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YET AGAIN ON TWO EXAMPLES BY IYAMA AND YOSHINO

DANIELE FAENZI

ABSTRACT. We give an elementary proof of Iyama-Yoshino's classification of rigid MCM modules on Veronese embeddings in \mathbb{P}^9 .

INTRODUCTION

The beautiful theory of cluster tilting in triangulated categories has been developed by Iyama and Yoshino; as an important outcome of this the authors gave in [10, Theorem 1.2 and Theorem 1.3] the classification of rigid indecomposable MCM modules over two Veronese embeddings in \mathbb{P}^9 given, respectively, by plane cubics and space quadrics. Another proof, that makes use of Orlov's singularity category, appears in [12], where the link between power series Veronese rings and the graded rings of the corresponding varieties is also explained. Also, [13] contains yet another argument.

The goal of this note is to present a simple proof of Iyama-Yoshino's classification of rigid MCM modules over the aforementioned Veronese rings, making use of vector bundles and Beilinson's theorem. This proof works over a field \mathbf{k} which is algebraically closed or finite.

Consider the embedding of the projective space \mathbb{P}^n , with $n \geq 2$ given by homogeneous forms of degree d , i.e. the d -fold Veronese variety. A coherent sheaf E on \mathbb{P}^n is arithmetically Cohen-Macaulay (ACM) with respect to this embedding if and only if E is locally free and has *no intermediate cohomology*:

$$(1) \quad H^i(\mathbb{P}^n, E(dt)) = 0, \quad \text{for all } t \in \mathbb{Z} \text{ and all } 0 < i < n.$$

This is equivalent to ask that the module of global sections associated with E is MCM over the corresponding Veronese ring. For d -fold Veronese embeddings of \mathbb{P}^n in \mathbb{P}^9 (i.e. $\{n, d\} = \{2, 3\}$), we are going to classify ACM bundles E which are *rigid*, i.e. $\text{Ext}_{\mathbb{P}^n}^1(E, E) = 0$. We set $\ell = \binom{n+1}{2}$.

To state the classification, we define the *Fibonacci numbers* $a_{\ell, k}$ by the relations: $a_{\ell, 0} = 0$, $a_{\ell, 1} = 1$ and $a_{\ell, k+1} = \ell a_{\ell, k} - a_{\ell, k-1}$. For instance $(a_{3, k})$ is given by the odd values of the usual Fibonacci sequence:

$$a_{3, k} = 0, 1, 3, 8, 21, 55, 144, \dots \quad \text{for } k = 0, 1, 2, 3, 4, 5, 6, \dots$$

Theorem 1. *Assume $\{n, d\} = \{2, 3\}$, let E be an indecomposable bundle on \mathbb{P}^n satisfying condition (1), namely $H^i(\mathbb{P}^n, E(dt)) = 0$ for all $t \in \mathbb{Z}$ and all $0 < i < n$.*

- i) If E has no non-trivial endomorphism factoring through $\mathcal{O}_{\mathbb{P}^n}(t)$, then there are $a, b \geq 0$ such that, up to a twist by $\mathcal{O}_{\mathbb{P}^n}(s)$, E or E^* is the cokernel of an injective map:*

$$(2) \quad \Omega_{\mathbb{P}^n}^2(1)^b \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a$$

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ii) If E is rigid, then there is $k \geq 1$ such that, up to tensoring with $\mathcal{O}_{\mathbb{P}^n}(s)$, E or E^* is the cokernel of an injective map:

$$\Omega_{\mathbb{P}^n}^2(1)^{a_{\ell,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}}$$

Conversely for any $k \geq 1$, there is a unique indecomposable bundle E_k fitting into:

$$0 \rightarrow \Omega_{\mathbb{P}^n}^2(1)^{a_{\ell,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}} \rightarrow E_k \rightarrow 0.$$

Finally, both E_k and E_k^* are ACM and exceptional.

In the previous statement, it is understood that a bundle E is *exceptional* if it is rigid, *simple* (i.e. $\mathrm{Hom}_{\mathbb{P}^n}(E, E) \simeq \mathbf{k}$) and $\mathrm{Ext}_{\mathbb{P}^n}^i(E, E) = 0$ for $i \geq 2$. We write $\Omega_{\mathbb{P}^n}^p = \wedge^p \Omega_{\mathbb{P}^n}^1$ for the bundle of differential p -forms on \mathbb{P}^n .

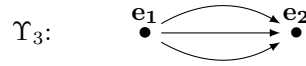
Remark 2. Part (i) of Theorem 1 is a version of Iyama-Yoshino's general results on Veronese rings [10, Theorem 9.1 and 9.3], to the effect that for $\{n, d\} = \{2, 3\}$ the stable category of MCM modules is equivalent to the category of representations of a certain Kronecker quiver. Our result is somehow algorithmic, for it provides the representation associated with an MCM module by computing a cohomology table of the corresponding ACM bundle.

Remark 3. The rank of the bundle E_k is given by the Fibonacci number between $a_{3,k-1}$ and $a_{3,k}$ in case $(n, d) = (2, 3)$. In this case E_{2k} (respectively, E_{2k+1}) is the k -th sheafified syzygy occurring in the resolution of $\mathcal{O}_{\mathbb{P}^2}(1)$ (respectively, of $\mathcal{O}_{\mathbb{P}^2}(2)$) over the Veronese ring, twisted by $\mathcal{O}_{\mathbb{P}^2}(3(k-1))$. A similar result holds for $(n, d) = (3, 2)$.

As for notation, we write small letters for the dimension of a space in capital letter, for instance $h^i(\mathbb{P}^n, E) = \dim_{\mathbf{k}} H^i(\mathbb{P}^n, E)$. We also write $\chi(E, F) = \sum (-1)^i \mathrm{ext}_{\mathbb{P}^n}^i(E, F)$ and $\chi(E) = \chi(\mathcal{O}_{\mathbb{P}^n}, E)$. The symbol $\delta_{i,j}$ is Kronecker's delta.

1. FIBONACCI BUNDLES

1.1. Representations of the Kronecker quiver. Fix an integer $\ell \geq 1$. Let us write Υ_{ℓ} for the ℓ -th Kronecker quiver, namely the oriented graph with two vertices \mathbf{e}_0 and \mathbf{e}_1 , and ℓ arrows from \mathbf{e}_0 to \mathbf{e}_1 . A representation R of Υ_{ℓ} , with dimension vector (a, b) is the choice of ℓ matrices of size $a \times b$.



We identify a basis of $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(2))$ with the set of $\ell = \binom{n+1}{2}$ arrows of Υ_{ℓ} . Then the derived category of finite-dimensional representations of Υ_{ℓ} embeds into the derived category of $\mathcal{O}_{\mathbb{P}^n}$ -modules by sending R to the cone $\Phi(R)$ of the morphism e_R associated with R according to this identification:

$$\Phi(R)[-1] \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{e_R} \Omega_{\mathbb{P}^n}(1)^b,$$

where we denote by $[-1]$ the shift to the right of complexes. It is clear that for all pairs of representations R, R' we have:

$$(3) \quad \mathrm{Ext}_{\mathbb{P}^n}^i(\Phi(R), \Phi(R')) \simeq \mathrm{Ext}_{\Upsilon_{\ell}}^i(R, R'), \quad \text{for all } i.$$

1.2. Rigid representations and Schur roots. We will use Kac's classification of rigid Υ_ℓ -modules as Schur roots (hence the restriction on \mathbf{k}), which is also one of the main ingredients in Iyama-Yoshino's proof. By [11, Theorem 4], any non-zero rigid Υ_ℓ -module is a direct sum of rigid simple representations of the form R_k , for some $k \in \mathbb{Z}$, where R_k is defined as the unique (up to isomorphism) indecomposable representation of Υ_ℓ with dimension vector $(a_{\ell, k-1}, a_{\ell, k})$ for $k \geq 1$, or $(a_{\ell, 1-k}, a_{\ell, -k})$ for $k \leq 0$.

1.3. Fibonacci bundles and their cohomology. Set $F_k = \Phi(R_k)$ for $k \geq 1$, and $F_k = \Phi(R_k)[-1]$ for $k \leq 0$. It turns out that F_k is an exceptional locally free sheaf, called a *Fibonacci bundle*, cf. [4]. We rewrite the defining exact sequences of F_k :

$$(4) \quad \begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell, k-1}} \rightarrow \Omega_{\mathbb{P}^n}(1)^{a_{\ell, k}} \rightarrow F_k \rightarrow 0, & \quad \text{for } k \geq 1, \\ 0 \rightarrow F_k \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell, 1-k}} \rightarrow \Omega_{\mathbb{P}^n}(1)^{a_{\ell, -k}} \rightarrow 0, & \quad \text{for } k \leq 0. \end{aligned}$$

Here is a lemma on the intermediate cohomology of Fibonacci bundles. Its statement should be read with a grain of salt, namely for $k = 1$ also $H^{n-1}(\mathbb{P}^n, F_1(-n))$ vanishes.

Lemma 4. *For $k \geq 1$, the only non-vanishing intermediate cohomology of F_k is:*

$$h^1(\mathbb{P}^n, F_k(-1)) = a_{\ell, k} \quad h^{n-1}(\mathbb{P}^n, F_k(-n)) = a_{\ell, k-1}.$$

Proof. All the terminology and results we need on exceptional collections in order to establish this lemma are contained in [2, 3]. We consider the left and right mutation endofunctors of the derived category of coherent sheaves on \mathbb{P}^n . These associate with a pair (E, F) of complexes two complexes, denoted respectively by $R_F E$ and $L_E F$, defined as the cones of the natural evaluation maps $f_{E, F}$ and $g_{E, F}$:

$$E \xrightarrow{f_{E, F}} \mathrm{RHom}_{\mathbb{P}^n}(E, F)^* \otimes F \rightarrow R_F E, \quad L_E F \rightarrow \mathrm{RHom}_{\mathbb{P}^n}(E, F) \otimes E \xrightarrow{g_{E, F}} F.$$

It is well-known (cf. [4]) that the Fibonacci bundles F_k can be defined recursively from $F_0 = \mathcal{O}_{\mathbb{P}^n}(-1)$ and $F_1 = \Omega_{\mathbb{P}^n}(1)$ by setting:

$$(5) \quad F_{k+1} = R_{F_k} F_{k-1}, \quad \text{for } k \geq 1,$$

$$(6) \quad F_{k-1} = L_{F_k} F_{k+1}, \quad \text{for } k \leq 0.$$

This way, for any $k \in \mathbb{Z}$ we get a natural exact sequence:

$$(7) \quad 0 \rightarrow F_{k-1} \rightarrow (F_k)^\ell \rightarrow F_{k+1} \rightarrow 0.$$

Over \mathbb{P}^n , we start with the standard collection:

$$(\mathcal{O}_{\mathbb{P}^n}(1-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(1)).$$

By the mutation $\Omega_{\mathbb{P}^n}(-1) \simeq L_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(1)$, we replace this with:

$$(\mathcal{O}_{\mathbb{P}^n}(1-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), \Omega_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}),$$

By (5), we can replace the previous exceptional sequence with:

$$(\mathcal{O}_{\mathbb{P}^n}(1-n), \dots, \mathcal{O}_{\mathbb{P}^n}(-1), F_{k-1}, F_k).$$

By [3, Theorem 4.1], the iterated mutation of F_k through this last full exceptional collection must give back $F_k \otimes \omega_{\mathbb{P}^n}$, i.e.:

$$(8) \quad L_{\mathcal{O}_{\mathbb{P}^n}(1-n)} \cdots L_{\mathcal{O}_{\mathbb{P}^n}(-1)} L_{F_{k-1}} F_k \simeq F_k(-n-1).$$

Also, since the first sequence we started with is strongly exceptional of vector bundles, and the same happens to its Koszul dual collection, in view of [3, Theorem 8.3] we know that all objects obtained by the mutations in the previous display consist of vector bundles. This provides us with integers u_1, \dots, u_n and with a long exact sequence:

$$(9) \quad 0 \rightarrow F_k(-n-1) \rightarrow \mathcal{O}_{\mathbb{P}^n}(1-n)^{u_1} \rightarrow \dots \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{u_{n-1}} \rightarrow F_{k-1}^{u_n} \rightarrow F_k \rightarrow 0,$$

where each short exact sequence extracted from the long one is given by one mutation in the sequence of mutations (8). Now by (4) we get:

$$H^i(\mathbb{P}^n, F_k(t)) = 0 \quad \text{for:} \quad \begin{cases} 2 \leq i \leq n-2, & \forall t, \\ i = 0, & t \leq 0, \\ i = 1, & t \neq -1, n \geq 3, \text{ and } t \geq 0, n = 2, \\ i = n-1, & t \geq 1-n, n \geq 3. \end{cases}$$

Also, the required non-vanishing cohomology of F_k appears again from (4). Summing up, we only have to show $H^{n-1}(\mathbb{P}^n, F_k(t)) = 0$ for $t \leq -n-1$. For this we use (9). Indeed, the kernel K of the map $F_{k-1}^{u_n} \rightarrow F_k$ appearing in (9) is $L_{F_{k-1}} F_k$, so that $u_n = \ell$ and $K \simeq F_{k-2}$ in view of (7). Then, using $H^0(\mathbb{P}^n, F_{k-2}(t)) = 0$ for $t \leq 0$ (which of course holds also for $k = 1, 2$) and chasing cohomology in (7) we see that $H^{n-1}(\mathbb{P}^n, F_k(t)) = 0$ for $t \leq -n-1$. \square

1.4. Ext groups among Fibonacci bundles. We would like to compute now the Ext groups between pairs of Fibonacci bundles. Of course when F_k and F_j are “close”, i.e. $|k-j| \leq 1$, we already know what happens, as the two Fibonacci bundles then form an exceptional pair. But what if $|k-j| \geq 2$? The next lemma gives the answer. Note the second formula holds also for $k = j$ since F_k is exceptional.

Lemma 5. *For any pair of integers $j \geq k+1$ we have:*

$$(10) \quad \text{ext}_{\mathbb{P}^n}^i(F_j, F_k) = \delta_{1,i} a_{\ell, j-k-1}, \quad \text{ext}_{\mathbb{P}^n}^i(F_k, F_j) = \delta_{0,i} a_{\ell, j-k+1}.$$

Proof. Set $\Upsilon = \Upsilon_\ell$ and $a_{\ell, j} = a_j$. We easily compute $\chi(F_j, F_k) = -a_{j-k-1}$ and $\chi(F_k, F_j) = a_{j-k+1}$ by calculating χ of Υ -modules via the Cartan form and using faithfulness of Φ .

Therefore, the second formula is proved once we show $\text{Ext}_{\mathbb{P}^n}^i(F_k, F_j) = 0$ for $i \neq 0$, and this is of course true for $i < 0$. For $k \leq 0$ and $j \geq 1$ we have $F_k \simeq \Phi(R_k)[-1]$ and thus $\text{Ext}_{\mathbb{P}^n}^i(F_k, F_j) \simeq \text{Ext}_{\Upsilon}^{i+1}(R_k, R_j)$, which is zero for $i \neq -1, 0$ since the category of Υ -representations is hereditary. Further, if $j, k \geq 1$ or $j, k \leq 0$ then it suffices to prove $\text{Ext}_{\mathbb{P}^n}^1(F_k, F_j) = 0$. Using (7), this vanishing holds for j if it does for $j-1$. By induction, it suffices to check $\text{Ext}_{\mathbb{P}^n}^1(F_k, F_k) = 0$, which in turn is obvious.

Let us now look at the first formula. To prove it, we consider the Auslander-Reiten translate τ ; we refer for instance to [1] for the definition and properties of τ . This satisfies:

$$(11) \quad \text{ext}_{\Upsilon}^i(R_j, \tau R_{k+2}) = \text{ext}_{\Upsilon}^{1-i}(R_{k+2}, R_j).$$

The functor τ operates on dimension vectors via the Coxeter transform, encoded by the matrix:

$$\begin{pmatrix} \ell^2 - 1 & \ell \\ -\ell & -1 \end{pmatrix}.$$

This means that, if R has dimension vector (a, b) , then τR has dimension vector:

$$(|(\ell^2 - 1)a - \ell b|, |\ell a - b|).$$

A straightforward computation involving the recursive definition of a_k now says that (a_{k+1}, a_{k+2}) is sent to (a_{k-1}, a_k) . This in turn implies that $\tau R_{k+2} \simeq R_k$. Therefore, the second part of (10) follows from the first part, by combining it with (3) and (11). \square

Remark 6. This lemma holds more generally (with the same proof) for any exceptional pair (F_0, F_1) of objects on a projective \mathbf{k} -variety X , with $\text{hom}_X(F_0, F_1) = \ell$, by defining recursively F_k for all $k \in \mathbb{Z}$ by (5) and (6).

2. RIGID ACM BUNDLES ON THE THIRD VERONESE SURFACE

We prove here Theorem 1 in case $(n, d) = (2, 3)$.

2.1. The Beilinson complex and the proof of (i). Let us first prove (i). So let E be an indecomposable vector bundle on \mathbb{P}^2 satisfying (1). Without loss of generality, we may replace E with $G = E(s_0)$, where s_0 is the smallest integer s such that $h^0(\mathbb{P}^2, E(s)) \neq 0$. Define the integers:

$$\alpha_{i,j} = h^i(\mathbb{P}^2, G(-j)).$$

Since we defined $G = E(s_0)$, we have $\alpha_{0,j} = 0$ if and only if $j \geq 1$. The Beilinson complex F associated with G (see for instance [9, Chapter 8]) reads:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{d_0} \begin{array}{c} \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \end{array} \xrightarrow{d_1} \begin{array}{c} \Omega_{\mathbb{P}^2}(1)^{\alpha_{2,1}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,0}} \end{array} \xrightarrow{d_2} \mathcal{O}_{\mathbb{P}^2}^{\alpha_{2,0}} \rightarrow 0.$$

The term consisting of three summands in the above complex sits in degree 0 (we call it *middle term*), and the cohomology of this complex is G . By condition (1), at least one of the $\alpha_{1,j}$ is zero, for $j = 0, 1, 2$.

If $\alpha_{1,2} = 0$, then $d_0 = 0$. By minimality of the Beilinson complex the restriction of d_1 to the summand $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ of the middle term is also zero. Therefore $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ is a direct summand of G , so $G \simeq \mathcal{O}_{\mathbb{P}^2}$ by indecomposability of E (and hence of G).

If $\alpha_{1,1} = 0$, then the non-zero component of d_0 is just a map $\mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$, and a direct summand of G is the cokernel of this map. By indecomposability of G , in this case $G(-1)$ has a resolution of the desired form with $a = \alpha_{0,0}$ and $b = \alpha_{1,2}$.

So in the cases $\alpha_{1,2} = 0$ or $\alpha_{1,1} = 0$, the bundle E has the required resolution up to twist. In the case $\alpha_{1,0} = 0$, we shall see that this holds for E^* . We thus call this case the “dual” one.

2.2. The dual case and the end of the proof of (i). It remains to look at the case $\alpha_{1,0} = 0$, actually $\alpha_{1,3k} = 0$ for all integers k . Note that the restriction of d_1 to $\Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ is zero, which implies that a direct summand of G (hence all of G by indecomposability) has the resolution:

$$(12) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{d_0} \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow G \rightarrow 0$$

and $\alpha_{2,j} = 0$ for $j = 0, 1, 2$. Recall also $\alpha_{1,3} = 0$, and obviously $\alpha_{0,3} = 0$. We compute $\chi(G(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}$, so:

$$\alpha_{2,3} = h^0(\mathbb{P}^2, G^*) = h^2(\mathbb{P}^2, G(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}.$$

If this value is positive, then there is a non-trivial morphism $g : G \rightarrow \mathcal{O}_{\mathbb{P}^2}$, and since $\alpha_{0,0} \neq 0$ there also exists $0 \neq f : \mathcal{O}_{\mathbb{P}^2} \rightarrow G$. So G (and hence E) has a non-trivial endomorphism factoring through $\mathcal{O}_{\mathbb{P}^2}$, a contradiction.

Summing up, we may assume $3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0} = 0$, in other words $\alpha_{2,3} = 0$. Therefore, the Beilinson complex associated with $G(-1)$ gives a resolution:

$$0 \rightarrow G(-1) \rightarrow \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \rightarrow 0.$$

It is easy to convert this resolution into the form we want by the diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & G(-1) & \longrightarrow & \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{3\alpha_{1,2} - \alpha_{1,1}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{3\alpha_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & = & \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

From the leftmost column, it follows that G^* has a resolution of the desired form, with $a = 3\alpha_{1,2} - \alpha_{1,1}$ and $b = \alpha_{1,2}$. Claim (i) is thus proved.

2.3. The proof of (ii) and the rigid representation associated with E . The first step to prove (ii) is to associate with a rigid ACM sheaf E (i.e. with G) a rigid representation R of Υ_3 . This is obvious whenever the conclusion of (i) holds, as G is associated with a representation R via Φ , and R is rigid since Φ is fully faithful. However, looking back to the proof of (i), we see that the conclusion of (i) holds if $\alpha_{1,1} = 0$ or $\alpha_{1,2} = 0$, as we did not use the condition on the endomorphisms of E for those cases. So we only have to work out the *dual case*, and we assume $\alpha_{1,0} = 0$.

Consider (12), let e be the restricted map $e : \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}}$ extracted from d_0 and let F be its cone, shifted by 1:

$$(13) \quad F \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{e} \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}}.$$

This is a complex with two terms, and its cohomology is concentrated in degrees zero and one, namely $\mathcal{H}^0 F \simeq \ker(e)$ and $\mathcal{H}^1 F \simeq \operatorname{coker}(e)$. From (12) we easily see that F fits into a distinguished triangle:

$$(14) \quad F \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow G.$$

Also, by (13) there is a representation R of Υ_3 such that $\Phi(R) \simeq F[1]$.

Applying $\operatorname{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, -)$ to (13), we get $\operatorname{Ext}_{\mathbb{P}^2}^i(\mathcal{O}_{\mathbb{P}^2}, F) = 0$ for all i . So (14) gives:

$$\operatorname{Ext}_{\mathbb{P}^2}^i(F, F) \simeq \operatorname{Ext}_{\mathbb{P}^2}^i(G, F[1]), \quad \text{for all } i.$$

Also, we have $\operatorname{Ext}_{\mathbb{P}^2}^2(G, \mathcal{O}_{\mathbb{P}^2}) \simeq H^0(\mathbb{P}^2, G(-3))^* = 0$, so applying $\operatorname{Hom}_{\mathbb{P}^2}(G, -)$ to (14) we get:

$$\operatorname{Ext}_{\mathbb{P}^2}^1(G, G) \rightarrow \operatorname{Ext}_{\mathbb{P}^2}^1(G, F[1]) \rightarrow 0.$$

Putting this together, we obtain a surjection:

$$\mathrm{Ext}_{\mathbb{P}^2}^1(E, E) \simeq \mathrm{Ext}_{\mathbb{P}^2}^1(G, G) \rightarrow \mathrm{Ext}_{\mathbb{P}^2}^1(F, F) \simeq \mathrm{Ext}_{\mathbb{T}_3}^1(R, R).$$

We understand now that, if E is rigid, then so is R .

2.4. The 4-term sequence and of the proof of (ii). If R is rigid, then by §1.2, R is a direct sum of rigid simple representations of the form R_k . Therefore, the rigid object F is a direct sum of (shifted) copies of the bundles F_k obtained from the rigid representations R_k . Taking cohomology of (14), we obtain the 4-term exact sequence:

$$0 \rightarrow \bigoplus_{i \leq 0} F_i^{r_i} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow G \rightarrow \bigoplus_{i \geq 1} F_i^{r_i} \rightarrow 0,$$

for some integers r_i .

If only R_i with $i \leq 0$ appear, this sequence says that G is globally generated. Then, whenever $\alpha_{2,3} = h^0(\mathbb{P}^2, G^*) \neq 0$, composing $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow G$ with a non-trivial map $G \rightarrow \mathcal{O}_{\mathbb{P}^2}$ we get a surjection $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow \mathcal{O}_{\mathbb{P}^2}$ which splits a direct summand $\mathcal{O}_{\mathbb{P}^2}$ of G . In this case $G \simeq \mathcal{O}_{\mathbb{P}^2}$ by indecomposability of G .

This case being settled, we may assume $\alpha_{2,3} = h^0(\mathbb{P}^2, G^*) = 0$, so that the end of proof of the dual case §2.2 works and says that G^* has a resolution of the form (2) for some integers a, b . Since the representation associated with G^* is rigid, we know that there is $k \geq 1$ such that $a = a_{3,k-1}$ and $b = a_{3,k}$.

If some R_i appears with $i \geq 1$, we call I the (non-zero) image of the middle map in the previous exact sequence, and we show $\mathrm{Ext}_{\mathbb{P}^2}^1(F_j, I) = 0$ for all $j \geq 1$, which contradicts G being indecomposable. To check this, note that $\mathrm{Hom}_{\mathbb{P}^2}(F_j, -)$ gives an exact sequence:

$$\mathrm{Ext}_{\mathbb{P}^2}^1(F_j, \mathcal{O}_{\mathbb{P}^2})^{\alpha_{0,0}} \rightarrow \mathrm{Ext}_{\mathbb{P}^2}^1(F_j, I) \rightarrow \bigoplus_{i \leq 0} \mathrm{Ext}_{\mathbb{P}^2}^2(F_j, F_i)^{r_i}.$$

The leftmost term vanishes by Serre duality and Lemma 4. The rightmost term is zero by Lemma 5. Part (ii) is now proved.

Relying on [11, Theorem 4], we see that the last statement of Theorem 1 is clear by Lemma 4 and by exceptionality of Fibonacci bundles. The fact that E^* is also ACM is obvious by Serre duality.

Remark 7. If \mathbf{k} is algebraically closed of characteristic zero, we may apply [5, Corollaire 7], to the effect that a rigid bundle is a direct sum of exceptional bundles. So, at the price of relying on this result, from (i) we may deduce directly (ii) via Kac's theorem.

3. ACM BUNDLES ON THE SECOND VERONESE THREEFOLD

The techniques we have just seen apply to the embedding of \mathbb{P}^3 in \mathbb{P}^9 by quadratic forms. Again we replace E with the $G = E(s_0)$, where s is the smallest integer s such that $E(s)$ has non-zero global sections, and set $\alpha_{i,j} = h^i(\mathbb{P}^3, G(-j))$. If (1) gives $\alpha_{1,1} = \alpha_{2,1} = 0$, then $G(-1)$ has the desired resolution. On the other hand, if (1) tells $\alpha_{1,0} = \alpha_{2,0} = \alpha_{1,2} = \alpha_{2,2} = 0$, then we are left with a resolution of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\alpha_{1,3}} \xrightarrow{d_0} \Omega_{\mathbb{P}^3}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^3}^{\alpha_{0,0}} \rightarrow G \rightarrow 0.$$

We also have $\alpha_{i,4} = 0$ for $i = 0, 1, 2$ again by (1). The fact that G has no endomorphism factoring through $\mathcal{O}_{\mathbb{P}^3}$ this time gives $\alpha_{3,4} = h^0(\mathbb{P}^3, G^*) = 0$. So $G(-1)$ has a resolution like:

$$0 \rightarrow G(-1) \rightarrow \Omega_{\mathbb{P}^3}^2(2)^{\alpha_{1,3}} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\alpha_{1,1}} \rightarrow 0.$$

Then, using the same trick as in the proof of the previous theorem, we see that G^* has the desired resolution, with $a = 6\alpha_{1,3} - \alpha_{1,1}$ and $b = \alpha_{1,3}$.

This proves the first statement. The rest follows by the same path. Drezet's theorem as shortcut for (i) \Rightarrow (ii) may be replaced by [7].

Remark 8. It should be noted that, in [10, Theorem 1.2 and Theorem 1.3], the ACM bundle E on the given Veronese variety is assumed to have a rigid module of global sections. This implies, respectively, $\text{Ext}_{\mathbb{P}^2}^1(E, E(3t)) = 0$, or $\text{Ext}_{\mathbb{P}^3}^1(E, E(2t)) = 0$, for all $t \in \mathbb{Z}$. A priori, this is a stronger requirement than just $\text{Ext}_{\mathbb{P}^n}^1(E, E) = 0$. However, our proof shows that the two conditions are equivalent for ACM bundles.

4. RIGID ACM BUNDLES ON HIGHER VERONESE SURFACES

Assume \mathbf{k} algebraically closed. The next result shows that, for $d \geq 4$, the class of rigid ACM bundles on d -fold Veronese surfaces contains the set of exceptional bundles on \mathbb{P}^2 , which is quite a rich class, cf. [6]. At least if $\text{char}(\mathbf{k}) = 0$, the two classes coincide by [5, Corollaire 7].

Theorem 9. *Let F be an exceptional bundle on \mathbb{P}^2 and fix $d \geq 4$. Then there is an integer t such that $E = F(t)$ satisfies (1).*

Proof. It is known that F is actually stable by [6]. This implies that F has natural cohomology by [8], i.e. for all $t \in \mathbb{Z}$ there is at most one i such that $H^i(\mathbb{P}^2, F(t)) \neq 0$. Then, $H^1(\mathbb{P}^2, F(t)) \neq 0$ if and only if $\chi(F(t)) < 0$.

Let now r, c_1, c_2 be the rank and the Chern classes of F . Riemann-Roch shows that $\chi(F(t))$ is a polynomial of degree 2 in t , of dominant term $r/2$, whose discriminant is:

$$\Delta = c_1^2(1 - r) + r(2c_2 + r/4) = -\chi(F, F) + 5r^2/4.$$

So, $\chi(F, F) = 1$ implies $\Delta = -1 + 5r^2/4$. Hence, the roots of $\chi(F(t))$ differ by:

$$\frac{2\sqrt{\Delta}}{r} = \frac{\sqrt{5r^2 - 4}}{r} < 3.$$

Therefore, there is an integer t_0 such that $\chi(F(t))$ is non-negative except when t takes one of the values $t_0, t_0 + 1, t_0 + 2$. Hence only these values of t may give $H^1(\mathbb{P}^2, F(t)) \neq 0$. This means that $E = F(t_0 - 1)$ satisfies (1) for any choice of $d \geq 4$. \square

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