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# NON-FLAT EXTENSION OF FLAT VECTOR BUNDLES

INDRANIL BISWAS AND VIKTORIA HEU

ABSTRACT. We construct a pair  $(E, F)$ , where  $E$  is a holomorphic vector bundle over a compact Riemann surface and  $F \subset E$  a holomorphic subbundle, such that both  $F$  and  $E/F$  admit holomorphic connections, but  $E$  does not.

## 1. INTRODUCTION

Let  $X$  be a compact connected Riemann surface. Let  $E$  be a holomorphic vector bundle over  $X$ . We say that  $E$  is *flat* if it can be endowed with a holomorphic connection. Such a holomorphic connection is automatically flat (in the usual sense) because there are no nonzero  $(2, 0)$ -forms on  $X$ . Conversely, given a  $C^\infty$  vector bundle  $E$  on  $X$ , a flat connection on  $V$  defines a holomorphic structure on  $E$  as well as a holomorphic connection on it. A criterion due to Atiyah and Weil says that a holomorphic vector bundle  $E$  on  $X$  is flat if and only if for every holomorphic subbundle  $0 \neq F \subseteq E$  such that there is another holomorphic subbundle  $F' \subset E$  with  $F \oplus F' = E$ , the degree of  $F$  is zero [At], [We]. In particular, any semistable vector bundle on  $X$  of degree zero admits a holomorphic connection.

Let  $E$  be a holomorphic vector bundle on  $X$  and

$$0 \subset F_1 \subset F_2 \subset \cdots \subset F_\ell = E$$

a filtration by holomorphic subbundles of  $E$ . It is natural to ask for conditions that ensure that  $E$  admits a holomorphic connection that preserves this filtration. An obvious necessary condition is that each successive quotient  $F_i/F_{i-1}$ ,  $1 \leq i \leq \ell$ , should admit a holomorphic connection. One might expect that this necessary condition is also sufficient. One reason for this expectation is the following: if  $E$  is semistable of degree zero, then indeed  $E$  admits a filtration preserving holomorphic connection by Simpson correspondence [Si, p. 40, Corollary 3.10] (see also [BH, p. 1474]). Note that if  $E$  is semistable of degree zero, and all successive quotients admit holomorphic connection, then each  $F_i$  is semistable of degree zero.

Our aim here is to show that flatness of vector bundles over curves does not behave well under extensions. More precisely, we produce a short exact sequence of holomorphic vector bundles

$$0 \longrightarrow F \longrightarrow E \longrightarrow \mathcal{Q} \longrightarrow 0$$

on any compact Riemann surface  $X$  of genus  $g \geq 2$  such that both  $F$  and  $\mathcal{Q}$  admit holomorphic connections but  $E$  does not.

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Note that such a vector bundle cannot exist in genus 0 and 1. Indeed, in that case, all flat vector bundles are semistable of degree 0, and extension of a semistable vector bundle of degree 0 by a semistable vector bundle of degree 0 is again semistable of degree 0.

The vector bundle  $E$  we construct is of rank 3. Note that this also is a minimal condition since a vector bundle  $E$  of rank at most two fitting in an exact sequence as above is automatically semistable of degree zero (and thus flat).

## 2. CONSTRUCTION OF $E$

Let  $X$  be a compact connected Riemann surface of genus  $g$ , with  $g \geq 2$ . Denote by  $K_X$  the canonical divisor on  $X$ . The linear equivalence class of  $K_X$  can be expressed as

$$(2.1) \quad K_X = P + D,$$

where  $P$  is a single point and  $D$  is an effective divisor of degree  $2g - 3$  such that  $P$  is disjoint from the support of  $D$ . Indeed, this follows from the fact that

$$\dim H^0(X, K_X) = g \geq 2.$$

Fix  $P$  and  $D$  as in (2.1). Let us now split the divisor  $D$  in two parts, namely

$$(2.2) \quad D = D_Q + D_R,$$

where  $D_Q$  and  $D_R$  are effective divisors with

$$(2.3) \quad \deg(D_Q) + 1 = \deg(D_R) = g - 1.$$

By Serre duality, we have

$$(2.4) \quad H^1(X, \mathcal{O}_X(D_Q + D_R)) \simeq H^0(X, \mathcal{O}_X(P)) \simeq \mathbb{C}.$$

In particular, we can choose a nonzero element (which is actually unique up to multiplication by a nonzero scalar)

$$(2.5) \quad \theta \in H^1(X, \mathcal{O}_X(D_Q + D_R)) \setminus \{0\}.$$

Since  $\mathcal{O}_X(D_Q + D_R) = \text{Hom}_{\mathcal{O}_X}(\mathcal{O}_X(-D_R), \mathcal{O}_X(D_Q))$ , the cohomology class  $\theta$  in (2.5) produces a short exact sequence of vector bundles

$$(2.6) \quad 0 \longrightarrow \mathcal{O}_X(D_Q) \longrightarrow V \longrightarrow \mathcal{O}_X(-D_R) \longrightarrow 0$$

on  $X$ . This exact sequence does not split since  $\theta \neq 0$ . From (2.3) it follows that  $\deg(V) = -1$ . Consider the holomorphic vector bundle

$$(2.7) \quad E := \mathcal{O}_X(P) \oplus V$$

on  $X$  of rank three and degree zero. We have

$$(2.8) \quad \mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q) \subset E$$

because  $\mathcal{O}_X(D_Q) \subset V$  (see (2.6)). Let  $s^P$  (respectively,  $s^Q$ ) be the holomorphic section of  $\mathcal{O}_X(P)$  (respectively,  $\mathcal{O}_X(D_Q)$ ) given by the constant function 1 on  $X$ . So  $s^P$  (respectively,  $s^Q$ )

vanishes over  $P$  (respectively the support of  $D_Q$ ) of order one, and is nonzero everywhere else. Now consider the holomorphic section

$$(2.9) \quad \sigma_1 : \mathcal{O}_X \longrightarrow \mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q)$$

defined by  $x \mapsto (s^P(x), s^Q(x))$ . Note that  $\sigma_1$  does not vanish anywhere because  $P$  is disjoint from the support of  $D_Q$  according to (2.1) and (2.2). Let  $\sigma$  be the composition

$$(2.10) \quad \mathcal{O}_X \xrightarrow{\sigma_1} \mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q) \hookrightarrow E$$

(see (2.8)). Since  $\sigma$  is nowhere vanishing, we get a short exact sequence of holomorphic vector bundles on  $X$

$$(2.11) \quad 0 \longrightarrow F := \mathcal{O}_X \xrightarrow{\sigma} E \longrightarrow \mathcal{Q} := E/\sigma(F) \longrightarrow 0.$$

By construction,  $\mathcal{O}_X(P)$  is a direct summand of  $E$ . Since  $\deg(\mathcal{O}_X(P)) \neq 0$ , from the criterion of Atiyah–Weil we conclude that  $E$  is not flat. We will now prove that both  $F$  and  $\mathcal{Q}$  in (2.11) are flat.

### 2.1. Flatness of $F$ and $\mathcal{Q}$ .

**Lemma 2.1.** *The holomorphic vector bundle  $\mathcal{Q}$  in (2.11) is a nontrivial extension of the line bundle  $\mathcal{O}_X(-D_R)$  by  $\mathcal{O}_X(P + D_Q)$ .*

*Proof.* On one hand, we have  $\bigwedge^2(\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q)) = \mathcal{O}_X(P + D_Q)$ . On the other hand,  $\bigwedge^2(\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q)) = (\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q))/\sigma_1(\mathcal{O}_X)$  (see (2.9)). It follows that

$$(2.12) \quad (\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q))/\sigma_1(\mathcal{O}_X) = \mathcal{O}_X(P + D_Q).$$

The inclusion of  $\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q)$  in  $E$  (see (2.10)) produces an inclusion of the quotient  $(\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q))/\sigma_1(\mathcal{O}_X)$  in  $E/\sigma(F) = \mathcal{Q}$  (see (2.11)). Therefore, from (2.12) we have

$$(2.13) \quad \mathcal{O}_X(P + D_Q) \subset \mathcal{Q}$$

as a subbundle. Using (2.6), (2.7) we have

$$\bigwedge^2 \mathcal{Q} = \bigwedge^3 E = \mathcal{O}_X(P) \otimes \bigwedge^2 V = \mathcal{O}_X(P + D_Q - D_R).$$

Note that its degree is zero (2.3). Therefore, from (2.13),

$$\mathcal{O}_X(P + D_Q - D_R) = \bigwedge^2 \mathcal{Q} = \mathcal{O}_X(P + D_Q) \otimes (\mathcal{Q}/\mathcal{O}_X(P + D_Q)).$$

So,  $\mathcal{Q}/\mathcal{O}_X(P + D_Q) = \mathcal{O}_X(-D_R)$ . Consequently, from (2.13), we get a short exact sequence of vector bundles

$$(2.14) \quad 0 \longrightarrow \mathcal{O}_X(P + D_Q) \longrightarrow \mathcal{Q} \longrightarrow \mathcal{O}_X(-D_R) \longrightarrow 0.$$

To complete the proof of the lemma, we need to show that the short exact sequence in (2.14) does not split. Let

$$(2.15) \quad \omega \in H^1(X, \text{Hom}(\mathcal{O}_X(-D_R), \mathcal{O}_X(P + D_Q))) = H^1(X, \mathcal{O}_X(P + D_Q + D_R))$$

be the extension class for the exact sequence in (2.14). We will now compute  $\omega$ .

From (2.6) and (2.7) we have the short exact sequence

$$0 \longrightarrow \mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q) \longrightarrow E \longrightarrow \mathcal{O}_X(-D_R) \longrightarrow 0$$

of holomorphic vector bundles on  $X$ . Let

$$\begin{aligned} \theta' &\in \mathrm{H}^1(X, (\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q)) \otimes \mathcal{O}_X(D_R)) \\ &= \mathrm{H}^1(X, \mathcal{O}_X(P + D_R)) \oplus \mathrm{H}^1(X, \mathcal{O}_X(D_Q + D_R)) \end{aligned}$$

be the cohomology class for this exact sequence. Evidently,  $\theta'$  coincides with

$$(0, \theta) \in \mathrm{H}^1(X, \mathcal{O}_X(P + D_R)) \oplus \mathrm{H}^1(X, \mathcal{O}_X(D_Q + D_R)),$$

where  $\theta$  is the class in (2.5).

Next, consider the homomorphism  $\gamma$  defined by the composition

$$\mathcal{O}_X(D_Q) \hookrightarrow \mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q) \twoheadrightarrow (\mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q))/\sigma_1(F) = \mathcal{O}_X(P + D_Q)$$

(see (2.12)), where the homomorphism  $\mathcal{O}_X(D_Q) \hookrightarrow \mathcal{O}_X(P) \oplus \mathcal{O}_X(D_Q)$  is the inclusion of the second factor. Clearly, this composition  $\gamma$  coincides with the natural inclusion of the coherent sheaf  $\mathcal{O}_X(D_Q)$  in  $\mathcal{O}_X(P + D_Q)$ . Therefore, the cohomology classes  $\omega$  and  $\theta$  (constructed in (2.15) and (2.5)) satisfy the equation

$$(2.16) \quad \omega = \rho(\theta),$$

where

$$\rho : \mathrm{H}^1(X, \mathcal{O}_X(D_Q + D_R)) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X(P + D_Q + D_R))$$

is the homomorphism induced by the natural inclusion of the coherent sheaf  $\mathcal{O}_X(D_Q + D_R)$  in  $\mathcal{O}_X(P + D_Q + D_R)$ . Consider the short exact sequence of coherent sheaves

$$0 \longrightarrow \mathcal{O}_X(D_Q + D_R) \longrightarrow \mathcal{O}_X(P + D_Q + D_R) \longrightarrow \mathcal{O}_X(P + D_Q + D_R)_P \longrightarrow 0,$$

where  $\mathcal{O}_X(P + D_Q + D_R)_P$  is the torsion sheaf supported at  $P$  with its stalk being the fiber of the line bundle  $\mathcal{O}_X(P + D_Q + D_R)$  over  $P$ . Let

$$(2.17) \quad \begin{array}{c} 0 \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X(D_Q + D_R)) \longrightarrow \mathrm{H}^0(X, \mathcal{O}_X(P + D_Q + D_R)) \\ \xrightarrow{\alpha_1} \mathcal{O}_X(P + D_Q + D_R)_P \xrightarrow{\alpha_2} \mathrm{H}^1(X, \mathcal{O}_X(D_Q + D_R)) \\ \xrightarrow{\rho} \mathrm{H}^1(X, \mathcal{O}_X(P + D_Q + D_R)) \end{array}$$

be the long exact sequence of cohomologies associated to it. We have

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(P + D_Q + D_R)) = \dim \mathrm{H}^0(X, \mathcal{O}_X(\mathbb{K}_X)) = g$$

and, by Riemann-Roch and (2.4),

$$\dim \mathrm{H}^0(X, \mathcal{O}_X(D_Q + D_R)) = g - 1.$$

These imply that  $\alpha_1$  in (2.17) is surjective. Therefore,  $\alpha_2$  in (2.17) is the zero homomorphism. This implies that  $\rho$  in (2.17) is injective.

Since  $\rho$  is injective, from (2.16) it follows that  $\omega \neq 0$ , because  $\theta \neq 0$  (see (2.5)). The exact sequence in (2.14) does not split because  $\omega \neq 0$ .  $\square$

**Proposition 2.2.** *The holomorphic vector bundle  $\mathcal{Q}$  in (2.14) admits a holomorphic connection.*

*Proof.* Assume that  $\mathcal{Q}$  does not admit any holomorphic connection. Since  $\text{degree}(\mathcal{Q}) = 0$ , and  $\mathcal{Q}$  does not admit any holomorphic connection, the criterion of Atiyah–Weil says that  $\mathcal{Q}$  holomorphically decomposes as

$$(2.18) \quad \mathcal{Q} = L \oplus M,$$

where  $\text{degree}(L) = -\text{degree}(M) > 0$ . Let  $p_M : \mathcal{Q} \rightarrow M$  be the projection given by the decomposition in (2.18). Let  $\beta$  denote the composition

$$\mathcal{O}_X(P + D_Q) \hookrightarrow \mathcal{Q} \xrightarrow{p_M} M,$$

where the inclusion is constructed in (2.13). Since

$$\text{degree}(\mathcal{O}_X(P + D_Q)) = g - 1 > 0 > \text{deg}(M),$$

there is no nonzero homomorphism from  $\mathcal{O}_X(P + D_Q)$  to  $M$ . In particular,  $\beta = 0$ .

We have  $\mathcal{O}_X(P + D_Q) \subset L$  because  $\beta = 0$ . Since both  $\mathcal{O}_X(P + D_Q)$  and  $L$  are line subbundles on  $\mathcal{Q}$ , this implies that the two subbundles  $\mathcal{O}_X(P + D_Q)$  and  $L$  coincide. Hence

$$M = \mathcal{Q}/L = \mathcal{Q}/\mathcal{O}_X(P + D_Q) = \mathcal{O}_X(-D_R)$$

(see Lemma 2.1). Therefore, the decomposition  $\mathcal{Q} = L \oplus M$  in (2.18) produces a splitting of the short exact sequence in (2.14). But we know from Lemma 2.1 that the short exact sequence in (2.14) does not split. In view of the above contradiction we conclude that  $\mathcal{Q}$  admits a holomorphic connection.  $\square$

As we have seen,  $E$  is not flat by construction. On the other hand, consider the short exact sequence in (2.11). The trivial holomorphic line bundle  $F = \mathcal{O}_X$  admits the trivial holomorphic connection. The quotient bundle  $\mathcal{Q}$  is flat by Proposition 2.2. Therefore, we have the following:

**Theorem 2.3.** *Let  $X$  be a compact connected Riemann surface of genus  $g \geq 2$ . The vector bundle  $E$  in (2.7) has a holomorphic subbundle such that both the subbundle and the quotient bundle admit holomorphic connections. But  $E$  does not admit a holomorphic connection.*

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