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Existence of periodic points near an isolated fixed point with Lefschetz index 1 and zero rotation for area preserving surface homeomorphisms

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Abstract

Let f be an orientation and area preserving diffeomorphism of an oriented surface M with an isolated degenerate fixed point z_0 with Lefschetz index one. Le Roux conjectured that z_0 is accumulated by periodic orbits. In this article, we will approach Le Roux's conjecture by proving that if f is isotopic to the identity by an isotopy fixing z_0 and if the area of M is finite, then z_0 is accumulated not only by periodic points, but also by periodic orbits in the measure sense. More precisely, the Dirac measure at z_0 is the limit in weak-star topology of a sequence of invariant probability measures supported on periodic orbits. Our proof is purely topological and will works for homeomorphisms and is related to the notion of local rotation set.

Keywords. periodic point, rotation set, transverse foliation, degenerate fixed point

Mathematical Subject Classification. 37E30 37E45 37A05

1 Introduction

The goal of this article is to give a result of existence of periodic orbits for area preserving homeomorphisms of surfaces. We will begin this introduction by explaining our result in the more general case of homeomorphisms, then will explain what does it mean in the case of diffeomorphisms and will conclude by giving its significance in the symplectic formalism.

Let f be a homeomorphism of an oriented surface M that is isotopic to the identity. We say that f is *area preserving* if it preserves a Borel measure without atom such that the measure of each open set is positive and that the measure of each compact set is finite. We call $I = (f_t)_{t\in[0,1]}$ an *identity isotopy of* f if it is an isotopy from the identity to f, and call a fixed point of f a *contractible fixed point* associated to I if its trajectory along I is a loop homotopic to zero in M. We say that a fixed point of f is a *fixed point of* I if it is fixed along the isotopy, and denote by $\mathrm{Fix}(I)$ the set of fixed points of I.

Suppose that f is an area preserving homeomorphism of M, that I is an identity isotopy of f, and that z is a fixed point of I. We say that f can be *blown-up* at z if we can replace z with a circle and extend f continuously to this circle. In particular, when f is a diffeomorphism near z, the blow-up can be induced by $Df(z)$. We can define the *blow-up rotation number*

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$\rho(I,z)$ to be a representative of the Poincaré rotation number of the homeomorphism on the added circle. (The precise definitions can be found in Section 2.10.)

Suppose that f can be blown-up at $z \in \mathrm{Fix}(I)$, and that M (resp. $M \setminus (\mathrm{Fix}(I) \setminus \{z\})$) is neither a sphere nor a plane. When the blow-up rotation number $\rho(I,z)$ is different from 0, we lift I (resp. $I|_{M\setminus(\mathrm{Fix}(I)\setminus\{z\})}$) to the universal covering space and get an identity isotopy of a lift of f (resp. $f|_{M\setminus(\mathrm{Fix}(I)\setminus\{z\})}$). If we fix a pre-image of z, we can blow up this point and get an area preserving homeomorphism $\widetilde{f}$ of the half-open annulus. Every other pre-image of z in the annulus is a fixed point of $\widetilde{f}$ with rotation number 0, and the rotation number of $\widetilde{f}$ on the boundary is different from 0. (A formal definition of the rotation number can be found in Section 2.11.) By a generalization of Poincaré-Birkhoff Theorem, we can deduce that $\widetilde{f}$ has infinitely many periodic points that correspond to different contractible periodic points of f. So, we can prove the existence of infinitely many contractible periodic points.

When the blow-up rotation number $\rho(I,z)$ is 0, the problem is much more difficult. We will be interested in this article in the case where $\rho(I,z) = 0$ and where the Lefschetz index is equal to 1. It must be noticed that this situation does not occur in the case of a diffeomorphism with no degenerate[1] fixed point. It is a critical case, but an interesting one for the following reason: there are many situations where existence of a fixed point of index one can be proven. For example, every orientation and area preserving homeomorphism of the sphere with finitely many fixed point has at least two fixed points of Lefschetz index 1. It is a consequence of Lefschetz formula and of the fact that the Lefschetz index of an orientation and area preserving homeomorphism at an isolated fixed point is always not bigger that 1 (see [PS87] and [LC99]). Existence of at least one fixed point of Lefschetz index 1 can be proven for an area preserving homeomorphism f of a closed surface of positive genus in the case where f is isotopic to the identity, f has finitely many fixed points, and the mean rotation vector vanishes. (See [Fra96] in the case of a diffeomorphism and [Mat01] in the more general case.)

More precisely, suppose that $f : M \to M$ is an area preserving homeomorphism of an oriented surface M, that z_0 is an isolated fixed point of f with a Lefschetz index $i(f,z_0) = 1$, and that I is an identity isotopy of f fixing z_0. The homeomorphism f can not always be blown-up at z_0, nevertheless, Frédéric Le Roux [LR13] generalized the rotation number and defined a *local rotation set* $\rho_s(I,z_0)$ (see Section 2.10). In particular, when f can be blown-up at z_0, the local rotation set is just reduced to the rotation number. We will prove that if the total area of M is finite and if $\rho_s(I,z_0)$ is reduced to 0, then z_0 is accumulated by contractible periodic points of f. More generally, the result is still valid if we relax the condition that $\rho_s(I,z_0)$ is reduced to 0 to the condition that $\rho_s(I,z_0)$ is reduced to an integer k. We will prove a stronger result: the Dirac measure δ_{z_0} at the fixed point z_0 is a limit, in the weak-star topology, of a sequence of invariant probability measures supported on periodic orbits. We can be more precise. Let us say that a contractible periodic orbit of period q has *type* (p,q) if its trajectory along the isotopy is homotopic to $p\Gamma$ in $M \setminus \mathrm{Fix}(I)$, where Γ is the boundary of a sufficiently small Jordan domain containing z_0. Then, there exists an open interval L containing an integer k in its boundary such that for all irreducible $p/q \in L$ there exists a contractible periodic orbit $O_{p/q}$ of type (p,q) and such that δ_{z_0} is the limit, in the weak-star topology, of any sequence $(\mu_{O_{p_n/q_n}})_{n\geq 1}$ such that $\lim_{n\to\infty} \frac{p_n}{q_n} = k$, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$.

Formally, we have the following theorem, which is the main result of this article:

[1] Here, degenerate means that 1 is an eigenvalue of the Jacobian matrix of f at z_0.

Theorem 1.1. *Let $f : M \to M$ be an area preserving homeomorphism of an oriented surface M, z_0 be an isolated fixed point of f with a Lefschetz index $i(f, z_0) = 1$, and I be an identity isotopy of f fixing z_0 and satisfying $\rho_s(I, z_0) = \{k\}$. Suppose that one of the following situations occurs,*

i) M is a plane, f has only one fixed point z_0 and has a periodic orbit besides z_0;

ii) the total area of M is finite.

Then, z_0 is accumulated by periodic points. More precisely, the following property holds:

***P)**: There exists $\varepsilon > 0$, such that either for all irreducible $p/q \in (k, k+\varepsilon)$, or for all irreducible $p/q \in (k-\varepsilon, k)$, there exists a contractible periodic orbit $O_{p/q}$ of type (p, q), such that $\mu_{O_{p/q}} \to \delta_{z_0}$ as $p/q \to k$, in the weak-star topology, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$,*

Remark 1.2. Le Roux [LR13] gave the following conjecture: if $f : (W, z_0) \to (W', z_0)$ is an orientation and area preserving homeomorphism between two neighborhoods of $z_0 \in M$, and if z_0 is an isolated fixed point of f such that $i(f, z_0)$ is equal to 1 and that $\rho_s(I, z_0)$ is reduced to 0 for a local isotopy I of f, then z_0 is accumulated by periodic orbits of f. Although we can not give a complete answer to this conjecture in this article, we approach it by the previous theorem.

When f is a diffeomorphism, we will give several versions of the theorem whose conditions are more easy to understand.

The rotation set of a local isotopy (see Section 2.10) at a degenerate fixed point of an orientation preserving diffeomorphism is reduced to an integer. So, given an area-preserving diffeomorphism f of a surface M with finite area that is isotopic to the identity, if z_0 is a degenerate fixed point whose Lefschetz index is equal to 1, the assumptions of the previous theorem are satisfied, and hence z_0 is accumulated by contractible periodic points. Formally, we have the following corollary:

Corollary 1.3. *Let f be an orientation and area preserving diffeomorphism of an oriented surface M with finite total area, and z_0 be a degenerate isolated fixed point such that $i(f, z_0) = 1$. If f is isotopic to the identity by an isotopy I that fixes z_0[2], then z_0 is accumulated by contractible periodic points. Moreover, the property **P)** holds.*

Let f be a $\mathcal{C}^1$ diffeomorphism of $\mathbb{R}^2$. A function $g : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^2$ is called a *generating function* of f if $\partial^2_{12} g < 1$, and

$$f(x, y) = (X, Y) \Leftrightarrow \begin{cases} X - x = & \partial_2 g(X, y), \\ Y - y = & -\partial_1 g(X, y). \end{cases}$$

We know that the previous diffeomorphism f is orientation and area preserving by a direct computation.

Generating functions are usual objects in symplectic geometry. We will give the following version of our result whose conditions are described by generating functions.

[2] If there exists an identity isotopy of f such that the trajectory of z_0 along the isotopy is homotopic to zero, there always exists an identity isotopy of f fixing z_0.

Corollary 1.4. *Let f be an orientation and area preserving diffeomorphism of an oriented surface M with finite area. We suppose that f is isotopic to the identity by an isotopy that fixes z_0. Suppose that in a neighborhood of z_0, f is conjugate to a local diffeomorphism at 0 that is generated by a generating function g, that 0 is a local extremum of g, and that the Hessian matrix of g at 0 is degenerate. Then z_0 is accumulated by periodic points, and the property **P)** holds.*

Let us explain now what is our result in the symplectic formalism.

A time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ is called a *Hamiltonian vector field* if it is defined by the equation:

$$dH_t = \omega(X_t, \cdot),$$

where (M, ω) is a symplectic manifold, and $H : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function. The Hamiltonian vector field induces a flow $(\varphi_t)_{t\in\mathbb{R}}$ on M, which is the solution of the following equation

$$\frac{\partial}{\partial t}\varphi_t(z) = X_t(\varphi_t(z)).$$

We say that a diffeomorphism f of M is a *Hamiltonian diffeomorphism* if it is the time-1 map of a Hamiltonian flow.

In particular, a Hamiltonian diffeomorphism f of the torus $\mathbb{T}^2$ that is close to the identity in $\mathcal{C}^1$ topology can be lifted to the plane $\mathbb{R}^2$, and the lifted diffeomorphism can be defined by a generating function g. If z_0 is a local maximum of g, and if the Hessian of g at z_0 is degenerate, we are in the case of the previous corollary, and the image of z_0 in $\mathbb{T}^2$ is a fixed point of f that is accumulated by contractible periodic points.

In 1984, Conley conjectured that a Hamiltonian diffeomorphism f of the standard symplectic torus (T^{2d}, ω) has infinitely many contractible periodic points. The result was proved later in the case where f has no degenerate fixed point, by Conley and Zehnder[CZ86], in the weakly non-degenerate case by Salamon and Zehnder [SZ92], and finally in the general case by Hinston[Hin09]. Recently, Mazzucchelli[Maz13] gave a simpler argument based on generating functions for the second part of the proof of Hinston, and noticed that the existence of a symplectical degenerate extremum, that will be defined in the section 4.2, implies the existence of infinitely many other periodic points. In the same paper, he asked whether a symplectical degenerate extremum actually corresponds to a fixed point accumulated by periodic points. In this article, we will give a positive answer in the case where $d = 1$. More precisely, we have the following result.

Theorem 1.5. *Let $f : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian diffeomorphism, and z_0 be a symplectical degenerate extremum. Then z_0 is accumulated by periodic points, and the property **P)** holds.*

Now, we will give a plan of this article. In Section 2, we will recall some definitions and results that we will use in the proofs of our results. In Section 3, we will prove Theorem 1.1, which is the main result of this paper. In Section 4, we will study a particular case where f is a diffeomorphism, and will give several versions of our results whose conditions are described in different ways: Corollary 1.4 and Theorem 1.5.

2 Preliminaries

2.1 A classification of isolated fixed points

In this section, we will give a classification of isolated fixed points. More details can be found in [LC03].

Let $f : (W, z_0) \to (W', z_0)$ be a local homeomorphism with an isolated fixed point z_0. We say that z_0 is an *accumulated point* if every neighborhood of z_0 contains a periodic orbit besides z_0. Otherwise, we say that z_0 is a *non-accumulated point*.

We define a *Jordan domain* to be a bounded domain whose boundary is a simple closed curve. We say that z_0 is *indifferent* if there exists a neighborhood $V \subset \overline{V} \subset W$ of z_0 such that for every Jordan domain $U \subset V$ containing z_0, the connected component of $\cap_{k\in\mathbb{Z}} f^{-k}(\overline{U})$ containing z_0 intersects the boundary of U.

We say that z_0 is *dissipative* if there exists a fundamental system $\{U_\alpha\}_{\alpha\in J}$ of the neighborhood of z_0 such that each U_α is a Jordan domain and that $f(\partial U_\alpha) \cap \partial U_\alpha = \emptyset$.

We say that z_0 is a *saddle* point if it is neither indifferent nor dissipative.

Note that if f is area preserving, an isolated fixed point of f is either an indifferent point or a saddle point.

2.2 Lefschetz index

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at an isolated fixed point $0 \in \mathbb{R}^2$. Denote by S^1 the unit circle. If $C \subset W$ is a simple closed curve which contains no fixed point of f, then we can define the *index* $i(f, C)$ of f along the curve C to be the Brouwer degree of the map

$$\begin{array}{rcl} \varphi : S^1 & \to & S^1 \\ t & \mapsto & \dfrac{f(\gamma(t)) - \gamma(t)}{||f(\gamma(t)) - \gamma(t)||}, \end{array}$$

where $\gamma : S^1 \to C$ is a parametrization compatible with the orientation, and $\|\cdot\|$ is the usual Euclidean norm. Let U be a Jordan domain containing 0 and contained in a sufficiently small neighborhood of 0. We define the *Lefschetz index* of f at 0 to be $i(f, \partial U)$, which is independent of the choice of U. We denote it by $i(f, 0)$.

More generally, if $f : (W, z_0) \to (W', z_0)$ is an orientation preserving local homeomorphism at a fixed point z_0 on a surface M, we can conjugate it topologically to an orientation preserving local homeomorphism g at 0 and define the *Lefschetz index* of f at z_0 to be $i(g, 0)$, which is independent of the choice of the conjugation. We denote it by $i(f, z_0)$.

2.3 Local isotopies and the index of local isotopies

Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at $z_0 \in M$. A *local isotopy* I of f at z_0 is a family of homeomorphisms $(f_t)_{t\in[0,1]}$ such that

- every f_t is a homeomorphism between the neighborhoods $V_t \subset W$ and $V'_t \subset W'$ of z_0, and $f_0 = \mathrm{Id}_{V_0}$, $f_1 = f|_{V_1}$;
- for all t, one has $f_t(z_0) = z_0$;
- the sets $\{(z, t) \in M \times [0, 1] : z \in V_t\}$ and $\{(z, t) \in M \times [0, 1] : z \in V'_t\}$ are both open in $M \times [0, 1]$;

- the maps $(z,t) \mapsto f_t(z)$ and $(z,t) \mapsto f_t^{-1}(z)$ are both continuous.

We say that two local isotopies of f are *equivalent* if they are locally homotopic.

Let us introduce the index of a local isotopy which was defined by Le Roux [LR13] and Le Calvez [LC08].

Let $f : (W,0) \to (W',0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $I = (f_t)_{i\in[0,1]}$ be a local isotopy of f. We denote by D_r the disk with radius r and centered at 0. Then the isotopy f_t is well defined in the disk D_r if r is sufficiently small. Let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta}, \end{array}$$

be the universal covering projection, and $\widetilde{I} = (\widetilde{f_t})_{t\in[0,1]}$ be the lift of $I|_{D_r\setminus\{0\}}$ to $\mathbb{R} \times (-r,0)$ such that f_0 is the identity. Let $\widetilde{\gamma} : [0,1] \to \mathbb{R} \times (-r,0)$ be a path from $\widetilde{z'} \in \mathbb{R} \times (-r,0)$ to $\widetilde{z'} + (1,0)$. The map

$$t \mapsto \frac{\widetilde{f_1}(\widetilde{\gamma}(t)) - \widetilde{\gamma}(t)}{||\widetilde{f_1}(\widetilde{\gamma}(t)) - \widetilde{\gamma}(t)||}$$

takes the same value at both 0 and 1, and hence descends to a continuous map $\varphi : [0,1]/_{0\sim 1} \to S^1$. We define the *index of the isotopy* I at 0 to be the Brouwer degree of φ, which does not depend on the choice of $\widetilde{\gamma}$ when r is sufficiently small. We denote it by $i(I,0)$.

More generally, we consider an orientation preserving local homeomorphism on an oriented surface. Let $f : (W,z_0) \to (W',z_0)$ be an orientation preserving local homeomorphism at a fixed point z_0 in a surface M. Let $h : (U,z_0) \to (U',0)$ be a local homeomorphism. Then $h \circ I \circ h^{-1} = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$ is a local isotopy at 0, and we define the *index of* I at z_0 to be $i(h \circ I \circ h^{-1}, 0)$, which is independent of the choice of h. We denote it by $i(I,z_0)$.

Let $I = (f_t)_{t\in[0,1]}$ and $I' = (g_t)_{t\in[0,1]}$ be two identity isotopies (resp. local isotopies). We denote by I^{-1} the isotopy (resp. local isotopy) $(f_t^{-1})_{t\in[0,1]}$, by $I'I$ the isotopy (resp. local isotopy) $(\varphi_t)_{t\in[0,1]}$ such that

$$\varphi_t = \begin{cases} f_{2t} & \text{for } t \in [0, \frac{1}{2}], \\ g_{2t-1} \circ f & \text{for } t \in [\frac{1}{2}, 0], \end{cases}$$

and by I^n the isotopy (resp. local isotopy) $\underbrace{I \cdots I}_{n \text{ times}}$ for every $n \geq 1$.

The Lefschetz index at an isolated fixed point and the indices of the local isotopies are related. We have the following result:

Proposition 2.1. *([LC08][LR13]) Let $f : W \to W'$ be an orientation preserving homeomorphism with an isolated fixed point z. Then, we have the following results:*

- *if $i(f,z) \neq 1$, there exists a unique homotopy class of local isotopies such that $i(I,z) = i(f,z)-1$ for every local isotopy I in this class, and the indices of the other local isotopies are equal to 0;*

- *if $i(f,z) = 1$, the indices of all the local isotopies are equal to 0.*

2.4 Brouwer plane translation theorem

In this section, we will recall the Brouwer plane translation theorem. More details can be found in [Bro12], [Gui94] and [Fra92].

Let f be an orientation preserving homeomorphism of $\mathbb{R}^2$. If f does not have any fixed point, the Brouwer plane translation theorem asserts that every $z \in \mathbb{R}^2$ is contained in a translation domain for f, i.e. an open connected set of $\mathbb{R}^2$ whose boundary is $L \cup f(L)$, where L is the image of a proper embedding of $\mathbb{R}$ in $\mathbb{R}^2$ such that L separates $f(L)$ and $f^{-1}(L)$.

As an immediate corollary, one knows that if f is an orientation and area preserving homeomorphism of a plane[3] with finite area, it has at least one fixed point.

2.5 Transverse foliations and its index at an isolated end

In this section, we will introduce the index of a foliation at an isolated end. More details can be found in [LC08].

Let M be an oriented surface and $\mathcal{F}$ be an oriented topological foliation on M. For every point z, there is a neighborhood V of z and a homeomorphism $h : V \to (0,1)^2$ preserving the orientation such that the images of the leaves of $\mathcal{F}|_V$ are the vertical lines oriented upward. We call V a *trivialization neighborhood* of z, and h a *trivialization chart*.

Let z_0 be an isolated end of M. We choose a small annulus $U \subset M$ such that z_0 is an end of U. Let $h : U \to \mathbb{D} \setminus \{0\}$ be a homeomorphism which sends z_0 to 0 and preserves the orientation. Let $\gamma : \mathbb{T}^1 \to \mathbb{D} \setminus \{0\}$ be a simple closed curve homotopic to $\partial\mathbb{D}$. We can cover the curve by finite trivialization neighborhoods $\{V_i\}_{1 \le i \le n}$ of the foliation $\mathcal{F}_h$, where $\mathcal{F}_h$ is the image of $\mathcal{F}|_U$. For every $z \in V_i$, we denote by $\phi^+_{V_i,z}$ the positive half leaf of the the leaf in V_i containing z. Then we can construct a continuous map ψ from the curve γ to $\mathbb{D} \setminus \{0\}$, such that $\psi(z) \in \phi^+_{V_i,z}$ for all $0 \le i \le n$ and for all $z \in V_i$. We define the *index* $i(\mathcal{F}, z_0)$ of $\mathcal{F}$ at z_0 to be the Brouwer degree of the application

$$\theta \mapsto \frac{\psi(\gamma(\theta)) - \gamma(\theta)}{\|\psi(\gamma(\theta)) - \gamma(\theta)\|},$$

which depends neither on the choice of ψ, nor on the choice of V_i, nor on the choice of γ, nor on the choice of h.

We say that a path $\gamma : [0,1] \to M$ is *positively transverse* to $\mathcal{F}$, if for every $t_0 \in [0,1]$, there exists a trivialization neighborhood V of $\gamma(t_0)$ and $\varepsilon > 0$ such that $\gamma([t_0-\varepsilon, t_0+\varepsilon] \cap [0,1]) \subset V$ and $h \circ \gamma|_{[t_0-\varepsilon,t_0+\varepsilon]\cap[0,1]}$ intersects the vertical lines from left to right, where $h : V \to (0,1)^2$ is the trivialization chart.

Let f be a homeomorphism on M isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. We say that an oriented foliation $\mathcal{F}$ on M is a *transverse foliation* of I if for every $z \in M$, there is a path that is homotopic to the trajectory $t \to f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

Suppose that $I = (f_t)_{t\in[0,1]}$ is a local isotopy at z_0. We say that $\mathcal{F}$ is *locally transverse* to I if for every sufficiently small neighborhood U of z_0, there exists a neighborhood $V \subset U$ such that for all $z \in V \setminus \{z_0\}$, there exists a path in $U \setminus \{z_0\}$ that is homotopic to the trajectory $t \mapsto f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

Proposition 2.2. *[LC08] Suppose that I is an identity isotopy on a surface M with an isolated end z and $\mathcal{F}$ is a transverse foliation of I. If M is not a plane, $\mathcal{F}$ is also locally transverse to the local isotopy I at z.*

[3] Here, a plane is an open set homeomorphic to $\mathbb{R}^2$.

Proposition 2.3. *[LC08] Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at an isolated fixed point z_0, I be a local isotopy of f at z_0, and $\mathcal{F}$ be a foliation that is locally transverse to I, then*

- $i(\mathcal{F}, z_0) = i(I, z_0) + 1$;
- $i(f, z_0) = i(\mathcal{F}, z_0)$ *if* $i(\mathcal{F}, z_0) \neq 1$.

2.6 Existence of a transverse foliation and Jaulent's preorder

Let f be a homeomorphism of M isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. A *contractible fixed point* z of f associated to I is a fixed point of f such that the trajectory of z along I, that is the path $t \mapsto f_t(z)$, is a loop homotopic to zero in M. One has the following generalization of Brouwer's translation theorem.

Theorem 2.4. *[LC05] Let M be an oriented surface. If $I = (f_t)_{t\in[0,1]}$ is an identity isotopy of a homeomorphism f of M such that there exists no contractible fixed point of f associated to I, then there exists a transverse foliation $\mathcal{F}$ of I.*

One can extend this result to the case where there exist contractible fixed points by defining the following preorder of Jaulent [Jau14].

Let us denote by $\mathrm{Fix}(f)$ the set of fixed points of f, and for every identity isotopy $I = (f_t)_{t\in[0,1]}$ of f, by $\mathrm{Fix}(I) = \cap_{t\in[0,1]}\mathrm{Fix}(f_t)$ the set of fixed points of I. Let X be a closed subset of $\mathrm{Fix}(f)$. We denote by (X, I_X) the couple that consists of a closed subset $X \subset \mathrm{Fix}(f)$ such that $f|_{M\setminus X}$ is isotopic to the identity and an identity isotopy I_X of $f|_{M\setminus X}$.

Let $\pi_X : \widetilde{M}_X \to M \setminus X$ be the universal cover, and $\widetilde{I}_X = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I_X. We say that $\widetilde{f}_X = \widetilde{f}_1$ is *the lift of f associated to I_X*. We say that a path $\gamma : [0, 1] \to M \setminus X$ from z to $f(z)$ is *associated* to I_X if there exists a path $\widetilde{\gamma} : [0, 1] \to \widetilde{M}_X$ that is the lift of γ and satisfies $\widetilde{f}_X(\widetilde{\gamma}(0)) = \widetilde{\gamma}(1)$. We write $(X, I_X) \precsim (Y, I_Y)$, if

- $X \subset Y \subset (X \cup \pi_X(\mathrm{Fix}(\widetilde{f}_X)))$;
- all the paths in $M \setminus Y$ associated to I_Y are also associated to I_X.

The preorder $\precsim$ is well defined. Moreover, if one has $(X, I_X) \precsim (Y, I_Y)$ and $(Y, I_Y) \precsim (X, I_X)$, then one knows that $X = Y$ and that I_X is homotopic to I_Y. In this case, we will write $(X, I_X) \sim (Y, I_Y)$. Jaulent proved the following result:

Theorem 2.5. *[Jau14] Let M be an oriented surface and I be an identity isotopy of a homeomorphism f on M. Then, there exists a maximal $(X, I_X) \in \mathcal{I}$ such that $(\mathrm{Fix}(I), I) \precsim (X, I_X)$. Moreover, $f|_{M\setminus X}$ has no contractible fixed point associated to I_X, and there exists a transverse foliation $\mathcal{F}$ of I_X on $M \setminus X$.*

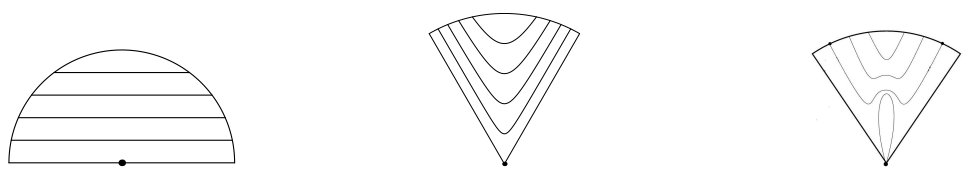
Remark 2.6. Here, we can also consider the previous foliation $\mathcal{F}$ to be a singular foliation on M whose singularities are the points in X. In particular, if I_X is the restriction to $M \setminus X$ of an identity isotopy I' on M, we will say that $\mathcal{F}$ a transverse (singular) foliation of I'.

We call $(Y, I_Y) \in \mathcal{I}$ a *maximal extension* of (X, I_X) if $(X, I_X) \precsim (Y, I_Y)$ and if (Y, I_Y) is maximal in Jaulent's preorder; we call (Y, I_Y) a *maximal extension* of I if (Y, I_Y) is a maximal extension of $(\mathrm{Fix}(I), I)$.

2.7 Dynamics of an oriented foliation in a neighborhood of an isolated singularity

In this section, we consider singular foliations. A *sink* (resp. a *source*) of $\mathcal{F}$ is an isolated singular point of $\mathcal{F}$ such that there is a homeomorphism $h : U \to \mathbb{D}$ which sends z_0 to 0 and sends the restricted foliation $\mathcal{F}|_{U\setminus\{z_0\}}$ to the radial foliation of $\mathbb{D}\setminus\{0\}$ with the leaves toward (resp. backward) 0, where U is a neighborhood of z_0 and $\mathbb{D}$ is the unit disk. A *petal* of $\mathcal{F}$ is a closed topological disk whose boundary is the union of a leaf and a singularity. Let $\mathcal{F}_0$ be the foliation on $\mathbb{R}^2\setminus\{0\}$ whose leaves are the horizontal lines except the $x-$axis which is cut into two leaves. Let $S_0 = \{y \geq 0 : x^2 + y^2 \leq 1\}$ be the half-disk. We call a closed topological disk S a *hyperbolic sector* if there exist

- a closed set $K \subset S$ such that $K \cap \partial S$ is reduced to a singularity z_0 and $K \setminus \{z_0\}$ is the union of the leaves of $\mathcal{F}$ that are contained in S,
- a continuous map $\phi : S \to S_0$ that maps K to 0 and the leaves of $\mathcal{F}|_{S\setminus K}$ to the leaves of $\mathcal{F}_0|_{S_0}$.



(a) the hyperbolic sector model S_0

(b) a pure hyperbolic sector

(c) a strange hyperbolic sector

Figure 1: The hyperbolic sectors

Le Roux gives a description of the dynamics of an oriented foliation $\mathcal{F}$ at an isolated singularity z_0.

Proposition 2.7. *[LR13] We have one of the following cases:*

i) (sink or source) there exists a neighborhood of z_0 that contains neither a closed leaf, nor a petal, nor a hyperbolic sector;

ii) (cycle) every neighborhood of z_0 contains a closed leaf;

iii) (petal) every neighborhood of z_0 contains a petal, and does not contain any hyperbolic sector;

iv) (saddle) every neighborhood of z_0 contains a hyperbolic sector, and does not contain any petal;

v) (mixed) every neighborhood of z_0 contains both a petal and a hyperbolic sector.

Moreover, $i(\mathcal{F}, z_0)$ is equal to 1 in the first two cases, is strictly bigger than 1 in the petal case, and is strictly smaller than 1 in the saddle case.

Remark 2.8. In particular, let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0, I be a local isotopy of f, $\mathcal{F}$ be an oriented foliation that is locally transverse to I, and z_0 be an isolated singularity of $\mathcal{F}$. If P is a petal in a small neighborhood

of z_0 and Φ is the leaf in ∂P, then $\Phi \cup \{z_0\}$ divides M into two parts. We denote by $L(\Phi)$ the one to the left and $R(\Phi)$ the one to the right. By definition, P contains the positive orbit of $R(\Phi) \cap L(f(\Phi))$ or the negative orbit of $L(\Phi) \cap R(f^{-1}(\Phi))$. Then, a petal in a small neighborhood of z_0 contains the positive or the negative orbit of a wandering open set. So does the topological disk whose boundary is a closed leaf in a small neighborhood of z_0. Therefore, if f is area preserving, then z_0 is either a sink, a source, or a saddle of $\mathcal{F}$.

2.8 The local rotation type of a local isotopy

In this section, suppose that $f : (W, z_0) \to (W', z_0)$ is an orientation and area preserving local homeomorphism at an isolated fixed point z_0, and that I is a local isotopy of f. We say that I has a *positive rotation type* (resp. *negative rotation type*) if there exists a locally transverse foliation $\mathcal{F}$ of I such that z_0 is a sink (resp. source) of $\mathcal{F}$. Shigenori Matsumoto [Mat01] proved the following result:

Proposition 2.9. *[Mat01] If $i(f, z_0)$ is equal to 1, I has unique one of the two kinds of rotation types.*

Remark 2.10. By considering the index of foliation, one deduces the following corollary: *if $\mathcal{F}$ and $\mathcal{F}'$ are two locally transverse foliations of I, and if 0 is a sink (resp. source) of $\mathcal{F}$, then 0 is a sink (resp. a source) of $\mathcal{F}'$.*

2.9 Prime-ends compactification and rotation number

In this section, we first recall some facts and definitions from Carathéodory's prime-ends theory, and then give the definition of the prime-ends rotation number. More details can be found in [Mil06] and [KLCN14].

Let $U \subsetneq \mathbb{R}^2$ be a simply connected domain, then there exists a natural compactification of U by adding a circle, that can be defined in different ways. One explanation is the following: we can identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider a conformal diffeomorphism

$$h : U \to \mathbb{D},$$

where $\mathbb{D}$ is the unit disk. We endow $U \sqcup S^1$ with the topology of the pre-image of the natural topology of $\overline{\mathbb{D}}$ by the application

$$\overline{h} : U \sqcup S^1 \to \overline{\mathbb{D}},$$

whose restriction is h on U and the identity on S^1 .

Any arc in U which lands at a point z of ∂U corresponds, under h, to an arc in $\mathbb{D}$ which lands at a point of S^1, and arcs which land at distinct points of ∂U necessarily correspond to arcs which land at distinct points of S^1. We define an *end-cut* to be the image of a simple arc $\gamma : [0, 1) \to U$ with a limit point in ∂U. Its image by h has a limit point in S^1. We say that two end-cuts are *equivalent* if their images have the same limit point in S^1. We say that a point $z \in \partial U$ is *accessible* if there is an end-cut that lands at z. Then the set of points of S^1 that are limit points of an end-cut is dense in S^1, and accessible points of ∂U are dense in ∂U. We define a *cross-cut* by the image of a simple arc $\gamma : (0, 1) \to U$ which extends to an arc $\overline{\gamma} : [0, 1] \to \overline{U}$ joining two points of ∂U and such that each of the two components of $U \setminus \gamma$ has a boundary point in $\partial U \setminus \overline{\gamma}$.

Let f be an orientation preserving homeomorphism of U. We can extend f to a homeomorphism of the prime-ends compactification $U \sqcup S^1$, and denote it by $\overline{f}$. In fact, for a point

$z \in S^1$ which is a limit point of an end-cut γ, we can naturally define $\overline{f}(z)$ to be the limit point of $f \circ \gamma$. Then we can define the *prime-ends rotation number* $\rho(f, U) \in \mathbb{T}^1 = \mathbb{R}/\mathbb{Z}$ to be the Poincaré's rotation number of $\overline{f}|_{S^1}$. In particular, if f fixes every point in ∂U, $\rho(f, U) = 0$.

Let $K \subset \mathbb{R}^2$ be a continuum, and U_K be the unbounded component of $\mathbb{R}^2 \setminus K$. Then, U_K is an annulus and becomes a simply connected domain of the Riemann sphere if we identify $\mathbb{R}^2$ with $\mathbb{C}$ and add a point at infinity. The prime-ends compactification also gives us a compactification of the end of U_K corresponding to K by adding the circle of prime-ends. We can define *end-cuts* and *cross-cuts* similarly.

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $K \subset W$ be an invariant continuum containing 0. Similarly, we can naturally extend $f|_{U_K \cap W}$ to a homeomorphism $f_K : U_K \cap W \cup S^1 \to U_K \cap W' \cup S^1$, and define the *rotation number* $\rho(f, K) \in \mathbb{R}/\mathbb{Z}$ to be the Poincaré's rotation number of $f_K|_{S^1}$.

Furthermore, if $I = (f_t)_{t\in[0,1]}$ is a local isotopy of f at 0, we consider the universal covering projections

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

and

$$\begin{array}{rcl} \pi' : \mathbb{R} & \to & S^1 \\ \theta & \mapsto & e^{i2\pi\theta}. \end{array}$$

Let $\widetilde{U}_K = \pi^{-1}(U_K)$, $\widetilde{W} = \pi^{-1}(W)$, and $\widetilde{W'} = \pi^{-1}(W')$. Let

$$\pi_K : \widetilde{U}_K \sqcup \mathbb{R} \to U_K \sqcup \mathbb{T}^1$$

be the map such that $\pi_K = \pi$ in $\widetilde{U}_K$ and $\pi_K = \pi'$ on $\mathbb{R}$. We endow the topology on $\widetilde{U}_K \sqcup R$ such that π_K is a universal cover. Let $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the lift of $(f_t|_{V\setminus\{0\}})_{t\in[0,1]}$ such that $\widetilde{f}_0$ is the identity, where V is a small neighborhood of z. Let $\widetilde{f} : \widetilde{W} \to \widetilde{W'}$ be the lift of $f|_{W\setminus\{0\}}$ such that $\widetilde{f} = \widetilde{f}_1$ in $\pi^{-1}(V)$, we call it the *lift of f associated to I*. Let $\widetilde{f}_K : (\widetilde{W} \cap \widetilde{U}_K) \sqcup \mathbb{R} \to (\widetilde{W'} \cap \widetilde{U}_K) \sqcup \mathbb{R}$ be the lift of f_K such that $\widetilde{f}_K = \widetilde{f}$ in $\widetilde{W} \cap \widetilde{U}_K$, we call it *the lift of f_K associated to I*. We define the *rotation number* $\rho(I, K) = \lim_{n\to\infty} \dfrac{\widetilde{f}_K^n(\theta) - \theta}{n}$ which is a real number that does not depend on the choice of θ. We know that $\rho(I, K)$ is a representative of $\rho(f, K)$ in $\mathbb{R}$.

We have the following property:

Proposition 2.11. *[LC03] Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at a non-accumulated indifferent point 0. Let $U \subset \overline{U} \subset W$ be a Jordan domain such that $\overline{U}$ does not contain any periodic orbit except 0, and that for all $V \subset U$, the connected component of $\cap_{n\in\mathbb{Z}} f^{-n}(\overline{V})$ containing 0 intersects the boundary of V. Let K_0 be the connected component of $\cap_{n\in\mathbb{Z}} f^{-n}(\overline{U})$ containing 0. Then for every local isotopy I of f, and for every invariant continuum $K \subset \overline{U}$ containing 0, one has $\rho(I, K) = \rho(I, K_0)$.*

This proposition implies that if $f : (W, 0) \to (W', 0)$ is an orientation preserving local homeomorphism at a non-accumulated indifferent point 0, we can define the *rotation number* $\rho(I, 0)$ for every local isotopy I of f at 0, by writing $\rho(I, 0) = \rho(I, K)$ where K is a non-trivial invariant continuum sufficiently close to 0.

More generally, if $f : (W, z_0) \to (W', z_0)$ is an orientation preserving local homeomorphism at a non-accumulated indifferent point $z_0 \in M$, we can conjugate it to a local homeomorphism at 0, and get the previous definitions and results similarly.

2.10 The local rotation set

In this section, we will give a definition of the local rotation set and will describe the relations between the rotation set and the rotation number. More details can be found in [LR13].

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f. Given two neighborhoods $V \subset U$ of 0 and an integer $n \geq 1$, we define

$$E(U, V, n) = \{z \in U : z \notin V, f^n(z) \notin V, f^i(z) \in U \text{ for all } 1 \leq i \leq n\}.$$

We define the *rotation set* of I relative to U and V by

$$\rho_{U,V}(I) = \cap_{m\geq 1}\overline{\cup_{n\geq m}\{\rho_n(z), z \in E(U, V, n)\}} \subset [-\infty, \infty],$$

where $\rho_n(z)$ is the average change of angular coordinate along the trajectory of z. More precisely, let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal covering projection, $\widetilde{f} : \pi^{-1}(W) \to \pi^{-1}(W')$ be the lift of f associated to I, and $p_1 : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ be the projection onto the first factor. We define

$$\rho_n(z) = \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n},$$

where $\widetilde{z}$ is any point in $\pi^{-1}\{z\}$.

We define the *local rotation set* of I to be

$$\rho_s(I, 0) = \cap_U\overline{\cup_V \rho_{U,V}(I)} \subset [-\infty, \infty],$$

where $V \subset U \subset W$ are neighborhoods of 0.

We say that f can be *blown-up* at 0 if there exists an orientation preserving homeomorphism $\Phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{T}^1 \times (-\infty, 0)$, such that $\Phi f \Phi^{-1}$ can be extended continuously to $\mathbb{T}^1 \times \{0\}$. We denote this extension by h. Suppose that f is not conjugate a contraction or an expansion. We define the *blow-up rotation number* $\rho(f, 0)$ of f at 0 to be the Poincaré rotation number of $h|_{\mathbb{T}^1}$. Let $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f, $(\widetilde{h}_t)$ be the natural lift of $\Phi|_{\mathbb{T}^1\times(0,r)} \circ f_t|_{D_r\setminus\{0\}} \circ \Phi^{-1}|_{\mathbb{T}^1\times(0,r)}$, where D_r is a sufficiently small disk with radius r and centered at 0, and $\widetilde{h}$ be the lift of h such that $\widetilde{h} = \widetilde{h}_1$ in a neighborhood of $\mathbb{R} \times \{0\}$. We define the *blow-up rotation number* $\rho(I, 0)$ *of* I *at* 0 to be the rotation number of $h|_{\mathbb{T}^1}$ associated to the lift $\widetilde{h}|_{\mathbb{R}\times\{0\}}$, which is a representative of $\rho(f, 0)$ on $\mathbb{R}$. Jean-Marc Gambaudo, Le Calvez, and Elisabeth Pécou [GLCP96] proved that neither $\rho(f, 0)$ nor $\rho(I, 0)$ depend on the choice of Φ, which generalizes a previous result of Naĭshul′ [Naĭ82]. In particular, if f is a diffeomorphism, f can be blown-up at 0 and the extension of f on $\mathbb{T}^1$ is induced by the map

$$v \mapsto \frac{Df(0)v}{\|Df(0)v\|}$$

on the space of unit tangent vectors.

More generally, if $f : (W, z_0) \to (W', z_0)$ is an orientation preserving local homeomorphism at z_0 that is not conjugate to the contraction or the expansion, we can give the previous definitions for f by conjugate it to an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$.

The local rotation set can be empty. However, due to Le Roux [LR08], we know that the rotation set is not empty if f is area preserving.

We say that z is a *contractible* fixed point of f associated to a local isotopy $I = (f_t)_{t\in[0,1]}$ if the trajectory $t \mapsto f_t(z)$ of z along I is a loop homotopic to zero in $W \setminus \{z_0\}$.

The local rotation set satisfies the following properties:

Proposition 2.12. *[LR13] Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0, and I be a local isotopy of f at z_0. One has the following results:*

i) For all integer p, q, $\rho_s(J^p I^q, z_0) = q\rho_s(I, z_0)+p$, where J is a local isotopy of the identity such that $\rho(J, z_0) = 1$.

ii) If z_0 is accumulated by contractible fixed points of f associated to I, then $0 \in \rho_s(I, z_0)$.

iii) If $\rho_s(I, z_0)$ is a non-empty set that is contained in $(0, +\infty]$ (resp. $[-\infty, 0)$), then I has a positive (resp.negative) rotation type.

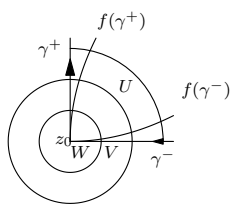
iv) If f can be blown-up at z_0, and if $\rho_s(I, z_0)$ is not empty, then $\rho_s(I, z_0)$ is reduced to the single real number $\rho(I, z_0)$.

v) If z_0 is a non-accumulated indifferent point, $\rho_s(I, z_0)$ is reduced to $\rho(I, z_0)$ (the rotation number defined in Section 2.9).

Remark 2.13. Le Roux also gives several criteria implying that f can be blown-up at z_0. The one we need in this article is due to Béguin, Crovisier and Le Roux [LR13]

If there exists an arc γ at z_0 whose germ is disjoint with the germs of $f^n(\gamma)$ for all $n \neq 0$, then f can be blown-up at z_0.

In particular, if there exists a leaf γ^+ from z_0 and a leaf γ^- toward z_0 (we are in this case if z_0 is a petal, a saddle, or a mixed singularity of $\mathcal{F}$), we can choose a sector U as in the picture. Let V be a small neighborhood of z_0. There exists a neighborhood $W \subset V$ of z_0 such



that

$$f(\overline{U} \cap W) \subset (\mathrm{Int}(U) \cap V) \cup \{z_0\}.$$

So, the germs at z_0 of $f^n(\gamma^+)$ are pairwise disjoint, and hence f can be blown-up at z_0. Moreover, $\rho(I, z_0)$ is equal to 0 in this case.

Le Roux also studied the dynamics near a non-accumulated saddle point, and proved the following result:

Proposition 2.14. *[LR13] If z_0 is a non-accumulated saddle point, then f can be blown-up at z_0 and $\rho_s(I, z_0)$ is reduced to a rational number. Moreover, if $i(f, z_0)$ is equal to 1, this rational number is not an integer.*

2.11 Some generalizations of Poincaré-Birkhoff theorem

In this section, we will introduce several generalizations of Poincaré-Birkhoff theorem. An *essential* loop in the annulus is a loop that is not homotopic to zero.

We first consider the homeomorphisms of closed annuli. Let f be a homeomorphism of $\mathbb{T}^1 \times [0,1]$ isotopic to the identity, $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. Let $\pi : \mathbb{R}\times[0,1] \to \mathbb{T}^1 \times \mathbb{R}$ be the universal cover, $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I, $\widetilde{f} = \widetilde{f}_1$ be the lift of f associated to I, and $p_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first factor. The limits

$$\lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(x,0) - x}{n} \quad \text{and} \quad \lim_{n\to\infty} \frac{p_1 \circ \widetilde{f}^n(x,1) - x}{n}$$

exists for all $x \in \mathbb{R}$, and do not depend on the choice of x. We define the *rotation number* of f on each boundary to be the corresponding limits. We define the *rotation number* of $z \in \mathbb{T}^1 \times \mathbb{R}$ associated to I to be the limit

$$\lim_{n\to+\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \in [-\infty, \infty],$$

if this limit exists. We say that f satisfies the *intersection property* if $f \circ \gamma$ intersects γ, for every simple essential loop $\gamma \subset \mathbb{T}^1 \times (0,1)$. We have the following generalizations of Poincaré-Birkhoff theorem:

Proposition 2.15. *[Bir26] Let f be a homeomorphism of $\mathbb{T}^1 \times [0,1]$ isotopic to the identity and satisfying the intersection property. If the rotation number of f on the two boundaries are different, then there exists a q-periodic orbit of rotation number p/q for all irreducible rational $p/q \in (\rho_1, \rho_2)$, where ρ_1 and ρ_2 are the rotation numbers of f on the boundaries.*

We also consider homeomorphisms of open annuli. Let $f : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a homeomorphism isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. Let $\pi : \mathbb{R} \times \mathbb{R} \to \mathbb{T}^2 \times \mathbb{R}$ be the universal cover, $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I, $\widetilde{f} = \widetilde{f}_1$ be the lift of f associated to I, and $p_1 : \mathbb{R}^2 \to \mathbb{R}$ be the projection on the first factor. Similarly, we define the *rotation number* of a recurrent point $z \in \mathbb{T}^1 \times \mathbb{R}$ associated to I to be the limit

$$\lim_{n\to+\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \in [-\infty, \infty],$$

if this limit exists. We say that f satisfies the *intersection property* if $f \circ \gamma$ intersects γ, for every simple essential loop $\gamma \subset \mathbb{T}^1 \times \mathbb{R}$. Then, we have the following generalization of Poincaré-Birkhoff theorem:

Proposition 2.16 ([Fra88], [LC05])**.** *Let $f : \mathbb{T}^1 \times \mathbb{R} \to \mathbb{T}^1 \times \mathbb{R}$ be a homeomorphism isotopic to the identity and satisfying the intersection property. If there exist two recurrent points of rotation numbers $\rho_1, \rho_2 \in [-\infty, +\infty]$ respectively such that $\rho_1 < \rho_2$, then there exists a q-periodic orbit of rotation number p/q for all irreducible rational $p/q \in (\rho_1, \rho_2)$.*

Remark 2.17. The result is also true for area preserving homeomorphisms of the closed or half closed annulus by considering a symmetry.

2.12 Topologically monotone periodic orbits for annulus homeomorphisms

In this section, we will recall the braid type of a periodic orbit and the existence of the topologically monotone periodic orbits for annulus homeomorphisms under some conditions. More details can be found in [Boy92].

Denote by $\mathbb{A}$ the closed annulus $\mathbb{T}^1 \times [0,1]$. Let f be a homeomorphism of $\mathbb{A}$ that preserves the orientation and each boundary circle, and $\widetilde{f}$ be a lift of f to the universal cover $\widetilde{\mathbb{A}} = \mathbb{R} \times [0,1]$. Given $\widetilde{z} \in \widetilde{\mathbb{A}}$, we define its rotation number under $\widetilde{f}$ as

$$\rho(\widetilde{z}, \widetilde{f}) = \lim_{n\to\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z})) - p_1(\widetilde{z})}{n},$$

if this limit exists, where p_1 is the projection onto the first factor. We define the rotation set of $\widetilde{f}$ to be

$$\rho(\widetilde{f}) = \{\rho(\widetilde{z}, \widetilde{f}), \widetilde{z} \in \widetilde{A}\}.$$

In particular, if I is an identity isotopy of f and $\widetilde{f}$ is the lift of f associated to I, this definition of the rotation number coincides with the definition of the rotation number in Section 2.11.

Fix a copy of the closed annulus minus n interior points, and denote it by $\mathbb{A}_n$. Let G_n be the group of isotopy classes of orientation preserving homeomorphism of $\mathbb{A}_n$. If O is an n-periodic orbit of f in the interior of $\mathbb{A}$, then there is an orientation preserving homeomorphism $h : \mathbb{A} \setminus O \to \mathbb{A}_n$. Philip Boyland defined the *braid type* $bt(O, f)$ to be the isotopy class of $h \circ f|_{\mathbb{A}\setminus O} \circ h^{-1}$ in G_n, this isotopy class is independent of the choice of h. If O is an n-periodic orbit of f contained in a boundary circle of $\mathbb{A}$, he extends f near this boundary and gets a homeomorphism $\overline{f}$ also on a closed annulus. Then O is in the interior of this new annulus. The braid type $bt(O, \overline{f})$ is independent of the choice of the extension, and Boyland defined $bt(O, f) = bt(O, \overline{f})$.

Let p/q be an irreducible positive rational, and $\widetilde{T}_{p/q}$ be the homeomorphism of $\widetilde{\mathbb{A}}$ defined by $(x, y) \mapsto (x + p/q, y)$. It descends to a homeomorphism $T_{p/q}$ of $\mathbb{A}$. We denote by $\alpha_{p/q}$ the braid type $bt(O, T_{p/q})$, where O is any periodic orbit of $T_{p/q}$. We say that a q-periodic orbit O of f is a (p, q)-periodic orbit if $\rho(\widetilde{z}, \widetilde{f}) = p/q$ for any $\widetilde{z}$ in the lift of O. We say that a (p, q)-periodic orbit O is *topologically monotone* if $bt(O, f) = \alpha_{p/q}$. We define the *Farey interval* $I(p/q)$ of p/q to be the closed interval

$$[\max\{r/s : r/s < p/q, 0 < s < q, \text{ and } (r, s) = 1\}, \min\{r/s : r/s > p/q, 0 < s < q, \text{ and } (r, s) = 1\}].$$

In particular, the Farey interval of $1/q$ is equal to $[0, 1/(q-1)]$.

Boyland proved the following result:

Proposition 2.18 ([Boy92])**.** *If f is an orientation and boundary preserving homeomorphism of the closed annulus, and $p/q \in \rho(\widetilde{f})$ is an irreducible positive rational, then f has a (p, q)-topologically monotone periodic orbit. If f has a (p, q)-orbit that is not topologically monotone, then $I(p/q) \subset \rho(\widetilde{f})$.*

2.13 Annulus covering projection

Let M be an oriented surface, $X_0 \subset M$ be a closed set, and $z_0 \in M \setminus X_0$. Denote by M_0 the connected component of $M \setminus X_0$ containing z_0. Let $V \subset U \subset M_0$ be two small Jordan domains containing z_0. Write $\dot{U} = U \setminus \{z_0\}$ and $\dot{V} = V \setminus \{z_0\}$. Fix $z_1 \in \dot{V}$. Let $\gamma \subset \dot{V}$ be a simple loop at z_1 such that the homotopic class $[\gamma]$ of γ in $\dot{V}$ generates $\pi_1(\dot{V}, z_1)$. Let $i : \dot{U} \to M_0 \setminus \{z_0\}$ be the inclusion, then $i_*\pi_1(\dot{U}, z_1)$ is a subgroup of $\pi_1(M_0 \setminus \{z_0\}, z_1)$. Then, there exists a covering

projection $\pi : (\widetilde{M}, \widetilde{z}_1) \to (M_0 \setminus \{z_0\}, z_1)$ such that $\pi_*\pi_1(\widetilde{M}, \widetilde{z}_1) = i_*\pi_1(\dot{U}, z_1)$ by Theorem 2.13 in [Spa66]. Moreover, the fundamental group of $\widetilde{M}$ is isomorphic to $\mathbb{Z}$, so $\widetilde{M}$ is an annulus.

Let $\widetilde{\dot{U}}$ be the component of $\pi^{-1}(\dot{U})$ containing $\widetilde{z}_1$. Then $\pi_*\pi_1(\widetilde{\dot{U}}, \widetilde{z}_1) = \pi_1(\dot{U}, z_1)$ and the restriction of π to $\widetilde{\dot{U}}$ is a homeomorphism between $\widetilde{\dot{U}}$ and $\dot{U}$ by Theorem 2.9 in [Spa66]. Consider the ideal-ends compactification of $\widetilde{M}$, and denote by $\star$ the end in $\widetilde{\dot{U}}$. Then $\pi|_{\widetilde{\dot{U}}}$ can be extended continuously to a homeomorphism between $\widetilde{\dot{U}} \cup \{\star\}$ and U. We denote it by h.

If f is an orientation preserving homeomorphism of M_0, and z_0 is a fixed point of f. By choosing sufficiently small V, we can suppose that $f(V) \subset U$. We know that $(f \circ \pi)_*\pi_1(\widetilde{M}, \widetilde{z}_1) = i_*\pi_1(\dot{U}, f(z_1)) = \pi_*\pi_1(\widetilde{M}, h^{-1}(f(z_1)))$, then we deduce by Theorem 2.5 of [Spa66] that there is a lift $\widetilde{f}$ of f to $\widetilde{M}$ that sends $\widetilde{z}_1$ to $h^{-1}(f(z_1))$. This map $\widetilde{f}$ is an homeomorphism because $\widetilde{f}_*\pi_1(\widetilde{M}, \widetilde{z}_1) = \pi_1(\widetilde{M}, h^{-1}(f(z_1))$ (see Corollary 2.7 in [Spa66]). Moreover, $\widetilde{f}$ can be extend continuously to a homeomorphism of $\widetilde{M} \cup \{\star\}$ that fixes $\star$.

In particular, if f is isotopic to the identity, and if $I = (f_t)_{t\in[0,1]}$ is an identity isotopy of f fixing z_0, then there exists a lift $\widetilde{f}_\cdot(\cdot) : I \times \widetilde{M} \to \widetilde{M}$ of the continuous map $(t, \widetilde{z}) \mapsto f_t(\pi(\widetilde{z}))$ such that $\widetilde{f}_0$ is equal to the identity, because π is a covering projection. Moreover, by choosing V small enough, we know that $\widetilde{f}_t|_{\widetilde{\dot{V}}}$ is conjugate to $f_t|_{\dot{V}}$ for $t \in [0, 1]$, where $\widetilde{\dot{V}}$ is the component of $\pi^{-1}(\dot{V})$ containing $\widetilde{z}_1$. Then $(\widetilde{f}_t)_*\pi_1(\widetilde{M}, \widetilde{z}_1) = \pi_1(\widetilde{M}, h^{-1}(f_t(z_1))$, therefore $\widetilde{f}_t$ is a homeomorphism by Corollary 2.7 in [Spa66]. We have indeed lifted I to an identity isotopy $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$. Moreover, $\widetilde{f}_t$ can be extended continuously to a homeomorphism of $\widetilde{M} \cup \{\star\}$ that fixes $\star$, and we get an isotopy on $\widetilde{M} \cup \{\star\}$ that fixes $\star$. We still denote by $\widetilde{f}_t$ the homeomorphism of $\widetilde{M} \cup \{\star\}$ and by $\widetilde{I}$ the identity isotopy on $\widetilde{M} \cup \{\star\}$ when there is no ambiguity. We call $\widetilde{I}$ the natural lift of I to $\widetilde{M} \cup \{\star\}$, and $\widetilde{f} = \widetilde{f}_1$ the lift of f to $\widetilde{M} \cup \{\star\}$ associated to I.

Moreover, if I is a maximal isotopy, $\widetilde{f}$ has no contractible fixed point associated to $\widetilde{I}$ on $\widetilde{M}$ and $\widetilde{I}$ is also a maximal isotopy. Recall that $\pi_*\pi_1(\widetilde{M}, \widetilde{z}_1) = i_*\pi_1(\dot{U}, z_1)$. So, $\pi(O)$ is a periodic orbit of type (p, q) associated to I at z_0 for all periodic orbit O of type (p, q) associated to $\widetilde{I}$ at $\star$, where $\frac{p}{q} \in \mathbb{Q}$ is irreducible.

Let $\mathcal{F}$ be an oriented foliation on M_0 such that z_0 is a sink (resp. source). Then there exists a lift $\widetilde{\mathcal{F}}$ of $\mathcal{F}|_{M_0\setminus\{z_0\}}$ to $\widetilde{M}$, and $\star$ is a sink (resp. source) of $\widetilde{\mathcal{F}}$. Denote by W the attracting (resp. repelling) basin of z_0 for $\mathcal{F}$, and by $\widetilde{W}$ the attracting (resp. repelling) basin of $\star$ for $\widetilde{\mathcal{F}}$. Write $\dot{W} = W \setminus \{z_0\}$, $\widetilde{\dot{W}} = \widetilde{W} \setminus \{\star\}$. Let $\widetilde{z}_1 \in \widetilde{\dot{W}}$ be a point sufficient close to $\star$. Then $(\pi|_{\widetilde{\dot{W}}})_*\pi_1(\widetilde{\dot{W}}, \widetilde{z}_1) = \pi_1(\dot{W}, \pi(z_1))$, and hence $\pi|_{\widetilde{\dot{W}}}$ is a homeomorphism between $\widetilde{\dot{W}}$ and $\dot{W}$ by Corollary 2.7 in [Spa66], and can be extended continuously to a homeomorphism between $\widetilde{W}$ and W.

2.14 Extend lifts of a homeomorphism to the boundary

In this section, let M be a plane, f be an orientation preserving homeomorphism of M, and X be a invariant, discrete, closed subset of M with at least 2 points.

We consider the Poincaré's disk model for the hyperbolic plane H, in which model, H is identified with the interior of the unit disk and the geodesics are segments of Euclidean circles and straight lines that meet the boundary perpendicularly. A choice of hyperbolic structure on $M \setminus X$ provides an identification of the universal cover of $M \setminus X$ with H. A detailed

description of the hyperbolic structures can be found in [CB88]. The compactification of the interior of the unit disk by the unit circle induces a compactification of H by the circle S_∞. Let $\pi : H \to M \setminus X$ be the universal cover. Then, $f|_{M\setminus X}$ can be lifted to homeomorphisms of H. Moreover, we have the following result:

Proposition 2.19. *[Han99] Each lift $\widehat{f}$ of $f|_{M\setminus X}$ extends uniquely to a homeomorphism of $H \cup S_\infty$.*

Remark 2.20. When X has infinitely many points, Michael Handel gave a proof in Section 3 of [Han99]; when X has finitely many points, the situation is easier and Handel's proof still works.

In particular, suppose that z_0 is an isolated point in X and is a fixed point of f. Let γ be a sufficiently small circle near z_0 whose lifts to H are horocycles. Fix one lift $\widehat{\gamma}$ of γ. Denote by P the end point of $\widehat{\gamma}$ in S_∞. Fix $z_1 \in \gamma$ and a lift $\widehat{z}_1$ of z_1 in $\widehat{\gamma}$. Let Γ be the group of parabolic covering translations that fix $\widehat{\gamma}$, and T be the parabolic covering translations that generates Γ. Then, π descends to a annulus cover $\pi' : (H/\Gamma, \widetilde{z}_1) \to (M \setminus X, z_1)$, where $\widetilde{z}_1 = \{T^n(\widehat{z}_1) : n \in \mathbb{Z}\}$. Also, $\widehat{z} \mapsto \{T^n(\widehat{z}) : n \in \mathbb{Z}\}$ defines a universal cover $\pi'' : H \to H/\Gamma$.

Let V be the disk containing z_0 and bounded by γ, $\widehat{V}$ be the disk bounded by $\widehat{\gamma}$ which is a component of $\pi^{-1}(V \setminus \{z_0\})$. We know that $\pi''(\widehat{V})$ is an annulus with $\pi''(\widehat{\gamma})$ as one of its boundary. We add a point $\star$ at the other end, and get a disk $\widetilde{V} = \pi''(\widehat{V}) \cup \{\star\}$. As in the previous section, $\pi'|_{\pi''(\widehat{V})}$ extends continuously to a homeomorphism between $\widetilde{V}$ and V, and f can be lifted to a homeomorphism $\widetilde{f}$ of $H/\Gamma \cup \{\star\}$ fixing $\star$. Let $\widehat{f}$ be a lift of $\widetilde{f}|_{H/\Gamma}$ to H, it is also a lift of $f|_{M\setminus X}$ and satisfies $\widehat{f} \circ T = T \circ \widehat{f}$. Moreover, both $\widehat{f}$ and T extend continuously to homeomorphisms of $H \cup S_\infty$ fixing P. We denote still by $\widehat{f}$ and T the two extensions respectively. The formula $\widehat{f} \circ T = T \circ \widehat{f}$ is still satisfied. So, $\widehat{f}|_{H\cup S_\infty\setminus\{P\}}$ descends to a homeomorphism of $(H \cup S_\infty \setminus \{P\})/\Gamma$. Because $(H \cup S_\infty \setminus \{P\})/\Gamma$ is homeomorphic to a compactification of $H/\Gamma \cup \{\star\}$ by adding a circle at infinity S_∞, one knows that $\widetilde{f}$ extends continuously to a homeomorphism of $H/\Gamma \cup \{\star\} \cup S_\infty$.

3 Proof of the main theorem

Let M be an oriented surface, $f : M \to M$ be an area preserving homeomorphism of M isotopic to the identity, and z_0 be an isolated fixed point of f such that $i(f, z_0) = 1$. Let I be an identity isotopy of f fixing z_0 such that its rotation set, which was defined in section 2.10, is reduced to an integer k.

We say that the property **P)** holds for (f, I, z_0) if there exists $\varepsilon > 0$, such that either for all irreducible $p/q \in (k, k+\varepsilon)$, or for all irreducible $p/q \in (k-\varepsilon, k)$, there exists a contractible

periodic orbit $O_{p/q}$ of type (p,q) associated to I at z_0, such that $\mu_{O_{p/q}} \to \delta_{z_0}$ as $p/q \to k$, in the weak-star topology, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$,

Our aim of this section is to prove the following result:

Theorem 3.1 (Theorem 1.1)**.** *Under the previous assumptions, if one of the following situations occurs,*

a) M is a plane, f has only one fixed point z_0, and has another periodic orbit besides z_0;

b) the total area of M is finite,

*then the property **P)** holds for (f, I, z_0).*

Remark 3.2. Let I' be a local isotopy of f at z_0 such that $\rho_s(I', z_0)$ is reduced to 0. Since f is area preserving and $i(f, z_0) = 1$, by Proposition 2.9, I' has either a positive or a negative rotation type. Let $\mathcal{F}'$ be a locally transverse foliation of I'. If I' has a positive rotation type, then z_0 is a sink of $\mathcal{F}'$ and the interval in Property **P)** is $(k, k+\varepsilon)$; if I' has a negative rotation type, then z_0 is a source and the interval in Property **P)** is $(k-\varepsilon, k)$.

We suppose that I' has a positive rotation type in this section, the other case can be treated similarly.

Remark 3.3. If z_0 is not accumulated by periodic orbits, since the rotation set is reduced to an integer and $i(f, z_0) = 1$, z_0 is an indifferent fixed point by Proposition 2.14. Then, by the assertion v) of Proposition 2.12, one deduces that $\rho(I, z_0)$ is equal to this integer.

We will prove the theorem in several cases.

3.1 The case where M is a plane

In this section, we suppose that M is a plane, and that I is a maximal identity isotopy of f such that $\mathrm{Fix}(I)$ is reduced to z_0. We will prove the following result in this section and get the proof of the first part of Theorem 1.1 as a corollary.

Theorem 3.4. *Under the previous assumption, if $\rho_s(I, z_0)$ is reduced to 0, and if f has another periodic orbit besides z_0, then the property **P)** holds for (f, I, z_0).*

This result is an important one in the proof of Theorem 1.1. In the latter cases, we will always reduce the problem to this case and get the result as a corollary. Before proving this result, we first prove the first case of Theorem 1.1 as a corollary.

Proof of the first case of Theorem 1.1. We only need to deal with the case where $\rho_s(I, z_0)$ is reduced to a non-zero integer k. Let J be an identity isotopy of the identity fixing z_0 such that the blow-up rotation number $\rho(J, z_0)$ is equal to 1. Write $I' = J^{-k}I$. By the first assertion of Proposition 2.12, $\rho_s(I', z_0)$ is reduced to 0. Since f has exactly one fixed point, I' is maximal and the property **P)** holds for (f, I', z_0). A periodic orbit in the annulus $M \setminus \{z_0\}$ with rotation number p/q associated to I' is a periodic orbit with rotation number $k + p/q$ associated to I. Therefore, the property **P)** holds for (f, I, z_0). □

Now we begin the proof of Theorem 3.4 by some lemmas.

Lemma 3.5. *Let g be a homeomorphism of $\mathbb{R}^2$, I' be a maximal identity isotopy of g that fixes z_0, and $\mathcal{F}'$ be a transverse foliation of I'. Suppose that z_0 is an isolated fixed point of g and a sink of $\mathcal{F}'$. Let W' be the attacting basin of z_0 for $\mathcal{F}'$. Suppose that either W' is equal*

to $\mathbb{R}^2$ or W' is a proper subset of $\mathbb{R}^2$ whose boundary is the union of some proper leaves of $\mathcal{F}'$. Let U be a Jordan domain containing z_0 that satisfies $U \subset W'$ and $g(U) \subset W'$.

If there exist a compact subset $K \subset U$ and $\varepsilon > 0$ such that K contains a q-periodic orbit $O_{p/q}$ with rotation number p/q in the annulus $\mathbb{R}^2 \setminus \{z_0\}$ for all irreducible $p/q \in (0, \varepsilon)$, then $\mu_{O_{p/q}}$ converges, in the weak-star topology, to the Dirac measure δ_{z_0} as $p/q \to 0$, where $\mu_{O_{p/q}}$ is the invariant probability measure supported on $O_{p/q}$.

Proof. We only need to prove that for every continuous function $\varphi : W' \to \mathbb{R}$, for every $\eta > 0$, there exists $\delta > 0$, such that for every $q-$periodic orbit $O \subset K$ with irreducible rotation number $p/q < \delta$, we have

$$|\int \varphi d\mu_O - \varphi(z_0)| < \eta,$$

where μ_O is the invariant probability measure supported on O.

Let V be a neighborhood of z_0 such that $|\varphi(z) - \varphi(z_0)| < \eta/2$ for all $z \in V$. Let $\pi : \mathbb{R} \times (-\infty, 0) \to W' \setminus \{z_0\}$ be the universal cover which sends the vertical lines upward to the leaves of $\mathcal{F}'$, and $p_1 : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ be the projection onto the first coordinate. Let $\widetilde{U} = \pi^{-1}(U \setminus \{z_0\})$, $\widetilde{K} = \pi^{-1}(K \setminus \{z_0\})$ and $\widetilde{g}$ be the lift of g to $\widetilde{U}$ associated to I'. By the assumptions about W', we know that any arc that is positively transverse to $\mathcal{F}'$ cannot come back into W' once it leaves W'. So

$$p_1(\widetilde{g}(z)) - p_1(z) > 0, \text{ for all } z \in \widetilde{K}.$$

Therefore there exists $\eta_1 > 0$ such that for all $z \in \pi^{-1}(K \setminus V)$, one has

$$p_1(\widetilde{g}(z)) - p_1(z) > \eta_1.$$

One deduces that for all $q-$periodic orbit $O \subset K$ with irreducible rotation number $p/q < \delta = \frac{\eta\eta_1}{4|\sup_K \varphi|}$,

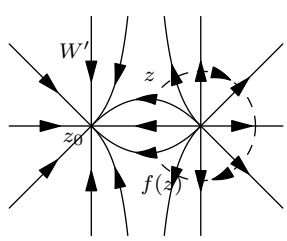
$$\frac{\#(O \setminus V)\eta_1}{q} < \frac{p}{q},$$

hence,

$$|\int \varphi d\mu_O - \varphi(z_0)| < \eta/2 + 2\sup_K |\varphi| \frac{\#(O \setminus V)}{q} < \eta.$$

□

Remark 3.6. In this lemma, the homeomorphism g do not need to be area preserving. The assumptions about W' prohibit the following bad situation:



Lemma 3.7. *If $\rho_s(I, z_0)$ is reduced to 0, and if f can be blown-up at ∞ such that the blow-up rotation number at ∞, that is defined in Section 2.10, is different from 0, then the property **P)** holds for (f, I, z_0).*

In order to to prove this lemma, we need the following sublemma:

Sublemma 3.8. *Under the conditions of the previous Lemma, $\rho(I,\infty)$ is negative, and there exists $\varepsilon > 0$ such that for all irreducible $p/q \in (0,\varepsilon)$, there exists a q-periodic orbit with rotation number p/q in the annulus $M \setminus \{z_0\}$.*

Proof. We consider a transverse foliation $\mathcal{F}$ of I. It has exactly two singularities z_0 and ∞. Since f is area preserving, $f|_{M\setminus\{z_0\}}$ satisfies the intersection property, and using the remark that follows Proposition 2.7, one can deduce that $\mathcal{F}$ does not have any closed leaf. Because f can be blown-up at ∞ and the blow-up rotation number $\rho(I,\infty)$ is different from 0, we deduce that ∞ is either a sink or a source. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}$, so ∞ is a source of $\mathcal{F}$, and hence $\rho(I,\infty)$ is smaller than 0. Write $\rho = -\rho(I,\infty)$. We denote by S_∞ the circle added at ∞ when blowing-up f at ∞, and still by f the extension of f to $M \sqcup S_\infty$. One has to consider the following two cases:

- Suppose that z_0 is accumulated by periodic orbits. Let z_1 be a periodic point of f in the annulus $M \setminus \{z_0\}$. Its rotation number is strictly positive. We denote by ε this number. Because the rotation set $\rho_s(I,z_0)$ is equal to 0, the rotation number of a periodic orbit tends to 0 as the periodic orbit tend to z_0. Hence for all irreducible $p/q \in (0,\varepsilon)$, there exists a periodic orbit near z_0 with rotation number $r/s \in (0,p/q)$. The restriction of the homeomorphism f to the annulus $M \setminus \{z_0\}$ satisfies the intersection property, then by Proposition 2.16, there exists a q-periodic orbit with rotation number p/q in the annulus for all irreducible $p/q \in (0,\varepsilon)$.

- Suppose that z_0 is not accumulated by periodic orbits. Then, z_0 is an indifferent fixed point by Proposition 2.14, and $\rho(I,z_0)$, which was defined in Section 2.9, is equal to 0. Let K_0 be a small enough invariant continuum at z_0 such that $\rho(I,K_0) = 0$ (see Section 2.9). We denote by $(M \setminus K_0) \sqcup \mathbb{T}^1 \sqcup S_\infty$ the prime-ends compactification at the ends K_0 and the compactification at ∞, which is an annulus. We can extend f to both boundaries and get a homeomorphism of the closed annulus satisfying the intersection condition. Moreover, the rotation number of f on the upper boundary $\mathbb{T}^1$ is equal to 0, and on the lower boundary S_∞ is equal to ρ. So, by Proposition 2.15, for all irreducible p/q between 0 and ρ, there exists a periodic orbit in the annulus with rotation number p/q.

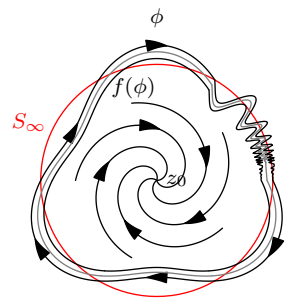
□

Remark 3.9. In the first case of the proof, it is natural to think that we can prove by a generalization of Poincaré-Birkhoff theorem that there exists a periodic orbit in the annulus with rotation number p/q for all irreducible p/q between 0 and ρ. But in fact, the annulus in this case is half-open, and we do not know whether there exists such a generalization of Poincaré-Birkhoff theorem. So, we choose another periodic orbit to avoid treating the half-open annulus.

Proof of Lemma 3.7. Paste two copies of the closed disk by S_∞. We get a sphere S and a homeomorphism f' that equals to f on each copy and has two fixed points z_0 and $\sigma(z_0)$, where σ is the natural involution. Let I' be an identity isotopy that fixes z_0 and $\sigma(z_0)$ and satisfies $\rho_s(I',z_0) = \{0\}$. Because I is a maximal isotopy, $f|_{M\setminus\{z_0\}}$ has no contractible fixed point associated to I. Because the blow-up rotation number $\rho(I,\infty)$ is different from 0, the extension of f to S_∞ does not have any fixed point with rotation number 0 (associated to I).

So, $f'|_{S\setminus\{z_0,\sigma(z_0)\}}$ has no contractible fixed point associated to $I'|_{S\setminus\{z_0,\sigma(z_0)\}}$. Therefore, I' is a maximal isotopy, and one knows $\mathrm{Fix}(I') = \{z_0, \sigma(z_0)\}$. Let $\mathcal{F}'$ be a transverse foliation of I'. Then $\mathcal{F}'$ has exactly two singularities z_0 and $\sigma(z_0)$. By the assumption, z_0 is a sink of $\mathcal{F}'$. Since the involution σ is orientation reversing, $\rho(I', \sigma(z_0)) = 0$ and I' has a negative rotation type at $\sigma(z_0)$. So $\sigma(z_0)$ is a source of $\mathcal{F}'$, and hence $\mathcal{F}'$ does not have any petal. One has to consider the following two cases:

- Suppose that all the leaves of $\mathcal{F}'$ are curves from $\sigma(z_0)$ to z_0. The compact set $M \sqcup S_\infty$ satisfies the conditions of Lemma 3.5, and we can deduce the result.



- Suppose that there exists a closed leaf in $\mathcal{F}'$. Since f is area preserving, similarly to the remark that follows Proposition 2.7, one can deduce that there does not exist any closed leaf in M or in $\sigma(M)$. So, each closed leaf intersects S_∞. Let W' be the basin of z_0 for $\mathcal{F}'$. Then $\partial W'$ is a closed leaf, and hence intersects S_∞. Denote this leaf by ϕ. We suppose that z_0 is to the right of ϕ, the other case can be treated similarly. Denote by $R(\phi)$ (resp. $L(\phi)$) the component of $S \setminus \phi$ to the right (resp. left) of ϕ. Since $f'(\phi)$ is included in $R(\phi)$, we know that both $R(\phi) \cap M$ and $R(\phi) \setminus (M \cup S_\infty)$ are not empty. Choose a suitable essential curve $\Gamma \subset (R(\phi) \cap L(f(\phi))) \subset W'$ that transversely intersects S_∞ at only finitely many points (see the gray curve between ϕ and $f(\phi)$ in the picture). Then, $(L(\Gamma) \cap M)$ has finitely many connected components, and so does $(L(\Gamma) \cap (M \cup S_\infty))$. Moreover, each component of $(L(\Gamma) \cap (M \cup S_\infty))$ contains a segment of S_∞.

  Since both M and S_∞ are invariant by f', one knows that $f'^{-1}(L(\Gamma) \cap (M \cup S_\infty))$ is included in $L(\Gamma) \cap (M \cup S_\infty)$. So, if V is a component of $(L(\Gamma) \cap (M \cup S_\infty))$, there exists $n > 0$ such that $f'^{-n}(V) \subset V$. Moreover, one knows that $f'^{-n}(V \cap S_\infty) \subset V \cap S_\infty$ and that the rotation number of each point in S_∞ is equal to ρ, so there exists $m > 0$ such that $\rho = m/n$ and the rotation number of every periodic point of f' in V is equal to ρ. Therefore, the rotation number of every periodic point $z \in (L(\Gamma) \cap M)$ of f' is equal to the rotation number of S_∞. So, all the periodic orbits in the annulus $M \setminus \{z_0\}$ with rotation number in $(0, \rho)$ is contained in $R(\Gamma) \cap M$. We find a compact set $\overline{R(\Gamma) \cap M}$ that satisfies the conditions of Lemma 3.5, and can deduce the result.

□

Lemma 3.10. *If $\rho_s(I, z_0)$ is reduced to 0, and if $O \subset M \setminus \{z_0\}$ is a periodic orbit of f, then the rotation number of O (associated to I) is positive.*

Proof. Let $\mathcal{F}$ be a transverse foliation of I. Then $\mathcal{F}$ has only one singularity z_0, and z_0 is a sink of $\mathcal{F}$ by the assumption in Remark 3.2. Since f is area preserving, by the remark

that follows 2.7 one knows that $\mathcal{F}$ does not have any closed leaf. Let W be the attracting basin of z_0 for $\mathcal{F}$. It is either M or a proper subset of M whose boundary is the union of some proper leaves. In the first case, any periodic orbit of $f|_{M\setminus\{z_0\}}$ has a positive rotation number associated to I, and the proof is finished. In the second case, note that each connected component of $M \setminus \overline{W}$ is a disk foliated by proper leaves, and hence does not contain any loop that is transverse to $\mathcal{F}$. Moreover, any loop transverse to $\mathcal{F}$ can not meet a boundary leaf of W, and hence is contained in W. One deduces that every periodic orbit of f distinct from $\{z_0\}$ is contained in W, and its trajectory along the isotopy is homotopic to a transverse loop in W. So, its rotation number is positive. □

Lemma 3.11. *If $\rho_s(I, z_0)$ is reduced to 0, and if f has another periodic orbit besides z_0, then there exist an interger $q \geq 1$ and a q-periodic orbit O with rotation number $1/q$ (associated to I) such that $f|_{M\setminus O}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying $R^q_{1/q} = \mathrm{Id}$.*

Proof. Let O_0 be a periodic orbit of f distinct from $\{z_0\}$. By the previous lemma, the rotation number ρ of O_0 in the annulus $M \setminus \{z_0\}$ associated to I is positive. Similarly to the proof of Sublemma 3.8, there exists a q-periodic orbit with rotation number p/q in the annulus $M \setminus \{z_0\}$ for all irreducible $p/q \in (0, \rho)$. Let $\mathcal{F}$ be a transverse foliation of I. One knows by the assumption in Remark 3.2 that z_0 is a sink of $\mathcal{F}$. Let W be the attracting basin of z_0 for $\mathcal{F}$. One has to consider the following three cases:

i) Suppose that W is equal to M.

Let $T : (x, y) \mapsto (x + 1, y)$ be the translation of $\mathbb{R}^2$. It induces a universal covering map $\pi : \mathbb{R}^2 \to \mathbb{R}^2/T \simeq \mathbb{T}^1 \times \mathbb{R}$. Let $h : M \setminus \{z_0\} \to \mathbb{T}^1 \times \mathbb{R}$ be an orientation preserving map that maps the leaves of $\mathcal{F}$ to the vertical lines $\{\pi(\{x\} \times \mathbb{R}) : x \in \mathbb{R}\}$ upward. Write $I' = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$, and $f' = h \circ f \circ h^{-1}$. We will prove that there exists a positive integer q, and a q-periodic orbit O of f' with rotation number $1/q$ (associated to I') such that $f'|_{(\mathbb{T}^1\times\mathbb{R})\setminus O}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying that $R^q_{1/q} = \mathrm{Id}$, and hence $h^{-1}(O)$ is a q-periodic orbit of f with rotation number $1/q$ (associated to I) such that $f|_{M\setminus h^{-1}(O)}$ is isotopic to $h^{-1} \circ R_{1/q} \circ h$.

Fix a q-periodic orbit O of f' with rotation number $1/q$ in the annulus $M \setminus \{z_0\}$ for $1/q \in (0, \rho)$. Choose $0 < M_1 < M_2$ such that

$$O \subset \mathbb{T}^1 \times (-M_1, M_1), \quad \text{and } \Big(\bigcup_{t\in[0,1]} f'_t(\mathbb{T}^1 \times [-M_1, M_1])\Big) \subset \mathbb{T}^1 \times (-M_2, M_2).$$

Let $\widetilde{f}$ be the lift of f' associated to I'. One knows that

$$p_1(\widetilde{f}(\widetilde{z})) - p_1(\widetilde{z}) > 0 \quad \text{for all } \widetilde{z} \in \mathbb{R}^2,$$

where p_1 is the projection to the first factor. Let φ_1 be the homeomorphism of $\mathbb{T}^1 \times \mathbb{R}$ whose lift to $\mathbb{R}^2$ is defined by

$$\widetilde{\varphi}_1(x, y) = \begin{cases} (x, y), & \text{for } |y| \leq M_2, \\ (x + |y| - M_2, y), & \text{for } |y| > M_2. \end{cases}$$

We know that $\eta(y) = \sup_{x\in\mathbb{R}} |p_2(\widetilde{f}'(x, y)) - y|$ is a continuous function, where p_2 is the projection onto the second factor. So, there exist $M_3 > M_2$ and a homeomorphism φ_2 of $\mathbb{T}^1 \times \mathbb{R}$ whose lift $\widetilde{\varphi}_2$ to $\mathbb{R}^2$ satisfies $p_1 \circ \widetilde{\varphi}_2 = \mathrm{Id}$ and

$$\widetilde{\varphi}_2(x, y) = \begin{cases} (x, y), & \text{for } |y| \leq M_2, \\ (x, y + \mathrm{sign}(y)(\eta(y) + 1)), & \text{for } |y| \geq M_3. \end{cases}$$

Let $f'' = \varphi_2 \circ \varphi_1 \circ f'$. It is a contraction near each end and hence can be blown-up at each end by adding a circle. Moreover, by choosing suitable blow-up, the rotation numbers at the boundary can be any real number, and we get a homeomorphism $\overline{f''}$ of closed annulus and a lift $\widetilde{f''}$ of $\overline{f''}$ such that O is a $(1,q)$-periodic orbit and $\rho(\widetilde{f''})$ (see Section 2.12 for the definition) is a closed interval in $(0,\infty)$. One deduces by Proposition 2.18 that O is topologically monotone (Otherwise $I(1/q) = [0, 1/(q-1)] \subset \rho(\widetilde{f''})$). Therefore, $f''|_{(\mathbb{T}^1\times\mathbb{R})\setminus O}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying $R_{1/q}^q = \mathrm{Id}$, and so is $f'|_{(\mathbb{T}^1\times\mathbb{R})\setminus O}$. The lemma is proved.

ii) Suppose that W is a proper subset of M whose boundary is the union of some proper leaves, and that z_0 is not accumulated by periodic orbits.

In this case, one knows by Remark 2.13 that f can be blown-up at ∞, and that the blow-up rotation number $\rho(I,\infty)$ is equal to 0. One knows by Remark 3.3 that z_0 is an non-accumulated indifferent point, and that $\rho(I, z_0)$ is equal to 0.

Recall that there exists a q-periodic orbit with rotation number p/q in the annulus $M \setminus \{z_0\}$ for all irreducible $p/q \in (0,\rho)$. We fix a q-periodic orbit O of f with rotation number $1/q$ in the annulus $M\setminus\{z_0\}$ for $1/q \in (0,\rho)$. Let γ_1 be a simple closed curve that separates O and z_0. Denote by U^- the component of $M\setminus\gamma_1$ containing O. We deduce by the assertions i) and ii) of Proposition 2.12 that there exists a neighborhood of z_0 that does not contain any q-periodic point of f with rotation number $1/q$. So, by choosing γ_1 sufficiently close to z_0, we can suppose that all the q-periodic points of f with rotation number $1/q$ are contained in U^-. Let γ_2 be a simple closed curve that separate γ_1 and z_0 such that $\cup_{t\in[0,1]} f_t(\overline{U^-})$ is in the component of $M \setminus \gamma_2$ containing γ_1. Denote by U the component of $M\setminus\gamma_2$ containing z_0. Let $V \subset U$ be a small Jordan domain containing z_0 such that $\cup_{t\in[0,1]} f_t(\overline{V}) \subset U$, and $K \subset V$ be a sufficiently small invariant continuum at z_0 such that $\rho(I,K) = 0$. Let $M \setminus K \cup S_\infty \cup S^1$ be a compactification of $M \setminus K$, where S_∞ is the circle added when blowing f at ∞ and S^1 is the circle added when blowing $f|_{M\setminus K}$ at the end K. It is a closed annulus, and $f|_{M\setminus K}$ extends continuously to a homeomorphism $\overline{f}$ of $M \setminus K \cup S_\infty \cup S^1$. The homeomorphism $\overline{f}$ has a $(1,q)$ periodic orbit O and hence by Proposition 2.18 has a $(1,q)$ topologically monotone periodic orbit O' (It could be equal to O or different from O). Since the rotation number of $\overline{f}$ at both boundary is equal to 0, O' is included in $M \setminus K$ and hence in U^-. So, $f|_{M\setminus K}$ is isotopic to a homeomorphism $R_{1/q}$ satisfying that $R_{1/q}^q = \mathrm{Id}$.

Let $h : M \setminus K \to M \setminus \{z_0\}$ be a homeomorphism whose restriction to $M \setminus U$ is equal to the identity. Then, $f' = h \circ f|_{M\setminus K} \circ h^{-1}$ is a homeomorphism of $M \setminus \{z_0\}$ which coincides f in $M \setminus U$. The restriction of $f' \circ f^{-1}$ to $M \setminus U$ is equal to the identity, using Alexander's trick one deduces that $f' \circ f^{-1}|_{M\setminus O'}$ is isotopic to the identity. So, $f'|_{M\setminus O'}$ and $f|_{M\setminus O'}$ are isotopic. Therefore, $f|_{M\setminus O'}$ is isotopic to $R'_{1/q} = h|_{M\setminus O'} \circ R_{1/q} \circ h^{-1}|_{M\setminus O'}$ which satisfies of course $R'^q_{1/q} = \mathrm{Id}$.

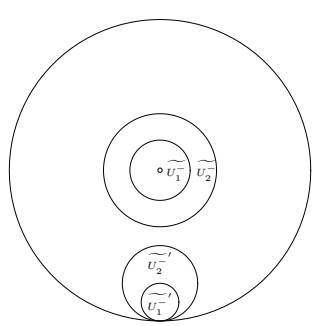
iii) Suppose that W is a proper subset of M whose boundary is the union of some proper leaves, and that z_0 is accumulated by periodic orbits.

As in case ii), f can be blown-up at ∞ and the blow-up rotation number $\rho(I,\infty)$ is equal to 0. Recall that there exists a q-periodic orbit with rotation number p/q in the annulus $M \setminus \{z_0\}$ for all irreducible $p/q \in (0,\rho)$. Fix two prime integers q_1 and q_2 such that $1/q_2 < 1/q_1 < \rho$. Choose a q_1-periodic orbit O_1 and a q_2-periodic orbit O_2 in $M \setminus \{z_0\}$ with rotation number (associated to I) $1/q_1$ and $1/q_2$ respectively. Recall that

the rotation number of every periodic orbit in $M \setminus \{z_0\}$ is positive and that $\rho_s(I, z_0)$ is reduced to 0. One deduces by the assertion i) and ii) that for any given integer $q > 1$, there is a neighborhood of z_0 that does not contain any q-periodic point of f. So, there exists a Jordan domain U containing z_0 that does not contain any periodic point of f with period not bigger that q_2 except z_0. Let $\gamma_1 \subset U$ be a simple closed curve that separates z_0 and $O_1 \cup O_2$. Denote by U_1^- the component of $M \setminus \gamma_1$ containing $O_1 \cup O_2$. Let γ_2 be a simple closed curve that separates γ_1 and z_0 such that the trajectory of each $z \in \overline{U_1^-}$ along I^{q_2} is in the component U_2^- of $M \setminus \gamma_2$ containing γ_1. Let γ_3 be a simple closed curve that separates γ_2 and z_0 such that the trajectory of each $z \in \overline{U_2^-}$ along I^{q_2} is in the component U_3^- of $M \setminus \gamma_3$ containing γ_2. Since $\gamma_3 \subset U$, there does not exist any periodic points of f with periodic not bigger that q_2 in γ_3. We can perturb f in $M \setminus (\overline{U_3^-} \cup \{z_0\})$ and get a homeomorphism f' such that f' has finitely many periodic points with periods not bigger than q_2 in $M \setminus U_3^-$.

Let X be the union of periodic orbits of f' with periodic not bigger than q_2 that intersects $M \setminus \overline{U_3^-}$. It is a finite set containing z_0. We consider the annulus covering $\pi : \widetilde{M} \to M \setminus X$ such that the restriction of π to a sufficiently small annulus near one end is a homeomorphism between this annulus and a small annulus near ∞ in $M \setminus X$. As in Section 2.13, we add a point $\star$ at this end of $\widetilde{M}$. Let $\widetilde{U_i^-}$ be the component of $\pi^{-1}(U_i^-)$ that has an end $\star$ and $\widetilde{O}_i'$ be the lift of O_i' in $\widetilde{U_2^-}$ for $i = 1, 2$. Let $\widetilde{f}'$ be the lift of $f'|_{M \setminus X}$. It extends continuously to a homeomorphism of $\widetilde{M} \cup \{\star\}$, and the dynamics of $\widetilde{f}'$ near $\star$ is conjugate to the dynamics of f' near ∞. So, $\widetilde{f}'$ can be blown-up at $\star$, and by choosing a suitable isotopy $\widetilde{I}'$ of $\widetilde{f}'$, the blow-up rotation number $\rho(\widetilde{I}', \star)$ is equal to 0. Moreover, $\widetilde{O}_i'$ is a q_i-periodic orbit of $\widetilde{f}'$ with rotation number $1/q_i$ (associated to $\widetilde{I}'$), for $i = 1, 2$. Referring to Section 2.14, one knows that $\widetilde{f}'$ can be blown-up at the other end.

We blow-up $\widetilde{f}$ at both ends and get a homeomorphism $\overline{\widetilde{f}'}$ of a closed annulus. For $i = 1, 2$, the homeomorphism $\overline{\widetilde{f}'}$ has a $(1, q_i)$ periodic orbit, so one can deduce by Proposition 2.18 that $\overline{\widetilde{f}'}$ has a $(1, q_i)$ topologically monotone periodic orbit $\widetilde{O}_i''$. The circle we added at $\star$ does not contain any periodic points with rotation number different from 0, so it does not contain $\widetilde{O}_1''$ or $\widetilde{O}_2''$. The rotation number of $\overline{\widetilde{f}'}$ at the circle we added at the other end is different from $1/q_1$ or $1/q_2$. Suppose that it is different from $1/q_1$, the other case can be treated similarly. Then, $\widetilde{O}_1''$ is included in $\widetilde{M}$, and $\pi(\widetilde{O}_1'')$ is a periodic orbit of f' of period not bigger than q_1. So, $\pi(\widetilde{O}_1'')$ is included in $\overline{U_3^-}$, and hence is a periodic orbit of f in U_1^-.



We will prove by contradiction that $\widetilde{O}_1''$ is included in $\widetilde{U_1^-}$. Otherwise, suppose that

there exists $\widetilde{z} \in \widetilde{O}_1''$ in another component $\widetilde{U_1^-}'$ of $\pi^{-1}(U_1^-)$. Then, $\widetilde{z}$ is a fixed point of $\widetilde{f}^{q_1}$. Let $\widetilde{U_2^-}'$ be the component of $\pi^{-1}(U_2^-)$ containing $\widetilde{U_1^-}'$. Since $f'^{q_1}(U_1^-) \subset U_2^-$, one deduces that $\widetilde{f}'^{q_1}(\widetilde{U_1^-}') \subset \widetilde{U_2^-}'$. Recall that the rotation number of $\widetilde{O}_1''$ is $1/q_1$. So, the rotation number of $\widetilde{f}'$ at the outer boundary is $1/q_1$, which contradicts our assumption.

Let $h : \widetilde{M} \to M \setminus \{z_0\}$ be a homeomorphism whose restriction to $\widetilde{U_2^-}$ is equal to π. As in the end of case ii), we deduce that $f|_{M\setminus\pi(\widetilde{O}_1'')}$ is isotopic to a homeomorphism R_{1/q_1} satisfying that $R_{1/q_1}^{q_1} = \mathrm{Id}$. The lemma is proved.

□

Proof of Theorem 3.4. By the previous lemma, there exists a q'-periodic orbit O with rotation number $1/q' > 0$ (associated to I) such that $f|_{M\setminus O}$ is isotopic to a homeomorphism $R_{1/q'}$ satisfying $R_{1/q'}^{q'} = \mathrm{Id}$. Let $I' = (\varphi_t)_{t\in[0,1]}$ be an identity isotopy of $f^{q'}$ that fixes every point in $O \cup \{z_0\}$. Since the rotation number of O associated to I is $1/q'$, each point in O is a fixed point of $f^{q'}$ and its rotation number associated to $I^{q'}$ is 1. Because I' fixes $O \cup \{z_0\}$, $I'|_{M\setminus\{z_0\}}$ is homotopic to $J_{z_0}^{-1} I^{q'}|_{M\setminus\{z_0\}}$, where J_{z_0} is an identity isotopy of the identity fixing z_0 such that $\rho(J_{z_0}, z_0) = 1$. By the first assertion of Proposition 2.12, one knows that $\rho_s(I', z_0)$ is reduced to -1.

Let $\pi' : \widehat{M} \to M \setminus O$ be the universal cover. Since $M \setminus O$ is a surface of finite type, we can endow it a hyperbolic structure, and $\widehat{M}$ can be viewed to be the hyperbolic plane. Fix $\widehat{z}_0 \in \pi'^{-1}(z_0)$. Let $\widehat{f}$ be the lift of $f|_{M\setminus O}$ that fixes $\widehat{z}_0$. Then, $\widehat{f}$ can be blown-up at ∞.

Let $\widehat{I}' = (\widehat{\varphi}_t)_{t\in[0,1]}$ be the identity isotopy of $\widehat{f}^{q'}$ that lifts I'. Then, $\rho_s(\widehat{I}', \widehat{z}_0)$ is reduced to -1. On the other hand, ∞ is accumulated by the points of $\pi'^{-1}\{z_0\}$ which are fixed points of $\widehat{I}'$, so by the assertion ii) of Proposition 2.12, one knows that 0 is belong to $\rho_s(\widehat{I}', \infty)$. But $\widehat{f}$ can be blown-up at ∞, by the assertion iv) of Proposition 2.12, we know that $\rho_s(\widehat{I}', \infty)$ is reduced to 0.

Let $\widehat{I}_0$ be an identity isotopy of $\widehat{f}$ that fixes $\widehat{z}_0$ and satisfies $\rho_s(\widehat{I}_0, \widehat{z}_0) = \{0\}$. Then $\rho_s(\widehat{I}_0^{q'}, \widehat{z}_0)$ is reduced to 0, and hence $\widehat{I}_0^{q'}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ is homotopic to $J_{\widehat{z}_0}\widehat{I}'|_{\widehat{M}\setminus\{\widehat{z}_0\}}$. So, $\rho_s(\widehat{I}_0^{q'}, \infty)$ is reduced to -1, and by the assertion i) of Proposition 2.12, we deduce that $\rho_s(\widehat{I}_0, \infty)$ is reduced to $-1/q'$. Since $\widehat{f}$ can be blown-up at ∞, by the assertion iv) of Proposition 2.12, one knows that the blow-up rotation number $\rho(\widehat{I}_0, \infty)$ is equal to $-1/q'$.

Every $\widehat{z}_0' \in \pi'^{-1}\{z_0\}\setminus\{\widehat{z}_0\}$ is a contractible fixed point of $\widehat{f}^{q'}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ associated to $\widehat{I}'|_{\widehat{M}\setminus\{\widehat{z}_0\}}$, so it is not a contractible fixed point of $\widehat{f}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ associated to $\widehat{I}_0|_{\widehat{M}\setminus\{\widehat{z}_0\}}$.

Let $\widehat{O}'$ be a periodic orbit of $\widehat{f}$ in the annulus $\widehat{M} \setminus \{\widehat{z}_0\}$ such that $z_0 \notin \pi'(\widehat{O}')$ and the rotation number of $\widehat{O}'$ associated to $\widehat{I}_0|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ is p/q. Then $\widehat{O}'$ is a periodic orbit of $\widehat{f}^{q'}$ in the annulus $\widehat{M} \setminus \{\widehat{z}_0\}$ and the rotation number associated to $\widehat{I}'|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ is $\frac{pq'}{q} - 1$. So, $\pi'(\widehat{O}')$ is a periodic orbit of f in the annulus $M \setminus \{z_0\}$, the rotation number associated to I' is $\frac{pq'}{q} - 1$, the rotation number associated to $I^{q'}$ is $\frac{pq'}{q}$, and the rotation number associated to I is p/q. In particular, if $\widehat{z}'$ is a contractible fixed point of $\widehat{f}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ associated to $\widehat{I}_0|_{\widehat{M}\setminus\{\widehat{z}_0\}}$, $\pi'(\widehat{z}')$ is a contractible fixed point of $f|_{M\setminus\{z_0\}}$ associated to $I|_{M\setminus\{z_0\}}$. So, $\widehat{f}|_{\widehat{M}\setminus\{\widehat{z}_0\}}$ does not have any contractible fixed point associated to $\widehat{I}_0$.

Moreover, if p/q is irreducible, and if $\widehat{O}'$ is a periodic orbit of $\widehat{f}$ of type (p, q) associated to $\widehat{I}_0$ at $\widehat{z}_0$ such that $z_0 \notin \pi'(\widehat{O}')$, then $\pi'(\widehat{O}')$ is a periodic orbit of f of type (p, q) associated

to I at z_0.

By Lemma 3.7, the property **P)** holds for $(\widehat{f}, \widehat{I}_0, \widehat{z}_0)$, and then holds for (f, I, z_0). $\square$

3.2 The case where the total area of M is finite

In this section, we assume that the area of M is finite. Recall that f is an area preserving homeomorphism of M, that z_0 is an isolated fixed point of f satisfying $i(f, z_0) = 1$, that I is an identity isotopy of f that fixes z_0 and satisfies $\rho_s(I, z_0) = \{k\}$. Let (X, I_X) be a maximal extension of I that satisfies $\rho_s(I_X, z_0) = \rho_s(I, z_0)$. Write $X_0 = X \setminus \{z_0\}$. Then, X_0 is a closed subset of $\mathrm{Fix}(f)$, and I_X can be extended to a maximal identity isotopy on $M \setminus X_0$ that fixes z_0. To simplify the notation, we still denote by I_X this extension. Moreover, by definition of Jaulent's preorder, we know that a periodic orbit of type (p, q) associated to I_X at z_0 is a periodic orbit of type (p, q) associated to I at z_0. Let M_0 be the connected component of $M \setminus X_0$ that contains z_0. Of course the total area of M_0 is also finite. When M is a sphere, $f|_{M\setminus\{z_0\}}$ has at least one fixed point (see Section 2.4), and hence X_0 is not empty. So, M_0 is not a sphere. To simplify the notations, we denote by f_0 the restriction of f to M_0, and by I_0 the restriction of I_X to M_0. If the property **P)** holds for (f_0, I_0, z_0), it holds for (f, I, z_0). So, we will prove the following proposition, and the second part of Theorem 1.1 is also proved.

Proposition 3.12. *Under the previous assumptions, the property **P)** holds for* (f_0, I_0, z_0).

We will prove this proposition in the following four cases:

- the component M_0 is a plane and $\rho_s(I, z_0)$ is reduced to 0;
- the component M_0 is neither a sphere nor a plane and $\rho_s(I, z_0)$ is reduced to 0;
- the component M_0 is a plane and $\rho_s(I, z_0)$ is reduced to an non-zero integer k;
- the component M_0 is neither a sphere nor a plane and $\rho_s(I, z_0)$ is reduced to an non-zero integer k.

We will use some results that will be deduced in the first two cases to obtain the last two cases.

3.2.1 The case where M_0 is a plane and $\rho_s(I, z_0)$ is reduced to 0

In this case, I_0 is a maximal identity isotopy on the plane M_0 that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{0\}$. The result of Proposition 3.12 is just a corollary of Theorem 3.4 and the following lemma:

Lemma 3.13. *Under the previous assumptions, there exists a periodic orbit of f in the annulus $M_0 \setminus \{z_0\}$.*

Proof. Of course, we can assume that z_0 is not accumulated by periodic orbits. As in Remark 3.3, one knows that z_0 is an indifferent fixed point with rotation number $\rho(I, z_0) = 0$.

Let $\mathcal{F}$ be a transverse foliation of I_0. One knows that $\mathcal{F}$ has a unique singularity z_0 and an end ∞. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}$. Since f_0 is area preserving and the total area of M_0 is finite, ∞ is a source of $\mathcal{F}$ and all the leaves of $\mathcal{F}$ are lines from ∞ to z_0. Let $\pi : \mathbb{R} \times (0, 1) \to M_0 \setminus \{z_0\}$ be the universal cover such that the leaves of the lift

$\widetilde{\mathcal{F}}$ of $\mathcal{F}$ are the vertical lines oriented upward. Let $\widetilde{f}$ be the lift of f_0 associated to I_0, and $p_1 : \mathbb{R} \times (0,1) \to \mathbb{R}$ be the projection onto the first factor. Then we know that

$$p_1(\widetilde{f}(\widetilde{z}) - \widetilde{z}) > 0, \text{ for all } \widetilde{z} \in \mathbb{R} \times (0,1).$$

Let V be a small Jordan domain in the annulus $M_0 \setminus \{z_0\}$ such that $f(V) \cap V = \emptyset$. Let $\widetilde{V}$ be one of the connected components of $\pi^{-1}(V)$. By choosing V small enough, one can suppose that

$$|p_1(\widetilde{z}) - p_1(\widetilde{z}')| < \frac{1}{2} \text{ for all } \widetilde{z}, \widetilde{z}' \in \widetilde{V}.$$

Then, for every $z \in V$ and $\widetilde{z} \in \pi^{-1}\{z\}$, we know that

$$\frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \geq \frac{\left(\sum_{k=1}^n \chi_V(f^k(z))\right) - \frac{1}{2}}{n}.$$

We define $U = \cup_{k \in \mathbb{Z}} f^k(V)$. By Poincaré Recurrence Theorem, almost all points in U are recurrent. By Birkhoff-Khinchen Theorem, for almost all $z \in U$, and every $\widetilde{z} \in \pi^{-1}\{z\}$, both of the two limits

$$\lim_{n\to\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n} \text{ and } \lim_{n\to\infty} \sum_{k=1}^{n} \frac{\chi_V(f^k(z))}{n}$$

exist, and there exists a non negative measurable function φ on U that satisfies $\varphi \circ f = \varphi$ and

$$\lim_{n\to\infty} \sum_{k=0}^{n-1} \frac{\chi_V(f^k(z))}{n} = \varphi(z) \text{ for almost all } z \in U.$$

Moreover, by Lebesgue's dominated convergence theorem,

$$\int_U \varphi = \int \chi_V = \text{Area}(V) > 0.$$

Therefore, there exist a recurrent point $z \in V$ and $\widetilde{z} \in \pi^{-1}\{z\}$ such that the limit

$$\lim_{n\to\infty} \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n}$$

exists and is positive. So, the rotation number of z is positive. We denote it by ρ.

On the other hand, let K_0 be a small enough continuum at z_0 whose rotation number is 0. We denote by $(W \setminus K_0) \sqcup \mathbb{T}^1$ the prime-ends compactification at the end K_0, which is an annulus. We can extend f to $\mathbb{T}^1$ and know that the rotation number on $\mathbb{T}^1$ is 0. Then, there exists a fixed point on $\mathbb{T}^1$ whose rotation number is 0.

By the remark that follows Proposition 2.16, there exists a q-periodic orbit of rotation number p/q in the annulus $(W \setminus K_0)$, for all irreducible $p/q \in (0, \rho)$. □

3.2.2 The case where M_0 is neither a sphere nor a plane and $\rho_s(I, z_0)$ is reduced to 0

Recall that f_0 is an area preserving homeomorphism of M_0, that z_0 is an isolated fixed point of f_0 satisfying $i(f_0, z_0) = 1$, that I_0 is a maximal identity isotopy that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{0\}$.

As in Section 2.13, let $\pi : \widetilde{M} \to M_0 \setminus \{z_0\}$ be the annulus covering projection, $\widetilde{I}$ be the natural lift of I_0 to $\widetilde{M} \cup \{\star\}$, $\widetilde{f}$ be the lift of f_0 to $\widetilde{M} \cup \{\star\}$ associated to I_0. Then $\widetilde{I}$ is a

maximal identity isotopy and $\mathrm{Fix}(\widetilde{I})$ is reduced to $\star$. For all irreducible $p/q \in \mathbb{Q}$, if O is a periodic orbit of type (p,q) associated to $\widetilde{I}$ at $\star$, then $\pi(O)$ is a periodic orbit of type (p,q) associated to I_0 at z_0. So, if the property **P)** holds for $(\widetilde{f}, \widetilde{I}, \star)$, then it holds for (f_0, I_0, z_0). The result of Proposition 3.12 is a corollary of Theorem 3.4 and the following Proposition 3.14, which is the most difficult part of this article.

Proposition 3.14. *There exists a periodic orbit of $\widetilde{f}$ besides $\star$.*

The idea of the proof of the proposition is the following: we will first consider several simple situations such that there exists a periodic orbit of $\widetilde{f}$ besides $\star$, then we suppose that we are not in these situations and follow the idea of Le Calvez (see Section 11 of [LC05]) to get a contradiction.

Let us begin with some necessary assumptions and lemmas. Of course, we can suppose that $\star$ is not accumulated by periodic orbits of $\widetilde{f}$. As in Remark 3.3, $\star$ is an indifferent fixed point of $\widetilde{f}$ and the rotation number $\rho(\widetilde{I}, \star)$ is equal to 0. Let $\mathcal{F}$ be a transverse foliation of I_0, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$. By the assumption in Remark 3.2, z_0 is a sink of $\mathcal{F}$, and $\star$ is a sink of $\widetilde{\mathcal{F}}$. Denote by W the attracting basin of z_0 for $\mathcal{F}$, and by $\widetilde{W}$ the attracting basin of $\star$ for $\widetilde{\mathcal{F}}$. Write $\dot{W} = W \setminus \{z_0\}$ and $\dot{\widetilde{W}} = \widetilde{W} \setminus \{\star\}$. Recall that $\pi|_{\dot{\widetilde{W}}}$ is a homeomorphism between $\dot{\widetilde{W}}$ and $\dot{W}$ and can be extended continuously to a homeomorphism between $\widetilde{W}$ and W. The area on M_0 induces an area on $\widetilde{M}$. So $\widetilde{f}$ is area preserving, and the area of $\widetilde{W}$ is finite.

Lemma 3.15. *Under the previous assumptions, if there exists an invariant continuum $K \subset \widetilde{W}$ with positive area, then there exists a periodic orbit besides $\star$.*

Proof. The proof is similar to the proof of Lemma 3.13 except some small modifications when we try to find a recurrent point with positive rotation number. We will give a more precise description.

Since $\widetilde{W}$ is different from $\widetilde{M} \cup \{\star\}$, we can not get a lift of $\widetilde{f}$ as in the proof of Lemma 3.13. Instead, we will get a similar one by the following procedure. Let $\pi' : \mathbb{R}^2 \to \dot{\widetilde{W}}$ be a universal cover which sends the vertical lines upwards to the leaves of $\widetilde{F}|_{\dot{\widetilde{W}}}$. Since K is an invariant subset of $\widetilde{W}$, we can lift $\widetilde{f}|_{K\setminus\{\star\}}$ to a homeomorphism $\widehat{f}$ of $\pi'^{-1}(K \setminus \{\star\})$ such that

$$p_1(\widehat{f}(\widehat{z}) - \widehat{z}) > 0, \text{ for all } \widehat{z} \in \pi'^{-1}(K \setminus \{\star\}),$$

where p_1 is the projection onto the first factor.

Also, we should replace the small Jordan domain V in the proof of Lemma 3.13 with $V \cap K$ by choosing suitable V such that the area of $V \cap K$ is positive, that $f(V) \cap V = \emptyset$, and that for every component $\widehat{V}$ of $\pi'^{-1}(V)$, one has

$$|p_1(\widehat{z}) - p_1(\widehat{z}')| < 1/2 \quad \text{for all } \widehat{z}, \widehat{z}' \in \widehat{V}.$$

We can always find such a set because the area of K is positive. □

Lemma 3.16. *Under the previous assumptions, if there exists an invariant continuum $K \subset \widetilde{W}$ such that $\rho(\widetilde{I}, K) \neq 0$, then there exists a periodic orbit in $\widetilde{M}$.*

Proof. Recall that $\pi|_{\dot{\widetilde{W}}}$ is a homeomorphism between $\dot{\widetilde{W}}$ and $\dot{W}$. So, $\widetilde{W}$ is a proper subset of $\widetilde{M} \cup \{\star\}$, and the boundary of $\widetilde{W}$ is the union of some proper leaves. By Remark 2.13, one knows that $\widetilde{f}$ can be blown-up at ∞ and the blow-up rotation number $\rho(\widetilde{I}, \infty)$ is equal to 0.

We consider the prime-ends compactification of $\widetilde{M} \setminus K$ at the end K, and extend $\widetilde{f}$ continuously to a homeomorphism of $(\widetilde{M} \setminus K) \sqcup S^1$. We get a homeomorphism g of the closed annulus $S_\infty \sqcup (\widetilde{M} \setminus K) \sqcup S^1$ that coincides with $\widetilde{f}$ on $\widetilde{M} \setminus K$, where S_∞ is the circle we added when blowing-up $\widetilde{f}$ at ∞.

Moreover, g satisfies the intersection property and has different rotation numbers at each boundary, then by Proposition 2.15, there exists a periodic orbit in $\widetilde{M} \setminus K$, which is also a periodic orbit of $\widetilde{f}$. □

Lemma 3.17. *Suppose that there exists a closed disk $D \subset \widetilde{W}$ containing $\star$ as an interior point such that the connected component of $\bigcap_{k\in\mathbb{Z}} \widetilde{f}^{-k}(D)$ containing $\star$ is contained in the interior of D. Then $\widetilde{f}$ has another periodic orbit besides $\star$.*

Proof. We will proof this lemma by contradiction. Suppose that $\widetilde{f}$ does not have any other periodic orbit. Let K be the connected component of $\bigcap_{k\in\mathbb{Z}} \widetilde{f}^{-k}(D)$ containing $\star$. We identify K as a point $\{K\}$, and still denote by $\widetilde{f}$ the reduced homeomorphism. The fixed point $\{K\}$ is a non-accumulated saddle-point of $\widetilde{f}$ with index $i(\widetilde{f}, \{K\}) = i(\widetilde{f}, K) = i(\widetilde{f}, 0) = 1$. By Proposition 2.14, $\widetilde{f}$ can be blown-up at $\{K\}$ and $\rho(\widetilde{f}, \{K\})$ is different from $0 \in \mathbb{R}/\mathbb{Z}$. So, $\rho(\widetilde{I}, K)$ is different from 0. By the previous lemma, $\widetilde{f}$ has another periodic orbit besides $\star$, which is a contradiction. □

Now we begin the proof of Proposition 3.14.

Proof of Proposition 3.14. We will prove this proposition by contradiction. Suppose that there does not exist any other periodic orbit except $\star$. Let $(D_n)_{n\in N}$ be an increasing sequence of closed disks containing $\star$ as an interior point such that D_n is contained in the interior of D_{n+1} for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} D_n = \widetilde{W}$. Let K_n be the connected component of $\cap_{k\in\mathbb{Z}} \widetilde{f}^{-k}(D_n)$ containing $\star$. By Lemma 3.16, we know that $\rho(\widetilde{I}, K_n)$ is equal to 0 for every $n \in \mathbb{N}$. By Lemma 3.17, each K_n intersects the boundary of D_n. Let $K = \overline{\cup_{n\in\mathbb{N}} K_n} \subset \widetilde{M} \cup \{\star\}$. It is an invariant set of $\widetilde{f}$. The boundary of $\widetilde{W}$ is the union of proper leaves, so for every point in $\partial\widetilde{W}$, either its image or its pre-image by $\widetilde{f}$ will leave $\widetilde{W}$. Therefore, K can not touch the boundary of $\widetilde{W}$, and is included in $\widetilde{W}$. But each K_n intersects the boundary of D_n, so K intersects every neighborhood of ∞.

Lemma 3.18. *There does not exist any connected component of $\widetilde{M} \setminus K$ that is included in $\widetilde{W}$.*

Proof. We will give a proof by contradiction. Suppose that there exists a component $\widetilde{U}$ of $\widetilde{M} \setminus K$ such that $\widetilde{U} \subset \widetilde{W}$. Then ∂U is a subset of K, which is invariant by $\widetilde{f}$. So, $\partial(\widetilde{f}^n(U))$ is a subset of K for every $n \in \mathbb{Z}$, and one deduces that $\cup_{n\in\mathbb{Z}} \widetilde{f}^n(\widetilde{U}) \subset \widetilde{W}$. Moreover, $\star$ is not an interior point of $\widetilde{U}$, and $\widetilde{U}$ is homeomorphic to a disk. We know that the area of $\widetilde{W}$ is finite, so there exists $q \in \mathbb{N}$ such that $\widetilde{f}^q(\widetilde{U}) = \widetilde{U}$. Then, one knows that $\widetilde{f}^q$ has a fixed point (see Section 2.4), and hence $\widetilde{f}$ has a periodic point different from $\star$. We get a contradiction. □

Let $\pi' : \widehat{M} \to \widetilde{M}$ be the universal cover, and T be a generator of the group of covering automorphisms. Let $\widehat{I} = (\widehat{f}_t)_{t\in[0,1]}$ be the natural lift of $\widetilde{I}$, and $\widehat{\mathcal{F}}$ be the lift of $\widetilde{\mathcal{F}}$. Write $\widehat{f} = \widehat{f}_1$. It is the lift of $\widetilde{f}$ associated to $\widetilde{I}$. Write $\widehat{K} = \pi'^{-1}(K \setminus \{\star\})$, and $\widehat{W} = \pi'^{-1}(\dot{\widetilde{W}})$.

Because K is connected, each connected component of $\widetilde{M} \setminus K$ is simply connected. So, if $\widetilde{U}$ is one of the connected components of $\widetilde{M} \setminus K$, and if $\widehat{U}$ is one of the components of $\pi'^{-1}(\widetilde{U})$, then $\widehat{U}$ does not intersect $T(\widehat{U})$. Therefore, $\widehat{M} \setminus \widehat{K}$ is not connected and has infinitely many

components. By Lemma 3.18, each component of $\widehat{M} \setminus \widehat{K}$ contains a proper leaf in $\partial\widehat{W}$, and hence a disk bounded by this leaf. As in the following picture, this disk contains the image

Figure 2: Each component $\widehat{U}$ of $\widehat{M} \setminus \widehat{K}$ is invariant by $\widehat{f}$

or the pre-image of this proper leaf by $\widehat{f}$. So, every component of $\widehat{M} \setminus \widehat{K}$ is invariant by $\widehat{f}$.

Lemma 3.19. *Each leaf in $\widetilde{W}$ is an arc from ∞ to $\star$.*

Proof. Recall that the area of $\widetilde{W}$ is finite. So, there exist a leaf included in $\partial\widetilde{W}$ such that $\widetilde{W}$ is to its right and a leaf included in $\partial\widetilde{W}$ such that $\widetilde{W}$ is to its left. (Otherwise, $\widetilde{W}$ contains the positive or negative orbit of a wandering open set $\widetilde{W} \setminus \widetilde{f}(\widetilde{W})$ or $\widetilde{W} \setminus \widetilde{f}^{-1}(\widetilde{W})$ respectively.) Therefore, a leaf in $\widetilde{W}$ is an arc from ∞ to $\star$. □

Every leaf $\widehat{\Phi} \subset \widehat{W}$ divides $\widehat{M}$ into two part. We denote by $R(\widehat{\Phi})$ the component of $\widehat{M} \setminus \widehat{\Phi}$ to the right of $\widehat{\Phi}$ and by $L(\widehat{\Phi})$ the component to the left.

Lemma 3.20. *There does not exist any leaf $\widehat{\Phi} \subset \widehat{W}$ such that $\widehat{\Phi} \subset \widehat{K}$.*

Proof. We can prove this lemma by contradiction. Suppose that $\widehat{\Phi} \subset \widehat{K}$. Then a component of $\widehat{M} \setminus \widehat{K}$ is either to the left or to the right of $\widehat{\Phi}$. Moreover, if it is to the right (resp. left) of $\widehat{\Phi}$, it is to the right (resp. left) of $\widehat{f}(\widehat{\Phi})$. Therefore, $R(\widehat{\Phi}) \cap L(\widehat{f}(\widehat{\Phi}))$ is included in $\widehat{K}$, and so the interior of $\widehat{K}$ is not empty. We deduce that K is an invariant set of $\widetilde{f}$ with non-empty interior and finite area. By Lemma 3.15, there exists a periodic orbit of $\widetilde{f}$ in $\widetilde{M}$, which is a contradiction. □

Lemma 3.21. *Let $\widehat{\Phi}$ be a leaf in $\widehat{W}$, $t \mapsto \widehat{\Phi}(t)$ be an oriented parametrization of $\widehat{\Phi}$, and $\widehat{U}$ be a component of $\widehat{M} \setminus \widehat{K}$. If $\widehat{\Phi}$ intersects $\widehat{U}$, then both the area of $L(\widehat{\Phi}) \cap \widehat{U}$ and the area of $R(\widehat{\Phi}) \cap \widehat{U}$ are infinite, and there exists t_0 such that $\widehat{\Phi}(t) \in \widehat{U}$ for all $t \leq t_0$.*

Proof. We will first give a proof of the first statement by contradiction. We suppose that the area of $L(\widehat{\Phi}) \cap \widehat{U}$ is finite, the other case can be treated similarly. Then, $L(\widehat{\Phi}) \cap R(\widehat{f}^{-1}(\widehat{\Phi})) \cap \widehat{U}$ is a wandering open set whose negative orbit is contained in $L(\widehat{\Phi}) \cap \widehat{U}$. It contradicts the fact that $\widehat{f}$ is area preserving.

Let us prove the second statement. We know that both the area of $L(\widehat{\Phi}) \cap \widehat{U}$ and the area of $R(\widehat{\Phi}) \cap \widehat{U}$ are infinite. Since $\pi'|_{\widehat{U}}$ is injective, both the area of $\pi'(L(\widehat{\Phi}) \cap \widehat{U})$ and the area of $\pi'(R(\widehat{\Phi}) \cap \widehat{U})$ are infinite. The area of $\widetilde{W}$ is finite, so both $\pi'(L(\widehat{\Phi}) \cap \widehat{U})$ and $\pi'(R(\widehat{\Phi}) \cap \widehat{U})$ intersect $\widetilde{M} \setminus \widetilde{W}$, and hence both $L(\widehat{\Phi}) \cap \widehat{U}$ and $R(\widehat{\Phi}) \cap \widehat{U}$ intersect $\widehat{M} \setminus \widehat{W}$. Therefore, there exists a proper leaf $\widehat{\Phi}_1$ in $L(\widehat{\Phi}) \cap \widehat{U}$ and a proper leaf $\widehat{\Phi}_2$ in $R(\widehat{\Phi}) \cap \widehat{U}$. Fix a parametrization $t \mapsto \widehat{\Phi}_1(t)$ of $\widehat{\Phi}_1$ and a parametrization $t \mapsto \widehat{\Phi}_2(t)$ of $\widehat{\Phi}_2$, and draw a path γ in $\widehat{U}$ from a point of $\widehat{\Phi}_1$ to a

point of $\widehat{\Phi}_2$. Let $s_1 = \inf\{t : \widehat{\Phi}_1(t) \in \gamma\}$, $s_2 = \sup\{t : \widehat{\Phi}_2(t) \in \gamma\}$, and γ' be the sub-path of γ connecting $\widehat{\Phi}_1(s_1)$ to $\widehat{\Phi}_2(s_2)$. Then, as in the following picture, $\Gamma = \widehat{\Phi}|_{(-\infty,s_1]}\gamma'\widehat{\Phi}_2|_{[s_2,\infty)}$ is an

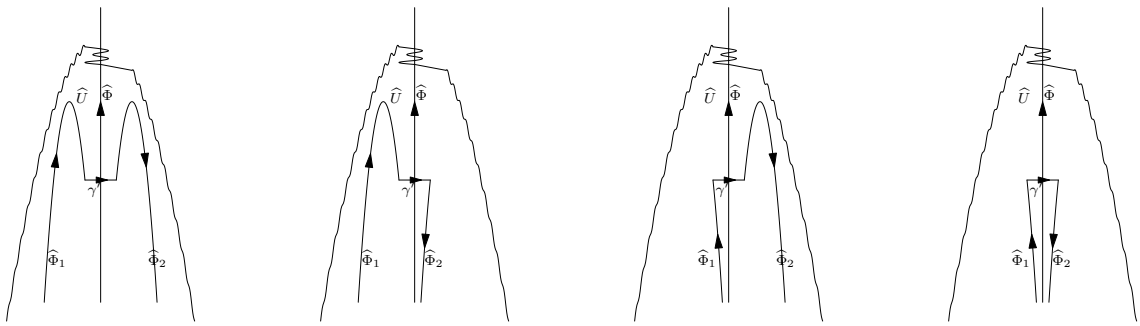


Figure 3: Four possible cases in the proof of Lemma 3.21

oriented proper arc and satisfies $R(\Gamma) \subset \widehat{U}$. We know that $\widehat{\Phi}$ intersects γ'. Let t_0 be a lower bound of the set $\{t : \widehat{\Phi}(t) \in \gamma'\}$. We know that $\widehat{\Phi}|_{(-\infty,t_0]} \subset \widehat{U}$. □

Let $\delta : \mathbb{T}^1 \to \widetilde{W}$ be an embedding that intersect $\widetilde{F}$ transversely, and $\widehat{\delta} : \mathbb{R} \to \widehat{W}$ be the lift of δ. Then $\widehat{\delta}$ intersects every leaf in $\widehat{W}$, and intersects each leaf at only one point. Moreover, if $\widehat{\delta}$ intersects $\widehat{\Phi}$ and $\widehat{\Phi}'$ at $\widehat{\delta}(t)$ and $\widehat{\delta}(t')$ respectively, and if $t < t'$, then $\widehat{\Phi}$ is to the left of $\widehat{\Phi}'$, and $\widehat{\Phi}'$ is to the right of $\widehat{\Phi}$. We define a map $h : \mathbb{R} \to \widehat{\mathcal{F}}|_{\widehat{W}}$ by $h(t) = \widehat{\Phi}$ if $\widehat{\delta}(t) \in \widehat{\Phi}$.

Lemma 3.22. *The set of points $t \in \mathbb{R}$ such that $h(t) \cap \widehat{U} \neq \emptyset$ is open for each component $\widehat{U}$ of $\widehat{W} \setminus \widehat{K}$.*

Proof. We fix a component $\widehat{U}$ of $\widehat{M} \setminus \widehat{K}$, and will first prove that the set $\{t : h(t) \cap \widehat{U} \neq \emptyset\}$ is open. Given a real number t such that $h(t)$ intersects $\widehat{U}$ and $z \in h(t) \cap \widehat{U}$, there is a trivialization neighborhood V of z such that $V \subset (\widehat{U} \cap \widehat{W})$. Moreover, $h^{-1}(V)$ is an open interval containing t. So, the set $\{t : h(t) \cap \widehat{U} \neq \emptyset\}$ is open. □

By Lemma 3.20, each leaf of $\widehat{\mathcal{F}}$ in $\widehat{W}$ intersects at least a component of $\widehat{M} \setminus \widehat{K}$. By lemma 3.21, each leaf of $\widehat{\mathcal{F}}$ in $\widehat{W}$ intersects at most one component of $\widehat{M} \setminus \widehat{K}$. So, each leaf of $\widehat{\mathcal{F}}$ in $\widehat{W}$ intersects exactly one component of $\widehat{M} \setminus \widehat{K}$. Since $\widehat{M} \setminus \widehat{K}$ has countable components,

$$\mathbb{R} = \cup_{\widehat{U}}\{t : h(t) \cap \widehat{U} \neq \emptyset\}$$

is a disjoint union of countable many open sets. This is impossible. □

3.2.3 The case where M_0 is a plane and $\rho_s(I, z_0)$ is reduced to a non-zero integer k

Recall that f_0 is an area preserving homeomorphism of M_0, that z_0 is an isolated fixed point of f_0 satisfying $i(f_0, z_0) = 1$, and that I_0 is a maximal identity isotopy that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{k\}$. In this case, one can easily deduce that the result of Proposition 3.12 is just a corollary of the result in the previous two cases. We will give a brief explanation. Let J be the identity isotopy of the identity map on M_0 fixing z_0 and satisfying $\rho_s(J, z_0) = 1$. Write $I_0' = J^{-k}I_0$. It is an identity isotopy of f_0 that satisfies $\rho_s(I_0', z_0) = \{0\}$. By the result of Proposition 3.12 in the two cases we have already proved, the property **P)** holds for (f_0, I_0', z_0). A periodic orbit in M_0 of type (p, q) associated to I_0' at z_0 is a periodic orbit of type $(kq + p, q)$ associated to I_0 at z_0. So, the property **P)** holds for (f_0, I_0, z_0).

3.2.4 The case where M_0 is neither a sphere nor a plane and $\rho_s(I, z_0)$ is reduced to a non-zero integer k

Recall that f_0 is an area preserving homeomorphism of M_0, that z_0 is an isolated fixed point of f_0 satisfying $i(f_0, z_0) = 1$, that I_0 is a maximal identity isotopy that fixes only one point z_0 and satisfies $\rho_s(I_0, z_0) = \{k\}$.

As in Section 2.13, let $\pi : \widetilde{M} \to M_0 \setminus \{z_0\}$ be the annulus covering projection, $\widetilde{I}$ be the natural lift of I_0 to $\widetilde{M} \cup \{\star\}$, and $\widetilde{f}$ be the lift of f_0 associated to I_0 to $\widetilde{M} \cup \{\star\}$. Then $\widetilde{I}$ is a maximal identity isotopy and $\mathrm{Fix}(\widetilde{I})$ is reduced to $\star$. As before, if the property **P)** holds for $(\widetilde{f}, \widetilde{I}, \star)$, it holds for (f_0, I_0, z_0).

Let $\mathcal{F}$ be a transverse foliation of I_0, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}$. Since $\rho_s(I_0, z_0)$ is reduced to a non-zero integer, by the assertion iv) of Proposition 2.12, z_0 is a sink or a source of $\mathcal{F}$ and $\star$ is a sink or a source of $\widetilde{\mathcal{F}}$. Let W be the attracting or repelling basin of z_0 for $\mathcal{F}$, and $\widetilde{W}$ be the attracting or repelling basin of $\star$ for $\widetilde{\mathcal{F}}$. Recall that $\pi|_{\widetilde{W}\setminus\{\star\}}$ is a homeomorphism be tween $\widetilde{W} \setminus \{\star\}$ and $W \setminus \{z_0\}$. So, $\widetilde{W}$ is a strict subset of $\widetilde{M} \cup \{\star\}$, and its boundary is the union of some proper leaves. By Remark 2.13, one knows that $\widetilde{f}$ can be blown-up at ∞ and $\rho(\widetilde{I}, \infty)$ is equal to 0.

Let J be the identity isotopy of the identity map of $\widetilde{M} \cup \{\star\}$ fixing $\star$ and satisfying $\rho_s(J, \star) = 1$. Write $\widetilde{I}' = J^{-k}\widetilde{I}$. We know that $\rho_s(\widetilde{I}', \star)$ is reduced to 0, and that the blow-up rotation number $\rho(\widetilde{I}', \infty)$ is equal to k. One deduces by the assertion ii) of Proposition 2.12 that there exists a neighborhood of ∞ that does not contain any contractible fixed points of $\widetilde{f}|_{\widetilde{M}}$ associated to $\widetilde{I}'|_{\widetilde{M}}$. Let $(Y, \widetilde{I}_Y)$ be a maximal extension of $(\{\star\}, \widetilde{I}')$ (see Section 2.6). One knows that Y is a closed subset of the union of $\{\star\}$ and the set of contractible fixed points of $\widetilde{f}|_{\widetilde{M}}$ associated to $\widetilde{I}'|_{\widetilde{M}}$. So, there is a neighborhood of ∞ that does not intersect Y, and hence Y is a compact set in $\widetilde{M} \cup \{\star\}$. One knows also that $\rho_s(\widetilde{I}_Y, \star)$ is reduced to 0, and that the blow-up rotation number $\rho(\widetilde{I}_Y, \infty)$ is equal to k. As in the previous subsection, in order to prove the result of Proposition 3.12, we only need to prove that the property **P)** holds for $(\widetilde{f}, \widetilde{I}_Y, \star)$, which is the aim of this subsection.

Proposition 3.23. *Under the previous assumptions, the property **P)** holds for $(\widetilde{f}, \widetilde{I}_Y, \star)$.*

Proof. To get this result, one has to consider two cases: Y is reduced to a single point $\star$ or it contains at least two points. In the first case, the proposition is a corollary of Lemma 3.7. Now, we will prove the proposition in the second case.

Suppose that Y contains at least two points and write $Y_0 = Y \setminus \{\star\}$. Let $\widetilde{M}_0$ be the connected component of $\widetilde{M} \cup \{\star\} \setminus Y_0$ containing $\star$. Recall that Y is a compact subset of $\widetilde{M} \cup \{\star\}$. So, one has to consider the following two cases:

- $\widetilde{M}_0$ is a bounded plane,
- $\widetilde{M}_0$ is neither a sphere nor a plane.

In the first case, the area of $\widetilde{M}_0$ is finite, and the problem is reduced to the case of Section 3.2.1; while in the second case, we will prove the result of Proposition 3.12 like in Section 3.2.2.

Now, we suppose that $\widetilde{M}_0$ is neither a sphere nor a plane. Let $\pi'' : \breve{M} \to \widetilde{M}_0$ be an annulus covering map, $\breve{I}$ be the natural lift of $\widetilde{I}_Y|_{\widetilde{M}_0}$ to $\breve{M} \cup \{\breve{\star}\}$, and $\breve{f}$ be the lift of $\widetilde{f}|_{\widetilde{M}_0}$ to $\breve{M} \cup \{\breve{\star}\}$ associated to $\widetilde{I}_Y|_{\widetilde{M}_0}$. As before, if the Property **P)** holds for $(\breve{f}, \breve{I}, \breve{\star})$, then it holds also for $(\widetilde{f}, \widetilde{I}_Y, \star)$. So, the proposition is a corollary of Theorem 3.4 and the following Lemma 3.24. □

Lemma 3.24. *There exists a periodic orbit of $\breve{f}$ besides $\breve{\star}$.*

Proof. The proof is similar to the proof of Proposition 3.14.

Let $\widetilde{\mathcal{F}}_Y$ be a transverse foliation of $\widetilde{I}_Y$, and $\breve{\mathcal{F}}$ be the lift of $\mathcal{F}_Y|_{\widetilde{M}_0}$ to $\breve{M} \cup \{\breve{\star}\}$. Recall the assumption in Remark 3.2, one knows that $\star$ is a sink of $\widetilde{\mathcal{F}}_Y$ and that $\breve{\star}$ is a sink of $\breve{\mathcal{F}}$. Let $\widetilde{W}^*$ be the attracting basin of $\star$ for $\widetilde{\mathcal{F}}_Y$ and $\breve{W}$ be the attracting basin of $\breve{\star}$ for $\breve{\mathcal{F}}$. Recall that $\pi''|_{\breve{W}\setminus\{\breve{\star}\}}$ is a homeomorphism between $\breve{W} \setminus \{\breve{\star}\}$ and $\widetilde{W}^* \setminus \{\star\}$.

When $k \geq 1$, one deduces by Proposition 2.12 that the end ∞ is sink of $\widetilde{\mathcal{F}}_Y$. In this case $\widetilde{W}^*$ is a bounded subset of $\widetilde{M} \cup \{\star\}$, and hence the area of both $\widetilde{W}^*$ and $\breve{W}$ are finite. We can repeat the proof of Proposition 3.14, and get the result.

Now, we suppose that $k \leq -1$. In this case, the end ∞ is a source of of $\widetilde{\mathcal{F}}_Y$.

Sublemma 3.25. *Each leaf in $\breve{W}$ is an arc from infinite to $\breve{\star}$.*

Proof. When the area of $\breve{W}$ is finite, we deduces the result as in Lemma 3.19. Now suppose that the area of $\breve{W}$ is infinite. We consider the compactification of $\widetilde{M} \cup \{\star\}$ by adding a point ∞ at infinite, the added point ∞ is a source of $\widetilde{F}_Y$ and is at the boundary of $\widetilde{W}^*$. So, there exists a leaf in $\widetilde{W}^*$ from the singularity ∞ to $\star$ whose lift in $\breve{W}$ is a leaf from infinite to $\breve{\star}$. Therefore, each leaf in $\breve{W}$ is an arc from infinite to $\breve{\star}$. □

The difference between our case and the case of Proposition 3.14 is that the area of $\breve{W}$ may be infinite. But we did not use this condition except in the proof of Lemma 3.19 and Lemma 3.20. We have proven Sublemma 3.25 corresponding to Lemma 3.19. We will prove that the area of K is finite, so the result of Lemma 3.20 is still valid.

Formally, suppose that there does not exist any periodic orbits besides $\breve{\star}$. Let $(D_n)_{n\in N}$ be an increasing sequence of closed disks containing $\breve{\star}$ such that D_n is contained in the interior of D_{n+1} for all $n \in \mathbb{N}$ and $\cup_{n\in\mathbb{N}} D_n = \breve{W}$. Let K_n be the connected component of $\cap_{k\in\mathbb{Z}} \breve{f}^{-k}(D_n)$ containing $\breve{\star}$ and $K = \overline{\cup_{n\in\mathbb{N}} K_n} \subset \breve{M} \cup \{\star\}$. We will prove that the area of K is finite.

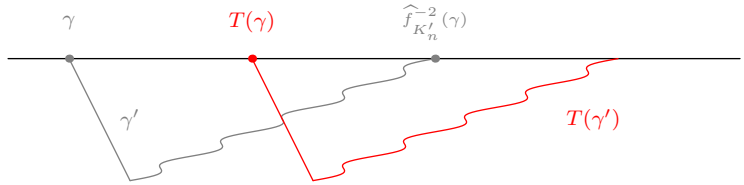
Let $K'_n = \pi''(K_n)$, and $K' = \overline{\cup_n K'_n} \subset \widetilde{M} \cup \{\star\} \cup S_\infty$, where S_∞ is the circle we added when blowing-up $\widetilde{f}$ at ∞. As before, we can deduce that $K \subset \breve{W}$. Recall that $\pi''|_{\breve{W}\setminus\{\breve{\star}\}}$ is a homeomorphism between $\breve{W} \setminus \{\breve{\star}\}$ and $\widetilde{W}^* \setminus \{\star\}$. Therefore, we know that $\pi''(K) \subset K'$, and that the area of K is not bigger than the area of K'. So, we only need to prove that the area of K' is finite.

We will prove it by contradiction. Suppose that the area of K' is infinite. One deduces that $K' \cap S_\infty \neq \emptyset$. As was proven in Section 3.2.2, one knows that $\rho(\breve{I}, K_n) = 0$, and so $\rho(\widetilde{I}_Y, K'_n) = 0$ for all $n \in \mathbb{N}$. Since $(Y, \widetilde{I}_Y)$ is a maximal extension of $(\{\star\}, J^{-k}\widetilde{I})$, one deduces that $\rho(J^{-k}\widetilde{I}, K'_n)$ is equal to 0 and that $\rho(\widetilde{I}, K'_n)$ is equal to k, for all $n \in \mathbb{N}$.

Since $K' \cap S_\infty$ is invariant by $\widetilde{f}$ and the blow-up rotation number $\rho(\widetilde{I}, \infty) = 0$, there exists a fixed point $\widetilde{z}_1 \in K' \cap S_\infty$, and the rotation number of $\widetilde{z}_1$ (associated to $\widetilde{I}$) in the annulus $\widetilde{M} \cup S_\infty$ is 0.

Let $\pi' : \widehat{M} \to \widetilde{M} \cup S_\infty$ be the universal cover, T be a generator of the group of covering automorphism, and $\widehat{f}$ the lift of $\widetilde{f}$ associated to $\widetilde{I}$. Fix one $\widehat{z}_1 \in \pi'^{-1}(\widetilde{z}_1)$. It is a fixed point of $\widehat{f}$. Let U be a small neighborhood of $\widehat{z}_1$ such that $T^n(U) \cap U = \emptyset$ for all $n \neq 0$. Let $V \subset U$ be a neighborhood of $\widehat{z}_1$ such that $\widehat{f}^2(V) \subset U$. Fix n large enough such that $K'_n \cap V \neq \emptyset$, and choose an arc γ in V connecting $\widehat{z}_1$ and an accessible point of K'_n such that $\gamma \cap K'_n$ has exactly one point. By choosing a sub-arc of $\gamma \cup \widehat{f}^{-2}(\gamma)$, we get a cross-cut γ'. On one hand, $T(\gamma') \cap \gamma' = \emptyset$ because $\gamma' \subset V$. On the other hand, we consider the prime-ends compactification of $\widetilde{M} \cup S_\infty \setminus K'_n$ at the end K'_n, and denote by $\widetilde{f}_{K'_n}$ the extension of $\widetilde{f}|_{\widetilde{M}\setminus K'_n}$.

As was in Section 2.9, let $\pi'_{K'_n} : \pi'^{-1}(\widetilde{M} \cup S_\infty \setminus K'_n) \cup \mathbb{R} \to (\widetilde{M} \cup S_\infty \setminus K'_n) \cup S^1$ be the universal cover, and $\widehat{f}_{K'_n}$ the lift of $\widetilde{f}_{K'_n}$ whose restriction to $\pi'^{-1}(\widetilde{M} \cup S_\infty \setminus K'_n)$ is equal to $\widehat{f}$. Recall



that $\rho(\widetilde{I}, K'_n)$ is equal to $k \leq -1$. So, the end-cut $\widehat{f}^{-2}_{K'_n}(\gamma) > T^{-2k-1}(\gamma) \geq T(\gamma)$, which means $\gamma' \cap T(\gamma') \neq \emptyset$. We get a contradiction. $\square$

4 The case of diffeomorphisms

4.1 The index at a degenerate fixed point that is an extremum of a generating function

Let f be a diffeomorphism of $\mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function, we call g a *generating function* of f if $\partial^2_{12} g < 1$, and if

$$f(x,y) = (X,Y) \Leftrightarrow \begin{cases} X - x = & \partial_2 g(X,y), \\ Y - y = & -\partial_1 g(X,y). \end{cases} \tag{1}$$

Every $\mathcal{C}^2$ function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfiying $\partial^2_{12} g \leq c < 1$ defines a diffeomorphism f of $\mathbb{R}^2$ by the previous equations. Moreover, the Jacobian matrix J_f of f is equal to

$$\frac{1}{1 - \partial^2_{12} g(X,y)} \begin{pmatrix} 1 & \partial^2_{22} g(X,y) \\ -\partial^2_{11} g(X,y) & -\partial^2_{11} g(X,y) \partial^2_{22} g(X,y) + (1 - \partial^2_{12} g(X,y))^2 \end{pmatrix}.$$

Since $\det J_f = 1$, the diffeomorphism f is orientation and area preserving. On the other hand, every orientation and area preserving diffeomorphism f of $\mathbb{R}^2$ satisfying $0 < \varepsilon \leq \partial_1(p_1 \circ f) \leq M < \infty$ can be generated by a generating function, where p_1 is the projection onto the first factor.

Moreover, we can naturally define an identity isotopy $I_0 = (f_t)_{t \in [0,1]}$ of f such that f_t is generated by tg. Precisely, the diffeomorphisms f_t are defined by the following equations:

$$f_t(x,y) = (X^t, Y^t) \Leftrightarrow \begin{cases} X^t - x = & t\partial_2 g(X^t,y), \\ Y^t - y = & -t\partial_1 g(X^t,y). \end{cases} \tag{2}$$

A point (x,y) is a fixed point of f if and only if it is a critical point of g. We say that a fixed point (x,y) of f is *degenerate* if 1 is an eigenvalue of $J_f(x,y)$. We will see later that a fixed point (x,y) of f is degenerate if and only if the Hessian matrix of g at (x,y) is degenerate.

We can also define a local generating function. Precisely, if (x,y) is a critical point of a $\mathcal{C}^2$ function g such that $\partial_{12} g(x,y) < 1$, then one can define an orientation and area preserving local diffeomorphism f at (x,y) by the equations (1). On the other side, if (x,y) is a fixed point of an orientation and area preserving diffeomorphism f such that $\partial_1(p_1 \circ f)(x,y) > 0$, where p_1 is the projection to the first factor, then one can find a $\mathcal{C}^2$ function g defined in a neighborhood of (x,y), that defines the germ of f at (x,y) by the equations (1). Moreover,

in both cases, we can define a local isotopy of f at (x, y) by the equations (2), and will call it the *local isotopy induced by* g.

In this section, suppose that $f : (W, 0) \to (W', 0)$ is a local diffeomorphism at $0 \in \mathbb{R}^2$, and that g is a local generating function of f. We will prove the following Proposition 4.1, and deduce Corollary 1.4 as an immediate consequence of Theorem 1.1 and Proposition 4.1.

Proposition 4.1. *If* 0 *is an isolated critical point of* g *and a local extremum of* g*, and if the Hessian matrix of* g *at* 0 *is degenerate, then* $i(f, 0)$ *is equal to* 1.

Proof. The idea is to compute the indices of the local isotopies, so that we can deduce the Lefschetz index by Proposition 2.1.

We denote by $I_0 = (f_t)_{t\in[0,1]}$ the local isotopy induced by g. We have the following lemma

Lemma 4.2. *The blow-up rotation number* $\rho(I_0, 0)$ *is equal to* 0.

Proof. Since $\mathrm{Hess}(g)(0)$ is degenerate, one deduces that 0 is an eigenvalue of $\mathrm{Hess}(g)(0)$. Let v be an eigenvector of $\mathrm{Hess}(g)(0)$ corresponding to the eigenvalue 0. We will prove that v is a common eigenvector of $J_{f_t}(0)$ corresponding to the eigenvalue 1 for $t \in [0, 1]$, and hence the blow-up rotation number $\rho(I_0, 0)$ is equal to 0.

Write

$$\mathrm{Hess}(g)(0) = \begin{pmatrix} \varrho & \sigma \\ \sigma & \tau \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} a \\ b \end{pmatrix}.$$

Then, one deduces that

$$\varrho\tau - \sigma^2 = 0, \quad \varrho a + \sigma b = 0, \quad \text{and } \sigma a + \tau b = 0.$$

By a direct computation, one knows that for every $t \in [0, 1]$,

$$J_{f_t}(0) = \frac{1}{1 - t\sigma} \begin{pmatrix} 1 & t\tau \\ -t\varrho & -t^2\varrho\tau + (1 - t\sigma)^2 \end{pmatrix} = \frac{1}{1 - t\sigma} \begin{pmatrix} 1 & t\tau \\ -t\varrho & 1 - 2t\sigma \end{pmatrix},$$

and then

$$J_{f_t}(0)v = \frac{1}{1 - t\sigma} \begin{pmatrix} a + t\tau b \\ -t\varrho a + b - 2t\sigma b \end{pmatrix} = \frac{1}{1 - t\sigma} \begin{pmatrix} a - t\sigma a \\ t\sigma b + b - 2t\sigma b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}.$$

□

Since f is area preserving, the rotation set at 0 is not empty. By the assertion iv) of Proposition 2.12, and the previous lemma, one can deduce that $\rho_s(I_0, 0)$ is reduced to 0, and that for all local isotopy I of f that is not equivalent to I_0, the rotation set $\rho_s(I, 0)$ is reduced to a non-zero integer.

Lemma 4.3. *If* I *is a local isotopy of* f *that is not equivalent to* I_0*, then* $i(I, 0)$ *is equal to* 0.

Proof. Let $\mathcal{F}$ be foliation locally transverse to I. Since $\rho_s(I, 0)$ is reduced to a non-zero integer, one can deduce by the assertion iii) of Proposition 2.12 that 0 is either a sink or a source of $\mathcal{F}$. By Proposition 2.3, one deduces that $i(I, 0) = i(\mathcal{F}, 0) - 1 = 0$. □

In order to compute the index of I_0, we will construct an isotopy I' that is equivalent to I_0, and prove that $i(I', 0) = 0$.

We define $I' = (f'_t)_{t\in[0,1]}$ in a neighborhood of 0 by

$$f'_t(x, y) = \begin{cases} (x, y) + 2t(X - x, 0) & \text{for } 0 \le t \le 1/2, \\ (X, y) + (2t - 1)(0, Y - y) & \text{for } 1/2 \le t \le 1, \end{cases}$$

where $(X, Y) = f(x, y)$.

Lemma 4.4. *The family $I' = (f'_t)_{t\in[0,1]}$ is a local isotopy of f.*

Proof. For every fixed $t \in [0, 1]$, We will prove that f'_t is a local diffeomorphism by computing the determinant of the Jacobian matrices, and then get the result.

Indeed, one knows

$$\partial_1 X = 1/(1 - \partial_{12} g) > 0.$$

Then for $t \in [0, 1/2]$,

$$\det J_{f'_t} = \det \begin{pmatrix} 1 + 2t(\partial_1 X - 1)) & 2t\partial_2 X \\ 0 & 1 \end{pmatrix} = 2t\partial_1 X + (1 - 2t) > 0;$$

and for $t \in [1/2, 1]$,

$$\det J_{f'_t} = \det \begin{pmatrix} \partial_1 X & \partial_2 X \\ (2t - 1)\partial_1 Y & (2 - 2t) + (2t - 1)\partial_2 Y \end{pmatrix} = (2t - 1) \det J_f + (2 - 2t)\partial_1 X > 0.$$

□

Lemma 4.5. *The blow-up rotation number $\rho(I', 0)$ is equal to 0, and hence I' is equivalent to I_0.*

Proof. As in the proof of Lemma 4.2, we will prove that an eigenvector of $\mathrm{Hess}(g)(0)$ corresponding to the eigenvalue 0 is a common eigenvector of $J_{f'_t}(0)$ corresponding to the eigenvalue 1 for $t \in [0, 1]$, and hence deduce the lemma.

We keep the notations in the proof of Lemma 4.2, and recall that

$$\varrho\tau - \sigma^2 = 0, \quad \varrho a + \sigma b = 0, \quad \text{and } \sigma a + \tau b = 0.$$

For $t \in [0, 1/2]$,

$$J_{f'_t}(0) = \mathrm{Id} + 2t \begin{pmatrix} \partial_1 X(0, 0) - 1 & \partial_2 X(0, 0) \\ 0 & 0 \end{pmatrix} = \mathrm{Id} + \frac{2t}{1 - \sigma} \begin{pmatrix} \sigma & \tau \\ 0 & 0 \end{pmatrix},$$

and

$$J_{f'_t}(0)v = v + \frac{2t}{1 - \sigma} \begin{pmatrix} \sigma a + \tau b \\ 0 \end{pmatrix} = v.$$

For $t \in [1/2, 1]$,

$$J_{f'_t}(0) = J_f(0) - (2 - 2t) \begin{pmatrix} 0 & 0 \\ \partial_1 Y(0, 0) & \partial_2 Y(0, 0) - 1 \end{pmatrix} = J_f(0) - \frac{2 - 2t}{1 - \sigma} \begin{pmatrix} 0 & 0 \\ -\varrho & -\sigma \end{pmatrix},$$

and

$$J_{f'_t}(0)v = J_f(0)v + \frac{2 - 2t}{1 - \sigma} \begin{pmatrix} 0 \\ \varrho a + \sigma b \end{pmatrix} = v.$$

We have verified that v is a common eigenvector of $J_{f'_t}(0)$ corresponding to the eigenvalue 1 for $t \in [0, 1]$. □

To conclude, we will define a locally transverse foliation $\mathcal{F}_0$ of I' such that 0 is a sink or a source of $\mathcal{F}_0$, and then deduce by Proposition 2.3 that $i(I', 0) = i(\mathcal{F}_0, 0) - 1 = 0$. Indeed, let $\mathcal{F}_0$ be the foliation in a neighborhood of 0 whose leaves are the integral curves of the gradient vector field[4] of g. One knows that 0 is a sink of $\mathcal{F}_0$ if 0 is a local maximum of g, and is a source of $\mathcal{F}_0$ if 0 is a minimum of g. We can finish our proof by the following lemma.

Lemma 4.6. *The foliation $\mathcal{F}_0$ is locally transverse to I'.*

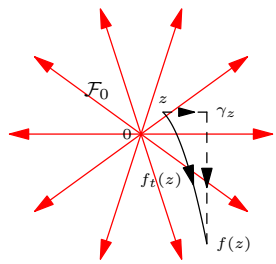


Figure 4: The dynamics and foliation generated by $g(x, y) = x^2 + y^2$

Proof. Let U be a sufficiently small Jordan domain containing 0 such that $\mathcal{F}_0$ is well defined on U, and $V \subset U$ be a sufficiently small neighborhood of 0 such that f'_t is well defined on V for $t \in [0, 1]$, that f does not have any other fixed point in V except 0, and that $\cup_{t\in[0,1]} f'_t(V) \subset U$. We will prove that for every $z = (x, y) \in V \setminus \{0\}$, the path $\gamma_z : t \mapsto f'_t(x, y)$ is positively transverse to $\mathcal{F}_0$, and then deduce the lemma.

Indeed, for $t \in [0, 1/2]$,

$$\begin{aligned}
& \det \begin{pmatrix} 2(X-x) & \partial_1 g(f'_t(x,y)) \\ 0 & \partial_2 g(f'_t(x,y)) \end{pmatrix} \\
= \; & 2(X-x)\partial_2 g(f'_t(x,y)) \\
= \; & 2(X-x)\partial_2 g(2tX + (1-2t)x, y) \\
= \; & 2(X-x)[\partial_2 g(X,y) + (2t-1)(X-x)\partial^2_{12} g(\xi, y)] \\
= \; & 2(X-x)^2[1 - (1-2t)\partial^2_{12} g(\xi, y)] \geq 0
\end{aligned}$$

where ξ is a real number between x and X, and the inequality is strict if $X \neq x$.

For $t \in [1/2, 1]$,

$$\begin{aligned}
& \det \begin{pmatrix} 0 & \partial_1 g(f'_t(x,y)) \\ 2(Y-y) & \partial_2 g(f'_t(x,y)) \end{pmatrix} \\
= \; & -2(Y-y)\partial_1 g(f'_t(x,y)) \\
= \; & -2(Y-y)\partial_1 g(X, (2-2t)y + (2t-1)Y) \\
= \; & -2(Y-y)[\partial_1 g(X,y) + (2t-1)(Y-y)\partial^2_{12} g(X, \eta)] \\
= \; & 2(Y-y)^2[1 - (2t-1)\partial^2_{12} g(X, \eta)] \geq 0
\end{aligned}$$

where η is a real number between y and Y, and the inequality is strict if $Y \neq y$.

Since $z = (x, y)$ is not a fixed point, either $X \neq x$ or $Y \neq y$. If both of the inequalities are satisfied, γ_z is positively transverse to $\mathcal{F}_0$; if $X \neq x$ and $Y = y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ is positively

[4] It means the vector field: $(x, y) \mapsto (\partial_1 g(x, y), \partial_2 g(x, y))$.

transverse to $\mathcal{F}_0$, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ is reduced to a point; if $X = x$ and $Y \neq y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ is reduced to a point, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ is positively transverse to $\mathcal{F}_0$. □

□

Remark 4.7. In the proof, we have indeed proven that $\mathcal{F}_0$ is locally transverse to any local isotopy of f that is equivalent to I_0.

4.2 Discrete symplectic actions and symplectically degenerate extrema

In this section, we will introduce symplectically degenerate extrema. More details can be found in [Maz13].

We say that a diffeomorphism $F : \mathbb{T}^2 \to \mathbb{T}^2$ is *Hamiltonian* if it is area preserving and if there exists a lift f satisfying

$$f(z+k) = f(z) + k \text{ for all } k \in \mathbb{Z}^2, \text{ and } \int_{\mathbb{T}^2} (f - \mathrm{Id}) dxdy = 0.$$

Refering to [MS98], this definition coincides with the usual definition of a Hamiltonian diffeomorphism of a symplectic manifold. More precisely, we call a time-dependent vector field $(X_t)_{t\in\mathbb{R}}$ a *Hamiltonian vector field* if it is defined by the equation:

$$dH_t = \omega(X_t, \cdot),$$

where (M, ω) is a symplectic manifold and $H : \mathbb{R} \times M \to \mathbb{R}$ is a smooth function. The Hamiltonian vector field induces a *Hamiltonian flow* $(\varphi_t)_{t\in\mathbb{R}}$ on M, which is the solution of the following equation

$$\frac{\partial}{\partial t}\varphi_t(z) = X_t(\varphi_t(z)).$$

We say that a diffeomorphism F of M is a *Hamiltonian diffeomorphism* if it is the time-1 map of a Hamiltonian flow. So, for a Hamiltonian diffeomorphism, there exists a natural identity isotopy I which is defined by the Hamiltonian flow. We say that a fixed point of a Hamiltonian diffeomorphism is *contractible* if its trajectory along I is a loop homotopic to zero in M, and that a q-periodic point of a Hamiltonian diffeomorphism is *contractible* if it is a contractible fixed point of F^q.

Let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian diffeomorphism. Then F is the time-1 map of a Hamiltonian flow, and we can factorize F by

$$F = F_{k-1} \circ \cdots \circ F_0,$$

where F_j is $\mathcal{C}^1$-close to the identity, for $j = 0, \cdots, k-1$. For every j, let f_j be the lift of F_j that is $\mathcal{C}^1$-close to the identity, and g_j be a generating function of f_j. We define the *discrete symplectic action*

$$g : \mathbb{R}^{2k} \to \mathbb{R}$$

by

$$g(z) := \sum_{j\in\mathbb{Z}_k} (< y_j, x_j - x_{j+1} > + g_j(x_{j+1}, y_j)),$$

where $z = (z_0, ..., z_{k-1})$ and $z_j = (x_j, y_j)$.

By a direct computation, we know that for every $j \in \mathbb{Z}_k$,

$$\frac{\partial}{\partial x_j} g(z) = y_j - y_{j-1} + \partial_1 g_{j-1}(x_j, y_{j-1}), \quad \text{and } \frac{\partial}{\partial y_j} g(z) = x_j - x_{j+1} + \partial_2 g_j(x_{j+1}, y_j).$$

So, $z \in \mathbb{R}^{2k}$ is a critical point of g if and only if $z_{j+1} = f_j(z_j)$ for every $j \in \mathbb{Z}_k$, and therefore if and only if $z_0 \in \mathbb{R}^2$ is a fixed point of $f = f_{k-1} \circ \cdots \circ f_0$.

In particular, each f_j commutes with the integer translation, and so g is invariant by the diagonal action of $\mathbb{Z}^2$ on $\mathbb{R}^{2k}$ and descends to a function

$$G : \mathbb{R}^{2k}/\mathbb{Z}^2 \to \mathbb{R}.$$

Moreover, $[z] \in \mathbb{R}^{2k}/\mathbb{Z}^2$ is a critical point of G if and only if $z_{j+1} = f_j(z_j)$ for every $j \in \mathbb{Z}_k$, and therefore if and only if $[z_0] \in \mathbb{T}^2$ is a contractible fixed point of F, where $F = F_{k-1} \circ \cdots \circ F_0$. In particular, critical points of G one-to-one correspond to contractible fixed points of F. Moreover, for any period $q \in \mathbb{N}$, contractible q-periodic points of F correspond to the equivalent classes in $\mathbb{R}^{2kp}/\mathbb{Z}^2$ of critical points of the discrete symplectic action $g^{\times q} : \mathbb{R}^{2kp}/\mathbb{Z}^2 \to \mathbb{R}$ defined by

$$g^{\times q}(z) := \sum_{j \in \mathbb{Z}_{kq}} (< y_j, x_j - x_{j+1} > + g_{(j \mod k)}(x_{j+1}, y_j)),$$

where $z = (z_0, ..., z_{kp-1})$ and $z_j = (x_j, y_j)$.

Moreover, if $[z_0] \in \mathbb{T}^2$ is a contractible fixed point of F, then by a suitable shift one can suppose that $[z_0]$ is fixed along the Hamiltonian flow, and hence the factors F_j fixes $[z_0]$ for $j = 0, \cdots, k-1$. So, z_0 is a fixed point of each f_j and a critical point of each g_j. We define the the graded group of local homology

$$C_*(z_0^{\times kn}) := H_*(\{g^{\times n} < g^{\times n}(z_0^{\times kn}) \cup \{z_0^{\times kn}\}, g^{\times n} < g^{\times n}(z_0^{\times kn})).$$

Then $C_j(z_0^{\times kn})$ are always trivial for $j < \mathrm{mor}(z_0^{\times kn})$ and $j > \mathrm{mor}(z_0^{\times kn}) + \mathrm{nul}(z_0^{\times kn})$, where $\mathrm{mor}(z_0^{\times kn})$ is the dimension of negative eigenvector space of Hessian matrix of $g^{\times n}$ at $z_0^{\times kn}$, and $\mathrm{nul}(z_0^{\times kn})$ is the dimension of the kernel of Hessian matrix of $g^{\times n}$ at $z_0^{\times kn}$.

We say that z_0 is a *symplectically degenerate maximum* (resp. *symplectically degenerate minimum*) if z_0 is an isolated local maximum (resp. minumum) of the generating functions $g_0, \cdots, g_{k-1}$, and the local homology $C_{kn+1}(z_0^{\times kn})$ is non-trivial for infinitely many $n \in \mathbb{N}$.

Proposition 4.8 ([Maz13][Rue85]). *Let $z = z_0^{\times k}$ be a critical point of g such that $C_{kn+1}(z^{\times n})$ is non-trivial for infinitely $n \in \mathbb{N}$. Then 1 is the only eigenvalue of $DF([z_0])$, and the blow-up rotation number $\rho(I, [z_0])$ is equal to 0 for any identity isotopy of F fixing $[z_0]$.*

Remark 4.9. In particular, a symplectically degenerate extremum satisfies the condition of the proposition, and hence is a degenerate fixed point of F.

4.3 The index at a symplectically degenerate extremum

As in the previous subsection, let $F : \mathbb{T}^2 \to \mathbb{T}^2$ be a Hamiltonian diffeomorphism, and $F = F_{k-1} \circ \cdots \circ F_0$ be a factorization by Hamiltonian diffeomorphisms F_i which are $\mathcal{C}^1$-close to the identity. Let f_j be the lift of F_j to $\mathbb{R}^2$ that is $\mathcal{C}^1$-close to the identity, and g_j be a generating function of f_j, for $j = 0, \cdots, k-1$. As was recalled in the previous subsection, if z_0 is a symplectically degenerate extremum, then the blow-up rotation number $\rho(I, [z_0])$ is equal to 0 for any identity isotopy of F fixing $[z_0]$. We will prove the following Proposition 4.10, and then can deduce Theorem 1.5 as an immediate corollary of Theorem 1.1.

Proposition 4.10. *If z_0 is a symplectically extremum, then $i(F,[z_0])$ is equal to 1.*

We will only deal with the case where 0 is a symplectically degenerated maximum, the other case can be treated similarly. Let us begin by some lemmas.

Lemma 4.11. *Suppose that g is a (local or global) generating function of a diffeomorphism f, and that 0 is a local maximum of g such that the Hessian matrix of g at 0 is degenerate. Let $I=(f_t)$ be the identity isotopy of f induced by g as in Section 4.1, and $\theta(t)$ be a continuous function such that*

$$\frac{J_{f_t}(0)\begin{pmatrix}\cos\theta(0)\\ \sin\theta(0)\end{pmatrix}}{\|J_{f_t}(0)\begin{pmatrix}\cos\theta(0)\\ \sin\theta(0)\end{pmatrix}\|}=\begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}.$$

Then, one can deduce that $\theta(1)\geq\theta(0)$.

Proof. As in Section 4.1, we denote the Hessian of g at 0 by

$$\mathrm{Hess}(g)(0)=\begin{pmatrix}\varrho & \sigma\\ \sigma & \tau\end{pmatrix}.$$

Since 0 is a local maximal point of g, Hess$g(0)$ is negative semi-definite. So, we knows that

$$\varrho\leq 0,\quad \tau\leq 0,\quad \text{and } \varrho\tau-\sigma^2=0.$$

As was proved in Section 4.1, if (a,b) is a unit eigenvector of $\mathrm{Hess}(g)(0)$ corresponding to the eigenvalue 0 , then it is a common eigenvector of $J_{f_t}(0)$ corresponding to the eigenvalue 1. Recall that

$$\varrho a+\sigma b=0,\quad \text{and } \sigma a+\tau b=0.$$

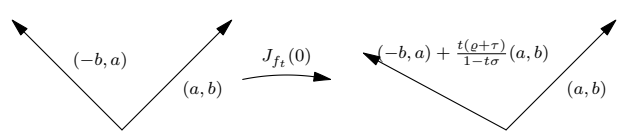
So,

$$J_{f_t}(0)\begin{pmatrix}-b\\ a\end{pmatrix}=\frac{1}{1-t\sigma}\begin{pmatrix}1 & t\tau\\ -t\varrho & 1-2t\sigma\end{pmatrix}\begin{pmatrix}-b\\ a\end{pmatrix}=\begin{pmatrix}-b\\ a\end{pmatrix}+\frac{t(\varrho+\tau)}{1-t\sigma}\begin{pmatrix}a\\ b\end{pmatrix}.$$

Therefore

$$J_{f_t}(0)\Omega=\Omega\begin{pmatrix}1 & \frac{t(\varrho+\tau)}{1-t\sigma}\\ 0 & 1\end{pmatrix},$$

where $\Omega=\begin{pmatrix}a & -b\\ b & a\end{pmatrix}$ is a normal matrix. Since $\frac{t(\varrho+\tau)}{1-t\sigma}\leq 0$, one can deduce that $\theta(1)\geq\theta(0)$. □



Lemma 4.12. *If 0 is a symplectically degenerate maximum, then there exists a normal matrix Ω such that*

$$\Omega^{-1}J_{f_j}(0)\Omega=\begin{pmatrix}1 & c_j\\ 0 & 1\end{pmatrix}$$

for $j=0,\cdots,k-1$, where c_j are non-positive real numbers.

Proof. Let $I_j = (f_{j,t})_{t\in[0,1]}$ be the local isotopy of f_j induced by g_j as in section 4.1. Let $\mathcal{F}_j$ be the foliation whose leaves are the integral curves of the gradient vector field of g_j. As in Section 4.1, one can deduce that 0 is a sink of $\mathcal{F}_j$ and that $\mathcal{F}_j$ is locally transverse to I_j. Therefore, one knows that $\rho(I_j, 0) \geq 0$, and that $\rho(I_j, 0) = 0$ if and only if 0 is a degenerate fixed point of f_j.

Let $\theta : [0, k] \to \mathbb{R}$ be a continuous function such that

$$\frac{J_{f'_{jt}}(0)\begin{pmatrix}\cos\theta(j)\\ \sin\theta(j)\end{pmatrix}}{\|J_{f'_{jt}}(j)\begin{pmatrix}\cos\theta(j)\\ \sin\theta(j)\end{pmatrix}\|} = \begin{pmatrix}\cos\theta(j+t)\\ \sin\theta(j+t)\end{pmatrix}.$$

One knows that $\theta(j+1) > \theta(j)$ if $\rho(I_j, 0) > 0$, and $\theta(j+1) \geq \theta(j)$ if $\rho(I_j, 0) = 0$. But we know that $\rho(I_{k-1}\cdots I_0, z_0) = \rho(I, [z_0]) = 0$, so there exists $\theta(0) \in \mathbb{R}$ and a continuous function θ as above such that $\theta(k) = \theta(0)$. Therefore, $\rho(I_j, z_0) = 0$ for $j = 0, \cdots, k-1$ and $\begin{pmatrix}\cos\theta(0)\\ \sin\theta(0)\end{pmatrix}$ is a common eigenvector of $J_{f_j}(0)$ corresponding to the eigenvalue 1. As in the proof of the previous lemma, we can prove this lemma by choosing

$$\Omega = \begin{pmatrix}\cos\theta(0) & -\sin\theta(0)\\ \sin\theta(0) & \cos\theta(0)\end{pmatrix}.$$

□

Lemma 4.13. *Suppose that g is a (local or global) generating function of a diffeomorphism f, that 0 is a local maximum of g, and that the Hessian matrix of g at 0 is degenerate. If Ω is a normal matrix, and if $f' = \Omega^{-1} f \Omega$ is generated by g' in a neighborhood of 0, then 0 is a local maximum of g' and* $\mathrm{Hess}(g')(0)$ *is degenerate.*

Proof. Since $\mathrm{Hess}(g)(0)$ is degenerate, 1 is an eigenvalue of $J_f(0)$ and hence an eigenvalue of $J_{f'}(0)$. So, $\mathrm{Hess}(g')(0)$ is degenerate.

Let $\mathcal{F}$ be the foliation whose leaves are integral curves of the gradient vector field of g, and $\mathcal{F}'$ be the foliation whose leaves are integral curves of the gradient vector field of g'. Let I_0 be a local isotopy of f satisfies $\rho(I_0, 0) = 0$, and I'_0 be a local isotopy of f satisfies $\rho(I'_0, 0) = 0$. As was proved in Section 4.1, $\mathcal{F}$ is locally transverse to I_0 and $\mathcal{F}'$ is locally transverse to I'_0. Therefore, $\Omega \circ \mathcal{F}'$ is locally transverse to I_0. Since 0 is a maximal point of g, it is a sink of $\mathcal{F}$. By the remark that follows Proposition 2.9, one deduces that 0 is a sink of $\Omega \circ \mathcal{F}'$, and hence a sink of $\mathcal{F}'$. Therefore, 0 is a local maximum of g'. □

Lemma 4.14. *Let g_0 and g_1 be local generating functions of f_0 and f_1 respectively such that 0 is a local maximal point of both g_0 and g_1, and that the Hessian matrices satisfy*

$$Hess(g_i)(0) = \begin{pmatrix}0 & 0\\ 0 & c_i\end{pmatrix},$$

where $c_i \leq 0$ for $i = 0, 1$. Then there exists a function g which is a generating function of $f = f_1 \circ f_0$ in a neighborhood of 0. Moreover, 0 is a local maximal point of g, and the Hessian matrix satisfies

$$Hess(g)(0) = \begin{pmatrix}0 & 0\\ 0 & c_0 + c_1\end{pmatrix}.$$

Proof. Suppose that $g_0(0) = g_1(0) = 0$. Since 0 is a local maximal point of both g_0 and g_1, it is a critical point of both g_0 and g_1. So,

$$\partial_1 g_0(0,0) = \partial_2 g_0(0,0) = \partial_1 g_1(0,0) = \partial_2 g_1(0,0) = 0.$$

Write $(x_1, y_1) = f_0(x_0, y_0)$ and $(x_2, y_2) = f_1(x_1, y_1)$. By definition of generating functions, one knows that

$$y_1 - y_0 + \partial_1 g_0(x_1, y_0) = 0, \quad \text{and } x_1 - x_2 + \partial_2 g_1(x_2, y_1) = 0. \tag{3}$$

Note that

$$\det \begin{pmatrix} \partial^2_{11} g_0(0,0) & 1 \\ 1 & \partial^2_{22} g_1(0,0) \end{pmatrix} = -1.$$

So, by implicit function theorem, there exists a $\mathcal{C}^1$ diffeomorphism $\varphi : W \to W'$ such that $(x_1, y_1) = \varphi(x_2, y_0)$, where W and W' are sufficiently small neighborhoods of 0 in $\mathbb{R}^2$. Moreover,

$$J_\varphi(0,0) = -\begin{pmatrix} \partial^2_{11} g_0(0,0) & 1 \\ 1 & \partial^2_{22} g_1(0,0) \end{pmatrix}^{-1} \begin{pmatrix} 0 & \partial^2_{12} g_0(0,0) - 1 \\ \partial^2_{12} g_1(0,0) - 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & -c_1 \\ 0 & 1 \end{pmatrix}.$$

Let

$$g(x_2, y_0) = g_0(x_1, y_0) + g_1(x_2, y_1) + (x_2 - x_1)(y_0 - y_1),$$

where $(x_1, y_1) = \varphi(x_2, y_0)$. We know that $g(0,0) = 0$. In a neighborhood of 0, by a direct computation and equations (3), one knows that

$$\begin{aligned} \partial_1 g(x_2, y_0) =& \partial_1 g_0(x_1, y_0)\partial_1 x_1(x_2, y_0) + \partial_1 g_1(x_2, y_1) + \partial_2 g_1(x_2, y_1)\partial_1 y_1(x_2, y_0) \\ &+ (1 - \partial_1 x_1(x_2, y_0))(y_0 - y_1) - \partial_1 y_1(x_2, y_0)(x_2 - x_1) \\ =& \partial_1 g_0(x_1, y_0) + \partial_1 g_1(x_2, y_1). \end{aligned}$$

Similarly, one gets

$$\partial_2 g(x_2, y_0) = \partial_2 g_0(x_1, y_0) + \partial_2 g_1(x_2, y_1).$$

So, g is a $\mathcal{C}^2$ function near 0. Moreover,

$$\partial^2_{12} g(0,0) = \partial^2_{11} g_0(0,0)\partial_2 y_1(0,0) + \partial^2_{12} g_0(0,0) + \partial^2_{12} g_1(0,0)\partial_2 y_1(0,0) = 0.$$

Because g_0 and g_1 locally generate f_0 and f_1 respectively, one deduces

$$\partial_1 g(x_2, y_0) = -(y_2 - y_0) \quad \text{and} \quad \partial_2 g(x_2, y_0) = x_2 - x_0.$$

Therefore, g is a generating function of f in a neighborhood of 0.

By a direct computation, one gets

$$\partial^2_{11} g(0,0) = \partial^2_{11} g_0(0,0)\partial_1 x_1(0,0) + \partial^2_{11} g_1(0,0) + \partial^2_{12} g_1(0,0)\partial_1 y_1(0,0) = 0,$$

and

$$\partial^2_{22} g(0,0) = \partial^2_{12} g_0(0,0)\partial_2 x_1(0,0) + \partial^2_{22} g_0(0,0) + \partial^2_{22} g_1(0,0) = c_0 + c_1.$$

So,

$$\mathrm{Hess}(g)(0) = \begin{pmatrix} 0 & 0 \\ 0 & c_0 + c_1 \end{pmatrix}.$$

We will conclude by proving that 0 is a locally maximum of g. Let $\varepsilon > 0$ be a small real number such that $|\varepsilon c_1| < 1$. We will prove that in a sufficiently small neighborhood of 0,

$$g(x_2, y_0) \leq g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) + g_1(x_2, y_1 + \varepsilon(x_2 - x_1)) \leq 0,$$

and hence 0 is a locally maximum of g because the second inequality is strict for $(x_2, y_0) \neq 0$. Indeed, by Taylor's theorem and equations (3), one knows that in a sufficiently small neighborhood of 0,

$$\begin{aligned} g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) &= g_0(x_1, y_0) + \frac{1}{\varepsilon}\partial_1 g_0(x_1, y_0)(y_0 - y_1) + \frac{1}{2\varepsilon^2}\partial_{11}^2 g_0(\xi, y_0)(y_0 - y_1)^2 \\ &= g_0(x_1, y_0) + \frac{1}{\varepsilon}(y_0 - y_1)^2 + \frac{1}{2\varepsilon^2}\partial_{11}^2 g_0(\xi, y_0)(y_0 - y_1)^2, \end{aligned}$$

where ξ is a real number between x_1 and $x_1 + \frac{1}{\varepsilon}(y_0 - y_1)$. Similarly, one deduces that in sufficiently small neighborhood of 0,

$$g_1(x_2, y_1 + \varepsilon(x_2 - x_1)) = g_1(x_2, y_1) + \varepsilon(x_2 - x_1)^2 + \frac{\varepsilon^2}{2}\partial_{22}^2 g_1(x_2, \eta)(x_2 - x_1)^2,$$

where η is a real number between y_1 and $y_1 + \varepsilon(x_2 - x_1)$. So,

$$\begin{aligned} g(x_2, y_0) =& g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) + g_1(x_2, y_1 + \varepsilon(x_2 - x_1)) \\ &- \frac{1}{2\varepsilon}(y_0 - y_1)^2 - \frac{\varepsilon}{2}(x_2 - x_1)^2 + (x_2 - x_1)(y_0 - y_1) \\ &- \frac{1}{2\varepsilon}(1 + \frac{1}{\varepsilon}\partial_{11} g_0(\xi, y_0))(y_0 - y_1)^2 - \frac{\varepsilon}{2}(1 + \varepsilon\partial_{22} g_1(x_2, \eta))(x_2 - x_1)^2. \end{aligned}$$

For $(x_2, y_0) \neq 0$ that is in a sufficiently small neighborhood of 0, one can suppose that

$$|\frac{1}{\varepsilon^2}\partial_{11} g_0(\xi, y_0)| < 1, \quad \text{and } |\varepsilon\partial_{22} g_1(x_2, \eta)| < 1.$$

So,

$$g(x_2, y_0) \leq g_0(x_1 + \frac{1}{\varepsilon}(y_0 - y_1), y_0) + g_1(x_2, y_1 + \varepsilon(x_2 - x_1)).$$

□

Now, we begin the proof of Proposition 4.10.

Proof of Propostion 4.10. Suppose that z_0 is a symplectically degenerated maximum. By Lemma 4.12, there exists a coordinate transformation such that in the new coordinate system the Jacobian of each f_j at z_0 has the form

$$\begin{pmatrix} 1 & c_j \\ 0 & 1 \end{pmatrix}$$

where c_j is a non-positive real number. We consider everything in the new coordinate system. Each f_j can be locally generated by a generating function g_j', and the Hessian of g_j' at z_0 has the form

$$\begin{pmatrix} 0 & 0 \\ 0 & c_j \end{pmatrix}.$$

By Lemma 4.13, z_0 is a local maximum of each g_j'. So, by Lemma 4.14, we can construct a generating function g' such that

- z_0 is a local maximum of g',
- $\mathrm{Hess}(g')(z_0)$ is degenerate,
- g' generates $f = f_{k-1} \cdots f_0$ in a neighborhood of z_0.

So, by Proposition 4.1, we know $i(f, z_0)$ is equal to 1, and hence $i(F, [z_0])$ is equal to 1. $\square$

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