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# YET AGAIN ON TWO EXAMPLES BY IYAMA AND YOSHINO

DANIELE FAENZI

ABSTRACT. We give an elementary proof of Iyama-Yoshino's classification of rigid MCM modules on Veronese embeddings in  $\mathbb{P}^9$ .

## INTRODUCTION

The beautiful theory of cluster tilting in triangulated categories has been developed by Iyama and Yoshino; as an important outcome of this the authors gave in [IY08, Theorem 1.2 and Theorem 1.3] the classification of rigid indecomposable MCM modules over two Veronese embeddings in  $\mathbb{P}^9$  given, respectively, by plane cubics and space quadrics. Another proof, that makes use of Orlov's singularity category, appears in [KMVdB11], where the link between power series Veronese rings and the graded rings of the corresponding varieties is also explained. Also, [KR08] contains yet another argument.

The goal of this note is to present a simple proof of Iyama-Yoshino's classification of rigid MCM modules over the aforementioned Veronese rings, making use of vector bundles and Beilinson's theorem. This proof works over a field  $k$  which is algebraically closed or finite.

Consider the embedding of the projective space  $\mathbb{P}^n$  given by homogeneous forms of degree  $d$ , i.e. the  $d$ -fold Veronese variety. A coherent sheaf  $E$  on  $\mathbb{P}^n$  is arithmetically Cohen-Macaulay (ACM) with respect to this embedding if and only if  $E$  is locally free and has *no intermediate cohomology*:

$$(1) \quad H^i(\mathbb{P}^n, E(dt)) = 0, \quad \text{for all } t \in \mathbb{Z} \text{ and all } 0 < i < n.$$

This is equivalent to ask that the module of global sections associated with  $E$  is MCM over the corresponding Veronese ring. For  $d$ -fold Veronese embeddings of  $\mathbb{P}^n$  in  $\mathbb{P}^9$  (i.e.  $\{n, d\} = \{2, 3\}$ ), we are going to classify ACM bundles  $E$  which are *rigid*, i.e.  $\text{Ext}_{\mathbb{P}^n}^1(E, E) = 0$ . We set  $\ell = \binom{n+1}{2}$ .

To state the classification, we define the *Fibonacci numbers*  $a_{\ell, k}$  by the relations:  $a_{\ell, 0} = 0$ ,  $a_{\ell, 1} = 1$  and  $a_{\ell, k+1} = \ell a_{\ell, k} - a_{\ell, k-1}$ . For instance  $(a_{3, k})$  is given by the odd values of the usual Fibonacci sequence:

$$a_{3, k} = 0, 1, 3, 8, 21, 55, 144, \dots \quad \text{for } k = 0, 1, 2, 3, 4, 5, 6, \dots$$

**Theorem 1** ( $\{n, d\} = \{2, 3\}$ ). *Let  $E$  be an indecomposable bundle on  $\mathbb{P}^n$  satisfying (1).*

i) *If  $E$  has no endomorphism factoring through  $\mathcal{O}_{\mathbb{P}^n}(t)$ , then there are  $a, b \geq 0$  such that, up to a twist by  $\mathcal{O}_{\mathbb{P}^n}(s)$ ,  $E$  or  $E^*$  is the cokernel of an injective map:*

$$\Omega_{\mathbb{P}^n}^2(1)^b \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a$$

ii) *If  $E$  is indecomposable and rigid, then there is  $k \geq 1$  such that, up to tensoring with  $\mathcal{O}_{\mathbb{P}^n}(s)$ ,  $E$  or  $E^*$  is the cokernel of an injective map:*

$$\Omega_{\mathbb{P}^n}^2(1)^{a_{\ell, k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell, k}}$$

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iii) Conversely for any  $k \geq 1$ , there is a unique indecomposable bundle  $E_k$  having a resolution of the form:

$$0 \rightarrow \Omega_{\mathbb{P}^n}^2(1)^{a_{\ell,k-1}} \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k}} \rightarrow E_k \rightarrow 0,$$

and both  $E_k$  and  $E_k^*$  are ACM and exceptional.

In the previous statement, it is understood that a bundle  $E$  is *exceptional* if it is rigid, *simple* (i.e.  $\mathrm{Hom}_{\mathbb{P}^n}(E, E) \simeq \mathbf{k}$ ) and  $\mathrm{Ext}_{\mathbb{P}^n}^i(E, E) = 0$  for  $i \geq 2$ . Next, we write  $\Omega_{\mathbb{P}^n}^p = \wedge^p \Omega_{\mathbb{P}^n}^1$  for the bundle of differential  $p$ -forms on  $\mathbb{P}^n$ .

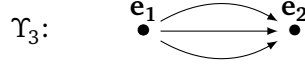
**Remark 2.** Part (i) of Theorem 1 is a version of Iyama-Yoshino's general results on Veronese rings [IY08, Theorem 9.1 and 9.3], to the effect that for  $\{n, d\} = \{2, 3\}$  the stable category of MCM modules is equivalent to the category of representations of a certain Kronecker quiver. However our result is algorithmic, for it provides the representation associated with an MCM module via Beilinson spectral sequence applied to the corresponding ACM bundle.

**Remark 3.** The rank of the bundle  $E_k$  is given by the Fibonacci number between  $a_{3,k-1}$  and  $a_{3,k}$  in case  $(n, d) = (2, 3)$ . In this case  $E_{2k}$  (respectively,  $E_{2k+1}$ ) is the  $k$ -th sheafified syzygy occurring in the resolution of  $\mathcal{O}_{\mathbb{P}^2}(1)$  (respectively, of  $\mathcal{O}_{\mathbb{P}^2}(2)$ ) over the Veronese ring, twisted by  $\mathcal{O}_{\mathbb{P}^2}(3(k-1))$ . A similar result holds for  $(n, d) = (3, 2)$ .

As for notation, we write small letters for the dimension of a space in capital letter, for instance  $h^i(\mathbb{P}^n, E) = \dim_{\mathbf{k}} H^i(\mathbb{P}^n, E)$ . We also write  $\chi(E, F) = \sum (-1)^i \mathrm{ext}_{\mathbb{P}^n}^i(E, F)$  and  $\chi(E) = \chi(\mathcal{O}_{\mathbb{P}^n}, E)$ .  $\delta_{i,j}$  is Kronecker's delta.

## 1. FIBONACCI BUNDLES

1.1. Let us write  $\Upsilon_{\ell}$  for the  $\ell$ -th Kronecker quiver, namely the oriented graph with two vertices  $\mathbf{e}_0$  and  $\mathbf{e}_1$ , and  $\ell$  arrows from  $\mathbf{e}_0$  to  $\mathbf{e}_1$ . A representation  $R$  of  $\Upsilon_{\ell}$ , with dimension vector  $(a, b)$  is the choice of  $\ell$  matrices of size  $a \times b$ .



We identify a basis of  $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(2))$  with the set of  $\ell = \binom{n+1}{2}$  arrows of  $\Upsilon_{\ell}$ . Then the derived category of finite-dimensional representations of  $\Upsilon_{\ell}$  embeds into the derived category of  $\mathcal{O}_{\mathbb{P}^n}$ -modules by sending  $R$  to the cone  $\Phi(R)$  of the morphism  $e_R$  associated with  $R$  according to this identification:

$$\Phi(R)[-1] \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^a \xrightarrow{e_R} \Omega_{\mathbb{P}^n}(1)^b,$$

where we denote by  $[-1]$  the shift to the right of complexes. It is clear that:

$$\mathrm{Ext}_{\mathbb{P}^n}^i(\Phi(R), \Phi(R)) \simeq \mathrm{Ext}_{\Upsilon_{\ell}}^i(R, R), \quad \text{for all } i.$$

1.2. We will use Kac's classification of rigid  $\Upsilon_{\ell}$ -modules as Schur roots (hence the restriction on  $\mathbf{k}$ ), which is also one of the main ingredients in Iyama-Yoshino's proof. By [Kac80, Theorem 4], any non-zero rigid  $\Upsilon_{\ell}$ -module is a direct sum of rigid simple representations of the form  $R_k$ , for some  $k \in \mathbb{Z}$ , where  $R_k$  is defined as the unique indecomposable representation of  $\Upsilon_{\ell}$  with dimension vector  $(a_{\ell,k-1}, a_{\ell,k})$  for  $k \geq 1$ , or  $(a_{\ell,-k}, a_{\ell,1-k})$  for  $k \leq 0$ .

Set  $F_k = \Phi(R_k)$  for  $k \geq 1$ , and  $F_k = \Phi(R_k)[-1]$  for  $k \leq 0$ . It turns out that  $F_k$  is an exceptional locally free sheaf, called a *Fibonacci bundle*, cf. [Bra08]. We rewrite the defining exact sequences of  $F_k$ :

$$(2) \quad \begin{aligned} 0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,k-1}} \rightarrow \Omega_{\mathbb{P}^n}(1)^{a_{\ell,k}} \rightarrow F_k \rightarrow 0, & \quad \text{for } k \geq 1, \\ 0 \rightarrow F_k \rightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^{a_{\ell,-k}} \rightarrow \Omega_{\mathbb{P}^n}(1)^{a_{\ell,1-k}} \rightarrow 0, & \quad \text{for } k \leq 0. \end{aligned}$$

1.3. Here is a lemma on the cohomology of Fibonacci bundles.

**Lemma 4.** *For  $k \geq 1$ , the only non-vanishing intermediate cohomology of  $F_k$  is:*

$$h^1(\mathbb{P}^n, F_k(-1)) = a_{\ell, k-1} \quad h^{n-1}(\mathbb{P}^n, F_k(-n)) = a_{\ell, k}.$$

*Proof.* We consider the left and right mutation endofunctors of the derived category of coherent sheaves on  $\mathbb{P}^n$ , that associate with a pair  $(E, F)$  of complexes, two complexes denoted respectively by  $R_F E$  and  $L_E F$ . These are the cones of the natural evaluation maps  $f_{E,F}$  and  $g_{E,F}$ :

$$E \xrightarrow{f_{E,F}} \mathrm{RHom}_{\mathbb{P}^n}(E, F)^* \otimes F \rightarrow R_F E, \quad L_E F \rightarrow \mathrm{RHom}_{\mathbb{P}^n}(E, F) \otimes E \xrightarrow{g_{E,F}} F.$$

It is well-known (cf. [Bra08]) that the Fibonacci bundles  $F_k$  can be defined recursively from  $F_0 = \mathcal{O}_{\mathbb{P}^n}(-1)$  and  $F_1 = \Omega_{\mathbb{P}^n}(1)$  by setting:

$$(3) \quad F_{k+1} = R_{F_k} F_{k-1}, \quad \text{for } k \geq 1,$$

$$(4) \quad F_{k-1} = L_{F_k} F_{k+1}, \quad \text{for } k \leq 0.$$

This way, for any  $k \in \mathbb{Z}$  we get a natural exact sequence:

$$(5) \quad 0 \rightarrow F_{k-1} \rightarrow (F_k)^\ell \rightarrow F_{k+1} \rightarrow 0.$$

Over  $\mathbb{P}^n$ , we consider the full exceptional sequence:

$$(\mathcal{O}_{\mathbb{P}^n}(-1), \Omega_{\mathbb{P}^n}(1), \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(2), \mathcal{O}_{\mathbb{P}^n}(3), \dots, \mathcal{O}_{\mathbb{P}^n}(n-1)),$$

obtained from the standard collection  $(\mathcal{O}_{\mathbb{P}^n}(-1), \dots, \mathcal{O}_{\mathbb{P}^n}(n-1))$ , by the mutation  $\Omega_{\mathbb{P}^n}(-1) \simeq L_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(1)$  (all the terminology and results we need on exceptional collections are contained in [Bon89]). By (3), we can replace the previous exceptional sequence with:

$$(F_{k-1}, F_k, \mathcal{O}_{\mathbb{P}^n}, \mathcal{O}_{\mathbb{P}^n}(2), \dots, \mathcal{O}_{\mathbb{P}^n}(n-1)),$$

Right-mutating  $F_{k-1}$  through the full collection, we must get back  $F_{k-1} \otimes \omega_{\mathbb{P}^n}^* \simeq F_{k-1}(n+1)$ . So, using (5), we get a long exact sequence:

$$(6) \quad 0 \rightarrow F_{k+1} \rightarrow \mathcal{O}_{\mathbb{P}^n}^{u_1} \rightarrow \mathcal{O}_{\mathbb{P}^n}(2)^{u_2} \rightarrow \dots \rightarrow \mathcal{O}_{\mathbb{P}^n}(n-1)^{u_{n-1}} \rightarrow F_{k-1}(n+1) \rightarrow 0,$$

for some integers  $u_i$ . Now by (2) we get:

$$H^i(\mathbb{P}^n, F_k(t)) = 0 \quad \text{for:} \quad \begin{cases} 2 \leq i \leq n-2, & \forall t, \\ i = 0, & t \leq 0, \\ i = 1, & t \neq -1, \\ i = n-1, & t \geq 1-n. \end{cases}$$

The required non-vanishing cohomology of  $F_k$  appears again from (2). So it only remains to check that  $H^{n-1}(\mathbb{P}^n, F_k(t)) = 0$  for  $t \leq -n-1$ . But this is clear by induction once we twist (6) by  $\mathcal{O}_{\mathbb{P}^n}(t)$ , and take cohomology.  $\square$

1.4. We compute the Ext groups between pairs of Fibonacci bundles.

**Lemma 5.** *For any pair of integers  $j \geq k+1$  we have:*

$$\mathrm{ext}_{\mathbb{P}^n}^i(F_j, F_k) = \delta_{1,i} a_{\ell, j-k-1}, \quad \mathrm{ext}_{\mathbb{P}^n}^i(F_k, F_j) = \delta_{0,i} a_{\ell, j-k+1}.$$

*Proof.* The formulas hold for  $k = j$  since  $F_k$  is exceptional, and we easily compute  $\chi(F_j, F_k) = -a_{\ell, j-k-1}$  and  $\chi(F_k, F_j) = a_{\ell, j-k+1}$  (for instance by computing  $\chi$  of  $\Upsilon_\ell$ -modules via the Cartan form and using faithfulness of  $\Phi$ ).

The second formula is proved once we show  $\mathrm{Ext}_{\mathbb{P}^n}^i(F_k, F_j) = 0$  for  $i \geq 1$ . In fact, since the category of  $\Upsilon_\ell$ -representations is hereditary, the second formula holds if  $k \leq 0$  for in this case  $F_k \simeq \Phi(R_k)[-1]$ . By the same reason, we only have to check it for  $i = 1$ . Using (5), this vanishing holds for  $j$  if it does for  $j-1$  and  $j-2$ . Since the statement is clear when extended to  $j = k$ , it suffices to check  $\mathrm{Ext}_{\mathbb{P}^n}^1(F_k, F_{k-1}) = 0$ .

Since  $\chi(F_k, F_{k-1}) = 0$ ,  $\text{Hom}_{\mathbb{P}^n}(F_k, F_{k-1}) = 0$  will do the job. However, any nonzero map  $F_k \rightarrow F_{k-1}$  would give, again by (5), a non-scalar endomorphism of  $F_k$ , which cannot exist since  $F_k$  is simple. The second formula is now proved.

As for the first formula, again we see that it holds if  $k \leq 0$  and  $j \geq 1$  once we check it for  $i = 0$ . However using repeatedly (5) we see that a non-zero map  $F_j \rightarrow F_k$  leads to an endomorphism of  $F_j$  which factors through  $F_k$  this is absurd for  $F_j$  is simple. When  $j, k$  have the same sign, the first formula has to be checked for  $i = 2$  only. Moreover, we have just proved the statement for  $j = k + 1$ , and using (5) and exceptionality of  $F_k$  we get it for  $j = k + 2$ . By iterating this argument we get the statement for any  $j \geq k + 1$ .  $\square$

**Remark 6.** This lemma holds more generally (with the same proof) for any exceptional pair  $(F_0, F_1)$  of objects on a projective  $k$ -variety  $X$ , with  $\text{hom}_X(F_0, F_1) = \ell$ , by defining recursively  $F_k$  for all  $k \in \mathbb{Z}$  by (3) and (4).

## 2. RIGID ACM BUNDLES ON THE THIRD VERONESE SURFACE

We prove here Theorem 1 in case  $(n, d) = (2, 3)$ .

2.1. Let us first prove (i). So let  $E$  be an indecomposable vector bundle on  $\mathbb{P}^2$  satisfying (1). Without loss of generality, we may replace  $E$  by  $E(s)$ , where  $s$  is the smallest integer such that  $h^0(\mathbb{P}^2, E(s)) \neq 0$ . Set  $\alpha_{i,j} = h^i(\mathbb{P}^2, E(-j))$ . Of course,  $\alpha_{0,j} = 0$  if and only if  $j \geq 0$ . The Beilinson complex  $F$  associated with  $E$  (see for instance [Huy06, Chapter 8]) reads:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{d_0} \begin{array}{c} \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{2,2}} \\ \oplus \\ \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \end{array} \xrightarrow{d_1} \begin{array}{c} \Omega_{\mathbb{P}^2}(1)^{\alpha_{2,1}} \\ \oplus \\ \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,0}} \end{array} \xrightarrow{d_2} \mathcal{O}_{\mathbb{P}^2}^{\alpha_{2,0}} \rightarrow 0.$$

The term consisting of three summands in the above complex sits in degree 0 (we call it *middle term*), and the cohomology of this complex is  $E$ . By condition (1), at least one of the  $\alpha_{1,j}$  is zero, for  $j = 0, 1, 2$ .

If  $\alpha_{1,2} = 0$ , then  $d_0 = 0$ . By minimality of the Beilinson complex the restriction of  $d_1$  to the summand  $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$  of the middle term is also zero. Therefore  $\mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$  is a direct summand of  $E$ , so  $E \simeq \mathcal{O}_{\mathbb{P}^2}$  by indecomposability of  $E$ .

If  $\alpha_{1,1} = 0$ , then the non-zero component of  $d_0$  is just a map  $\mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$ , and a direct summand of  $E$  is the cokernel of this map. By indecomposability of  $E$ , in this case  $E(-1)$  has a resolution of the desired form with  $a = \alpha_{0,0}$  and  $b = \alpha_{1,2}$ .

2.2. It remains to look at the case  $\alpha_{1,0} = 0$ . Note that the restriction of  $d_1$  to  $\Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}}$  is zero, which implies that a direct summand of  $E$  (hence all of  $E$  by indecomposability) has the resolution:

$$(7) \quad 0 \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{d_0} \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow E \rightarrow 0$$

and  $\alpha_{2,j} = 0$  for  $j = 0, 1, 2$ . We compute  $\chi(E(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}$ , so:

$$h^0(\mathbb{P}^2, E^*) = h^2(\mathbb{P}^2, E(-3)) = 3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}.$$

If this value is positive, then there is a non-trivial morphism  $g : E \rightarrow \mathcal{O}_{\mathbb{P}^2}$ , and since  $\alpha_{0,0} \neq 0$  there also exists  $0 \neq f : \mathcal{O}_{\mathbb{P}^2} \rightarrow E$ . So  $E$  has an endomorphism factoring through  $\mathcal{O}_{\mathbb{P}^2}$ , a contradiction.

Hence we may assume  $3\alpha_{1,2} - 3\alpha_{1,1} + \alpha_{0,0}$ , in other words  $\alpha_{0,3} = 0$ . Therefore, the Beilinson complex associated with  $E(-1)$  gives a resolution:

$$0 \rightarrow E(-1) \rightarrow \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \rightarrow 0.$$

It is easy to convert this resolution into the form we want by the diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E(-1) & \longrightarrow & \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \parallel \\ 0 & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{3\alpha_{1,2}-\alpha_{1,1}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{3\alpha_{1,2}} & \longrightarrow & \mathcal{O}_{\mathbb{P}^2}^{\alpha_{1,1}} \rightarrow 0 \\ & & \downarrow & & \downarrow & & \\ & & \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & = & \mathcal{O}_{\mathbb{P}^2}(1)^{\alpha_{1,2}} & & \\ & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & \end{array}$$

From the leftmost column, it follows that  $E^*$  has a resolution of the desired form, with  $a = 3\alpha_{1,2} - \alpha_{1,1}$  and  $b = \alpha_{1,2}$ . Claim (i) is thus proved.

2.3. Let us now prove (ii). We may assume that  $\text{rk}(E) > 1$ . We check that  $E$  being indecomposable and rigid forces  $E$  to be simple, so that it has no endomorphism factoring through  $\mathcal{O}_{\mathbb{P}^2}(t)$  and thus (i) applies. In the previous proof, we used this condition only for  $\alpha_{1,0} = 0$ , and (i) will apply if  $\alpha_{0,3} = 0$ .

2.4. We work with  $\alpha_{1,0} = 0$ . Let  $e$  be the restricted map  $e : \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \rightarrow \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}}$  extracted from  $d_0$  and let  $F$  be its cone, shifted by 1:

$$(8) \quad F \rightarrow \mathcal{O}_{\mathbb{P}^2}(-1)^{\alpha_{1,2}} \xrightarrow{e} \Omega_{\mathbb{P}^2}(1)^{\alpha_{1,1}}.$$

This is a complex with two terms, and its cohomology is concentrated in degrees zero and one, namely  $\mathcal{H}^0 F \simeq \ker(e)$  and  $\mathcal{H}^1 F \simeq \text{coker}(e)$ . From (7) we easily see that  $F$  fits into a distinguished triangle:

$$(9) \quad F \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow E.$$

Applying  $\text{Hom}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}, -)$  to (8), we get  $\text{Ext}_{\mathbb{P}^2}^i(\mathcal{O}_{\mathbb{P}^2}, F) = 0$  for all  $i$ , so:

$$\text{Ext}_{\mathbb{P}^2}^i(F, F) \simeq \text{Ext}_{\mathbb{P}^2}^i(E, F[1]), \quad \text{for all } i.$$

Also, we know that  $H^2(\mathbb{P}^2, E^*) = 0$ , so applying  $\text{Hom}_{\mathbb{P}^2}(E, -)$  to (9) we get:

$$\text{Ext}_{\mathbb{P}^2}^1(E, E) \rightarrow \text{Ext}_{\mathbb{P}^2}^1(E, F[1]) \rightarrow 0.$$

Putting this together, we obtain a surjection:

$$\text{Ext}_{\mathbb{P}^2}^1(E, E) \twoheadrightarrow \text{Ext}_{\mathbb{P}^2}^1(F, F) \simeq \text{Ext}_{\mathbb{P}^2}^1(R, R),$$

with  $F \simeq \Phi(R)$ . We understand now that, if  $E$  is rigid, then also  $R$  is.

2.5. If  $R$  is rigid, then by §1.2,  $R$  is a direct sum of rigid simple representations of the form  $R_k$ . Therefore, cohomology of (9) gives an exact sequence:

$$0 \rightarrow \bigoplus_{i \leq 0} F_i^{r_i} \rightarrow \mathcal{O}_{\mathbb{P}^2}^{\alpha_{0,0}} \rightarrow E \rightarrow \bigoplus_{i \geq 1} F_i^{r_i} \rightarrow 0,$$

for some integers  $r_i$ . If only  $R_i$  with  $i \leq 0$  appear, then we are done by §2.3. Indeed, in that case  $E$  is globally generated, so  $H^0(\mathbb{P}^2, E) \neq 0$  implies  $H^0(\mathbb{P}^2, E^*) = 0$  for otherwise  $\mathcal{O}_{\mathbb{P}^2}$  would be a direct summand of  $E$ .

If some  $R_i$  appears with  $i \geq 1$ , we call  $I$  the (non-zero) image of the middle map in the previous exact sequence, and we check  $\text{Ext}_{\mathbb{P}^2}^1(F_j, I) = 0$  for all  $j \geq 1$ , which

contradicts  $E$  being indecomposable. To check this, note that  $\mathrm{Hom}_{\mathbb{P}^2}(F_j, -)$  gives an exact sequence:

$$\mathrm{Ext}_{\mathbb{P}^2}^1(F_j, \mathcal{O}_{\mathbb{P}^2})^{\alpha_{0,0}} \rightarrow \mathrm{Ext}_{\mathbb{P}^2}^1(F_j, I) \rightarrow \bigoplus_{i \leq 0} \mathrm{Ext}_{\mathbb{P}^2}^2(F_j, F_i)^{r_i}.$$

The leftmost term vanishes by Serre duality and Lemma 4. The rightmost term is zero by Lemma 5. Part (ii) is now proved.

2.6. The statement (iii) is clear by Lemma 4 and by exceptionality of Fibonacci bundles. The fact that  $E^*$  is also ACM is obvious by Serre duality.

**Remark 7.** If  $k$  is algebraically closed of characteristic zero, we may apply [Dre86, Corollaire 7], to the effect that a rigid bundle is a direct sum of exceptional bundles. So, at the price of relying on this result, from (i) we may deduce directly (ii) via Kac's theorem.

### 3. ACM BUNDLES ON THE SECOND VERONESE THREEFOLD

The techniques we have just seen apply to the embedding of  $\mathbb{P}^3$  in  $\mathbb{P}^9$  by quadratic forms. Again we replace  $E$  with the  $E(s)$ , where  $s$  is the smallest integer such that  $E$  has non-zero global sections, and set  $\alpha_{i,j} = h^i(\mathbb{P}^3, E(-j))$ . If (1) gives  $\alpha_{1,1} = \alpha_{2,1} = 0$ , then  $E(-1)$  has the desired resolution. On the other hand, if (1) tells  $\alpha_{1,0} = \alpha_{2,0} = \alpha_{1,2} = \alpha_{2,2} = 0$ , then we are left with a resolution of the form:

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^{\alpha_{1,3}} \xrightarrow{d_0} \Omega_{\mathbb{P}^3}(1)^{\alpha_{1,1}} \oplus \mathcal{O}_{\mathbb{P}^3}^{\alpha_{0,0}} \rightarrow E \rightarrow 0.$$

This time we also have  $\alpha_{0,4} = 0$ , and  $\alpha_{1,4} = \alpha_{2,4} = 0$  again by (1), and simplicity of  $E$  gives  $\alpha_{3,4} = h^0(\mathbb{P}^3, E^*) = 0$ . So  $E(-1)$  has a resolution like:

$$0 \rightarrow E(-1) \rightarrow \Omega_{\mathbb{P}^3}^2(2)^{\alpha_{1,3}} \rightarrow \mathcal{O}_{\mathbb{P}^3}^{\alpha_{1,1}} \rightarrow 0.$$

Then, using the same trick as in the proof of the previous theorem, we see that  $E^*$  has the desired resolution, with  $a = 6\alpha_{1,3} - \alpha_{1,1}$  and  $b = \alpha_{1,3}$ .

This proves the first statement. The rest follows by the same path. Drezet's theorem as shortcut for (i)  $\Rightarrow$  (ii) may be replaced by [HZ13].

**Remark 8.** It should be noted that, in [IY08, Theorem 1.2 and Theorem 1.3], the ACM bundle  $E$  on the given Veronese variety is assumed to have a rigid module of global sections. This implies, respectively,  $\mathrm{Ext}_{\mathbb{P}^2}^1(E, E(3t)) = 0$ , or  $\mathrm{Ext}_{\mathbb{P}^3}^1(E, E(2t)) = 0$ , for all  $t \in \mathbb{Z}$ . A priori, this is a stronger requirement than just  $\mathrm{Ext}_{\mathbb{P}^n}^1(E, E) = 0$ . However, our proof shows that the two conditions are equivalent for ACM bundles.

### 4. RIGID ACM BUNDLES HIGHER VERONESE SURFACES

Assume  $k$  algebraically closed. The next result shows that, for  $d \geq 4$ , the class of rigid ACM bundles on  $d$ -fold Veronese surfaces contains the class of exceptional bundles on  $\mathbb{P}^2$ , which is indeed quite complicated, cf. [DLP85]. At least if  $\mathrm{char}(k) = 0$ , the two classes coincide by [Dre86, Corollaire 7].

**Theorem 9.** *Let  $F$  be an exceptional bundle on  $\mathbb{P}^2$  and fix  $d \geq 4$ . Then there is an integer  $t$  such that  $E = F(t)$  satisfies (1).*

*Proof.* It is known that  $F$  is actually stable by [DLP85]. This implies that  $F$  has natural cohomology by [HL93], i.e. for all  $t \in \mathbb{Z}$  there is at most one  $i$  such that  $H^i(\mathbb{P}^2, F(t)) \neq 0$ . Then,  $H^1(\mathbb{P}^2, F(t)) \neq 0$  if and only if  $\chi(F(t)) < 0$ .

Let now that  $r$ ,  $c_1$  and  $c_2$  be the rank and the Chern classes of  $F$ . Computing  $\chi$  by additivity, we see that  $\chi(F(t))$  is a polynomial of degree 2 in  $t$ , of dominant term  $r/2$ , whose discriminant is:

$$\Delta = c_1^2(1-r) + r(2c_2 + r/4) = -\chi(F, F) + 5r^2/4.$$

So, using  $\chi(F, F) = 1$ , we get  $\Delta = -1 + 5r^2/4$ . Therefore, the roots of  $\chi(F(t))$  differ at most by:

$$\lceil \frac{2\sqrt{\Delta}}{r} \rceil = \lceil \frac{\sqrt{5r^2-4}}{r} \rceil \leq 3.$$

Then, there are at most three consecutive integers  $t_0$ ,  $t_0 + 1$ ,  $t_0 + 2$  such that  $H^1(\mathbb{P}^2, F(t_0 + j)) \neq 0$  for  $j = 0, 1, 2$ . This means that  $E = F(t_0 - 1)$  satisfies (1) for any choice of  $d \geq 4$ .  $\square$

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