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Existence of torsion-low maximal identity isotopies for area preserving surface homeomorphisms

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Abstract

The paper concerns area preserving homeomorphisms of surfaces that are isotopic to the identity. The purpose of the paper is to find a maximal identity isotopy such that we can give a fine descriptions of the dynamics of its transverse foliation. We will define a kind of identity isotopies: torsion-low isotopies. In particular, when f is a diffeomorphism with finitely many fixed points such that every fixed point is not degenerate, an identity isotopy I of f is torsion-low if and only if for every point z fixed along the isotopy, the (real) rotation number $\rho(I, z)$, which is well defined when one blows-up f at z, is contained in $(-1, 1)$. We will prove the existence of torsion-low maximal identity isotopies, and we will deduce the local dynamics of the transverse foliations of any torsion-low maximal isotopy near any isolated singularity.

Keywords. transverse foliation, maximal isotopy, rotation set, rotation number, torsion-low

Mathematical Subject Classification. 37E30 37E45

1 Introduction and definitions

Let M be an oriented and connected surface, $f : M \to M$ be a homeomorphism of M that is isotopic to the identity, and I be an isotopy from the identity to f. We call I an *identity isotopy* of f. Let us denote by $\mathrm{Fix}(f)$ the set of fixed points of f, and for every identity isotopy $I = (f_t)_{t\in[0,1]}$ of f, by $\mathrm{Fix}(I) = \cap_{t\in[0,1]}\mathrm{Fix}(f_t)$ the set of fixed points of I. We say that $z \in \mathrm{Fix}(f)$ is a *contractible* fixed point associated to I if the trajectory $\gamma : t \mapsto f_t(z)$ of z along I is a loop homotopic to zero in M.

Suppose that there exist (non singular) oriented topological foliations on M, and fix such a foliation $\mathcal{F}$. We say that a path $\gamma : [0, 1] \to M$ is positively transverse to $\mathcal{F}$ if it locally meets transversely every leaf from the left to the right. We say that $\mathcal{F}$ is a *transverse foliation* of I, if for every $z \in M$, there exists a path that is homotopic to the trajectory of z along I and is positively transverse to $\mathcal{F}$.

Of course the existence of a transverse foliation prohibits the existence of fixed points of I but also contractible fixed points of f associated to I. Patrice Le Calvez [LC05] proved that if f does not have any contractible fixed point associated to I, there exists a transverse foliation of I. Later, Olivier Jaulent [Jau14] generalized this result to the case where there exist contractible fixed points, and obtained singular foliations. He proved that there exist a

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closed subset $X \subset \mathrm{Fix}(f)$ and an identity isotopy I_X on $M \setminus X$ such that $f|_{M\setminus X}$ does not have any contractible fixed point associated to I_X. It means that there exists a singular foliation on M whose set of singularities is X and whose restriction to $M \setminus X$ is transverse to I_X. Recently, François Béguin, Sylvain Crovisier, and Frédéric Le Roux [BCLR] generalized Jaulent's result, and proved that there exists an identity isotopy I of f such that $f|_{M\setminus \mathrm{Fix}(I)}$ does not have any contractible fixed point associated to $I|_{M\setminus \mathrm{Fix}(I)}$. Then, there exists a singular foliation on M whose set of singularities is the set of fixed points of I and whose restriction to $M \setminus \mathrm{Fix}(I)$ is transverse to $I|_{M\setminus \mathrm{Fix}(I)}$. We call such an identity isotopy I a *maximal identity isotopy*, and such a singular foliation a *transverse foliation* of I.

Transverse foliations are fruitful tools in the study of homeomorphisms of surfaces. For example, one can prove the existence of periodic orbits in several cases [LC05], [LC06]; one can give precise descriptions of the dynamics of some homeomorphisms of the torus $\mathbb{R}^2/\mathbb{Z}^2$ [Dáv13], [KT14] ; .... It is a natural question whether we can get a more efficient tool by choosing suitable maximal identity isotopies.

The primary idea is to choose a maximal isotopy that fixes as many fixed points as possible. When $f : M \to M$ is an orientation preserving diffeomorphism, and I is an identity isotopy of f fixing z_0, we can give a natural blow-up at z_0 by replacing z_0 with the unit circle of the tangent space $U_{z_0}M$, where M is equipped with a Riemannian structure. The extension of f to this circle can be induced by the derivative $Df(z_0)$. We define the *blow-up rotation number* $\rho(f, z_0) \in \mathbb{R}/\mathbb{Z}$ to be the Poincaré's rotation number of this homeomorphism on the circle, and can define the *blow-up rotation number* $\rho(I, z_0) \in \mathbb{R}$, that is a representative of $\rho(f, z_0)$ (see Section 2.7). Moreover, if the diffeomorphism f is area preserving, and if there exists a fixed point $z_0 \in \mathrm{Fix}(I)$ such that $|\rho(I, z_0)| > 1$ and that the connected component M_0 of $M \setminus (\mathrm{Fix}(I) \setminus \{z_0\})$ containing z_0 is not homeomorphic to a sphere or a plane, we can find another fixed point of f that is not a fixed point of I as a corollary of a generalized version of Poincaré-Birkhoff theorem. Let us explain briefly the reason: it is easy to prove that z_0 is isolated in $\mathrm{Fix}(I)$ in this case. We consider the universal cover $\pi : \widetilde{M} \to M_0$, and the lift $\widetilde{f}$ of $f|_{M_0}$ that fixes every point in $\pi^{-1}\{z_0\}$. Fix $\widetilde{z}_0 \in \pi^{-1}(z_0)$, and consider the blow-up of $\widetilde{f}$ at $\widetilde{z}_0$. One gets a homeomorphism of the annulus $(\widetilde{M} \setminus \{\widetilde{z}_0\}) \sqcup U_{z_0}\widetilde{M}$. By a generalized version of Poincaré-Birkhoff theorem, this homeomorphism has two fixed points $\widetilde{z}$ and $\widetilde{z}'$ such that $\pi(\widetilde{z})$ and $\pi(\widetilde{z}')$ are distinct fixed points of f but are not fixed points of I. Moreover, if $\mathrm{Fix}(I)$ is finite, by a technical construction, one can find another identity isotopy that fixes $\mathrm{Fix}(I) \setminus \{z_0\}$ and has no less (probably more) fixed points than I (see Section 4). Then, it is reasonable to think that a maximal identity isotopy I such that

$$-1 \leq \rho(I, z) \leq 1 \text{ for all } z \in \mathrm{Fix}(I),$$

fixes more fixed points than a usual one. In this article, we will study a more general case, and prove the existence of such an isotopy as a corollary.

More precisely, we will study orientation and area preserving homeomorphisms of an oriented surface isotopic to the identity, and prove the existence of a special kind of maximal identity isotopies: the torsion-low maximal identity isotopies. In this case, we also have more information about its transverse foliation: we can deduce the local dynamics of a transverse foliation near any isolated singularity.

Now, we give an exact description about what we will do in this article.

We write $f : (W, 0) \to (W', 0)$ for an orientation preserving homeomorphism between two neighborhoods W and W' of 0 in $\mathbb{R}^2$ such that $f(0) = 0$. We say that f is an *orientation preserving local homeomorphism at* 0. More generally, we write $f : (W, z_0) \to (W', z_0)$ for an

orientation preserving local homeomorphism between two neighborhoods W and W' of z_0 in any oriented surface M such that $f(z_0) = z_0$. We say that f is *area preserving* if it preserves a Borel measure without atom such that the measure of every open set is positive and the measure of every compact set is finite.

Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0. A *local isotopy* I of f is a continuous family of local homeomorphisms $(f_t)_{t\in[0,1]}$ fixing z_0. When f is not conjugate to a contraction or an expansion, we can give a preorder on the space of local isotopies such that for two local isotopies I and I', one has $I \lesssim I'$ if and only if there exists $k \geq 0$ such that I' is locally homotopic to $J_{z_0}^k I$, where $J_{z_0} = (R_{2\pi t})_{t\in[0,1]}$ is the local isotopy of the identity such that each $R_{2\pi t}$ is the counter-clockwise rotation through an angle $2\pi t$ about the center z_0. We will give the formal definitions in Section 2.2.

Let $\mathcal{F}$ be a singular oriented foliation on M. We say that $\mathcal{F}$ is *locally transverse* to a local isotopy $I = (f_t)_{t\in[0,1]}$ at z_0, if there exists a neighborhood U_0 of z_0 such that $\mathcal{F}|_{U_0}$ has exactly one singularity z_0, and if for every sufficiently small neighborhood U of z_0, there exists a neighborhood $V \subset U$ such that for all $z \in V \setminus \{z_0\}$, there exists a path in $U \setminus \{z_0\}$ that is homotopic in $U \setminus \{z_0\}$ to the trajectory $t \mapsto f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

We will generalize the definitions of "positive type" and "negative type" by Shigenori Matsumoto [Mat01]. We say that I has a *positive (resp. negative) rotation type* at z_0 if there exists a foliation $\mathcal{F}$ locally transverse to I such that z_0 is a sink (resp. source) of $\mathcal{F}$. We say that I has a *zero rotation type* at z_0 if there exists a foliation $\mathcal{F}$ locally transverse to I such that z_0 is an isolated singularity of $\mathcal{F}$ and is neither a sink nor a source of $\mathcal{F}$. We know that two local isotopies I and I' have the same rotation type if they are locally homotopic.

When z_0 is an isolated fixed point of f, a local isotopy of f has at least one of the previous rotation types. It is possible that a local isotopy of f has two rotation types. But if we assume that f is area preserving (or more generally, satisfies the condition that there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set), we will show in Section 3 that a local isotopy of f has exactly one of the three rotation types. We say that a local isotopy I of an orientation preserving local homeomorphism f at an isolated fixed point is *torsion-low* if

- every local isotopy $I' > I$ has a positive rotation type;
- every local isotopy $I' < I$ has a negative rotation type.

Under the previous assumptions, we will prove in Section 3 the existence of a torsion-low local isotopy of f. Formally, we have the following result:

Theorem 1.1. *Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at an isolated fixed point z_0 such that there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set, then*

- *a local isotopy of f has exactly one of the three kinds of rotation types;*
- *there exists a local isotopy I_0 that is torsion-low at z_0. Moreover, I_0 has a zero rotation type if the Lefschetz index $i(f, z_0)$ is different from 1, and has either a positive or a negative rotation type if the Lefschetz index $i(f, z_0)$ is equal to 1.*

When f is a diffeomorphism fixing z_0, and I is a local isotopy of f at z_0, we can blow-up f at z_0 and define the blow-up rotation number $\rho(I, z_0)$. We say that z_0 is a *degenerate* fixed

point of f if 1 is an eigenvalue of $Df(z_0)$. When f is a homeomorphism, one may fail to find a blow-up at z_0, and cannot define a rotation "number". However, we can generalize it and define a *local rotation set* $\rho_s(I, z_0)$ which was introduced by Le Roux and will be recalled in Section 2.7. A torsion-low local isotopy has the following properties:

Proposition 1.2. *Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving homeomorphism at an isolated fixed point z_0 such that there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set. If I is a torsion-low isotopy at z_0, then*

$$\rho_s(I, z_0) \subset [-1, 1].$$

In particular, if f can be blown-up at z_0, the rotation set is reduced to a single point in $[-1, 1]$. Moreover, if f is a diffeomorphism in a neighborhood of z_0, the blow-up rotation number satisfies

$$-1 \leq \rho(I, z_0) \leq 1,$$

and the inequalities are both strict when z_0 is not degenerate.

When z_0 is not an isolated fixed point and f is area preserving, we will generalize the definition of torsion-low isotopy by considering the local rotation set. We say a local isotopy I of an orientation and area preserving local homeomorphism f at a non-isolated fixed point z_0 is *torsion-low* if $\rho_s(I, z_0) \cap [-1, 1] \neq \emptyset$. One may fail to find a torsion-low local isotopy in some particular cases. In fact, there exists an orientation and area preserving local homeomorphism whose local rotation set is reduced to ∞, and hence there does not exist any torsion-low isotopy of this local homeomorphism. We will give such an example in Section 5.

However, if f is an area preserving homeomorphism of M that is isotopic to the identity, we can find a maximal identity isotopy I that is torsion-low at every fixed point of I. Formally, we will prove the following theorem in Section 4.1, which will be the main result of this article.

Theorem 1.3. *Let f be an area preserving homeomorphism of M that is isotopic to the identity. Then, there exists a maximal identity isotopy I such that I is torsion-low at z for every $z \in \mathrm{Fix}(I)$.*

Remark 1.4. In the proof of this theorem, we will use an unpublished yet result of Béguin, Le Roux and Crovisier, when $\mathrm{Fix}(f)$ is not totally disconnected; but we do not need their result when $\mathrm{Fix}(f)$ is totally disconnected.

Remark 1.5. The area preserving condition is necessary for the result of this theorem. Even if f has only finitely many fixed points and is area preserving near each fixed point, one may still fail to find a maximal isotopy I that is torsion-low at every $z \in \mathrm{Fix}(I)$. We will give such an example in Section 5.

We say that an identity isotopy is *torsion-low* if it is torsion-low at each of its fixed points. A torsion-low maximal isotopy gives more information than a usual one. We have the following three results related to the questions at the beginning of this section. The first two will be proved in Section 4.1, while the third is an immediately corollary of Proposition 1.2 and Theorem 1.3.

Proposition 1.6. *Let f be an area preserving homeomorphism of M that is isotopic to the identity and has finitely many fixed points. Let*

$$n = \max\{\#\mathrm{Fix}(I) : I \text{ is an identity isotopy of } f\}.$$

Then, there exists a torsion-low identity isotopy of f with n fixed points.

Proposition 1.7. *Let f be an area preserving homeomorphism of M that is isotopic to the identity, I be a maximal identity isotopy that is torsion-low at $z \in \mathrm{Fix}(I)$, and $\mathcal{F}$ be a transverse foliation of I. If z is isolated in the set of singularities of $\mathcal{F}$, then we have the following results:*

- *if z is an isolated fixed point of f such that $i(f,z) \neq 1$, then z is a saddle*[1] *of $\mathcal{F}$ and $i(\mathcal{F},z) = i(f,z)$;*
- *if z is an isolated fixed point of f such that $i(f,z) = 1$, or if z is not isolated in $\mathrm{Fix}(f)$, then z is a sink or a source of $\mathcal{F}$.*

Proposition 1.8. *Let f be an area preserving diffeomorphism of M that is isotopic to the identity. Then, there exists a maximal identity isotopy I, such that for all $z \in \mathrm{Fix}(I)$,*

$$-1 \leq \rho(I,z) \leq 1.$$

Moreover, the inequalities are both strict when z is not degenerate.

Now we give a plan of this article. In Section 2, we will introduce many definitions and will recall previous results that will be essential in the proofs of our results. In Section 3, we will study the local rotation types at an isolated fixed point of an orientation preserving homeomorphism and will prove Theorem 1.1 and Proposition 1.2. In Section 4, we will prove the existence of a global torsion-low maximal identity isotopy: Theorem 1.3 in two cases, and will study its properties: Proposition 1.6, 1.7 and 1.8. In Section 5, we will give some explicit examples to get the optimality of our results. In Appendix A, we will introduce a way to construct maximal isotopies and transverse foliations by generating functions, which will be used when constructing examples.

2 Preliminaries

2.1 Lefschetz index

Let $f : (W,0) \to (W',0)$ be an orientation preserving local homeomorphism at an isolated fixed point $0 \in \mathbb{R}^2$. Denote by S^1 the unit circle. If $C \subset W$ is a simple closed curve which does not contain any fixed point of f, then we can define the *index* $i(f,C)$ of f along the curve C to be the Brouwer degree of the map

$$\begin{array}{rcl} \varphi : S^1 & \to & S^1 \\ t & \mapsto & \dfrac{f(\gamma(t)) - \gamma(t)}{||f(\gamma(t)) - \gamma(t)||}, \end{array}$$

where $\gamma : S^1 \to C$ is a parametrization compatible with the orientation, and $\| \cdot \|$ is the usual Euclidean norm. We define a *Jordan domain* to be a bounded domain whose boundary is a simple closed curve. Let U be a Jordan domain containing 0 and contained in a sufficiently small neighborhood of 0. We define the *Lefschetz index* of f at 0 to be $i(f,\partial U)$, which is independent of the choice of U. We denote it by $i(f,0)$.

More generally, if $f : (W,z_0) \to (W',z_0)$ is an orientation preserving local homeomorphism at a fixed point z_0 on a surface M, we can conjugate it topologically to an orientation preserving local homeomorphism g at 0 and define the the *Lefschetz index* of f at z_0 to be $i(g,0)$, which is independent of the choice of the conjugation. We denote it by $i(f,z_0)$.

[1] The precise definitions of a saddle, a sauces and a sink will be given in Section 2.5.

2.2 Local isotopies and the index of local isotopies

Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at $z_0 \in M$. A *local isotopy* I of f at z_0 is a family of homeomorphisms $(f_t)_{t\in[0,1]}$ such that

- every f_t is a homeomorphism between the neighborhoods $V_t \subset W$ and $V'_t \subset W'$ of z_0, and $f_0 = \mathrm{Id}_{V_0}$, $f_1 = f|_{V_1}$;
- for all t, one has $f_t(z_0) = z_0$;
- the sets $\{(z,t) \in M \times [0,1] : z \in V_t\}$ and $\{(z,t) \in M \times [0,1] : z \in V'_t\}$ are both open in $M \times [0,1]$;
- the maps $(z,t) \mapsto f_t(z)$ and $(z,t) \mapsto f_t^{-1}(z)$ are both continuous.

Let us introduce the index of a local isotopy which was defined by Le Roux [LR13] and Le Calvez [LC08].

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f. We denote by D_r the disk with radius r and centered at 0. Then, each f_t is well defined in the disk D_r if r is sufficiently small. Let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta}, \end{array}$$

be the universal covering projection, and $\widetilde{I} = (\widetilde{f_t})_{t\in[0,1]}$ be the lift of $I|_{D_r\setminus\{0\}}$ to $\mathbb{R} \times (-r, 0)$ such that f_0 is the identity. Let $\widetilde{\gamma} : [0,1] \to \mathbb{R} \times (-r, 0)$ be a path from $\widetilde{z} \in \mathbb{R} \times (-r, 0)$ to $\widetilde{z} + (1, 0)$. The map

$$t \mapsto \frac{\widetilde{f_1}(\widetilde{\gamma}(t)) - \widetilde{\gamma}(t)}{||\widetilde{f_1}(\widetilde{\gamma}(t)) - \widetilde{\gamma}(t)||}$$

takes the same value at both 0 and 1, and hence descends to a continuous map $\varphi : [0,1]/_{0\sim 1} \to S^1$. We define the *index of the isotopy* I at 0 to be the Brouwer degree of φ, which does not depend on the choice of $\widetilde{\gamma}$ when r is sufficiently small. We denote it by $i(I, 0)$.

Suppose that f is not conjugate to a contraction or an expansion. We will give a preorder on the set of local isotopies of f at 0. Let $I' = (f'_t)_{t\in[0,1]}$ be another local isotopy of f at 0. For sufficiently small r, each f'_t is also well defined in D_r. Let $\widetilde{I'} = (\widetilde{f'_t})_{t\in[0,1]}$ be the lift of $I'|_{D_r\setminus\{0\}}$ on $\mathbb{R} \times (-r, 0)$ such that $\widetilde{f'_0}$ is the identity. We write $I \lesssim I'$ if

$$p_1\widetilde{f_1}(\theta, y) \leq p_1\widetilde{f'_1}(\theta, y) \text{ for all } (\theta, y) \in \mathbb{R} \times (-r, 0),$$

where p_1 is the projection onto the first factor. Thus $\lesssim$ is a preorder, and

$$I \lesssim I' \text{ and } I' \lesssim I \Longleftrightarrow I \text{ is locally homotopic to } I'.$$

In this case, we will say that I and I' are *equivalent* and write $I \sim I'$.

More generally, we consider an orientation preserving local homeomorphism on an oriented surface. Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at a fixed point z_0 in M. Let $h : (U, z_0) \to (U', 0)$ be a local homeomorphism. Then $h \circ I \circ h^{-1} = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$ is a local isotopy at 0, and we define the *index of* I at z_0 to be $i(h \circ I \circ h^{-1}, 0)$, which is independent of the choice of h. We denote it by $i(I, z_0)$. Similarly, we have a preorder on the set of local isotopies of f at z_0.

Let $I = (f_t)_{t\in[0,1]}$ and $I' = (g_t)_{t\in[0,1]}$ be two isotopies (resp. local isotopies). We denote by I^{-1} the isotopy (resp. local isotopy) $(f_t^{-1})_{t\in[0,1]}$, by $I'I$ the isotopy (resp. local isotopy) $(\varphi_t)_{t\in[0,1]}$ such that

$$\varphi_t = \begin{cases} f_{2t} & \text{for } t \in [0, \frac{1}{2}], \\ g_{2t-1} \circ f_t & \text{for } t \in [\frac{1}{2}, 0], \end{cases}$$

and by I^n the isotopy (resp. local isotopy) $\underbrace{I \cdots I}_{n \text{ times}}$ for every $n \geq 1$.

Let $J_{z_0} = (R_{2\pi t})_{t\in[0,1]}$ be the isotopy such that each $R_{2\pi t}$ is the counter-clockwise rotation through an angle $2\pi t$ about the center z_0, then

$$I \lesssim I' \text{ if and only if } I' \sim J^q I \text{ where } q \geq 0.$$

The Lefschetz index at an isolated fixed point and the indices of the local isotopies are related. We have the following result:

Proposition 2.1. *([LC08][LR13]) Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at an isolated fixed point z_0. Then, we have the following results:*

- *if $i(f, z_0) \neq 1$, there exists a unique homotopy class of local isotopies such that $i(I, z_0) = i(f, z_0) - 1$ for every local isotopy I in this class, and the indices of the other local isotopies are equal to 0;*
- *if $i(f, z_0) = 1$, the indices of all the local isotopies are equal to 0.*

2.3 Transverse foliations and index at an isolated end

In this section, we will introduce the index of a foliation at an isolated end. More details can be found in [LC08].

Let M be an oriented surface and $\mathcal{F}$ be an oriented topological foliation on M. For every point z, there is a neighborhood V of z and a homeomorphism $h : V \to (0, 1)^2$ preserving the orientation such that the images of the leaves of $\mathcal{F}|_V$ are the vertical lines oriented upward. We call V a *trivialization neighborhood* of z, and h a *trivialization chart*.

Let z_0 be an isolated end of M. We choose a small annulus $U \subset M$ such that z_0 is an end of U. Let $h : U \to \mathbb{D} \setminus \{0\}$ be an orientation preserving homeomorphism which sends z_0 to 0. Let $\gamma : S^1 \to \mathbb{D} \setminus \{0\}$ be a simple closed curve that is homotopic to $\partial\mathbb{D}$ in $\overline{\mathbb{D}} \setminus \{0\}$. We can cover the curve by finite trivialization neighborhoods $(V_i)_{1\leq i\leq n}$ of the foliation $\mathcal{F}_h$, where $\mathcal{F}_h$ is the image of $\mathcal{F}|_U$. For every $z \in V_i$, we denote by $\phi^+_{V_i,z}$ the positive half leaf of the leaf in V_i containing z. Then we can construct a continuous map ψ from the curve γ to $\mathbb{D} \setminus \{0\}$, such that $\psi(z) \in \phi^+_{V_i,z}$ for all $0 \leq i \leq n$ and for all $z \in V_i$. We define the *index* $i(\mathcal{F}, z_0)$ of $\mathcal{F}$ at z_0 to be the Brouwer degree of the map

$$\theta \mapsto \frac{\psi(\gamma(\theta)) - \gamma(\theta)}{\|\psi(\gamma(\theta)) - \gamma(\theta)\|},$$

which depends neither on the choice of ψ, nor on the choice of V_i, nor on the choice of γ, nor on the choice of h.

We say that a path $\gamma : [0, 1] \to M$ is *positively transverse* to $\mathcal{F}$, if for every $t_0 \in [0, 1]$, there exists a trivialization neighborhood V of $\gamma(t_0)$ and $\varepsilon > 0$ such that $\gamma([t_0-\varepsilon, t_0+\varepsilon]\cap[0, 1]) \subset V$

and $h \circ \gamma|_{[t_0-\varepsilon,t_0+\varepsilon]\cap[0,1]}$ intersects the vertical lines from left to right, where $h : V \to (0,1)^2$ is a trivialization chart.

Let f be a homeomorphism of M isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. We say that an oriented foliation $\mathcal{F}$ on M is a *transverse foliation* of I if for every $z \in M$, there is a path that is homotopic to the trajectory $t \to f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

Suppose that $I = (f_t)_{t\in[0,1]}$ is a local isotopy at z_0, we say that $\mathcal{F}$ is *locally transverse* to I if for every sufficiently small neighborhood U of z_0, there exists a neighborhood $V \subset U$ such that for all $z \in V \setminus \{z_0\}$, there exists a path in $U \setminus \{z_0\}$ that is homotopic in $U \setminus \{z_0\}$ to the trajectory $t \mapsto f_t(z)$ of z along I and is positively transverse to $\mathcal{F}$.

Proposition 2.2. *[LC08] Suppose that I is an identity isotopy on a surface M with an isolated end z_0 and that $\mathcal{F}$ is a transverse foliation of I. If M is not a plane, $\mathcal{F}$ is also locally transverse to the local isotopy I at z_0.*

Proposition 2.3. *[LC08] Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at an isolated fixed point z_0, I be a local isotopy of f at z_0, and $\mathcal{F}$ be a foliation that is locally transverse to I. Then, we have the following results:*

- $i(\mathcal{F}, z_0) = i(I, z_0) + 1$;
- $i(f, z_0) = i(\mathcal{F}, z_0)$ *if* $i(\mathcal{F}, z_0) \neq 1$.

2.4 The existence of a transverse foliation and Jaulent's preorder

Let f be a homeomorphism of M isotopic to the identity, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. A *contractible fixed point* z of f associated to I is a fixed point of f such that the trajectory of z along I, that is the path $t \mapsto f_t(z)$, is a loop homotopic to zero in M.

Theorem 2.4. *[LC05] If $I = (f_t)_{t\in[0,1]}$ is an identity isotopy of a homeomorphism f of M such that there exists no contractible fixed point of f associated to I, then there exists a transverse foliation $\mathcal{F}$ of I.*

One can extend this result to the case where there exist contractible fixed points by defining the following preorder of Jaulent [Jau14].

Let us denote by $\mathrm{Fix}(f)$ the set of fixed points of f, and for every identity isotopy $I = (f_t)_{t\in[0,1]}$ of f, by $\mathrm{Fix}(I) = \cap_{t\in[0,1]}\mathrm{Fix}(f_t)$ the set of fixed points of I. Let X be a closed subset of $\mathrm{Fix}(f)$. We denote by (X, I_X) the couple that consists of a closed subset $X \subset \mathrm{Fix}(f)$ such that $f|_{M\setminus X}$ is isotopic to the identity and an identity isotopy I_X of $f|_{M\setminus X}$.

Let $\pi_X : \widetilde{M}_X \to M \setminus X$ be the universal cover, and $\widetilde{I}_X = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I_X. We say that $\widetilde{f}_X = \widetilde{f}_1$ is *the lift of f associated to I_X*. We say that a path $\gamma : [0,1] \to M \setminus X$ from z to $f(z)$ is *associated* to I_X if there exists a path $\widetilde{\gamma} : [0,1] \to \widetilde{M}_X$ that is the lift of γ and satisfies $\widetilde{f}_X(\widetilde{\gamma}(0)) = \widetilde{\gamma}(1)$. We write $(X, I_X) \precsim (Y, I_Y)$, if

- $X \subset Y \subset (X \cup \pi_X(\mathrm{Fix}(\widetilde{f}_X)))$;
- all the paths in $M \setminus Y$ associated to I_Y are also associated to I_X.

The preorder $\precsim$ is well defined. Moreover, if one has $(X, I_X) \precsim (Y, I_Y)$ and $(Y, I_Y) \precsim (X, I_X)$, then one knows that $X = Y$ and that I_X is homotopic to I_Y. In this case, we will write $(X, I_X) \sim (Y, I_Y)$.

When the closed subset $X \subset \mathrm{Fix}(f)$ is totally disconnected, an identity isotopy I_X on $M \setminus X$ can be extended to an identity isotopy on M that fixes every point in X; but when X is not totally disconnected, one may fail to find such an extension. A necessary condition for the existence of such an extension is that for every closed subset $Y \subset X$, there exists (Y, I_Y) that satisfies $(Y, I_Y) \precsim (X, I_X)$. By a result (unpublished yet) due to Béguin, Le Roux and Crovisier, this condition is also sufficient to prove the existence of an identity isotopy I' of f on M that fixes every point in X and satisfies $(X, I_X) \sim (X, I'|_{M\setminus X})$ (here, we do not know whether I_X can be extended). Formally, we denote by $\mathcal{I}$ the set of couples (X, I_X) such that for all closed subset $Y \subset X$, there exists (Y, I_Y) that satisfies $(Y, I_Y) \precsim (X, I_X)$. Then, we have the following results:

Proposition 2.5. *[BCLR][2] For $(X, I_X) \in \mathcal{I}$, there exists an identity isotopy I' of f on M that fixes every point in X and satisfies $(X, I_X) \sim (X, I'|_{M\setminus X})$.*

Proposition 2.6. *[Jau14] Let $(X, I_X) \in \mathcal{I}$, and $\widetilde{f}_X$ be the lift of $f|_{M\setminus X}$ to $\widetilde{M}_X$ associated to I_X. If $z \in \mathrm{Fix}(f) \setminus \mathrm{Fix}(I)$ is a fixed point of f such that $\widetilde{f}_X$ fixes all the points in $\pi_X^{-1}\{z\}$, then there exists $(X \cup \{z\}, I_{X\cup\{z\}}) \in \mathcal{I}$ such that $(X, I_X) \precsim (X \cup \{z\}, I_{X\cup\{z\}})$. In particular, if (X, I_X) is maximal in $(\mathcal{I}, \precsim)$, $f|_{M\setminus X}$ has no contractible fixed point associated to I_X.*

Proposition 2.7. *[Jau14] If $\{(X_\alpha, I_{X_\alpha})\}_{\alpha\in J}$ is a totally ordered chain in $(\mathcal{I}, \precsim)$, then there exists $(X_\infty, I_{X_\infty}) \in \mathcal{I}$ that is an upper bound of the this chain, where $X_\infty = \overline{\cup_{\alpha\in J} X_\alpha}$*

Theorem 2.8. *[Jau14] If I is an identity isotopy of a homeomorphism f on M, then there exists a maximal $(X, I_X) \in \mathcal{I}$ such that $(\mathrm{Fix}(I), I) \precsim (X, I_X)$. Moreover, $f|_{M\setminus X}$ has no contractible fixed point associated to I_X, and there exists a transverse foliation $\mathcal{F}$ of I_X on $M \setminus X$.*

Remark 2.9. Here, we can also consider the previous foliation $\mathcal{F}$ to be a singular foliation on M whose singularities are the points in X. In particular, if I_X is the restriction to $M \setminus X$ of an identity isotopy I' on M, we will say that $\mathcal{F}$ a transverse (singular) foliation of I'.

Remark 2.10. In this article, we denote also by I_X an identity isotopy on M that fixes all the points in X, when there is no ambiguity. Proposition 2.6, 2.7 and Theorem 2.8 are still valid if we replace the definition of $\mathcal{I}$ with the set of couples (X, I_X) of a closed subset $X \subset \mathrm{Fix}(f)$ and an identity isotopy I_X on M that fixes every point in X. When $\mathrm{Fix}(f)$ is totally disconnected, it is obvious; when $\mathrm{Fix}(f)$ is not totally disconnected, we should admit Proposition 2.5.

We call $(Y, I_Y) \in \mathcal{I}$ an *extension* of (X, I_X) if $(X, I_X) \precsim (Y, I_Y)$; we call I' an *extension* of $(X, I_X) \in \mathcal{I}$ if $(X, I_X) \precsim (\mathrm{Fix}(I'), I')$; we call I' an *extension* of I if I' is an extension of $(\mathrm{Fix}(I), I)$. We say that I' is a *maximal extension* if $(\mathrm{Fix}(I'), I')$ is maximal in $(\mathcal{I}, \precsim)$.

In particular, when M is a plane, Béguin, Le Roux and Crovisier proved the following result (unpublished yet).

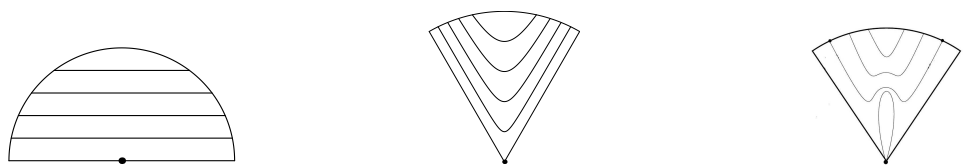
Proposition 2.11. *[BCLR] If f is an orientation preserving homeomorphism of the plane, and if $X \subset \mathrm{Fix}(f)$ is a connected and closed subset, then there exists an identity isotopy I of f such that $X \subset \mathrm{Fix}(I)$.*

[2] It is a talk of Crovisier in the conference "Surfaces in Sao Paulo" in April, 2014.

2.5 Dynamics of an oriented foliation in a neighborhood of an isolated singularity

In this section, we consider singular foliations. A *sink* (resp. a*source*) of $\mathcal{F}$ is an isolated singular point of $\mathcal{F}$ such that there is a homeomorphism $h : U \to \mathbb{D}$ which sends z_0 to 0 and sends the restricted foliation $\mathcal{F}|_{U\setminus\{z_0\}}$ to the radial foliation of $\mathbb{D}\setminus\{0\}$ with the leaves toward (resp. backward) 0, where U is a neighborhood of z_0 and $\mathbb{D}$ is the unit disk. A *petal* of $\mathcal{F}$ is a closed topological disk whose boundary is the union of a leaf and a singularity. Let $\mathcal{F}_0$ be the foliation on $\mathbb{R}^2\setminus\{0\}$ whose leaves are the horizontal lines except the $x-$axis which is cut into two leaves. Let $S_0 = \{y \geq 0 : x^2 + y^2 \leq 1\}$ be the half-disk. We call a closed topological disk S a *hyperbolic sector* if there exist

- a closed set $K \subset S$ such that $K \cap \partial S$ is reduced to a singularity z_0 and $K \setminus \{z_0\}$ is the union of the leaves of $\mathcal{F}$ that are contained in S,
- and a continuous map $\phi : S \to S_0$ that maps K to 0 and the leaves of $\mathcal{F}|_{S\setminus K}$ to the leaves of $\mathcal{F}_0|_{S_0}$.



(a) the hyperbolic sector model S_0

(b) a pure hyperbolic sector

(c) a strange hyperbolic sector

Figure 1: The hyperbolic sectors

Le Roux gives a description of the dynamics of an oriented foliation $\mathcal{F}$ near an isolated singularity z_0.

Proposition 2.12. *[LR13] We have one of the following cases:*

i) (sink or source) there exists a neighborhood of z_0 that contains neither a closed leaf, nor a petal, nor a hyperbolic sector;

ii) (cycle) every neighborhood of z_0 contains a closed leaf;

iii) (petal) every neighborhood of z_0 contains a petal, and does not contain any hyperbolic sector;

iv) (saddle) every neighborhood of z_0 contains a hyperbolic sector, and does not contain any petal;

v) (mixed) every neighborhood of z_0 contains both a petal and a hyperbolic sector.

Moreover, $i(\mathcal{F}, z_0)$ is equal to 1 in the first two cases, is strictly bigger than 1 in the petal case, and is strictly smaller than 1 in the saddle case.

Remark 2.13. In particular, let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0, I be a local isotopy of f, $\mathcal{F}$ be an oriented foliation that is locally transverse to I, and z_0 be an isolated singularity of $\mathcal{F}$. If P is a petal in a small neighborhood

of z_0 and ϕ is the leaf in ∂P, then $\phi \cup \{z_0\}$ divides M into two parts. We denote by $L(\phi)$ the one to the left and $R(\phi)$ the one to the right. By definition, P contains the positive orbit of $R(\phi) \cap L(f(\phi))$ or the negative orbit of $L(\phi) \cap R(f^{-1}(\phi))$. Then, a petal in a small neighborhood of z_0 contains the positive or the negative orbit of a wandering open set. So does the topological disk whose boundary is a closed leaf in a small neighborhood of z_0. So, if f is area preserving, or if there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set, then z_0 is either a sink, a source, or a saddle of $\mathcal{F}$.

In some particular cases, the local dynamics of a transverse foliation can be easily deduced. We have the following result:

Proposition 2.14. *[LC08] Let I be a local isotopy at z_0 such that $i(I, z_0) \neq 0$. If I' is another local isotopy at z_0 and if $\mathcal{F}'$ is an oriented foliation that is locally transverse to I'. Then,*

- *the indices $i(I', z_0)$ and $i(I, z_0)$ are equal if $I' \sim I$;*
- *0 is a sink of $\mathcal{F}'$ if $I' > I$;*
- *0 is a source of $\mathcal{F}'$ if $I' < I$.*

2.6 Prime-ends compactification and rotation number

In this section, we first recall some facts and definitions from Carathéodory's prime-ends theory, and then give the definition of the prime-ends rotation number of an orientation preserving homeomorphism. More details can be found in [Mil06] and [KLCN14].

Let $U \subsetneq \mathbb{R}^2$ be a simply connected domain, then there exists a natural compactification of U by adding a circle, that can be defined in different ways. One explanation is the following: we can identify $\mathbb{R}^2$ with $\mathbb{C}$ and consider a conformal diffeomorphism

$$h : U \to \mathbb{D},$$

where $\mathbb{D}$ is the unit disk. We endow $U \sqcup S^1$ with the topology of the pre-image of the natural topology of $\overline{\mathbb{D}}$ by the application

$$\overline{h} : U \sqcup S^1 \to \overline{\mathbb{D}},$$

whose restriction is h on U and the identity on S^1 .

Any arc in U which lands at a point z of ∂U corresponds, under h, to an arc in $\mathbb{D}$ which lands at a point of S^1, and arcs which land at distinct points of ∂U necessarily correspond to arcs which land at distinct points of S^1. We define an *end-cut* to be the image of a simple arc $\gamma : [0, 1) \to U$ with a limit point in ∂U. Its image by h has a limit point in S^1. We say that two end-cuts are *equivalent* if their images have the same limit point in S^1. We say that a point $z \in \partial U$ is *accessible* if there is an end-cut that lands at z. Then the set of points of S^1 that are limit points of an end-cut is dense in S^1, and accessible points of ∂U are dense in ∂U.

Let f be an orientation preserving homeomorphism of U. We can extend f to a homeomorphism of the prime-ends compactification $U \sqcup S^1$, and denote it by $\overline{f}$. In fact, for a point $z \in S^1$ which is a limit point of an end-cut γ, we can naturally define $\overline{f}(z)$ to be the limit point of $f \circ \gamma$. Then we can define the *prime-ends rotation number* $\rho(f, U) \in \mathbb{R}/\mathbb{Z}$ to be the Poincaré's rotation number of $\overline{f}|_{S^1}$. In particular, if f fixes every point in ∂U, $\rho(f, U) = 0$.

2.7 The local rotation set

In this section, we will give a definition of the local rotation set and will describe the relations between the rotation set and the rotation number. More details can be found in [LR13].

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$, and $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f. Given two neighborhoods $V \subset U$ of 0 and an integer $n \geq 1$, we define

$$E(U, V, n) = \{z \in U : z \notin V, f^n(z) \notin V, f^i(z) \in U \text{ for all } 1 \leq i \leq n\}.$$

We define the *rotation set* of I relative to U and V by

$$\rho_{U,V}(I) = \cap_{m\geq 1}\overline{\cup_{n\geq m}\{\rho_n(z), z \in E(U, V, n)\}} \subset [-\infty, \infty],$$

where $\rho_n(z)$ is the average change of angular coordinate along the trajectory of z. More precisely, let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{C} \setminus \{0\} \simeq \mathbb{R}^2 \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal covering projection, $\widetilde{f} : \pi^{-1}(W) \to \pi^{-1}(W')$ be the lift of f associated to I, and $p_1 : \mathbb{R} \times (-\infty, 0) \to \mathbb{R}$ be the projection onto the first factor. We define

$$\rho_n(z) = \frac{p_1(\widetilde{f}^n(\widetilde{z}) - \widetilde{z})}{n},$$

where $\widetilde{z}$ is any point in $\pi^{-1}\{z\}$.

We define the *local rotation set* of I to be

$$\rho_s(I, 0) = \cap_U\overline{\cup_V \rho_{U,V}(I)} \subset [-\infty, \infty],$$

where $V \subset U \subset W$ are neighborhoods of 0.

Remark 2.15. Here, $\rho_s(I, 0)$ is a closed subset of $[-\infty, \infty]$. Jonathan Conejeros[3] proved that $\rho_s(I, 0)$ is indeed a closed interval.

We say that f can be *blown-up* at 0 if there exists an orientation preserving homeomorphism $\Phi : \mathbb{R}^2\setminus\{0\} \to \mathbb{T}^1\times(-\infty, 0)$, such that $\Phi f \Phi^{-1}$ can be extended continuously to $\mathbb{T}^1\times\{0\}$. We denote this extension by h. Suppose that f is not conjugate the contraction $z \mapsto \frac{z}{2}$ or the expansion $z \mapsto 2z$. We define the *blow-up rotation number* $\rho(f, 0)$ of f at 0 to be the Poincaré rotation number of $h|_{\mathbb{T}^1}$. Let $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f, $(\widetilde{h}_t)$ be the natural lift of $(\Phi f_t \Phi^{-1})|_{\mathbb{T}^1\times(0,r)}$, where r is a sufficiently small positive number, and $\widetilde{h}$ be the lift of h such that $\widetilde{h} = \widetilde{h}_1$ in a neighborhood of $\mathbb{R} \times \{0\}$. We define the *blow-up rotation number* $\rho(I, 0)$ *of* I *at* 0 to be the rotation number of $h|_{\mathbb{T}^1}$ associated to the lift $\widetilde{h}|_{\mathbb{R}\times\{0\}}$, which is a representative of $\rho(f, 0)$ on $\mathbb{R}$. Jean-Marc Gambaudo, Le Calvez and Elisabeth Pécou [GLCP96] proved that neither $\rho(f, 0)$ nor $\rho(I, 0)$ depend on the choice of Φ, which generalizes a previous result of Naĭshul′ [Naĭ82]. In particular, if f is a diffeomorphism, f can be blown-up at 0 and the extension of f on $\mathbb{T}^1$ is induced by the map

$$v \mapsto \frac{Df(0)v}{\|Df(0)v\|}$$

[3]This work have not been published. But in some cases we are most interested, this result can be easily deduced. We will give one in the following Remark 2.19.

on the space of unit tangent vectors.

More generally, if $f : (W, z_0) \to (W', z_0)$ is an orientation preserving local homeomorphism at z_0 that is not conjugate to the contraction or the expansion, we can give the previous definitions for f by conjugate it to an orientation preserving local homeomorphism at $0 \in \mathbb{R}^2$.

The local rotation set can be empty. However, due to Le Roux [LR08], we know that the rotation set is not empty if f is area preserving, or if there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set. More precisely, we have the following result:

Proposition 2.16. *[LR13] Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0, and $I = (f_t)_{t\in[0,1]}$ be a local isotopy of f. Then $\rho_s(I, z_0)$ is empty if and only if f is conjugate to one of the following maps*

- *the contraction $z \mapsto \frac{z}{2}$;*
- *the expansion $z \mapsto 2z$;*
- *a holomorphic function $z \mapsto e^{i2\pi\frac{p}{q}} z(1 + z^{qr})$ with $q, r \in \mathbb{N}$ and $p \in \mathbb{Z}$.*

Remark 2.17. In the three cases, f can be blown-up at z_0. But $\rho(f, z_0)$ is defined only in the third case. More precisely, $\rho(f, z_0)$ is equal to $\frac{p}{q} + \mathbb{Z}$. Moreover, if I is conjugate to $z \mapsto z^{i2\pi t\frac{p}{q}}(1 + tz^{qr})$, then $\rho(I, z_0)$ is equal to $\frac{p}{q}$.

We say that z is a *contractible* fixed point of f associated to the local isotopy $I = (f_t)_{t\in[0,1]}$ if the trajectory $t \mapsto f_t(z)$ of z along I is a loop homotopic to zero in $W \setminus \{z_0\}$. We say that f satisfies the *local intersection condition*, if there exists a neighborhood of z_0 that does not contain any simple closed curve which is the boundary of a Jordan domain containing z_0 and does not intersect its image by f. In particular, if f is area preserving or if there exists a neighborhood of z_0 that contains neither the positive nor the negative orbit of any wandering open set, f satisfies the local intersection condition.

The local rotation set satisfies the following properties:

Proposition 2.18. *[LR13] Let $f : (W, z_0) \to (W', z_0)$ be an orientation preserving local homeomorphism at z_0, and I be a local isotopy of f at z_0. One has the following results:*

i) for all integer p, q, $\rho_s(J^p_{z_0} I^q, z_0) = q\rho_s(I, z_0) + p$;

ii) if z_0 is accumulated by fixed points of I, then $0 \in \rho_s(I, z_0)$;

iii) if f satisfies the local intersection condition and if 0 is an interior point of the convex hull of $\rho_s(I, z_0)$, then z_0 is accumulated by contractible fixed points of f associated to I;

iv) if I has a positive (resp. negative) rotation type, then $\rho_s(I, z_0) \subset [0, +\infty]$ (resp. $\rho_s(I, z_0) \subset [-\infty, 0]$);

v) if $\rho_s(I, z_0)$ is a non-empty set that is contained in $(0, +\infty]$ (resp. $[-\infty, 0)$), then I has a positive (resp.negative) rotation type;

vi) if $\rho_s(I, z_0)$ is a non-empty set that is contained in $[0, \infty]$ (resp. $[-\infty, 0]$) and is not reduced to 0, and if z_0 is not accumulated by contractible fixed points of f associated to I, then I has a positive (resp. negative) rotation type;

vii) if f can be blown-up at z_0, and if $\rho_s(I, z_0)$ is not empty, then $\rho_s(I, z_0)$ is reduced to the single real number $\rho(I, z_0)$.

Remark 2.19. When f satisfies the local intersection condition, one can deduce that $\rho_s(I, z_0)$ is a closed interval as a corollary of the assertion i), ii), iii) of the proposition.

Remark 2.20. Le Roux also gives several criteria implying that f can be blown-up at z_0. The one we need in this article is due to Béguin, Crovisier and Le Roux [LR13]

If there exists an arc γ at z_0 whose germ is disjoint with the germs of $f^n(\gamma)$ for all $n \neq 0$, then f can be blown-up at z_0.

In particular, if there exists a petal at z_0, and Γ is the leaf in the boundary of this petal, we can find an arc in $L(\Gamma) \cap R(f(\Gamma))$ satisfying this criteria, then f can be blown-up at z_0.

2.8 The linking number

Let f be an orientation preserving homeomorphism of $\mathbb{R}^2$, and $I = (f_t)_{t\in[0,1]}$ be an identity isotopy of f. If z_0, z_1 are two fixed points of f, the map

$$t \mapsto \frac{f_t(z_0) - f_t(z_1)}{\|f_t(z_0) - f_t(z_1)\|}$$

descends to a continuous map from $[0,1]/_{0\sim 1}$ to S^1. We define the *linking number* between z_0 and z_1 associated to I to be the Brouwer degree of this map, and denote it by $L(I, z_0, z_1)$. We say that z_0 and z_1 are *linked* (relatively to I) if the linking number is not zero.

Suppose that I and I' are identity isotopies of f, and that z_0, z_1 are two fixed points of f. Note the following facts:

- if I and I' fixes z_0 and satisfies $I' \sim J^k_{z_0} I$ as local isotopies at z_0, then one can deduce

$$L(I', z_0, z_1) = L(I, z_0, z_1) + k;$$

- if both I and I' can be viewed as local isotopies at ∞, and if I is equivalent to I' as local isotopies at ∞, then one can deduce

$$L(I', z_0, z_1) = L(I, z_0, z_1).$$

2.9 A generalization of Poincaré-Birkhoff theorem

In this section, we will introduce a generalization of Poincaré-Birkhoff theorem. An *essential* loop in the annulus is a loop that is not homotopic to zero.

Proposition 2.21. *[Gui94] Let $f : \mathbb{T}^1 \times [-a, a] \to \mathbb{T}^1 \times [-b, b]$ be an embedding homotopic to the inclusion, where $0 < a < b$, and $\widetilde{f} : \mathbb{R} \times [-a, a] \to \mathbb{R} \times [-b, b]$ be a lift of f. If f does not have any fixed point, and if $\widetilde{f}$ satisfies*

$$(p_1(\widetilde{f}(x, a)) - x)(p_1(\widetilde{f}(x', -a)) - x') < 0, \text{ for all } x, x' \in \mathbb{R},$$

then there exists an essential loop γ in $\mathbb{T}^1 \times [-a, a]$ such that $f(\gamma) \cap \gamma = \emptyset$.

3 The rotation type at an isolated fixed point of an orientation preserving local homeomorphism

Let $f : (W, 0) \to (W', 0)$ be an orientation preserving local homeomorphism at the isolated fixed point $0 \in \mathbb{R}$. The main aim of this section is to detect the local rotation type of the local isotopies of f and prove Theorem 1.1.

Before proving the theorem, we will first prove the following lemma:

Lemma 3.1. *If f satisfies the local intersection condition, then a local isotopy $I = (f_t)_{t\in[0,1]}$ of f can not have both a positive and a negative rotation type.*

Proof. We will give a proof by contradiction. Suppose that $\mathcal{F}_1$ and $\mathcal{F}_2$ are two locally transverse foliations of I such that 0 is a sink of $\mathcal{F}_1$ and a source of $\mathcal{F}_2$. Then, there exist two orientation preserving local homeomorphisms $h_1 : (V_1, 0) \to (\mathbb{D}, 0)$ and $h_2 : (V_2, 0) \to (\mathbb{D}, 0)$ such that h_1 (resp. h_2) sends the restricted foliation $\mathcal{F}_1|_{V_1}$ (resp. $\mathcal{F}_2|_{V_2}$) to the radial foliation on $\mathbb{D}$ with the orientation toward (resp. backward) 0, where $\mathbb{D}$ is the unit disk centered at 0, and $V_i \subset U \subset W$ is a small neighborhood of 0 such that f does not have any fixed point in V_i except 0, and $f(\gamma) \cap \gamma \neq \emptyset$ for all essential closed curve γ in $V_i \setminus \{0\}$, for $i = 1, 2$. We denote by $\mathbb{D}_r$ the disk centered at 0 with radius r, and S_r the boundary of $\mathbb{D}_r$. Choose $0 < r_2 < 1$ such that for all $z \in h_2^{-1}(S_{r_2})$, there exists an arc in $V_2 \setminus \{0\}$ that is homotopic to $t \mapsto f_t(z)$ in $V_2 \setminus \{0\}$ and is positively transverse to $\mathcal{F}_2$; choose $0 < r_2' < r_2$ such that $h_2 \circ f \circ h_2^{-1}(S_{r_2}) \subset \mathbb{D} \setminus \mathbb{D}_{r_2'}$; choose $0 < r_1' < r_2'$ such that $h_1^{-1}(\overline{\mathbb{D}_{r_1'}}) \subset h_2^{-1}(\mathbb{D}_{r_2'})$; and choose $0 < r_1 < r_1'$ such that $h_1 \circ f \circ h_1^{-1}(\mathbb{D}_{r_1}) \subset \mathbb{D}_{r_1'}$, and for all $z \in h_1^{-1}(S_{r_1})$, there exists an arc in $V_1 \setminus \{0\}$ that is homotopic to $t \mapsto f_t(z)$ in $V_1 \setminus \{0\}$ and is positively transverse to $\mathcal{F}_1$. We consider a homeomorphism $h : (V_2, 0) \to (\mathbb{D}, 0)$ such that $h|_{h_1^{-1}(\mathbb{D}_{r_1'})} = h_1$ and $h|_{h_2^{-1}(\mathbb{D}\setminus\mathbb{D}_{r_2'})} = h_2$. Then, $h \circ f \circ h^{-1}$ does not have any fixed point except 0. Let

$$\begin{array}{rcl} \pi : \mathbb{R} \times (-\infty, 0) & \to & \mathbb{R}^2 \setminus \{0\} \simeq \mathbb{C} \setminus \{0\} \\ (\theta, y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal covering projection, and $\widetilde{f}'$ be the lift of $h \circ f \circ h^{-1}$ associated to $I' = (h \circ f_t \circ h^{-1})_{t\in[0,1]}$. Then, $p_1(\widetilde{f}'(\theta, -r_1)) - \theta > 0$ and $p_1(\widetilde{f}'(\theta, -r_2)) - \theta < 0$ for all $\theta \in \mathbb{R}$, where p_1 is the projection onto the first factor. Then, $h \circ f \circ h^{-1}$ is a map satisfying the conditions of Proposition 2.21. But we know that $h \circ f \circ h^{-1}(\gamma) \cap \gamma \neq \emptyset$ for all essential simple closed curve γ in $\mathbb{D} \setminus \{0\}$, which is a contradiction. □

Remark 3.2. In particular, a local homeomorphism satisfying the assumption of Theorem 1.1 also satisfies the condition of the previous lemma. But not all local isotopies can not have both a positive and a negative rotation type. As we can see in Section 5, there exist local isotopies that have both positive and negative rotation types.

Now, we begin the proof of Theorem 1.1.

Proof of Theorem 1.1 . To simplify the notations, we suppose that the local homeomorphism is at $0 \in \mathbb{R}^2$. One has to consider two cases: $i(f, 0)$ is equal to 1 or not.

a) Suppose that $i(f, 0) \neq 1$. By Proposition 2.1, there exists a unique homotopy class of local isotopies at 0 such that $i(I_0, 0) = i(f, 0) - 1 \neq 0$ for every local isotopy I_0 in this class. Let $\mathcal{F}$ be a locally transverse foliation of I_0. Then $i(\mathcal{F}, 0) = i(I_0, 0) + 1 \neq 1$ by Proposition 2.3, and therefore 0 is neither a sink nor a source of $\mathcal{F}$. This implies that I_0 has neither a positive nor a negative rotation type. So, I_0 has a zero rotation type at z_0. For a local isotopy I at 0 that is not in the homotopy class of I_0, by Proposition 2.14, it has only a positive rotation type if $I > I_0$, and has only a negative rotation type if $I < I_0$. Then, both statements of Theorem 1.1 are proved.

b) Suppose that $i(f, 0) = 1$. Let I be a local isotopy of f, and $\mathcal{F}$ be an oriented foliation that is locally transverse to I. Since there exists a neighborhood $U \subset W$ of 0 that contains neither the positive nor the negative orbit of any wandering open set, one knows (see the remark following Proposition 2.12) that 0 is either a sink, a source or a saddle of $\mathcal{F}$. As

recalled in Proposition 2.12, in the first two cases $i(\mathcal{F},0)$ is equal to 1, and in the last case $i(\mathcal{F},0)$ is not positive. By Proposition 2.3 one deduces that $i(\mathcal{F},0)=1$ because $i(f,0)=1$. So, 0 is a sink or a source. Therefore, I has exactly one of the three rotation types by Lemma 3.1.

Since there exists a neighborhood $U \subset W$ of 0 that contains neither the positive nor the negative orbit of any wandering open set, one deduces by Proposition 2.16 that $\rho_s(I,0)$ is not empty, and knows that f satisfies the local intersection condition. Moreover, 0 is an isolated fixed point, so one can deduce by the first three assertions of Proposition 2.18 that there exists $k \in \mathbb{Z}$ such that $\rho_s(I,0)$ is a subset of $[k,k+1]$. By the assertion i) of Proposition 2.18, there exists a local isotopy I_0 of f such that $\rho_s(I_0,0)$ is a nonempty subset of $[0,1]$ and is not reduced to 1. Then, as a corollary of the assertions iv)-vi) of Proposition 2.18,

- I has a positive rotation type if $I > I_0$,
- I has a negative rotation type if $I < I_0$.

□

Remark 3.3. It is easy to see that the condition that there exists a neighborhood $U \subset W$ of 0 that contains neither the positive nor the negative orbit of any wandering open set is necessary for the first assertion of the theorem. Indeed, if we do not require this condition, even if f satisfies the local intersection condition, there still exists local isotopies that have both positive (resp. negative) and zero rotation types. We will give one such example in Section 5.

Remark 3.4. Matsumoto [Mat01] defined a notion of positive and negative type for an orientation and area preserving local homeomorphism at an isolated fixed point with Lefschetz index 1. In this case, our definitions of "positive rotation type" (resp. "negative rotation type") is equivalent to his definition of "positive type" (resp. negative type").

Now, let us prove Proposition 1.2.

Proof of Proposition 1.2. The first statement is just a corollary of the definition of the torsion-low property and the assertions i), iv) of Proposition 2.18. Suppose now that f can be blown-up at z_0. If f satisfies the hypothesis, $\rho_s(I,z_0)$ is not empty by Proposition 2.16. So, using the assertion vii) of Proposition 2.18, one deduces that $\rho_s(I,z_0)$ is reduced to a single point in $[-1,1]$. Suppose now that f is a diffeomorphism in a neighborhood of z_0. The first part of the third statement is just a special case of the second statement.

To conclude, let us prove the last part of the third statement. To simplify the notations, we suppose that $z_0 = 0 \in \mathbb{R}^2$. Since there exists a neighborhood of 0 that contains neither the positive nor the negative orbit of any wandering open set, $Df(0)$ can not have two real eigenvalues such that the absolute values of both eigenvalues are strictly smaller (resp. bigger) than 1. Since 1 is not an eigenvalue of $Df(0)$, one has to consider the following three cases:

- Suppose that $Df(0)$ do not have any real eigenvalue. In this case, $\rho(I,0)$ is not an integer.
- Suppose that $Df(0)$ has two real eigenvalues λ_1 and λ_2 such that $\lambda_1 < -1 < \lambda_2 < 0$. In this case, $\rho(I,0)$ is equal to $\frac{1}{2}$ or $-\frac{1}{2}$, and is not an integer.
- Suppose that $Df(0)$ has two real eigenvalues λ_1 and λ_2 such that $0 < \lambda_1 < 1 < \lambda_2$. In this case, $i(f,0) = -1$, and I has a zero rotation type at 0. So, $\rho(I,0)$ is equal to 0.

Anyway, we know that $\rho(I,0)$ belongs to $(-1,1)$. □

4 The existence of a global torsion-low isotopy

Let f be an orientation and area preserving homeomorphism of a connected oriented surface M that is isotopic to the identity. The main aim of this section is to prove the existence of a torsion-low maximal isotopy of f, i.e. Theorem 1.3.

When $\mathrm{Fix}(f) = \emptyset$, the theorem is trivial, and so we suppose that $\mathrm{Fix}(f) \neq \emptyset$ in the following part of this section. Recall that $\mathcal{I}$ is the set of couples (X, I_X) that consists of a closed subset $X \subset \mathrm{Fix}(f)$ and an identity isotopy I_X of f on M that fixes all the points in X. We denote by $\mathcal{I}_0$ be the set of $(X, I_X) \in \mathcal{I}$ such that I_X is torsion-low at every $z \in X$. Recall that $\precsim$ is Jaulent's preorder defined in Section 2.4. Then, Theorem 1.3 is just an immediate corollary of the following theorem. Moreover, the proof do not need any other assumptions when $\mathrm{Fix}(f)$ is totally disconnected, while we should admit the yet unpublished results of Béguin, Le Roux and Crovisier stated in Section 2.4 when $\mathrm{Fix}(f)$ is not totally disconnected.

Theorem 4.1. *Given $(X, I_X) \in \mathcal{I}_0$, there exists a maximal extension $(X', I_{X'})$ of (X, I_X) that belongs to $\mathcal{I}_0$.*

Remark 4.2. We will see that, except in the case where M is a sphere and X is reduced to a point, $I_{X'}$ and I_X are equivalent as local isotopies at z, for every $z \in X$. In the case where M is a sphere and X is reduced to one point, this is not necessary the case. We will give an example in Section 5.

Remark 4.3. One may fail to find a torsion-low maximal identity isotopy I such that $0 \in \rho_s(I, z)$ for every $z \in \mathrm{Fix}(I)$ that is not isolated in $\mathrm{Fix}(f)$. We will give an example in Section 5. In particular, in this example, for every torsion-low maximal identity isotopy, there is a point that is isolated in $\mathrm{Fix}(I)$ but is not isolated in $\mathrm{Fix}(f)$.

Before proving this theorem, we will first state some properties of a torsion-low maximal isotopy.

Proposition (Proposition 1.7)**.** *Let f be an area preserving homeomorphism of M that is isotopic to the identity, I be a maximal identity isotopy that is torsion-low at $z \in \mathrm{Fix}(I)$, and $\mathcal{F}$ be a transverse foliation of I. If z is an isolated singularity of $\mathcal{F}$, then*

- *z is a saddle of $\mathcal{F}$ and $i(\mathcal{F}, z) = i(f, z)$, if z is an isolated fixed point of f such that $i(f, z) \neq 1$;*
- *z is a sink or a source of $\mathcal{F}$ if z is an isolated fixed point such that $i(f, z) = 1$ or if z is not isolated in $\mathrm{Fix}(f)$.*

Proof. One has to consider two cases: z is isolated in $\mathrm{Fix}(f)$ or not.

i) Suppose that z is isolated in $\mathrm{Fix}(f)$, then as a corollary of Theorem 1.1,

   - z is neither a sink nor a source of $\mathcal{F}$ if $i(f, z) \neq 1$;
   - z is a sink or a source of $\mathcal{F}$ if $i(f, z) = 1$.

   Moreover, in the first case, z is a saddle of $\mathcal{F}$ and $i(\mathcal{F}, z) = i(f, z)$ by Proposition 2.3 and the remark that follows Proposition 2.12.

ii) Suppose that z is not isolated in $\mathrm{Fix}(f)$. Let D be a small closed disk containing z as an interior point such that D does not contain any other fixed point of I, and $V \subset D$ be a neighborhood of z such that for every $z' \in V$, the trajectory of z' along I is contained in

D. We define the rotation number of a fixed point $z' \in V \setminus \{z\}$ to be the integer k such that its trajectory along I is homotopic to $k\partial D$ in $D \setminus \{z\}$. Then, by the maximality of I, the rotation number of a fixed point $z' \in V \setminus \{z\}$ is nonzero, and 0 is not an interior point of the convex hull of $\rho_s(I, z)$, as tells us the assertion iii) of Proposition 2.18. Since z is accumulated by fixed points of f, there exist $k_0 \subset \mathbb{Z} \setminus \{0\} \cup \{\pm\infty\}$ and a sequence of fixed points $\{z_n\}_{n\in\mathbb{N}}$ converging to z, such that their rotation numbers converge to k_0. Then, k_0 belongs to $\rho_s(I, z)$.

When $k_0 > 0$, $\rho_s(I, z)$ is included in $[0, +\infty]$ and not reduced to 0. By the assertion v) of Proposition 2.18, one deduces that z is a sink of $\mathcal{F}$. For the same reason, when $k_0 < 0$, we deduce that z is a source of $\mathcal{F}$.

□

The following result is an immediate corollary of Theorem 1.3 and Proposition 1.2.

Corollary 4.4 (Proposition 1.8)**.** *Let f be an area preserving diffeomorphism of M that is isotopic to the identity. Then, there exists a maximal isotopy I, such that for all $z \in \mathrm{Fix}(I)$, the rotation number satisfies*

$$-1 \leq \rho(I, z) \leq 1.$$

Moreover, the inequalities are both strict if z is not degenerate.

Remark 4.5. One may fail to get the strict inequalities without the assumption of nondegenerality. We will give an example in Section 5.

Now, we begin the proof of Theorem 4.1. We first note the following fact which results immediately from the definition:

If $(Y, I_Y) \in \mathcal{I}$ and $z \in Y$ is a point such that I_Y is not torsion-low at z, then z is isolated in Y.

Then, given such a couple $(Y, I_Y) \in \mathcal{I}$, we will try to find an extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$ such that $I_{Y'}$ is torsion-low at z'.

We will divide the proof into two cases. Unlike the second case, the first case does not use the result of Béguin, Le Roux and Crovisier stated in Section 2.4, but only use Jaulent's results.

4.1 Proof of Theorem 4.1 when $\mathrm{Fix}(f)$ is totally disconnected

When $\mathrm{Fix}(f)$ is totally disconnected, Theorem 4.1 is a corollary of Zorn's lemma and the following Propositions 4.6-4.9. We will explain first why the propositions imply the theorem, then we will prove the four propositions one by one. We will also give a proof of Proposition 1.6 at the end of this subsection.

Proposition 4.6. *If $\{(X_\alpha, I_{X_\alpha})\}_{\alpha\in J}$ is a totally ordered chain in $\mathcal{I}_0$, then there exists an upper bound $(X_\infty, I_{X_\infty}) \in \mathcal{I}_0$ of this chain, where $X_\infty = \overline{\cup_{\alpha\in J} X_\alpha}$*

Proposition 4.7. *For every maximal $(Y, I_Y) \in \mathcal{I}$ and $z \in Y$ such that I_Y is not torsion-low at z and $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane, there exist a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$ such that $I_{Y'}$ is torsion-low at z'.*

Proposition 4.8. *When M is a plane, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $X = \emptyset$.*

Proposition 4.9. *When M is a sphere, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $\#X \leq 1$.*

Remark 4.10. Proposition 4.8 and 4.9 deal with two special cases. The first is easy, while the second is more difficult. Indeed, to find an identity isotopy on a plane that is torsion-low at one point, we do not need to know the dynamics at infinity; but to find an identity isotopy on a sphere that is torsion-low at two points, we need check the properties of the isotopy near both points.

Proof of Theorem 4.1 when $\mathrm{Fix}(f)$ *is totally disconnected.* Fix $(X, I_X) \in \mathcal{I}_0$. Let $\mathcal{I}_*$ be the set of equivalent classes of the extensions $(X', I_{X'}) \in \mathcal{I}_0$ of (X, I_X). Then, the preorder $\precsim$ induces a partial order over $\mathcal{I}_*$. To simplify the notations, we still denote by $\precsim$ this partial order. By Proposition 4.6, $(\mathcal{I}_*, \precsim)$ is a partial ordered set satisfying the condition of Zorn's lemma, so $(\mathcal{I}_*, \precsim)$ contains at least one maximal element. Choose one representative $(X', I_{X'})$ of a maximal element of $(\mathcal{I}_*, \precsim)$. It is an extension of (X, I_X) and is maximal in $(\mathcal{I}_0, \precsim)$.

Using Proposition 4.7-4.9, we will prove by contradiction that a maximal couple $(X, I_X) \in (\mathcal{I}_0, \precsim)$ is also maximal in $(\mathcal{I}, \precsim)$. Suppose that there exists a couple $(X, I_X) \in \mathcal{I}_0$ that is maximal in $(\mathcal{I}_0, \precsim)$ but is not maximal in $(\mathcal{I}, \precsim)$. Fix a maximal extension (Y, I_Y) of (X, I_X) in $(\mathcal{I}, \precsim)$, and $z \in Y \setminus X$. Then, I_Y is not torsion-low at z, and so z is isolated in Y. Write $Y_0 = Y \setminus \{z\}$. By Proposition 4.8 and 4.9, $M \setminus Y_0$ is neither a sphere nor a plane. By Proposition 4.7, there exist a maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) and $z' \in Y'$, such that $I_{Y'}$ is torsion-low at z'. Then $(X \cup \{z'\}, I_{Y'}) \in \mathcal{I}_0$ is an extension of (X, I_X), which contradicts the maximality of (X, I_X) in $(\mathcal{I}_0, \precsim)$. □

Proof of Proposition 4.6. By Proposition 2.7, we know that there exists an upper bound $(X_\infty, I_{X_\infty}) \in \mathcal{I}$ of the chain, where $X_\infty = \overline{\cup_{\alpha \in J} X_\alpha}$. We only need to prove that $(X_\infty, I_{X_\infty}) \in \mathcal{I}_0$.

When J is finite, the result is obvious. We suppose that J is infinite. Fix $z \in X_\infty$. Either it is a limit point of X_∞, or there exists $\alpha_0 \in J$ such that z is an isolated point of X_α for all $\alpha \in J$ satisfying $(X_{\alpha_0}, I_{X_{\alpha_0}}) \precsim (X_\alpha, I_{X\alpha})$. In the first case, $0 \in \rho_s(I_{X_\infty}, z)$; in the second case, I_{X_∞} is locally homotopic to $I_{X_{\alpha_0}}$ at z. In both case, I_{X_∞} is torsion-low at z. □

Before proving Proposition 4.7, we will first prove the following two lemmas (Lemma 4.11 and 4.12). We will use Lemma 4.11 when proving Lemma 4.12, and we will use Lemma 4.12 when proving Proposition 4.7.

Lemma 4.11. *Let us suppose that* (Y, I_Y) *is maximal in* $(\mathcal{I}, \precsim)$*, that* I_Y *is not torsion-low at* $z \in Y$*, and that* $M \setminus (Y \setminus \{z\})$ *is neither a sphere nor a plane. If for every maximal extension* $(Y', I_{Y'})$ *of* $(Y \setminus \{z\}, I_Y)$ *and every point* $z' \in Y' \setminus (Y \setminus \{z\})$*,* $I_{Y'}$ *is not torsion-low at* z'*, then there exists a maximal extension* $(Y', I_{Y'}) \in \mathcal{I}$ *of* $(Y \setminus \{z\}, I_Y)$ *such that* $\#(Y' \setminus (Y \setminus \{z\})) > 1$.

Proof. Fix a couple (Y, I_Y) maximal in $(\mathcal{I}, \precsim)$ and $z_0 \in Y$ satisfying the assumptions of this lemma. Then, z_0 is an isolated point of Y. Write $Y_0 = Y \setminus \{z_0\}$. Then Y_0 is a closed subset and $M \setminus Y_0$ is neither a sphere nor a plane. Due to Remark 2.19, one has to consider the following four cases:

i) z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} > I_Y$ at z_0 which does not have a positive rotation type;

ii) z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset [-\infty, -1)$;

iii) z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} < I_Y$ at z_0 which does not have a negative rotation type;

iv) z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset (1, +\infty]$.

We will study the first two cases, the other ones can be treated in a similar way.

Let $\mathcal{F}_Y$ be a transverse foliation of I_Y. In case i), by Theorem 1.1, there exists a local isotopy I_0 at z_0 that is torsion-low at z_0, and we know that $I_Y < I'_{z_0} \lesssim I_0$, so I_Y has a negative rotation type at z_0; in case ii), we know that I_Y has a negative rotation type at z_0 by the assertion v) of Proposition 2.18 and the fact that $\rho_s(I_Y, z_0) \subset [-\infty, -1)$. Anyway, z_0 is a source of $\mathcal{F}_Y$. We denote by W the repelling basin of z_0 for $\mathcal{F}_Y$.

Let $\pi_{Y_0} : \widetilde{M}_{Y_0} \to M \setminus Y_0$ be the universal cover, $\widetilde{I} = (\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts $I_Y|_{M\setminus Y_0}$, $\widetilde{f} = \widetilde{f}_1$ be the induced lift of $f|_{M\setminus Y_0}$, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}_Y$. Then, $\widetilde{I}$ fixes every point in $\pi_{Y_0}^{-1}\{z_0\}$, and every point in $\pi_{Y_0}^{-1}\{z_0\}$ is a source of $\widetilde{\mathcal{F}}$. We fix one element $\widetilde{z}_0$ in $\pi_{Y_0}^{-1}\{z_0\}$, and denote by $\widetilde{W}$ the repelling basin of $\widetilde{z}_0$ for $\widetilde{\mathcal{F}}$. Let $J_{\widetilde{z}_0}$ be an identity isotopy of the identity map of $\widetilde{M}_{Y_0}$ that fixes $\widetilde{z}_0$ and satisfies $\rho_s(J_{\widetilde{z}_0}, \widetilde{z}_0) = \{1\}$. Let $\widetilde{I}^*$ be a maximal extension of $(\{\widetilde{z}_0\}, J_{\widetilde{z}_0}\widetilde{I})$, and $\widetilde{\mathcal{F}}^*$ be a transverse foliation of $\widetilde{I}^*$.

Because $M \setminus Y_0$ is neither a sphere not a plane, $\pi_{Y_0}^{-1}\{z_0\}$ is not reduced to one point, and $\widetilde{W}$ is a proper subset of $\widetilde{M}_{Y_0}$. Moreover, if we consider the end ∞ as a singularity, the disk bounded by the union of $\{\infty\}$ and a leaf of $\widetilde{\mathcal{F}}$ in the boundary of $\widetilde{W}$ is a petal. Consequently, $\widetilde{f}$ can be blown-up at ∞ by the criteria in Section 2.7. On the other hand, ∞ is accumulated by the points of $\pi_{Y_0}^{-1}\{z_0\}$, so 0 belongs to $\rho_s(\widetilde{I}, \infty)$. Therefore, $\rho_s(\widetilde{I}, \infty)$ is reduced to 0 by the assertion vii) of Proposition 2.18, and $\rho_s(\widetilde{I}^*, \infty)$ is reduced to -1 by the first assertion of Proposition 2.18.

We can assert that $\widetilde{I}^*$ has finitely many fixed points. We will prove it by contradiction. Suppose that $\widetilde{I}^*$ fixes infinitely many points. Because $\rho_s(\widetilde{I}^*, \infty)$ is reduced to -1, ∞ is not accumulated by fixed points of $\widetilde{I}^*$. Since $\widetilde{I}$ fixes each point in $\pi_{Y_0}^{-1}\{z_0\}$, $\widetilde{I}^*$ does not fix any point in $\pi_{Y_0}^{-1}\{z_0\} \setminus \{\widetilde{z}_0\}$. Since I_Y is not torsion-low at z_0, $\widetilde{z}_0$ is isolated in $\mathrm{Fix}(\widetilde{I}^*)$ (otherwise, z_0 is accumulated by fixed points of f and $-1 \in \rho_s(I_Y, z_0)$). Therefore, there exists a non-isolated point $\widetilde{z}'$ in $\mathrm{Fix}(\widetilde{I}^*)$ such that $z' = \pi_Y(\widetilde{z}') \neq z_0$, and one knows that 0 belongs to $\rho_s(\widetilde{I}^*, \widetilde{z}')$. Moreover, z' is a non-isolated fixed point of f. By Proposition 2.6, there exists an extension $(Y', I_{Y'})$ of (Y_0, I_Y) that fixes z'. Let $\widetilde{I}'$ be the identity isotopy that lifts $I_{Y'}|_{M\setminus Y_0}$. One knows that $\rho_s(\widetilde{I}', \infty) = 0$. Therefore $\widetilde{I}'$ and $J_{\widetilde{z}'}^{-1}\widetilde{I}^*$ are equivalent as local isotopies at $\widetilde{z}'$, which means that -1 belongs to $\rho_s(I_{Y'}, z')$. So, $I_{Y'}$ is torsion-low at z', which contradicts the assumption of this lemma.

Since $\rho_s(\widetilde{I}^*, \infty)$ is reduced to -1, the assertion v) of Proposition 2.18 tells us that ∞ is a source of $\widetilde{\mathcal{F}}^*$. We can assert that $\widetilde{z}_0$ is not a sink of $\widetilde{\mathcal{F}}^*$. Indeed, in case i), one knows that $\widetilde{I}^* \lesssim I'_{z_0}$ as a local isotopy at z_0, and that I'_{z_0} does not have a positive rotation type, so $\widetilde{I}^*$ does not have a positive rotation type; in case ii), one knows that $\rho_s(\widetilde{I}^*, \widetilde{z}_0) = \rho_s(J_{\widetilde{z}_0}\widetilde{I}, \widetilde{z}_0) \subset [-\infty, 0)$, and the result is a corollary of the assertion v) of Proposition 2.18.

In $\widetilde{M}_{Y_0} \sqcup \{\infty\}$, there does not exist any closed leaf or oriented simple closed curve that consists of leaves and singularities of $\mathcal{F}^*$ with the orientation inherited from the orientation of leaves. We can prove this assertion by contradiction. Let Γ be such a curve. Since ∞ is a source of $\mathcal{F}^*$, it does not belong to Γ. Let U be the bounded component of $\widetilde{M}_{Y_0} \setminus \Gamma$, then U contains the positive or the negative orbit of a wandering open set in $U \setminus \widetilde{f}(U)$ or $U \setminus \widetilde{f}^{-1}(U)$ respectively. This contradicts the area preserving property of $\widetilde{f}$.

Then, we can give a partial order $<$ over the set of singularities of $\mathcal{F}^*$ such that $\widetilde{z} < \widetilde{z}'$ if there exists a leaf or a connection of leaves and singularities with the orientation inherited from the orientation of leaves from $\widetilde{z}'$ to $\widetilde{z}$. Since $\mathcal{F}^*$ has only finitely many singularities,

there exists a minimal singularity $\widetilde{z}_1$. Moreover, this minimal singularity $\widetilde{z}_1$ is a sink of $\mathcal{F}^*$ by definition. Therefore, $\widetilde{f}$ fixes $\widetilde{z}_1$ and hence there exists a maximal extension (Y_1, I_{Y_1}) of (Y_0, I_Y) such that $Y_0 \cup \{z_1\} \subset Y_1$, where $z_1 = \pi_{Y_0}(\widetilde{z}_1)$.

Now, we will prove by contradiction that $Y_1 \setminus Y_0$ contains at least two points. Suppose that $Y_1 = Y_0 \sqcup \{z_1\}$. Let $\mathcal{F}_{Y_1}$ be a transverse foliation of I_{Y_1}, $\widetilde{I}_1$ be the identity isotopy that lifts $I_{Y_1}|_{M\setminus Y_0}$, and $\widetilde{\mathcal{F}}_1$ be the lift of $\mathcal{F}_{Y_1}$ to $\widetilde{M}_{Y_0}$. Since I_Y and I_{Y_1} are homotopic relative to Y_0, the lift of $f|_{M\setminus Y_0}$ to $\widetilde{M}_{Y_0}$ associated to I_{Y_1} is also $\widetilde{f}$. The set of singularities of $\widetilde{\mathcal{F}}_1$ is $\pi_{Y_0}^{-1}\{z_1\}$, and $\widetilde{z}_1$ is an isolated singularity of $\widetilde{\mathcal{F}}_1$, so it is a sink, or a source, or a saddle of $\widetilde{\mathcal{F}_1}$ by the remark that follows Proposition 2.12. We know that $\rho_s(\widetilde{I}^*, \infty)$ is reduced to -1 and that $\rho_s(\widetilde{I}_1, \infty)$ is reduced to 0, so $\widetilde{I}^*$ and $J_{\widetilde{z}_1}\widetilde{I}_1$ are equivalent as local isotopies at $\widetilde{z}_1$. By the assumption, I_{Y_1} is not torsion-low at z_1, so $\widetilde{z}_1$ is a sink of $\widetilde{\mathcal{F}}_1$, and z_1 is a sink of $\mathcal{F}_{Y_1}$. Let $\widetilde{W}_1$ be the attracting basin of $\widetilde{z}_1$ for $\widetilde{\mathcal{F}}_1$. A leaf in $\partial\widetilde{W}_1$ is a proper leaf. For every fixed point $\widetilde{z}$ of $\widetilde{f}$, there exists a loop δ that is homotopic to its trajectory along $\widetilde{I}_1$ in $\widetilde{M}_{Y_0} \setminus \pi_{Y_0}^{-1}\{\widetilde{z}_1\}$ (so in $\widetilde{M}_{Y_0} \setminus \{\widetilde{z}_1\}$) and is transverse to $\widetilde{\mathcal{F}}_1$. The linking number $L(\widetilde{I}_1, \widetilde{z}, \widetilde{z}_1)$ is the index of the trajectory of $\widetilde{z}$ along $\widetilde{I}_1$ relatively to $\widetilde{z}_1$, so it is equal to the index of δ relatively to $\widetilde{z}_1$. When $\widetilde{z}$ is in $\widetilde{W}_1$, the loop δ is included in $\widetilde{W}_1$ and is transverse to $\widetilde{\mathcal{F}}_1$, so $L(\widetilde{I}_1, \widetilde{z}, \widetilde{z}_1)$ is positive. When $\widetilde{z}$ is not in $\widetilde{W}_1$, it is in one of the connected component of $\widetilde{M}_{Y_0} \setminus \overline{\widetilde{W}_1}$, and so is δ, therefore $L(\widetilde{I}_1, \widetilde{z}, \widetilde{z}_1)$ is equal to 0. Since $\widetilde{I}^*$ fixes $\widetilde{z}_0$ and $\widetilde{z}_1$, the linking number $L(\widetilde{I}^*, \widetilde{z}_0, \widetilde{z}_1)$ is equal to 0. By Section 2.8, we know that

$$L(\widetilde{I}_1, \widetilde{z}_0, \widetilde{z}_1) = L(\widetilde{I}^*, \widetilde{z}_0, \widetilde{z}_1) - 1 = -1,$$

and find a contradiction. □

The following lemma is a consequence of the previous one.

Lemma 4.12. *Let us suppose that (Y, I_Y) is maximal in $(\mathcal{I}, \precsim)$, that I_Y is not torsion-low at $z \in Y$, and that $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every point $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) = \infty$.*

Proof. Fix a couple (Y_0, I_{Y_0}) maximal in $(\mathcal{I}, \precsim)$ and $z_0 \in Y$ satisfying the assumptions of the lemma. By the previous lemma, there exists a maximal extension (Y_1, I_{Y_1}) of $(Y_0 \setminus \{z_0\}, I_{Y_0})$ such that $\#(Y_1 \setminus (Y_0 \setminus \{z_0\})) > 1$. If $\#(Y_1 \setminus (Y_0 \setminus \{z_0\})) = \infty$, the proof is finished; if $\#(Y_1 \setminus (Y_0 \setminus \{z_0\}) < \infty$, we fix a point $z_1 \in Y_1 \setminus (Y_0 \setminus \{z_0\})$. By hypothesis, I_{Y_1} is not torsion-low at z_1 and $M \setminus (Y_1 \setminus \{z_1\})$ is neither a sphere nor a plane. Since a maximal extension of $(Y_1 \setminus \{z_1\}, I_{Y_1})$ is also a maximal extension of $(Y_0 \setminus \{z_0\}, I_{Y_0})$, the couple (Y_1, I_{Y_1}) and $z_1 \in Y_1$ satisfies the assumptions of the previous lemma. We apply the previous lemma, and deduce that there exists a maximal extension $(Y_2, I_{Y_2}) \in \mathcal{I}$ of $(Y_1 \setminus \{z_1\}, I_{Y_1})$ such that $\#(Y_2 \setminus (Y_1 \setminus \{z_1\})) > 1$. If $\#(Y_2 \setminus (Y_1 \setminus \{z_1\})) = \infty$, the proof is finished; if $\#(Y_2 \setminus (Y_1 \setminus \{z_1\})) < \infty$, we continue the construction...

Then, either we end the proof in finitely many steps, or we can construct a strictly increasing sequence

$$(Y_0 \setminus \{z_0\}, I_{Y_0}) \prec (Y_1 \setminus \{z_1\}, I_{Y_1}) \prec (Y_2 \setminus \{z_2\}, I_{Y_2}) \prec (Y_3 \setminus \{z_3\}, I_{Y_3}) \cdots$$

By Proposition 2.7, there exists an upper bound $(Y_\infty, I_{Y_\infty}) \in \mathcal{I}$ of this sequence, where $Y_\infty = \overline{\cup_{n\geq 1}(Y_n \setminus \{z_n\})}$. By Theorem 2.8, there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of (Y_∞, I_{Y_∞}). It is also a maximal extension of $(Y_0 \setminus \{z_0\}, I_{Y_0})$, and satisfies $\#(Y' \setminus (Y_0 \setminus \{z_0\})) = \infty$. □

Proof of Proposition 4.7. We will prove this proposition by contradiction. Fix $(Y, I_Y) \in \mathcal{I}$ and $z_0 \in Y$ such that I_Y is not torsion-low at z_0 and $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane. Write $Y_0 = Y \setminus \{z_0\}$, and suppose that for all maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) and $z' \in Y' \setminus Y_0$, $I_{Y'}$ is not torsion-low at z'. By the previous lemma, there exists a maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) such that $\#(Y' \setminus Y_0) = \infty$.

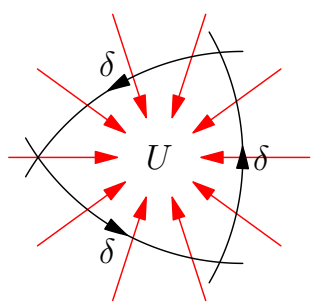
Let $\pi_{Y_0} : \widetilde{M}_{Y_0} \to M \setminus Y_0$ be the universal cover, $\widetilde{I}$ be the identity isotopy that lifts $I_Y|_{M\setminus Y_0}$, $\widetilde{I}'$ be the identity isotopy that lifts $I_{Y'}|_{M\setminus Y_0}$, and $\widetilde{f}$ be the lift of $f|_{M\setminus Y_0}$ associated to $I_Y|_{M\setminus Y_0}$. Since both I_Y and $I_{Y'}$ are maximal and $M \setminus Y_0$ is neither a sphere nor a plane, the point z_0 does not belong to Y'. Moreover, $I_Y|_{M\setminus Y_0}$ and $I_{Y'}|_{M\setminus Y_0}$ are homotopic, so $\widetilde{f}$ is also the lift of $f|_{M\setminus Y_0}$ associated to $I_{Y'}|_{M\setminus Y_0}$. In particular, $\widetilde{f}$ fixes every point in $\pi_{Y_0}^{-1}(\{z_0\} \cup Y' \setminus Y_0)$. Fix $\widetilde{z}_0 \in \pi_{Y_0}^{-1}\{z_0\}$.

Sublemma 4.13. *For every $z \in Y' \setminus Y_0$, there exists $\widetilde{z} \in \pi_{Y_0}^{-1}\{z\}$ such that $\widetilde{z}_0$ and $\widetilde{z}$ are linked relatively to $\widetilde{I}$.*

Proof. Let $\mathcal{F}$ be a transverse foliation of I_Y, and $\widetilde{\mathcal{F}}$ be the lift of $\mathcal{F}|_{M\setminus Y_0}$ to $\widetilde{M}_{Y_0}$. Fix $z \in Y' \setminus Y_0$ and $\widetilde{z} \in \pi_{Y_0}^{-1}\{z\}$. Since I_Y is a maximal identity isotopy, the trajectory of $\widetilde{z}$ along $\widetilde{I}$ is a loop that is not homotopic to zero in $\widetilde{M}_{Y_0} \setminus \pi_{Y_0}^{-1}\{z_0\}$. Let δ be a loop that is transverse to $\widetilde{\mathcal{F}}$, and is homotopic to the trajectory of $\widetilde{z}$ along $\widetilde{I}$ in $\widetilde{M}_{Y_0} \setminus \pi_{Y_0}^{-1}\{z_0\}$. By choosing suitable δ, we can suppose that δ intersects itself at most finitely many times, that each intersection point is a double point, and that the intersections are transverse. So, $\widetilde{M}_{Y_0} \setminus \delta$ has finitely many components, and we can define a locally constant function $\Lambda : \widetilde{M}_{Y_0} \setminus \delta \to \mathbb{Z}$ such that

- Λ is equal to 0 in the component of $\widetilde{M}_{Y_0} \setminus \delta$ that is not relatively compact;
- $\Lambda(\widetilde{z}') - \Lambda(\widetilde{z}'')$ is equal to the (algebraic) intersection number of δ and any arc from $\widetilde{z}''$ to $\widetilde{z}'$.

This function is not constant, and we have either $\max \Lambda > 0$ or $\min \Lambda < 0$. Suppose that we are in the first case (the other case can be treated similarly). Let U be a component of $\widetilde{M}_{Y_0} \setminus \delta$ such that Λ is equal to $\max \Lambda > 0$ in U. As in the picture, the boundary of U is a sub-curve



of δ with the orientation such that U is to the left of its boundary, and is also transverse to $\widetilde{\mathcal{F}}$. So, there exists a singularity of $\widetilde{\mathcal{F}}$ in U. Note the fact that the set of singularities of $\widetilde{\mathcal{F}}$ is $\mathrm{Fix}(\widetilde{I}) = \pi_{Y_0}^{-1}\{z_0\}$. So, there exists an automorphism T of the universal cover space such that $T(\widetilde{z}_0)$ belongs to U, and the index of δ relatively to $T(\widetilde{z}_0)$ is positive. Note also that the linking number $L(\widetilde{I}, \widetilde{z}, T(\widetilde{z}_0))$ is equal to the index of δ relatively to $T(\widetilde{z}_0)$. So, $T(\widetilde{z}_0)$ and $\widetilde{z}$ are linked relatively to $\widetilde{I}$. Consequently, $\widetilde{z}_0$ and $T^{-1}(\widetilde{z})$ are linked relatively to $\widetilde{I}$. □

As in the proof of Lemma 4.11, we know that $\widetilde{f}$ can be blown-up at ∞. Since ∞ is accumulated by both the points in $\pi_{Y_0}^{-1}\{z_0\}$ and the points in $\pi_{Y_0}^{-1}(Y' \setminus Y_0)$, both $\rho_s(\widetilde{I}, \infty)$ and $\rho_s(\widetilde{I}', \infty)$ contain 0. Then, both $\rho_s(\widetilde{I}, \infty)$ and $\rho_s(\widetilde{I}', \infty)$ are reduced to 0, so $\widetilde{I}$ and $\widetilde{I}'$ are equivalent as local isotopies at ∞. Therefore, for every point $z \in Y' \setminus Y_0$, there exists $\widetilde{z} \in \pi_{Y_0}^{-1}\{z\}$ such that $\widetilde{z}_0$ and $\widetilde{z}$ are linked relatively to $\widetilde{I}'$. Let us denote by L the set of points $\widetilde{z} \in \pi_{Y_0}^{-1}(Y' \setminus Y_0)$ such that $\widetilde{z}$ and $\widetilde{z}_0$ are linked relatively to $\widetilde{I}'$. It contains infinitely many points.

Let γ be the trajectory of $\widetilde{z}_0$ along the isotopy $\widetilde{I}'$, and V be the connected component of $\widetilde{M}_{Y_0} \setminus \gamma$ containing ∞. Then $K = \widetilde{M}_{Y_0} \setminus V$ is a compact set that contains all the fixed points of $\widetilde{I}'$ that are linked with $\widetilde{z}_0$ relatively to $\widetilde{I}'$. In particular, $L \subset K$. Then, there exists $\widetilde{z}' \in K$ that is accumulated by points of L. We know that $\mathrm{Fix}(\widetilde{I}')$ is a closed set. So, $\widetilde{z}'$ belongs to $\mathrm{Fix}(\widetilde{I}') = \pi^{-1}(Y' \setminus Y_0)$. We find a point $\widetilde{z}'$ that is not isolated in $\pi_{Y_0}^{-1}(Y' \setminus Y_0)$, and a point $z' = \pi_Y(\widetilde{z}')$ that is not isolated in Y'. This means that $I_{Y'}$ is torsion-low at z'. We get a contradiction. □

Proof of Proposition 4.8. We only need to prove that there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$, because one knows $(\emptyset, I) \precsim (X, I_X)$ for all $(X, I_X) \in \mathcal{I}$ when M is a plane.

One has to consider the following two cases:

- Suppose that $\mathrm{Fix}(f)$ is reduced to one point z_0. In this case, similarly to the proof of Theorem 1.1, we can find an isotopy I_0 that fixes z_0 and is torsion-low at z_0. Then, $(\{z_0\}, I_0)$ belongs to $\mathcal{I}_0$.

- Suppose that $\mathrm{Fix}(f)$ contains at least two points. In this case, there exists a maximal $(Y, I_Y) \in \mathcal{I}$ such that $\#Y \geq 2$. If I_Y is torsion-low at a point in Y, the proof is finished; if I_Y is not torsion-low at every $z \in Y$, we fix $z_0 \in Y$ and can find a maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z_0\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z_0\})$ such that $I_{Y'}$ is torsion-low at z' by Proposition 4.7. Consequently, $(\{z'\}, I_{Y'})$ belongs to $\mathcal{I}_0$.

□

Proof of Proposition 4.9. One knows $(X, I_X) \precsim (Y, I_Y)$ for all $(Y, I_Y) \in \mathcal{I}$ satisfying $X \subset Y$, when M is a sphere and $\#X \leq 1$. So, we only need to prove the following two facts:

i) there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$;

ii) given $(X, I_X) \in \mathcal{I}_0$ such that $\#X = 1$, there exists $(X', I_{X'}) \in \mathcal{I}_0$ such that $X \subsetneqq X'$.

One has to consider the following two cases:

- Suppose that $\#\mathrm{Fix}(f) = 2$. In this case, we will prove that there exists an identity isotopy that fixes both fixed points and is torsion-low at each fixed point, which implies both i) and ii).

  Denote by N and S the two fixed points. Since both N and S are isolated fixed points, we can find an identity isotopy I that fixes both N and S and is torsion-low at S. We will prove that I is also torsion-low at N.

  Let J_N (resp. J_S) be an identity isotopy of the identity map of the sphere that fixes both N and S and satisfies $\rho_s(J_N, N) = \{1\}$ (resp. $\rho_s(J_S, S) = \{1\}$). One knows that the restrictions to $M \setminus \{N, S\}$ of J_N and J_S^{-1} are equivalent.

For every $k \geq 1$, since I is torsion-low at S, $J_S^{-k}I$ has a negative rotation type as a local isotopy at S. Let $\mathcal{F}_k$ be a transverse foliation of $J_S^{-k}I$. Then S is a source of $\mathcal{F}_k$. Since f is area preserving and $\mathcal{F}_k$ has exactly two singularities, N is a sink of $\mathcal{F}_k$. Note the fact that the restrictions to $M \setminus \{S, N\}$ of $J_N^k I$ and $J_S^{-k}I$ are homotopic. So, $J_N^k I$ has a positive rotation type as a local isotopy at N.

Similarly, for every $k \geq 1$, $J_N^{-k}I$ has a negative rotation type as a local isotopy at N. Therefore, I is torsion-low at N.

- Suppose that $\#\mathrm{Fix}(f) \geq 3$.

  In this case, there exists $(Y, I_Y) \in \mathcal{I}$ such that $\#Y \geq 3$. We can prove i) by a similar discussion to the second part of the proof of Proposition 4.8. We can also give the following direct proof. Fix a maximal $(Y, I_Y) \in \mathcal{I}$ such that $\#Y \geq 3$. If Y is infinite, there exists a point $z \in Y$ that is not isolated in Y, and hence I_Y is torsion-low at z. If Y is finite, we consider a transverse foliation of I_Y and know that there is a saddle singulary point z of $\mathcal{F}$ by the Poincaré-Hopf formula, and hence I_Y is torsion-low at z. Anyway, there exists $z \in Y$ such that $(\{z\}, I_Y) \in \mathcal{I}_0$.

  To prove ii), we fix $(X, I_X) \in \mathcal{I}_0$ such that $X = \{S\}$. For a maximal extension $(Y, I_Y) \in \mathcal{I}$ of (X, I_X) such that I_Y is torsion-low at S, one knows that $Y \setminus X$ is not empty. If I_Y is torsion-low at another fixed point, the proof is finished; if I_Y is not torsion-low at any other fixed points and if $\#(Y \setminus X)$ is bigger that 1, we get the result as a corollary of Proposition 4.7. Then, we only need to prove that there exists a maximal extension $(Y, I_Y) \in \mathcal{I}$ of (X, I_X) such that I_Y is torsion-low at S and that satisfies one of the two conditions: I_Y is torsion-low at another fixed point or $\#(Y \setminus X) > 1$.

  Fix a maximal extension $(Y, I_Y) \in \mathcal{I}$ of (X, I_X) such that $\rho_s(I_Y, S) = \rho_s(I_X, S)$. Of course, I_Y is torsion-low at S. If I_Y is torsion-low at another fixed point or if $\#(Y \setminus X) > 1$, the proof is finished. Now, we suppose that $Y = \{S, N\}$ and I_Y is not torsion-low at N. One has to consider two cases: S is isolated in $\mathrm{Fix}(f)$ or not.

  a) Suppose that S is isolated in $\mathrm{Fix}(f)$. As in the proof of Lemma 4.11, one has two consider the following four cases:

     - N is an isolated fixed point of f and there exists a local isotopy $I'_N > I_Y$ at N which does not have a positive rotation type;
     - N is not an isolated fixed point of f and $\rho_s(I_Y, N) \subset [-\infty, -1)$;
     - N is an isolated fixed point of f and there exists a local isotopy $I'_N < I_Y$ at N which does not have a negative rotation type;
     - N is not an isolated fixed point of f and $\rho_s(I_Y, N) \subset (1, +\infty]$.

     As before, we study the first two cases.

     Let $\mathcal{F}_Y$ be a transverse foliation of I_Y. As in the proof of Lemma 4.11, N is a source of $\mathcal{F}_Y$. Since f is area preserving and $\mathcal{F}_Y$ has exactly two singularities, S is a sink of $\mathcal{F}_Y$.

     Let I' be a maximal extension of $(Y, J_N I_Y)$. Since I_Y is torsion-low at S and I' is equivalent to $J_S^{-1}I_Y$ as local isotopies at S, I' has a negative rotation type at S. Moreover, as local isotopies at S, $J_S^k I' \sim J_S^{k-1}I_Y$ has a positive rotation type at S for $k \geq 1$, and has a negative rotation type for $k \leq -1$. Therefore I' is torsion-low at S.

Let $\mathcal{F}'$ be a transverse foliation of I'. One knows that S is a source of $\mathcal{F}'$. As in the proof of Lemma 4.11, we can deduce that N is not a sink of $\mathcal{F}'$. Therefore, $\mathcal{F}'$ has another singularity, and hence one deduces that $\mathrm{Fix}(I') \geq 3$.

b) Suppose that S is not isolated in $\mathrm{Fix}(f)$. We know that $\rho_s(I_Y, S) \cap [-1, 1] \neq \emptyset$ by definition.

We define the rotation number of a fixed point near S as in the proof of Proposition 1.7. By the maximality of I_Y, the rotation number of a fixed point near S is not zero. Then, either there exists $k \in \mathbb{Z} \setminus \{0\}$ such that S is accumulated by fixed points of f with rotation number k, or $\rho_s(I_Y, S)$ intersects $\{\pm\infty\}$. In the second case, the interior of the convex hull of $\rho_s(I_Y, S)$ contains a non-zero integer k', and hence 0 is in the interior of the convex hull of $\rho_s(J_S^{-k'} I_Y, S)$. So, S is accumulated by contractible fixed points of $J_S^{-k'} I_Y$ by the assertion iii) of Proposition 2.18, and hence is accumulated by fixed points with rotation k' (associate to I_Y). Anyway, there exists $k \in \mathbb{Z} \setminus \{0\}$ such that S is accumulated by fixed points of f with rotation number k. We fix one such k.

Let I' be a maximal extension of $J_S^{-k} I_Y$. Then, I' fixes at least 3 fixed points, and 0 belongs to $\rho_s(I', S)$. Therefore, I' is torsion-low at S, and satisfies $\#\mathrm{Fix}(I') \geq 3$.

□

Remark 4.14. In both case a) and case b), we construct an identity isotopy I' that is torsion-low at S and has at least three fixed points. Even though $\rho_s(I_X, S)$ and $\rho_s(I', S)$ are different, I' is still an extension of (X, I_X) because M is a sphere and X is reduced to a single point. However, as was in Remark 4.2, for $(X', I_{X'}) \in \mathcal{I}_0$ that is an maximal extension of (X, I_X), $I_{X'}$ and I_X are not necessarily equivalent as local isotopies at S.

Now, let us prove Proposition 1.6.

Proof of Proposition 1.6. Let f be an area preserving homeomorphism of M that is isotopic to the identity and has finitely many fixed points. When $\mathrm{Fix}(f)$ is empty, the proposition is trivial. So, we suppose that $\mathrm{Fix}(f)$ is not empty. Let

$$n = \max\{\#\mathrm{Fix}(I) : I \text{ is an identity isotopy of } f\}.$$

One has to consider the following three cases:

- Suppose that M is a plane and f has exactly one fixed point. As in the first part of the proof of Proposition 4.8, there exists an identity isotopy that fixes this fixed point and is torsion-low at this fixed point.

- Suppose that M is a sphere and f has exactly two fixed points. As in the first part of the proof of Proposition 4.9, there exists an identity isotopy that fixes these two fixed points and is torsion-low at each fixed point.

- Suppose that we are not in the previous two cases. Let $\mathfrak{I}$ be the set of identity isotopies of f with n fixed points. It is not empty. We can give a preorder $\lhd$ over $\mathfrak{I}$ such that $I \lhd I'$ if and only if

$$\#\{z \in \mathrm{Fix}(I), I \text{ is torsion-low at } z\} \leq \#\{z \in \mathrm{Fix}(I'), I' \text{ is torsion-low at } z\}.$$

Since $\#\{z \in \mathrm{Fix}(I), I \text{ is torsion-low at } z\}$ is not bigger than n for all $I \in \mathfrak{I}$, $\mathfrak{I}$ has a maximal element. Fix a maximal element I of $\mathfrak{I}$. We will prove by contradiction that I is torsion-low at every $z \in \mathrm{Fix}(I)$.

Suppose that I is not torsion-low at $z_0 \in \mathrm{Fix}(I)$. Write $Y_0 = \mathrm{Fix}(I) \setminus \{z_0\}$. Since we are not in the previous two cases, $M \setminus Y_0$ is neither a plane nor a sphere. By Proposition 4.7, there exist a maximal extension I' of (Y_0, I) and $z' \in \mathrm{Fix}(I') \setminus Y_0$ such that I' is torsion-low at z'. This contradicts the fact that I is maximal in $(\mathfrak{I}, \lhd)$.

□

4.2 Proof of Theorem 4.1 when $\mathrm{Fix}(f)$ is not totally disconnected

When $\mathrm{Fix}(f)$ is not totally disconnected, the proof of Theorem 4.1 is similar to the one in the previous section except that we should consider more cases. More precisely, Theorem 4.1 is a corollary of Zorn's lemma and the following four similar propositions. The proof of Proposition 4.15 is just a copy of the one of Proposition 4.6; while the proofs of the others are the aim of this subsection.

Proposition 4.15. *If $\{(X_\alpha, I_{X_\alpha})\}_{\alpha \in J}$ is a totally ordered chain in $\mathcal{I}_0$, then there exists an upper bound $(X_\infty, I_{X_\infty}) \in \mathcal{I}_0$ of the chain, where $X_\infty = \overline{\cup_{\alpha \in J} X_\alpha}$*

Proposition 4.16. *For every maximal $(Y, I_Y) \in \mathcal{I}$ and $z \in Y$ such that I_Y is not torsion-low at z and $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane[4] whose boundary is empty or reduced to one point, there exist a maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$ such that $I_{Y'}$ is torsion-low at z'.*

Proposition 4.17. *When M is a plane, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $X = \emptyset$.*

Proposition 4.18. *When M is a sphere, $(X, I_X) \in \mathcal{I}_0$ is not maximal in $(\mathcal{I}_0, \precsim)$ if $\#X \leq 1$.*

To prove Proposition 4.16, we need the following Lemmas 4.19-4.21. Lemma 4.19 is almost the same as Lemma 4.11 except that we deal with the the connected component of $M \setminus (Y \setminus \{z\})$ containing z instead of $M \setminus (Y \setminus \{z\})$. Similarly to the proof of Lemma 4.12, we will get Lemma 4.21 by Lemma 4.19 and Lemma 4.20. Then, as in the proof of Proposition 4.7, we can give a similar proof of Proposition 4.16 as a corollary of Lemma 4.21. The new case is Lemma 4.20.

Lemma 4.19. *Suppose that (Y, I_Y) is maximal in $\mathcal{I}$, that I_Y is not torsion-low at $z \in Y$, and that the connected component of $M \setminus (Y \setminus \{z\})$ containing z is neither a sphere nor a plane. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every point $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) > 1$.*

Proof. The proof of Lemma 4.19 is just a copy of the one of Lemma 4.11 except that we should replace $M \setminus Y_0$ with the the connected component of $M \setminus Y_0$ containing z_0. □

Lemma 4.20. *Suppose that (Y, I_Y) is maximal in $\mathcal{I}$, that I_Y is not torsion-low at $z \in Y$, and that the connected component of $M \setminus (Y \setminus \{z\})$ containing z is a plane whose boundary contains more that two points. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) > 1$.*

[4] Here, a plane means an open set that is homeomophic to $\mathbb{R}^2$

Proof. Fix a maximal $(Y, I_Y) \in \mathcal{I}$ and $z_0 \in Y$ satisfying the assumptions of this lemma. Write $Y_0 = Y \setminus \{z_0\}$, and denote by M_{Y_0} the connected component of $M \setminus Y_0$ containing z_0. Then M_{Y_0} is a plane and $\#\partial M_{Y_0} > 1$. As in the proof of Lemma 4.11, since I_Y is not torsion-low at z_0, one has to consider the following four cases:

- z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} > I_Y$ at z_0 which does not have a positive rotation type;
- z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset [-\infty, -1)$;
- z_0 is an isolated fixed point of f and there exists a local isotopy $I'_{z_0} < I_Y$ at z_0 which does not have a negative rotation type;
- z_0 is not an isolated fixed point of f and $\rho_s(I_Y, z_0) \subset (1, +\infty]$.

As before, we only study the first two cases.

Let $\mathcal{F}_Y$ be a transverse foliation of I_Y. As in the proof of Lemma 4.11, we know that z_0 is a source of $\mathcal{F}_Y$.

Since $\#\partial M_{Y_0} > 1$, the plane M_{Y_0} can be blown-up by prime-ends at infinity. Because I_Y fixes ∂M_{Y_0} and z_0, $I_Y|_{M_{Y_0}}$ can be viewed as a local isotopy at ∞, and the blow-up rotation number $\rho(I_Y|_{M_{Y_0}}, \infty)$, that was defined in Section 2.7, is equal to 0.

Let I^* be a maximal extension of $(\{z_0\}, J_{z_0} I_Y|_{M_{Y_0}})$, and $\mathcal{F}^*$ be a transverse foliation of I^*. Note that I_Y is not torsion-low at z_0, by the same argument of the proof of Lemma 4.11, we know that z_0 is not a sink of $\mathcal{F}^*$.

We can assert that ∞ is a source of $\mathcal{F}^*$. Indeed, when the total area of M_{Y_0} is finite, $f|_{M_{Y_0}}$ is area preserving as a local homeomorphism at ∞, so $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is not empty by Proposition 2.16 and is reduced to 0 by the assertion vii) of Proposition 2.18. Then, by the assertion i) of Proposition 2.18, $\rho_s(I^*, \infty)$ is reduced to -1, and by the assertion v) of Proposition 2.18, ∞ is a source of $\mathcal{F}^*$. However, the total area of M_{Y_0} may be infinite. In this case, we can not get the result that $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is not empty. But anyway, we can prove the assertion by considering the following two cases:

- Suppose that $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is not empty. As in the case where the total area of M_{Y_0} is finite, $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is reduced to 0, and $\rho_s(I^*, \infty)$ is reduced to -1. Therefore, ∞ is a source of $\mathcal{F}^*$ by the assertion v) of Proposition 2.18.
- Suppose that $\rho_s(I_Y|_{M_{Y_0}}, \infty)$ is empty. Since $f|_{M_{Y_0}}$ is area preserving, $f|_{M_{Y_0}}$ is not conjugate to a contraction or a expansion at ∞. By Proposition 2.16, the germ of $f|_{M_{Y_0}}$ at ∞ is conjugate to a local homeomorphism $z \mapsto e^{i2\pi\frac{p}{q}} z(1 + z^{qr})$ at 0 with $q, r \in \mathbb{N}$ and $p \in \mathbb{Z}$. Since $\rho(I_Y|_{M_{Y_0}}, \infty) = 0$, we can deduce that $p \in q\mathbb{Z}$. Therefore, one has $i(f|_{M_{Y_0}}, \infty) > 1$. Let $I_0 = (g_t)_{t\in[0,1]}$ be a local isotopy at 0 such that $g_t(z) = z(1+tz^{qr})$. Then, $\rho(I_0, 0)$ is equal to 0 and $i(I_0, 0)$ is positive. Since $\rho(I_Y|_{M_{Y_0}}, \infty)$ is equal to 0, $I_Y|_{M_{Y_0}}$ is conjugate to a local isotopy that is in the same homotopy class of I_0. So, $i(I_Y|_{M_{Y_0}}, \infty) = i(I_0, 0)$ is positive. Therefore, ∞ is a source of $\mathcal{F}^*$ by Proposition 2.14.

Then, like in the proof of Lemma 4.11, we deduce that I^* fixes finitely many points, that there exists a sink z_1 of $\mathcal{F}^*$, and that there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of (Y_0, I_Y) such that $Y_0 \cup \{z_1\} \subset Y'$ and $\#(Y' \setminus Y_0) > 1$. □

Lemma 4.21. *Suppose that (Y, I_Y) is maximal in $\mathcal{I}$, that I_Y is not torsion-low at $z \in Y$, and that the connected component of $M \setminus (Y \setminus \{z\})$ is neither a sphere nor a plane whose boundary*

is empty or reduced to one point. If for every maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and every $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z', then there exists a maximal extension $(Y', I_{Y'}) \in \mathcal{I}$ of $(Y \setminus \{z\}, I_Y)$ such that $\#(Y' \setminus (Y \setminus \{z\})) = \infty$.

Proof. The proof is almost the same as the one of Lemma 4.12 except the following: every time we want to get a new couple, we should check that the previous couple satisfies the assumptions of Lemma 4.19 or Lemma 4.20 instead of the assumptions of Lemma 4.11. □

Now, we begin the proof of Proposition 4.16. The proof is similar to the one of Proposition 4.7.

Proof of Proposition 4.16. We will prove this proposition by contradiction. Fix a maximal $(Y, I_Y) \in \mathcal{I}$ and $z_0 \in Y$ such that I_Y is not torsion-low at z_0 and $M \setminus (Y \setminus z_0)$ is neither a sphere nor a plane whose boundary is empty or reduced to a single point. Write $Y_0 = Y \setminus \{z_0\}$, and denote by M_{Y_0} the connected component of $M \setminus Y_0$ containing z_0. Suppose that for all maximal extension $(Y', I_{Y'})$ of $(Y \setminus \{z\}, I_Y)$ and $z' \in Y' \setminus (Y \setminus \{z\})$, $I_{Y'}$ is not torsion-low at z'. By the previous lemma, there exists a maximal extension $(Y', I_{Y'})$ of (Y_0, I_Y) such that $\#(Y' \setminus Y_0) = \infty$.

Let us prove by contradiction that $Y' \setminus Y_0 \subset M_{Y_0}$. Suppose that there exists $z_1 \in Y' \setminus Y_0$ that is in another component of $M \setminus Y_0$. Since $I_Y|_{M \setminus Y_0}$ and $I_{Y'}|_{M \setminus Y_0}$ are homotopic, the trajectory of z_1 along I_Y is homotopic to zero in $M \setminus Y_0$. Moreover, because the trajectory of z_1 along I_Y is in another component of $M \setminus Y_0$, this trajectory is homotopic to zero in $M \setminus Y$, which contradicts the maximality of (Y, I_Y).

Then, one has to consider two cases:

- M_{Y_0} is neither a sphere nor a plane,
- M_{Y_0} is a plane whose boundary contains more than two points.

In the first case, we repeat the proof of Proposition 4.7 except that we should replace $M \setminus Y_0$ with M_{Y_0}. In the second case, the idea is similar, but we do not lift the isotopies to the universal cover because M_{Y_0} itself is a plane. □

Proof of Proposition 4.17. As in the proof of Proposition 4.8, we only need to prove that there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$.

Since $\mathrm{Fix}(f)$ is not totally disconnected, we can fix a connected component X of $\mathrm{Fix}(f)$ that is not reduced to a point. By Proposition 2.11, there exists a maximal identity isotopy I of f that fixes all the points in X. So, 0 belongs to $\rho_s(I, z)$ for all $z \in X$, and hence (X, I) belongs to $\mathcal{I}_0$. □

Proof of Proposition 4.18). As in the proof of Proposition 4.9, we only need to prove the following two facts:

i) there exists $(X, I_X) \in \mathcal{I}_0$ such that $X \neq \emptyset$;

ii) given $(X, I_X) \in \mathcal{I}_0$ such that $\#X = 1$, there exists $(X', I_{X'}) \in \mathcal{I}_0$ such that $X \subsetneqq X'$.

The proof of the first fact is the same to the proof of Proposition 4.17; while the proof of the second fact is similar to the proof of Proposition 4.9 in the case $\#\mathrm{Fix}(f) \geq 3$. □

5 Examples

In this section, we will give some explicit examples to get the optimality of previous results.

Example 1. (A local isotopy that has both positive and negative rotation types)

Write f_t for the homothety of factor $1+t$ of a plane. One can note that f_1 has an isolated fixed point 0, and $I = (f_t)_{t\in[0,1]}$ has both positive and negative rotation types at 0. In fact, let

$$\begin{array}{rcl} \pi : \mathbb{R}\times(-\infty,0) & \to & \mathbb{C}\setminus\{0\}\simeq\mathbb{R}^2\setminus\{0\} \\ (\theta,y) & \mapsto & -ye^{i2\pi\theta} \end{array}$$

be the universal cover. Let $\mathcal{F}_1'$ be the foliation on $\mathbb{R}\times(-\infty,0)$ whose leaves are the lines $y=\theta+c$ upward. It descends to an oriented foliation $\mathcal{F}_1$ on $\mathbb{C}\setminus\{0\}$ that is locally transverse to I. Moreover, 0 is a sink of $\mathcal{F}_1$. Let $\mathcal{F}_2'$ be the foliation on $\mathbb{R}\times(-\infty,0)$ whose leaves are the lines $y=-\theta+c$ downward. It descends to an oriented foliation $\mathcal{F}_2$ on $\mathbb{C}\setminus\{0\}$ that is locally transverse to I. Moreover, 0 is a source of $\mathcal{F}_2$.

(a) $\mathcal{F}_1$ (b) $\mathcal{F}_2$

Figure 2: The two foliations of Example 1

Example 2. (A local isotopy that has both positive and zero rotation types)

We define a flow[5] on $\mathbb{R}^2$ by

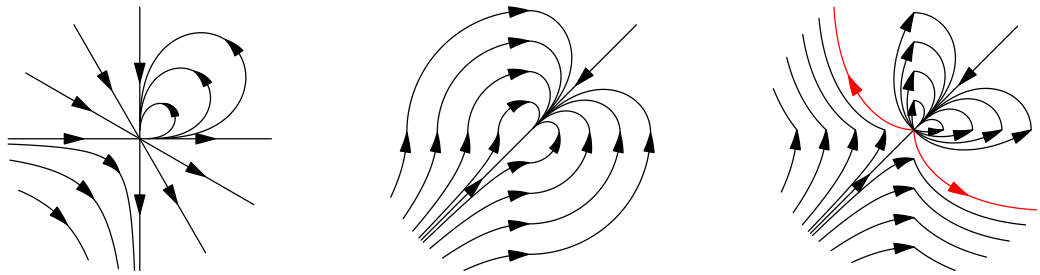
$$f_t(x,y)=\begin{cases} \frac{x^2+y^2}{x^2e^{-2t}+y^2e^{2t}}(xe^{-t},ye^{t}) & \text{for } x\geq 0, y\geq 0, \\ (xe^{-t},ye^{-t}) & \text{for } x\leq 0, y\geq 0, \\ (xe^{-t},ye^{t}) & \text{for } x\leq 0, y\leq 0, \\ (xe^{t},ye^{t}) & \text{for } x\geq 0, y\leq 0. \end{cases}$$

It is the flow of the (time-independent) continuous vector field V in the plane $\mathbb{R}^2$, where V is defined by

$$V(x,y)=\begin{cases} (\frac{x(x^2-3y^2)}{x^2+y^2},\frac{y(3x^2-y^2)}{x^2+y^2}) & \text{for } x>0, y>0, \\ (-x,-y) & \text{for } x\leq 0, y\geq 0, \\ (-x,y) & \text{for } x\leq 0, y\leq 0, \\ (x,y) & \text{for } x\geq 0, y\leq 0. \end{cases}$$

Then, $f=f_1$ has a unique fixed point 0, and $I=(f_t)_{t\in[0,1]}$ is an identity isotopy of f. We will prove that I has both positive and zero rotation types at 0 by constructing two transverse foliations $\mathcal{F}_1$, $\mathcal{F}_2$ of I such that 0 is a sink of $\mathcal{F}_1$ and is a mixed singularity of $\mathcal{F}_2$.

[5]The flow on the first quadrant is just $f_t(z)=\frac{1}{\varphi_t(1/z)}$, where $z=x+iy$ is the complex coordinate and φ is the flow defined by $\varphi_t(x,y)=(xe^{-t},ye^{t})$.



(a) The dynamics of the flow (b) $\mathcal{F}_1$ (c) $\mathcal{F}_2$

Figure 3: The dynamics and two foliations of Example 2

We will construct $\mathcal{F}_1$ by considering the integral curves of vector field. We define a continuous vector field ξ in the plane by

$$\xi(x,y) = \begin{cases} (-\frac{y(3x^2-y^2)}{x^2+y^2}, \frac{x(x^2-3y^2)}{x^2+y^2}) & \text{for } x>0, y>0, \\ (y,-x) & \text{for } x\leq 0, y\geq 0, \\ (-y,-x) & \text{for } x\leq 0, y\leq 0, \\ (-y,x) & \text{for } x\geq 0, y\leq 0. \end{cases}$$

One knows that ξ vanish at a unique point 0, is transverse to V, and satisfies $\det(V(x,y),\xi(x,y)) > 0$ for all $(x,y)\neq 0$. So, the the foliation $\mathcal{F}_1$ whose leaves are the integral curves of ξ is transverse to I. Moreover, by a direct computation, we can get the formulae of the integral curves of ξ and find that every integral curves go to 0 as in the picture. So, 0 is a sink of $\mathcal{F}_1$.

Let

$$\begin{array}{rcl} \pi : \mathbb{R}\times(-\infty,0) & \to & \mathbb{C}^2\setminus\{0\}\simeq\mathbb{R}^2\setminus\{0\} \\ (x,y) & \mapsto & -ye^{i2\pi x} \end{array}$$

be the universal cover, and $(\widetilde{f}_t)_{t\in[0,1]}$ be the identity isotopy that lifts I. We know that $\gamma_{(x,y)} : t\mapsto \widetilde{f}_t(x,y)$ is a vertical segment upward for every $(x,y)\in[\frac{1}{4},\frac{1}{2}]\times(-\infty,0)$, and is a vertical segment downward for every $(x,y)\in[-\frac{1}{4},0]\times(-\infty,0)$. We define an oriented foliation $\widetilde{\mathcal{F}}_{II}$ on the domain $(\frac{1}{4},\frac{1}{2})\times(-\infty,0)$ whose leaves are the the restriction to $(\frac{1}{4},\frac{1}{2})\times(-\infty,0)$ of the family of curves $(\ell_c)_{c\in(-1,\infty)}$ such that

- ℓ_c is the graph of $y=\log(4x-1-c)$ with the direction from right to left, for $c\in(-1,0]$,
- ℓ_c is the graph of $y=\log(4x-1)-c$ with the direction from right to left, for $c\in(0,\infty)$.

Then, $\gamma_{(x,y)}$ is positively transverse to $\widetilde{\mathcal{F}}_{II}$ for every $(x,y)\in(\frac{1}{4},\frac{1}{2})\times(-\infty,0)$. Similarly, we define an oriented foliation $\widetilde{\mathcal{F}}_{IV}$ on the domain $(-\frac{1}{4},0)\times(-\infty,0)$ whose leaves are the restriction to $(-\frac{1}{4},0)\times(-\infty,0)$ of the family of curves $(\ell'_c)_{c\in(-1,\infty)}$ such that

- ℓ'_c is the graph of $y=\log(-4x-c)$ with the direction from left to right, for $c\in(-1,0]$,
- ℓ'_c is the graph of $y=\log(-4x)-c$ with the direction from left to right, for $c\in(0,\infty)$.

Then, $\gamma_{(x,y)}$ is positively transverse to $\widetilde{\mathcal{F}}_{IV}$ for every $(x,y)\in(-\frac{1}{4},0)\times(-\infty,0)$.

Note the following facts:

- ℓ_c intersects $\{\frac{1}{4}\} \times (-\infty, 0)$, and does not intersect $\{\frac{1}{2}\} \times (-\infty, 0)$, for $c \in (-1, 0)$,
- ℓ_0 intersects neither $\{\frac{1}{4}\} \times (-\infty, 0)$ nor $\{\frac{1}{2}\} \times (-\infty, 0)$,
- ℓ_c intersects $\{\frac{1}{2}\} \times (-\infty, 0)$, and does not intersect $\{\frac{1}{4}\} \times (-\infty, 0)$, for $c \in (0, \infty)$,
- ℓ'_c intersects $\{0\} \times (-\infty, 0)$, and does not intersect $\{-\frac{1}{4}\} \times (-\infty, 0)$, for $c \in (-1, 0)$,
- ℓ'_0 intersects neither $\{0\} \times (-\infty, 0)$ nor $\{-\frac{1}{4}\} \times (-\infty, 0)$,
- ℓ'_c intersects $\{-\frac{1}{4}\} \times (-\infty, 0)$, and does not intersect $\{0\} \times (-\infty, 0)$, for $c \in (0, \infty)$.

We can define a transverse foliation $\mathcal{F}_2$ of I such that

- the restriction of $\mathcal{F}_2$ to the second quadrant II is equal to $\pi \circ \widetilde{\mathcal{F}}_{II}$,
- the restriction of $\mathcal{F}_2$ to the fourth quadrant IV is equal to $\pi \circ \widetilde{\mathcal{F}}_{IV}$,
- the restriction of $\mathcal{F}_2$ to $\mathbb{R}^2 \setminus (II \cup IV)$ is equal to the restriction of $\mathcal{F}_1$ to the same set.

Moreover, one can deduce that 0 is a mixed singularity of $\mathcal{F}_2$.

Example 3. (An orientation and area preserving local homeomorphism whose local rotation set is reduced to ∞)

Let g be a diffeomorphism of $\mathbb{R} \times (-\infty, 0)$ defined by

$$g(x, y) = (x - \frac{1}{y}, y).$$

It is area preserving and descends a diffeomorphism f of the annulus $\mathbb{T}^1 \times (-\infty, 0)$. Moreover, we can give a compactification of the annulus at the upper end by adding a point $\star$, and extend f continuously at this point. Denote by $\overline{f}$ this extension. Then, $\overline{f}$ is an area and orientation preserving homeomorphism that fixes $\star$, and $\rho_s(I, \star)$ is reduced to ∞ for every local isotopy I of $\overline{f}$ at $\star$.

Example 4. (Example of Remark 1.5)

We will construct an orientation preserving diffeomorphism f of the sphere with 2 fixed points such that f is area preserving in a neighborhood of each fixed point but there does not exist any torsion-low maximal identity isotopy of f.

Let φ be a diffeomorphism of $[0, 1]$ that satisfies

$$\begin{cases} \varphi(y) = y & \text{for } y \in [0, 1/6] \cup [5/6, 1], \\ \varphi(y) < y & \text{for } y \in (1/6, 5/6). \end{cases}$$

Let g be a diffeomorphism of $\mathbb{R} \times [0, 1]$ that is defined by

$$g(x, y) = (x + 3y, \varphi(y)).$$

We define an equivalent relation $\sim$ on $\mathbb{R} \times [0, 1]$ such that

$$\begin{cases} (x, y) \sim (x + 1, y) & \text{for all } (x, y) \in \mathbb{R} \times (0, 1) \\ (x, 0) \sim (x', 0) & \text{for all } x, x' \in \mathbb{R} \\ (x, 1) \sim (x', 1) & \text{for all } x, x' \in \mathbb{R}. \end{cases}$$

Then, $\mathbb{R} \times [0, 1]/_\sim$ is a sphere, and g descends to a diffeomorphism f of the sphere that has two fixed points and is area preserving near each fixed point. Note the facts that every maximal identity isotopy I fixes both fixed points of f, that the rotation number of I at each fixed point is an integer, and that the sum of the rotation numbers of I at every fixed point is 3. By Proposition 1.2, there does not exist any torsion-low maximal identity isotopy of f.

Example 5. (Example of Remark 4.2)

In this example, we will construct an isotopy I^* on the sphere such that I^* is torsion-low at a fixed point z, but there does not exist any torsion-low maximal isotopy that is equivalent to I^* as a local isotopy at z.

We will induce the isotopy by generating functions (see Appendix A).

Let φ be a smooth 1-periodic function on $\mathbb{R}$ that satisfies

$$\varphi(0) = \varphi(3/4) = \varphi(1) = 0 \text{ and } |\varphi| \leq \frac{1}{2\pi},$$

$$\begin{cases} \varphi(s) > 0 \text{ for } 0 < s < 3/4 \\ \varphi(s) < 0 \text{ for } 3/4 < s < 1 \end{cases}, \quad \text{and} \int_0^1 \varphi(s)ds = 0,$$

$$|\varphi(s)| < s \sin^2 \frac{\pi}{s} \quad \text{for} \quad 3/4 < s < 1.$$

Let

$$g(x,y) = \begin{cases} 0 & \text{for } y \leq 0, \\ \int_0^y s \sin^2 \frac{\pi}{s} + \varphi(s) \sin^2 \pi x ds & \text{for } 0 < y < 1, \\ \int_0^1 s \sin^2 \frac{\pi}{s} ds & \text{for } y \geq 1. \end{cases}$$

Then, g is constant on $\mathbb{R} \times (-\infty, 0]$ and on $\mathbb{R} \times [1, \infty)$ respectively, and satisfies $g(x+1, y) = g(x, y)$. Moreover, one knows

$$\partial_{12}^2 g(x,y) = \begin{cases} 0 & \text{for } y \leq 0 \text{ or } y \geq 1, \\ \pi\varphi(y) \sin(2\pi x) & \text{for } 0 < y < 1. \end{cases}$$

So, $\partial_{12}^2 g \leq \frac{1}{2} < 1$. Therefore, g defines an identity isotopy $I = (f_t)_{t \in [0,1]}$ by the following equations:

$$f_t(x,y) = (X^t, Y^t) \Leftrightarrow \begin{cases} X^t - x = & t\partial_2 g(X^t, y), \\ Y^t - y = & -t\partial_1 g(X^t, y), \end{cases}$$

For every $t \in [0,1]$, f_t is the identity on $\mathbb{R} \times (-\infty, 0] \cup \mathbb{R} \times [1, \infty)$, and satisfies $f_t(x+1, y) = f_t(x, y)$. Moreover, for every $t \in (0,1]$, a point (x,y) is a fixed point of f_t if and only if it is an critical point of g. Let $\mathcal{F}$ be the foliation whose leaves are the integral curves of the gradient vector field $(x,y) \mapsto (\partial_1 g(x,y), \partial_2 g(x,y))$ of g. As will be proved in Appendix A, $\mathcal{F}$ is a transverse foliation of I.

We know that

$$\partial_1 g(x,y) = \begin{cases} 0 & \text{for } y \leq 0 \text{ or } y \geq 1, \\ \pi \sin(2\pi x) \int_0^y \varphi(s)ds & \text{for } 0 < y < 1, \end{cases}$$

and that

$$\partial_2 g(x,y) = \begin{cases} 0 & \text{for } y \leq 0 \text{ or } y \geq 1, \\ y \sin^2 \frac{\pi}{y} + \varphi(y) \sin^2(\pi x) & \text{for } 0 < y < 1. \end{cases}$$

So, the set of critical points of g is

$$C = \{(n, \frac{1}{m}) : n \in \mathbb{Z}, m \in \mathbb{N}\} \cup \mathbb{R} \times (-\infty, 0] \cup \mathbb{R} \times [1, \infty),$$

and one deduces that $\partial_2 g(x,y) > 0$ for $(x,y) \notin C$.

We define an equivalent relation $\sim$ on $\mathbb{R}^2$ by

$$\begin{cases} (x,y) \sim (x',y') & \text{for } y,y' \leq 0, \\ (x,y) \sim (x+1,y) & \text{for } 0 < y < 1, \\ (x,y) \sim (x',y') & \text{for } y,y' \geq 1. \end{cases}$$

Then, $\mathbb{R}^2/_\sim$ is a sphere, f_1 descends to an area preserving homeomorphism f' of the sphere, I descends to an identity isotopy I' of f', and $\mathcal{F}$ descends to a transverse foliation $\mathcal{F}'$ of I'. Moreover, one knows that $\mathrm{Fix}(I') = \mathrm{Fix}(f') = \mathrm{Sing}(\mathcal{F}')$, where $\mathrm{Sing}(\mathcal{F}')$ is the set of singularities of $\mathcal{F}'$. We denote by S and N the two points $\mathbb{R} \times (-\infty, 0]$ and $\mathbb{R} \times [1, \infty)$ in the sphere respectively.

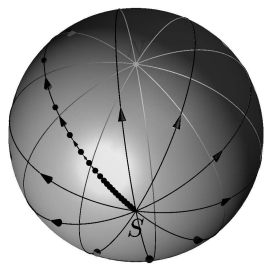


Figure 4: A sketch map of $\mathcal{F}'$

The fixed point S is not isolated in $\mathrm{Fix}(I')$, and so $\rho_s(I', S)$ is reduced to 0; N is isolated in $\mathrm{Fix}(f')$ and is a sink of $\mathcal{F}'$; and all the other fixed points of f' are isolated in $\mathrm{Fix}(f')$ and are saddles of $\mathcal{F}'$. Let I^* be an identity isotopy of f' fixing S such that $\rho_s(I^*, S)$ is reduced to -1. Then, I^* is torsion-low at S. We will prove that there does not exist any torsion-low maximal isotopy I'' such that $\rho_s(I'', S)$ is reduced to -1.

Indeed, a maximal identity isotopy of f' fixes either all the fixed points of f' (in which case, the isotopy is homotopic to I' relatively to $\mathrm{Fix}(f')$) or exactly two fixed points. If I'' is a maximal identity isotopy of f such that $\rho_s(I'', S)$ is reduced to -1, then I'' fixes exactly two fixed points. Denote by $\{S, z_1\}$ the set of fixed points of I''. One knows that z_1 is an isolated fixed point of f', and that $J_{z_1}^{-1} I''$ is equivalent to I' as local isotopies at z_1. Therefore, $J_{z_1}^{-1} I''$ does not have a negative rotation type at z_1, and hence I'' is not torsion-low at z_1.

Example 6. (Example of Remark 4.3)

In this example, we will construct an orientation and area preserving homeomorphism f of the sphere such that there does not exist any maximal identity isotopy I of f such that $0 \in \rho_s(I, z)$ for every $z \in \mathrm{Fix}(I)$ that is not an isolated fixed point of f.

Let g be a homeomorphism on $\mathbb{R} \times [0,1]$ that is defined by

$$g(x,y) = \begin{cases} (x,y) & \text{for } 0 \leq y \leq \frac{1}{3}, \\ (x+3y-1,y) & \text{for } \frac{1}{3} < y \leq \frac{2}{3}, \\ (x+1,y) & \text{for } \frac{2}{3} < y \leq 1. \end{cases}$$

We define an equivalent relation $\sim$ on $\mathbb{R} \times [0,1]$ such that

$$\begin{cases} (x,0) \sim (x',0) & \text{for } x,x' \in \mathbb{R} \\ (x,y) \sim (x+1,y) & \text{for } 0 < y < 1, \\ (x,1) \sim (x',1) & \text{for } x,x' \in \mathbb{R}. \end{cases}$$

Then, $\mathbb{R}\times[0,1]/_\sim$ is a sphere, g descends to an orientation and area preserving diffeomorphism f of the sphere that has infinitely many fixed points, and every fixed point of f is not isolated in $\mathrm{Fix}(f)$. We will prove that there does not exist any maximal isotopy I such that for all $z \in \mathrm{Fix}(I)$, one has $0 \in \rho_s(I,z)$.

By definition of f, one knows that f can be blown-up at each fixed point, and hence for every identity isotopy I of f and every $z \in \mathrm{Fix}(I)$, the rotation set $\rho_s(I,z)$ is reduced to $\rho(I,z)$. Then, we only need to prove that there does not exist any maximal identity isotopy I such that $\rho(I,z)=0$ for every $z \in \mathrm{Fix}(I)$.

Denote by N and S the two components of $\mathrm{Fix}(f)$ respectively. Note the following fact: for $z_1, z_2 \in \mathrm{Fix}(f)$ and every identity isotopy I fixing both z_1 and z_2, one can deduce

$$\rho(I,z_1)+\rho(I,z_2)=\begin{cases} 0 & \text{if } z_1,z_2\in S, \text{ or if } z_1,z_2\in N,\\ 1 & \text{if } z_1\in S, z_2\in N, \text{ or if } z_1\in N, z_2\in S.\end{cases}$$

Let us conclude the proof by observing the properties of any maximal identity isotopy of f. Indeed, if I is a maximal identity isotopy of f, it satisfies one of the following properties:

- The set of fixed points of I is the union of N (resp. S) and a point z in S (resp. N). In this case, $\rho(I,z)=1$.
- The set of fixed points of I is the union of a point z_1 in N (resp. S) and a point z_2 in S (resp. N), and the rotation numbers satisfy $\rho(I,z_i)\neq 0$ for $i=1,2$.
- The set of fixed points of I is a subset of N (resp. S) with exactly two points z_1 and z_2, and the rotation numbers satisfies

$$\rho(I,z_1)=-\rho(I,z_2)\in\mathbb{Z}\setminus\{0\}.$$

Example 7. (Example of Remark 4.5)

We will construct an orientation and area preserving diffeomorphism of the sphere such that there does not exist any maximal identity isotopy I satisfying

$$-1<\rho(I,z)<1, \text{ for every } z\in\mathrm{Fix}(F).$$

Let g be a diffeomorphism of $\mathbb{R}\times[0,1]$ that is defined by

$$g(x,y)=(x+y,y).$$

We define an equivalent relation $\sim$ on $\mathbb{R}\times[0,1]$ such that

$$\begin{cases} (x,0)\sim(x',0) & \text{for } x,x'\in\mathbb{R}\\ (x,y)\sim(x+1,y) & \text{for } 0<y<1,\\ (x,1)\sim(x',1) & \text{for } x,x'\in\mathbb{R}.\end{cases}$$

Then $\mathbb{R}\times[0,1]/_\sim$ is a sphere and g descends to an orientation and area preserving diffeomorphism f of the sphere that has exactly two fixed points. Note the facts that every maximal identity isotopy I fixes both fixed points of f, that the rotation number of I at each fixed point is an integer, and that the sum of the rotation numbers of I at both fixed point is 1. So, there does not exist any maximal isotopy I such that for all $z\in\mathrm{Fix}(I)$,

$$-1<\rho(I,z)<1.$$

A Construct a transverse foliation from the generating function

Let f be a diffeomorphism of $\mathbb{R}^2$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^2$ function, we call g a *generating function* of f if $\partial^2_{12} g < 1$, and if

$$f(x,y) = (X,Y) \Leftrightarrow \begin{cases} X - x = & \partial_2 g(X,y), \\ Y - y = & -\partial_1 g(X,y). \end{cases}$$

Every $\mathcal{C}^2$ function $g : \mathbb{R}^2 \to \mathbb{R}$ satisfiying $\partial^2_{12} g \leq c < 1$ defines a diffeomorphism f of $\mathbb{R}^2$ by the previous equations, and every area preserving diffeomorphism f of $\mathbb{R}^2$ satisfying $0 < \varepsilon \leq \partial_1(p_1 \circ f) \leq M < \infty$ can be generated by a generating function, where p_1 is the projection onto the first factor. Moreover, the Jacobian matrix J_f of f is equal to

$$\frac{1}{1 - \partial^2_{12} g(X,y)} \begin{pmatrix} 1 & \partial^2_{22} g(X,y) \\ -\partial^2_{11} g(X,y) & -\partial^2_{11} g(X,y) \partial^2_{22} g(X,y) + (1 - \partial^2_{12} g(X,y))^2 \end{pmatrix}.$$

Since $\det J_f = 1$, the diffeomorphism f is orientation and area preserving. A point (x,y) is a fixed point of f if and only if it is a critical point of g. We can naturally define an identity isotopy $I = (f_t)_{t \in [0,1]}$ of f such that f_t is generated by tg. Precisely, the diffeomorphisms f_t are defined by the following equations:

$$f_t(x,y) = (X^t, Y^t) \Leftrightarrow \begin{cases} X^t - x = & t\partial_2 g(X^t, y), \\ Y^t - y = & -t\partial_1 g(X^t, y). \end{cases}$$

In this section, we suppose that f is a diffeomorphism of $\mathbb{R}^2$, and that g is a generating function of f. We will construct a transverse foliation of I. More precisely, denote by $\mathcal{F}$ the foliation whose leaves are the integral curves of the gradient vector field $(x,y) \mapsto (\partial_1 g(x,y), \partial_2 g(x,y))$ of g, we will prove the following result:

Theorem A.1. *The foliation $\mathcal{F}$ is a transverse foliation of I.*

Proof. We will prove the theorem by constructing an identity isotopy I' of f that is homotopic to I relatively to $\mathrm{Fix}(f)$ and satisfies that for every $z \in \mathbb{R}^2 \setminus \mathrm{Fix}(f)$, the trajectory of z along I' is positively transverse to $\mathcal{F}$.

We define $I' = (f'_t)_{t \in [0,1]}$ by the following equations:

$$f'_t(x,y) = \begin{cases} (x,y) + 2t(X - x, 0) & \text{for } 0 \leq t \leq 1/2, \\ (X,y) + (2t-1)(0, Y - y) & \text{for } 1/2 \leq t \leq 1, \end{cases}$$

where $(X,Y) = f(x,y)$.

Lemma A.2. *One can verify that I' is an identity isotopy of f.*

Proof. We know that $\partial_1 X(x,y) = 1/(1 - \partial^2_{12} g(X,y)) > 0$. By computing the determinant of the Jacobian matrix of f'_t, we know that $\det J_{f'_t} > 0$ for every $t \in [0,1]$. Indeed, for $t \in [0,1/2]$,

$$\det J_{f'_t} = \det \begin{pmatrix} 1 + 2t(\partial_1 X - 1)) & 2t\partial_2 X \\ 0 & 1 \end{pmatrix} = 2t\partial_1 X + (1 - 2t) > 0;$$

for $t \in [1/2, 1]$,

$$\begin{aligned} \det J_{f'_t} &= \det \begin{pmatrix} \partial_1 X & \partial_2 X \\ (2t-1)\partial_1 Y & (2-2t) + (2t-1)\partial_2 Y \end{pmatrix} \\ &= (2t-1)\det J_f + (2-2t)\partial_1 X > 0. \end{aligned}$$

To prove that I' is an isotopy, we only need to check that f'_t is a bijection for every $t \in (0,1)$.

For $t \in (0, \frac{1}{2})$, write $f_t(x,y) = (\varphi_{t,y}(x), y)$. One deduces

$$\frac{\partial}{\partial x}\varphi_{t,y}(x) = 2t\partial_1 X(x,y) + (1-2t) > 1 - 2t > 0.$$

So, f'_t is a surjection. Now, we will prove f'_t is an injection. Suppose that $f'_t(x,y) = f'_t(x',y')$, and write $(X',Y') = f'(x',y')$. One knows $y = y'$ and

$$x + 2t(X - x) = x' + 2t(X' - x').$$

So, one knows

$$X - (1-2t)\partial_2 g(X,y) = X' - (1-2t)\partial_2 g(X',y),$$

and deduces

$$(X - X') - (1-2t)\partial^2_{12} g(\xi, y)(X - X') = 0,$$

where ξ is a real number between X and X'. So, one knows $X = X'$. By definition of the generating function, one deduces $(x,y) = (x',y')$. Therefore, f'_t is injective.

For $t \in [\frac{1}{2}, 1)$, write $\psi_{t,X}(y) = y + (2t-1)(Y - y)$. One deduces

$$\frac{\partial}{\partial y}\psi_{t,X}(y) = 1 - (2t-1)\partial^2_{12} g(X,y) > 2 - 2t > 0.$$

So, f'_t is a surjection. Now, we will prove f'_t is an injection. Suppose that $f'_t(x,y) = f'_t(x',y')$, and write $(X',Y') = f'(x',y')$. One knows $X = X'$ and

$$y + (2t-1)(Y - y) = y' + (2t-1)(Y' - y').$$

So, one knows

$$(y - y') - (2t-1)(\partial_1 g(X,y) - \partial_1 g(X,y')) = 0,$$

and then deduces

$$(y - y') - (2t-1)\partial^2_{12} g(X,\eta)(y - y'),$$

where η is a real number between y and y'. So, one knows $y = y'$, and then deduces $(x,y) = (x',y')$. We conclude that f'_t is an injection. □

By definition, we know $\mathrm{Fix}(I_0) = \mathrm{Fix}(I') = \mathrm{Fix}(f)$. If $\mathrm{Fix}(f)$ is empty or contains more than one point, I_0 and I' are homotopic relatively to $\mathrm{Fix}(f)$; if $\mathrm{Fix}(f)$ is reduced to one point, we can deduce the same result by the following lemma:

Lemma A.3. *If 0 is an isolated fixed point of f, one can deduce that $\rho(I,0) = \rho(I',0) \in [-1,1]$.*

Proof. Let $\theta : [0,1] \to \mathbb{R}$ and $\theta' : [0,1] \to \mathbb{R}$ be the continuous functions that satisfies $\theta(0) = \theta'(0) = 0$ and

$$\frac{J_{f_t}(0)\begin{pmatrix}1\\0\end{pmatrix}}{\|J_{f_t}(0)\begin{pmatrix}1\\0\end{pmatrix}\|} = \begin{pmatrix}\cos\theta(t)\\ \sin\theta(t)\end{pmatrix}, \quad \frac{J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix}}{\|J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix}\|} = \begin{pmatrix}\cos\theta'(t)\\ \sin\theta'(t)\end{pmatrix}.$$

To simplify the notations, we write

$$\mathrm{Hess}(g)(0) = \begin{pmatrix}\varrho, \sigma\\ \sigma, \tau\end{pmatrix}.$$

One knows

$$J_{f_t}(0)\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{1-t\sigma}\begin{pmatrix}1\\-t\varrho\end{pmatrix}.$$

We know $1 - t\sigma > 0$ for all $t \in [0,1]$, so $\theta(t)$ belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \in [0,1]$.

For $t \in [0, \frac{1}{2}]$, one knows

$$J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}(1-2t) + 2t\partial_1 X(0,0)\\ 0\end{pmatrix}.$$

We know $(1-2t) + 2t\partial_1 X(0,0) > 0$, so $\theta'(t)$ is equal to 0 for all $t \in [0, \frac{1}{2}]$.

For $t \in [\frac{1}{2}, 1]$, one knows

$$J_{f'_t}(0)\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\partial_1 X(0,0)\\ (2t-1)\partial_1 Y(0,0)\end{pmatrix}.$$

We know $\partial_1 X(0,0) > 0$, so $\theta'(t)$ belongs to $(-\frac{\pi}{2}, \frac{\pi}{2})$ for all $t \in [\frac{1}{2}, 1]$.

Therefore, one deduces $\theta(1) = \theta'(1) \in (-\frac{\pi}{2}, \frac{\pi}{2})$, and hence $\rho(I,0) = \rho(I',0) \in [-1,1]$. □

Lemma A.4. *For every $z = (x,y)$ that is not a fixed point of f, the path $\gamma_z : t \mapsto f'_t(x,y)$ is positively transverse to $\mathcal{F}$.*

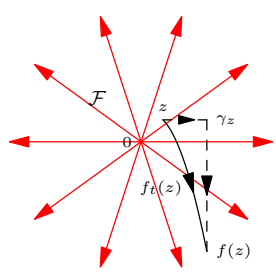


Figure 5: The dynamics and foliation generated by $g(x,y) = x^2 + y^2$

Proof. For $t \in [0, 1/2]$,

$$\begin{aligned}
& \det\begin{pmatrix}2(X-x) & \partial_1 g(f'_t(x,y))\\ 0 & \partial_2 g(f'_t(x,y))\end{pmatrix}\\
= \;& 2(X-x)\partial_2 g(f'_t(x,y))\\
= \;& 2(X-x)\partial_2 g(2tX + (1-2t)x, y)\\
= \;& 2(X-x)\partial_2 g(X,y) + (2t-1)(X-x)\partial^2_{12} g(\xi, y)]\\
= \;& 2(X-x)^2[1 - (1-2t)\partial^2_{12} g(\xi, y)] \geq 0
\end{aligned}$$

where ξ is a real number between x and X, and the inequality is strict if $X \neq x$.

For $t \in [1/2, 1]$,

$$\begin{aligned}
& \det \begin{pmatrix} 0 & \partial_1 g(f_t'(x,y)) \\ 2(Y-y) & \partial_2 g(f_t'(x,y)) \end{pmatrix} \\
= & -2(Y-y)\partial_1 g(f_t'(x,y)) \\
= & -2(Y-y)\partial_1 g(X, (2-2t)y + (2t-1)Y) \\
= & -2(Y-y)[\partial_1 g(X,y) + (2t-1)(Y-y)\partial_{12}^2 g(X,\eta)] \\
= & 2(Y-y)^2[1-(2t-1)\partial_{12}^2 g(X,\eta)] \geq 0
\end{aligned}$$

where η is a real number between y and Y, and the inequality is strict if $Y \neq y$.

Since z is not a fixed point, either $X \neq x$ or $Y \neq y$. If both of the inequalities are satisfied, γ_z intersects $\mathcal{F}$ positively transversely; if $X \neq x$ and $Y = y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ intersects $\mathcal{F}$ positively transversely, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ is reduced to a point; if $X = x$ and $Y \neq y$, $\gamma_z|_{t\in[0,\frac{1}{2}]}$ is reduced to a point, and $\gamma_z|_{t\in[\frac{1}{2},1]}$ intersects $\mathcal{F}$ positively transversely. □

□

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