric g and a chronology defined by a local stable field T , such that $g(T, T) = -1$. Each stable chronology can be used to foliate Ω into two orthogonal foliations for metric g . One is the space foliation, written \mathcal{E} , whose leaves are related sub-manifolds without edges, of dimension 3 of Ω , known as space leaves, the other is the chronological foliation written \mathcal{T} , whose leaves are related sub-manifolds without edges, of dimension 1 of Ω , known as chronological leaves. These chronological leaves can only be \mathbb{S}^1 or \mathbb{R} . Each field X is broken

down individually, under the form $X = X_E - tT$, where $X_E(\omega) \in T_\omega(E) = E_\omega$, $t \in C^\infty(\Omega)$ and E is the single foliation leaf of \mathcal{E} containing ω .

The notion of a local change of coordinates between two space-times defined by chronological fields T and S is the datum of a section θ , defined on an interrelated open set $U \subseteq \Omega$ of the fibre bundle $\text{Hom}(T\Omega) = \Lambda^1(\Omega) \otimes T\Omega$ such that $\theta \otimes T = S$ and $g(\theta \otimes X, \theta \otimes Y) = g(X, Y)$ on $U \subseteq \Omega$ for the whole field X, Y of Ω . We remember that for a field X of Ω defined on U , $\theta \otimes X(\omega) = \theta(\omega)(X(\omega))$ and the application $\omega \rightarrow \theta(\omega) : T_\omega\Omega \rightarrow T_\omega\Omega$ with, $\theta(\omega)$ is an $g(\omega)$ -isometry. We can observe that this space-time change is an evolution operator.

Theorem 4. *For any T and S chronological fields, there is a local change in coordinates.*

Proof. We take an open set U on which $T\Omega$ is trivial, we give ourselves four linearly independent fields A, B, C and D . We can build 3 fields X^1, Y^1 and Z^1 such as $\text{Vect}(X^1, Y^1, Z^1) = (\mathbb{R}T)^\perp$, by the Gram-Schmidt process, we can build 3 fields X_1, Y_1, Z_1 that form an orthonormal basis of $(\mathbb{R}T)^\perp$ for the Riemannian metric g_R which is the restriction of g to $(\mathbb{R}T)^\perp$. We write θ_1 , the application which sends X_1 on A, Y_1 on B, Z_1 on C and T on D . In an identical way, we build θ_2 the application which sends X_2 on A, Y_2 on B, Z_2 on C and S on D , the application $\theta = \theta_2^{-1} \circ \theta_1$ answers the question as the construction processes are C^∞ . \square

6. SCHRÖDINGER-DIRAC EQUATION

In relativistic quantum mechanics, the state of a free electron is represented by a wave function $\Psi(t, x)$ with $\Psi(t, \bullet) \in L^2(\mathbb{R}^3, \mathbb{C}^4)$ for any t . This function is the solution of the free Dirac equation $i\partial_t\Psi = H_0\Psi$, with

$$H_0 = i \sum_{k=1}^3 \alpha_k \partial_k + \beta$$

, the units chosen are $\hbar = c = 1$ and the electron's mass is $m_e = 1$. $\alpha_1, \alpha_2, \alpha_3$ and β are complex square order 4 matrices represented by blocks, $\beta = \begin{pmatrix} I & O \\ O & -I \end{pmatrix}$ and $\alpha_k = \begin{pmatrix} O & \sigma_k \\ \sigma_k & O \end{pmatrix}$, $k = 1, 2, 3$ and the matrices σ_k are Pauli matrices of order 2, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ which prove $\alpha_k \alpha_l + \alpha_l \alpha_k = 2\delta_{kl} I_4$ and $\alpha_k \beta + \beta \alpha_k = O_4$. These relations ensure that $H_0^2 = -\Delta + Id$ is a symmetric operator [3].

The Dirac equation is a natural generalisation. We focus on a real vector bundle $\zeta = (E, \Omega, \pi, \mathbb{R}^4)$ on the universe Ω , endowed with a connection ∇ .

A generalised wave function Ψ is a section of the complexified fibre bundle of ζ , written $\zeta_{\mathbb{C}} = (E_{\mathbb{C}}, \Omega, \pi_{\mathbb{C}}, \mathbb{C}^4)$. The complexified connection associated with ∇ is $\nabla(r + is) = \nabla(r) + i\nabla(s)$ for all sections r and s if ζ , ∇ is an application of $\Gamma(E_{\mathbb{C}})$ with values in $\Gamma(\Lambda^1\Omega \otimes E_{\mathbb{C}})$.

If $E = \Omega \times \mathbb{C}^4$, Ψ is a 4-scalar, we are in the "QRD" representation and if $E = T\Omega$, Ψ is a 4-vector, we are in the "TRD" representation of the Dirac equation ([1],[2]).

In a map (x^1, x^2, x^3, x^4) of the open set U of the space-time Ω , on which the fibre bundle $T\Omega$ is trivial, the Dirac-Einstein evolution equation,

$$\gamma^\nu D_\nu (\Psi) = -i \frac{mc}{\hbar} \Psi, \quad (6.1)$$

where m is the mass of the wave particle Ψ .

The operator $\Phi = \gamma^\nu D_\nu$, must be a Hermitian operator for the Hermitian product on $T_{\mathbb{C}}\Omega$ which can be defined by the metric g . The eigenvalues of $i\Phi$ are in the form $\frac{mc}{\hbar}$, where m covers all the particle masses in the universe, especially, the particle masses which exist in the space-time associated with the chronological field T .

In the hypothesis of the existence of a chronological field T , we can define a Schrödinger-Dirac equation, the waves become fields Ψ of space-time Ω and the Schrödinger equation $H\Psi = -i\partial_T\Psi$ is generalised by an equation of the form

$$H\Psi = -i\mathcal{L}_T\Psi = -i[T, \Psi], \quad (6.2)$$

\mathcal{L}_T is an extension of the Lie derivative ∂_T to the fields of Ω . The Hamiltonian H becomes an operator on complex fields, i.e., a section on Ω with values in $\Lambda_{\mathbb{C}}^1\Omega \otimes T_{\mathbb{C}}\Omega$.

Remark 3. $\mathcal{L}_T\Psi$ measures the lack of commutativity of a field Ψ compared to a chronological field T .

Theorem 5. The Hamiltonien H is written,

$$H = \frac{\hbar}{mc} \mathcal{L}_T \circ \Phi + i\partial_T(\log(m)) \text{Id}_{T_{\mathbb{C}}\Omega}, \quad (6.3)$$

where $\Phi = \gamma^\nu D_\nu$ is the Dirac-Einstein operator at rank 1. In particular, if the mass does not evolve along the chronological field T , the Hamiltonian is written,

$$H = \frac{\hbar}{mc} \mathcal{L}_T \circ \Phi. \quad (6.4)$$

Proof. $\mathcal{L}_T \circ \Phi (\Psi) = [T, \Phi (\Psi)] = [T, -i\frac{mc}{\hbar}\Psi] = -i\frac{mc}{\hbar} [T, \Psi] + \partial_T (-i\frac{mc}{\hbar}) \Psi$
 $= \frac{mc}{\hbar} H\Psi - i\frac{c}{\hbar} \partial_T(m) \Psi.$

$$H\Psi = \frac{\hbar}{mc} (\mathcal{L}_T \circ \Phi (\Psi) + i\frac{c}{\hbar} \partial_T(m) \Psi) = (\frac{\hbar}{mc} \mathcal{L}_T \circ \Phi + i\partial_T(\log(m)) \text{Id}_{T_{\mathbb{C}}\Omega}) (\Psi). \quad \square$$

Remark 4. $\partial_T(\log(m)) = \frac{\partial_T(m)}{m}$ represents the logarithmic derivative along the chronological field T of the mass m of the particle.

Remark 5. The Hamiltonian is dependent on the choice of chronological field T , therefore of the "local frame" as opposed to the evolution operator $\Phi = \gamma^\nu D_\nu$ which is only dependent on the Dirac section of the complexified fibre bundle of the fields and the chosen connection ∇ on this fibre bundle, therefore is only dependent on the connection ∇ .

Let X, Y and Z of the local sections for which the family $\{X, Y, Z\}$ is an orthogonal basis of $(\mathbb{R}T)^\perp$. We set out $\partial_0 = T, \partial_1 = X, \partial_2 = Y$ and $\partial_3 = Z$. In the "local frame" $\{\partial_0, \partial_1, \partial_2, \partial_3\}$,

Theorem 6.

$$H = \frac{\hbar}{mc} (\gamma_{\nu,0}^\nu D_\nu + \gamma^\nu (L_0 L_\nu + \Gamma_{\nu,0} + \Gamma_\nu L_0)) + i\partial_T(\log(m)) \text{Id}, \quad (6.5)$$

with $L_\nu \Psi = (\partial_\nu \Psi^\sigma) \partial_\sigma$ for $\Psi = \Psi^\sigma \partial_\sigma$, $\gamma_{\nu,0}^\nu$ and $\Gamma_{\nu,0}$ are endomorphisms whose matrices in the canonical basis are respectively $(\partial_0 \gamma_{\beta}^{\nu\sigma})$ and $(\partial_0 \Gamma_{\alpha\nu}^\beta)$.

Proof. $[T, \Phi(\Psi)] = \partial_0 \left(\gamma_{\beta}^{\nu\sigma} (\partial_{\nu} \Psi^{\beta} + \Gamma_{\alpha\nu}^{\beta} \Psi^{\alpha}) \right) \partial_{\sigma}$
 $\partial_0 \left(\gamma_{\beta}^{\nu\sigma} (\partial_{\nu} \Psi^{\beta} + \Gamma_{\alpha\nu}^{\beta} \Psi^{\alpha}) \right)$
 $= \partial_0 \gamma_{\beta}^{\nu\sigma} (\partial_{\nu} \Psi^{\beta} + \Gamma_{\alpha\nu}^{\beta} \Psi^{\alpha}) + \gamma_{\beta}^{\nu\sigma} (\partial_0 \partial_{\nu} \Psi^{\beta} + (\partial_0 \Gamma_{\alpha\nu}^{\beta}) \Psi^{\alpha} + \Gamma_{\alpha\nu}^{\beta} \partial_0 \Psi^{\alpha})$
 $= (\gamma_{\nu,0}^{\nu})_{\beta}^{\sigma} (D_{\nu} \Psi)^{\beta} + (\gamma^{\nu})_{\beta}^{\sigma} ((L_0 \circ L_{\nu} + \Gamma_{\nu,0} + \Gamma_{\nu} L_0) (\Psi))^{\beta}$
 $= ((\gamma_{\nu,0}^{\nu} D_{\nu} + \gamma^{\nu} (L_0 L_{\nu} + \Gamma_{\nu,0} + \Gamma_{\nu} L_0)) (\Psi))^{\sigma}$. Therefore,
 $H = \frac{\hbar}{mc} (\gamma_{\nu,0}^{\nu} D_{\nu} + \gamma^{\nu} (L_0 L_{\nu} + \Gamma_{\nu,0} + \Gamma_{\nu} L_0)) + i\partial_T (\log(m)) \text{Id}$.
We set out $L_{\nu} \Psi = (\partial_{\nu} \Psi^{\sigma}) \partial_{\sigma}$, $\gamma_{\nu,0}^{\nu}$ and $\Gamma_{\nu,0}$ are endomorphisms whose matrices in the canonical basis, are respectively, $(\partial_0 \gamma_{\beta}^{\nu\sigma})$ and $(\partial_0 \Gamma_{\alpha\nu}^{\beta})$. \square

7. CONCLUSION

Taking position on a related part Ω of the universe where no measurement has been made on the metric g and where there exists at least one chronological field T . Any particle is described by a section Ψ of $T_{\mathbb{C}}\Omega$. On the fibre bundle $T_{\mathbb{C}}\Omega$ there is also a connection ∇ and a Dirac section γ . The connection ∇ is not necessarily the Levi-Civita connection if we are on the entire universe.

The field Ψ solves the Dirac-Einstein field equation given by 3.1, with $\lambda = \frac{mc}{\hbar}$, where m is the mass of the particle associated with field Ψ . The general Dirac-Weyl-Fock equation, known as the Schrödinger-Dirac equation is given in 6.2, \mathcal{L}_T is the Lie derivative along the chronological field extended to the fields.

The Hamiltonian of the Dirac-Weyl-Fock equation is given in the equation 6.2 where Φ is the evolution operator of the Dirac-Einstein equation at rank 1. We can entirely describe the evolution of a particle with a connection of the universe, known as a fundamental connection ∇ . The Dirac section γ is the solution of the relativistic quantum equation 4.3 where $\Lambda = \Lambda_0$ is the cosmological constant and $\gamma^{\alpha\beta}$ is the Poisson tensor defined in 4.1.

The masses of particles are quantified, they are proportional to the eigenvalues of the operator Φ of the Einstein-Dirac field evolution equation. If we want to study the change along the chronological field, we simply need to take the Lie derivative of the field evolution equation, along this chronological field.

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