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Sampled output observer design for a class of nonlinear systems

M. Farza¹, I. Bouraoui^{1,2}, T. Ménard¹, R. Ben Abdennour², M. M'Saad¹

Abstract—This paper presents a continuous-discrete time observer for a class of uncertain nonlinear systems with non uniformly sampled measurements. Two features of the proposed observer are worth to be pointed out. The first one consists in the simplicity of its calibration, while the second one lies in its comprehensive convergence analysis. More specifically, it is shown that the observation error lies in a ball centered at the origin and whose radius is proportional to the bounds of the uncertainties and the sampling partition diameter. In the free uncertainties case, the exponential convergence to zero of the observation error is established under a well-defined condition on the maximum value of the sampling partition diameter. The effectiveness of the proposed observer is emphasized throughout simulation results involving the estimation of the reaction rate in a typical bioreactor.

Key words: Nonlinear systems, high gain observers, impulsive systems, continuous-discrete time observers, bioreactors, reaction rates estimation.

I. INTRODUCTION

Although a considerable research activity has been devoted to the observer design for nonlinear systems over the last decades, the available contributions deal mainly with continuous time measurements [6], [10], [5], [2], [14], [11]. The problem of sampled measurements has been particularly addressed by redesigning the available state observers that have been proposed for the continuous time measurements case [3], [8], [1], [12]. In all these contributions, a dynamical system, which is similar to the underlying system, is used to provide a state prediction over the sampling intervals. This state prediction is updated by the sampled measurements. An output predictor approach has been proposed in [9]. It consists in a continuous time observer incorporating a suitable predictor of the output over the sampling intervals. The output prediction is provided by the solution of an ordinary differential initialized with the sampled measurements. The underlying observer is a hybrid system which is able to recover the continuous time observer properties for relatively fast sampling. An impulsive continuous-discrete observer has been proposed in [13] using an appropriate LMI approach.

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The contribution of this paper consists in designing a continuous-discrete time observer for a class of uncertain nonlinear systems with non uniformly sampled measurements from a redesigned version of the high gain observer proposed in [4] for continuous time measurements. Though, the proposed observer shares the impulsive nature of the observer given in [13] up to an adequate modification of the observer gain, it can be rewritten under the form of a hybrid system similar to the continuous-discrete time observer proposed in [9]. There are two main features of the proposed continuous-discrete time observer that are worth to be emphasized with respect to the available ones. The first one concerns the ease with which the observer gain is updated at sampling instants together with the simplicity of its implementation. The second feature is related to the convergence analysis simplicity and its ability to provide precise expressions of the upper bounds of the sampling partition diameter as well as the rate of the observation error convergence. More specifically, it is shown that the observation error lies in a ball centered at the origin with a radius proportional to the magnitude of the bounds of the uncertainties and the maximum sampling partition diameter. Moreover, it is shown that the ultimate bound of the observation error can be made arbitrarily small, as in the case where the output measurements are continuously available, when the maximum sampling partition diameter tends to zero. A particular emphasis is put on the fact that the observation error converges exponentially to the origin in the absence of uncertainties. Of particular interest, the expression of the underlying decay rate is given.

The paper is organized as follows. The observer design problem is stated in section 2 with a technical lemma that will be used to establish the main result of the paper. Section 3 is devoted to the main contribution of the paper. The fundamental result is first given with a comprehensive proof thanks to the technical lemma derived in section 2. The effectiveness of the proposed observer is highlighted via simulation results in section 4. Some concluding remarks are given in section 5.

Throughout the paper, I_p and 0_p will denote the p -dimensional identity and zero matrices respectively and $\|\cdot\|$ denotes the euclidian norm; λ_M (resp. λ_m) is the maximum (resp. minimum) eigenvalue of a symmetric positive definite (SPD) matrix P and $\sigma = \frac{\lambda_M}{\lambda_m}$ is the ratio of the maximum

and minimum values of P .

II. PROBLEM STATEMENT AND PRELIMINARIES

A. Presentation of the considered class of systems

Consider the class of multivariable nonlinear systems that are diffeomorphic to the following block triangular form with specific uncertainties:

$$\begin{cases} \dot{x}(t) = Ax(t) + \varphi(u(t), x(t)) + B\varepsilon(t) \\ y(t_k) = Cx(t_k) = x^1(t_k) \end{cases} \quad (1)$$

with

$$x = \begin{pmatrix} x^1 \\ \vdots \\ x^{q-1} \\ x^q \end{pmatrix}, \quad \varphi(u, x) = \begin{pmatrix} \varphi^1(u, x^1) \\ \varphi^2(u, x^1, x^2) \\ \vdots \\ \varphi^{q-1}(u, x^1, \dots, x^{q-1}) \\ \varphi^q(u, x) \end{pmatrix} \quad (2)$$

$$A = \begin{pmatrix} 0_p & I_p & & 0_p \\ \vdots & \ddots & \ddots & \\ \vdots & & \ddots & I_p \\ 0_p & 0_p & \dots & 0_p \end{pmatrix}, \quad B = \begin{pmatrix} 0_p \\ \vdots \\ 0_p \\ I_p \end{pmatrix}$$

$$C = (I_p \quad \dots \quad 0_p \quad 0_p)$$

where $x \in \mathbb{R}^n$ is the state and $x^i \in \mathbb{R}^p$ for $i \in [1, q]$ are the state blocks, $u(t) \in U$ a compact subset of \mathbb{R}^m denotes the system input and $y \in \mathbb{R}^p$ denotes the system output which is available only at the sampling instants satisfying $0 \leq t_0 < \dots < t_k < t_{k+1} < \dots$ with time-varying sampling intervals $\tau_k = t_{k+1} - t_k$ and $\lim_{k \rightarrow \infty} t_k = +\infty$ and $\varepsilon : \mathbb{R}^+ \mapsto \mathbb{R}^p$ is unknown function representing the system uncertainties and may depend on the state, the input and uncertain parameters.

As it is mentioned in the introduction, our main objective is to design a continuous-discrete time observer providing a continuous time estimation of the entire state of system (1) by using the output measurements that are available only at the sampling instants. The time intervals τ_k 's are naturally bounded away from zero by τ_m and are upperly bounded by the upper bound of the sampling partition diameter τ_M , i.e.

$$0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M, \quad \forall k \geq 0 \quad (3)$$

Such a design will be carried out under the following assumptions on the functions φ^i and the unknown function ε .

A1. The functions φ^i for $i \in [1, q]$ are globally Lipschitz with respect to x uniformly in u , i.e. there exists $L > 0$ such that for all $u \in U$ and for all $x, \bar{x} \in \mathbb{R}^n$, the following inequality holds for $i = 1, \dots, q$:

$$\|\varphi^i(u, x) - \varphi^i(u, \bar{x})\| \leq L\|x - \bar{x}\| \quad (4)$$

A2. The unknown function ε is bounded, i.e.

$$\exists \delta > 0; \forall t \geq 0 : \|\varepsilon(t)\| \leq \delta \quad (5)$$

There are two remarks that are worth to be pointed out on the considered class of systems. Firstly, it may seem very

restrictive since it assumes a non prime dimension, i.e. $n = pq$, and the state blocks x^k have the same dimension p . This is by no means the case since it is shown in [7] that there exists a class of uniformly observable nonlinear systems that can be put under the form (1) via an injective map. Secondly, the Lipschitz assumption A1 which seems very restrictive can be relaxed in the case where the system state trajectory lies in a bounded set Ω by appropriately extending the nonlinearities $\varphi(u, x)$ as in [15], [2].

B. A technical lemma

In what follows, a technical lemma together with its proof is given. This lemma will be used to establish the main result of the paper in the next section.

Lemma 2.1: Consider a differentiable function $v : t \in \mathbb{R}^+ \mapsto v(t) \in \mathbb{R}^+$ satisfying the following inequality

$$\dot{v}(t) \leq -av(t) + b \int_{t_k}^t v(s)ds + c \quad \forall t \in [t_k, t_{k+1}[\quad (6)$$

where $0 < \tau_m \leq \tau_k = t_{k+1} - t_k \leq \tau_M < +\infty$ and a, b and c are positive reals satisfying

$$\frac{b\tau_M}{a} < 1 \quad (7)$$

Then, the function v satisfies

$$v(t) \leq e^{-\eta(t-t_0)}v(t_0) + c\tau_M \left(1 + \frac{1}{1 - e^{-\eta\tau_m}}\right) \quad (8)$$

with

$$0 < \eta = (a - b\tau_M)e^{-a\tau_M} \quad (9)$$

Proof of lemma 2.1. We shall first prove that $v(t) \leq v(t_k) + c(t - t_k) \quad \forall t \in [t_k, t_{k+1}[$ and use this property to establish the key inequality (8). By integrating (6) from t_k to t , we obtain

$$\begin{aligned} v(t) &\leq v(t_k) - a \int_{t_k}^t v(s)ds + b \int_{t_k}^t \int_{t_k}^s v(v)dvds + c(t - t_k) \\ &\leq v(t_k) + (b\tau_M - a) \int_{t_k}^t v(s)ds + c(t - t_k) \end{aligned} \quad (10)$$

And by using the condition (7), i.e. $b\tau_M - a < 0$, the last inequality (10) becomes

$$v(t) \leq v(t_k) + c(t - t_k) \quad (11)$$

Now, we shall establish the inequality (8) using the property (11). Indeed, inequality (6) becomes

$$\begin{aligned} \dot{v}(t) &\leq -av(t) + b \int_{t_k}^t (v(t_k) + c(s - t_k)) ds + c \\ &\leq -av(t) + b(v(t_k) + c(t - t_k)) \int_{t_k}^t ds + c \\ &\leq -av(t) + b\tau_M (v(t_k) + c(t - t_k)) + c \end{aligned} \quad (12)$$

Solving the above inequality from t_k to t , we get

$$\begin{aligned} v(t) &\leq e^{-a(t-t_k)}v(t_k) + (b\tau_M v(t_k) + c) \int_{t_k}^t e^{-a(t-s)} ds \\ &\quad + b\tau_M c \int_{t_k}^t (s - t_k)e^{-a(t-s)} ds \\ &\leq e^{-a(t-t_k)}v(t_k) + \frac{b\tau_M v(t_k) + c}{a} (1 - e^{-a(t-t_k)}) \\ &\quad + ca \int_{t_k}^t (s - t_k)e^{-a(t-s)} ds \end{aligned} \quad (13)$$

The last inequality results from condition (7), i.e. $b\tau_M < a$. Now, using an integration by part, one can show that

$$a \int_{t_k}^t (s - t_k) e^{-a(t-s)} ds = (t - t_k) - \frac{1}{a} \left(1 - e^{-a(t-t_k)}\right) \quad (14)$$

And substituting (14) in (13), one gets

$$\begin{aligned} v(t) &\leq e^{-a(t-t_k)} v(t_k) + \frac{b\tau_M v(t_k) + c}{a} \left(1 - e^{-a(t-t_k)}\right) \\ &\quad + c \left((t - t_k) - \frac{1}{a} \left(1 - e^{-a(t-t_k)}\right) \right) \\ &= g(t) v(t_k) + c(t - t_k) \end{aligned} \quad (16)$$

where $g(t) \triangleq e^{-a(t-t_k)} \left(1 - \frac{b\tau_M}{a}\right) + \frac{b\tau_M}{a}$. Let us show that $g(t) \leq e^{-\eta(t-t_k)}$ where η is given as in (8). To this end, it suffices to show that $\delta(t) = (g(t) - e^{-\eta(t-t_k)}) \leq 0$ for all $t \geq t_k$. This can be done by simply showing that $\dot{\delta}(t) \leq 0$ since one can easily check that $\delta(t_k) = 0$. This is actually the case since

$$\begin{aligned} \dot{\delta}(t) &= \dot{g}(t) + \eta e^{-\eta(t-t_k)} \\ &= -a e^{-a(t-t_k)} \left(1 - \frac{b\tau_M}{a}\right) + \eta e^{-\eta(t-t_k)} \\ &\leq -e^{-a\tau_M} (a - b\tau_M) + \eta = 0 \end{aligned} \quad (17)$$

The last equality results from the expression of η given by (8). Hence, inequality (16) becomes

$$v(t) \leq e^{-\eta(t-t_k)} v(t_k) + c(t - t_k), \quad \forall t \in [t_k, t_{k+1}], \quad k \in \mathbb{N} \quad (18)$$

Iterating inequality (18), one can show that

$$\begin{aligned} v(t_k) &\leq e^{-\eta(\tau_{k-1} + \dots + \tau_0)} v(t_0) + c(\tau_{k-1} + \tau_{k-2} e^{-\eta\tau_{k-1}} \\ &\quad + \tau_{k-3} e^{-\eta(\tau_{k-1} + \tau_{k-2})} + \dots + \tau_0 e^{-\eta(\tau_{k-1} + \dots + \tau_1)}) \\ &\leq e^{-\eta(t_k - t_0)} v(t_0) + c(\tau_{k-1} + \tau_{k-2} e^{-\eta\tau_{k-1}} + \\ &\quad \tau_{k-3} e^{-\eta(\tau_{k-1} + \tau_{k-2})} + \dots + \tau_0 e^{-\eta(\tau_{k-1} + \dots + \tau_1)}) \\ &\leq e^{-\eta(t_k - t_0)} v(t_0) + c\tau_M \left(\sum_{j=0}^{k-1} e^{-\eta j \tau_m} \right) \\ &\leq e^{-\eta(t_k - t_0)} v(t_0) + c\tau_M \left(\frac{1 - e^{-\eta k \tau_m}}{1 - e^{-\eta \tau_m}} \right) \\ &\leq e^{-\eta(t_k - t_0)} v(t_0) + c\tau_M \left(\frac{1}{1 - e^{-\eta \tau_m}} \right) \end{aligned} \quad (19)$$

And using (19), inequality (18) becomes

$$\begin{aligned} v(t) &\leq e^{-\eta(t-t_0)} v(t_0) + c(t - t_k) + \frac{c\tau_M}{1 - e^{-\eta \tau_m}} \\ &\leq e^{-\eta(t-t_0)} v(t_0) + c\tau_M \left(1 + \frac{1}{1 - e^{-\eta \tau_m}}\right) \end{aligned}$$

This ends the proof of the lemma.

III. THE CONTINUOUS-DISCRETE TIME OBSERVER

Let us consider the following impulsive system

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t) + \varphi(u(t), \hat{x}(t)) - \theta \Delta_\theta^{-1} K e^{-\theta K^1(t-t_k)} \times \\ &\quad (C\hat{x}(t_k) - y(t_k)) \text{ for } t \in [t_k, t_{k+1}] \end{aligned} \quad (20)$$

where $\hat{x} = \begin{pmatrix} \hat{x}^1 \\ \vdots \\ \hat{x}^q \end{pmatrix}$ is the state estimate, $K = \begin{pmatrix} K^1 \\ \vdots \\ K^q \end{pmatrix}$ is a gain matrix with K^i being $p \times p$ square matrices that

are specified such that the matrix $\bar{A} = A - KC$ is Hurwitz and Δ_θ is the block-diagonal matrix defined by:

$$\Delta_\theta = \text{diag} \left(I_p, \frac{1}{\theta} I_p, \dots, \frac{1}{\theta^{q-1}} I_p \right) \quad (21)$$

with $\theta \geq 1$. In the following, we shall show that this impulsive system is a continuous-discrete time observer for system (1) and exhibit its output prediction form.

A. The convergence result

The above impulsive system is actually a continuous-discrete time observer for system (1) as pointed out by the following result.

Theorem 3.1: Consider the system (1)-(2) subject to Assumption A1 and A2. Then, for every $u \in U$, for every $\theta > \theta_0$, there exists $\chi_\theta > 0$, $\eta_\theta(\tau_M) > 0$, $N_\theta(\tau_m, \tau_M) > 0$ such that if the upper bound of the sampling partition diameter τ_M is chosen such that $\tau_M < \chi_\theta$, then for every $\hat{x}(0) \in \mathbb{R}^n$, we have:

$$\|\hat{x}(t) - x(t)\| \leq \lambda \theta^{q-1} e^{-\eta_\theta(\tau_M)t} \|\hat{x}(0) - x(0)\| + N_\theta(\tau_m, \tau_M) \delta$$

where $x(t)$ is the unknown trajectory of (1) associated to the input u , $\hat{x}(t)$ is any trajectory of system (20) associated to (u, y) , δ is the upper bound of $\|\varepsilon\|$ and the parameters $\lambda, \theta_0, \mu_\theta, M_\theta$ are given in the proof while τ_m and τ_M are defined in (3).

Proof of Theorem 3.1. Let $\tilde{x}(t) = \hat{x}(t) - x(t)$ be the observation error, it can easily be checked that

$$\dot{\tilde{x}}(t) = A\tilde{x} + \Phi(u, \hat{x}, x) - \theta \Delta_\theta^{-1} K e^{-\theta K^1(t-t_k)} C\tilde{x}(t_k) - B\varepsilon(t)$$

where $\Phi(u, \hat{x}, x) = \varphi(u, \hat{x}) - \varphi(u, x)$. Setting $\bar{x} = \Delta_\theta \tilde{x}$ and using the following identities

$$\Delta_\theta A \Delta_\theta^{-1} = \theta A \text{ and } C \Delta_\theta^{-1} = C \quad (22)$$

allow to get

$$\begin{aligned} \dot{\tilde{x}} &= \Delta_\theta A \Delta_\theta^{-1} \bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) \\ &\quad - \theta K e^{-\theta K^1(t-t_k)} C \Delta_\theta^{-1} \bar{x}(t_k) - \Delta_\theta B \varepsilon(t) \\ &= \theta A \bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) \\ &\quad - \theta K e^{-\theta K^1(t-t_k)} C \bar{x}(t_k) - \frac{1}{\theta^{q-1}} B \varepsilon(t) \end{aligned} \quad (23)$$

Moreover, adding and subtracting the term $\theta K C \bar{x}$ in the last equation yields

$$\begin{aligned} \dot{\tilde{x}} &= \theta (A - KC) \bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) \\ &\quad + \theta K \left(C \bar{x} - e^{-\theta K^1(t-t_k)} C \bar{x}(t_k) \right) - \frac{1}{\theta^{q-1}} B \varepsilon(t) \\ &= \theta \bar{A} \bar{x} + \Delta_\theta \Phi(u, \hat{x}, x) + \theta K z - \frac{1}{\theta^{q-1}} B \varepsilon(t) \end{aligned}$$

where $z(t) = C \bar{x}(t) - e^{-\theta K^1(t-t_k)} C \bar{x}(t_k) = \bar{x}^1(t) - e^{-\theta K^1(t-t_k)} \bar{x}^1(t_k)$. Notice that $z(t_k) = 0$ and according to (23), the time derivative of z can be written as follows

$$\begin{aligned} \dot{z}(t) &= \dot{\bar{x}}^1(t) + K^1 \theta e^{-\theta K^1(t-t_k)} \bar{x}^1(t_k) \\ &= \theta \bar{x}^2 + \Phi^1(u, \hat{x}^1, x^1) \end{aligned} \quad (24)$$

where $\Phi^1(u, \hat{x}^1, x^1) = \varphi^1(u, \hat{x}^1) - \varphi^1(u, x^1)$.

Now, let us consider the following candidate quadratic Lyapunov function: $V(\bar{x}) = \bar{x}^T P \bar{x}$ where P is defined by

$$P\bar{A} + \bar{A}^T P \leq -2\mu I_n \quad (25)$$

Proceeding as in the continuous time case, one can show that

$$\begin{aligned} \dot{V}(\bar{x}) &= 2\bar{x}^T(t)P(\theta\bar{A}\bar{x} + \Delta_\theta\Phi(u, \hat{x}, x) + \theta Kz) \\ &\quad - \frac{2}{\theta^{q-1}}\bar{x}^T(t)PB\varepsilon(t) \\ &\leq -(2\mu\theta - 2L\sqrt{n}\lambda_M)\|\bar{x}\|^2 + 2\theta\bar{x}^T PKz \\ &\quad - \frac{2}{\theta^{q-1}}\bar{x}^T(t)PB\varepsilon(t) \end{aligned} \quad (26)$$

Choosing θ such that $2(\mu\theta - \sqrt{n}\lambda_M L) > \mu\theta$ i.e.

$$\theta > \theta'_0 \triangleq \frac{2L\sqrt{n}\lambda_M}{\mu} \quad (27)$$

the last inequality (26) becomes

$$\begin{aligned} \dot{V}(\bar{x}) &\leq -\frac{\mu\theta}{\lambda_M}V(\bar{x}) + 2\theta\|P\bar{x}\|\|K\|\|z(t)\| \\ &\quad + \frac{2}{\theta^{q-1}}\|P\bar{x}\|\|\varepsilon(t)\| \\ &\leq -\frac{\mu\theta}{\lambda_M}V(\bar{x}) + 2\theta\sqrt{\lambda_M}\sqrt{V(\bar{x})}\|K\|\|z(t)\| \\ &\quad + \frac{2\sqrt{\lambda_M}}{\theta^{q-1}}\delta\sqrt{V(\bar{x})} \end{aligned}$$

Furthermore, integrating equation (24) from t_k to t while using the fact that $z(t_k) = 0$ yields

$$z(t) = \int_{t_k}^t (\theta\bar{x}^2(s) + \Phi^1(u(s), \hat{x}^1(s), x^1(s))) ds$$

Bearing in mind that $\Phi^1(u, \hat{x}^1, x^1) = \varphi^1(u, \hat{x}^1) - \varphi^1(u, x^1)$, one gets

$$\begin{aligned} \|z(t)\| &\leq \int_{t_k}^t (\theta\|\bar{x}^2(s)\| + L\|\bar{x}^1(s)\|) ds \\ &\leq (\theta + L) \int_{t_k}^t \|\bar{x}(s)\| ds \leq \frac{\theta + L}{\sqrt{\lambda_m}} \int_{t_k}^t \sqrt{V(\bar{x}(s))} ds \end{aligned}$$

Combining the above inequalities, one obtains

$$\begin{aligned} \dot{V}(\bar{x}(t)) &\leq -\frac{\mu\theta}{\lambda_M}V(\bar{x}(t)) + \\ &\quad 2\theta\sqrt{\frac{\lambda_M}{\lambda_m}}\|K\|(\theta + L)\sqrt{V(\bar{x}(t))} \int_{t_k}^t \sqrt{V(\bar{x}(s))} ds \\ &\quad + \frac{2\sqrt{\lambda_M}}{\theta^{q-1}}\delta\sqrt{V(\bar{x})} \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{d}{dt}\sqrt{V(\bar{x}(t))} &\leq -\frac{\mu\theta}{2\lambda_M}\sqrt{V(\bar{x}(t))} + \\ &\quad + \theta\sqrt{\sigma}\|K\|(L + \theta) \int_{t_k}^t \sqrt{V(\bar{x}(s))} ds \\ &\quad + \frac{\sqrt{\lambda_M}}{\theta^{q-1}}\delta \end{aligned} \quad (28)$$

where $\sigma = \frac{\lambda_M}{\lambda_m}$. Finally, let

$$a_\theta = \frac{\mu\theta}{2\lambda_M}, \quad b_\theta = \theta\sqrt{\sigma}\|K\|(L + \theta) \quad \text{and} \quad c_\theta = \frac{\sqrt{\lambda_M}}{\theta^{q-1}}\delta \quad (29)$$

and assume that the upper diameter of the sampling partition τ_M satisfies the following condition $\tau_M < \frac{a_\theta}{b_\theta}$, i.e.

$$\frac{b_\theta\tau_M}{a_\theta} < 1 \quad \text{or more explicitly} \quad 2\tau_M(L + \theta) \frac{\|K\|\lambda_M\sqrt{\sigma}}{\mu} < 1 \quad (30)$$

Then, according to Lemma 1, one has

$$\sqrt{V(\bar{x}(t))} \leq e^{-\eta_\theta t} \sqrt{V(\bar{x}(0))} + c_\theta\tau_M \left(1 + \frac{1}{1 - e^{-\eta_\theta\tau_m}}\right) \quad (31)$$

where

$$0 < \eta_\theta(\tau_M) = (a_\theta - b_\theta\tau_M) e^{-a_\theta\tau_M} \quad (32)$$

Note that the constant χ_θ involved in the theorem is $\chi_\theta = \frac{a_\theta}{b_\theta} = \frac{\mu\theta}{2(L + \theta)\|K\|\lambda_M\sqrt{\sigma}}$. Now, coming back to the original coordinates of the observation error \tilde{x} and proceeding as in the continuous time measurements case, we get

$$\begin{aligned} \|\tilde{x}(t)\| &\leq \sqrt{\sigma}\theta^{q-1}e^{-\eta_\theta t}\|\bar{x}(0)\| \\ &\quad + \frac{\theta^{q-1}}{\sqrt{\lambda_m}}c_\theta\tau_M \left(1 + \frac{1}{1 - e^{-\eta_\theta(\tau_M)\tau_m}}\right) \end{aligned} \quad (33)$$

Substituting in the above inequality c_θ by its expression (29) leads to

$$\|\tilde{x}(t)\| \leq \sqrt{\sigma}\theta^{q-1}e^{-\eta_\theta(\tau_M)t}\|\bar{x}(0)\| \quad (34)$$

$$+ \sqrt{\sigma}\tau_M \left(1 + \frac{1}{1 - e^{-\eta_\theta(\tau_M)\tau_m}}\right) \delta \quad (35)$$

The functions $\eta_\theta(\tau_M)$ and $N_\theta(\tau_m, \tau_M)$ involved in theorem 3.1 are hence given by

$$\eta_\theta(\tau_M) = (a_\theta - b_\theta\tau_M) e^{-a_\theta\tau_M}$$

$$\text{and } N_\theta(\tau_m, \tau_M) = \sqrt{\sigma}\tau_M \left(1 + \frac{1}{1 - e^{-\eta_\theta(\tau_M)\tau_m}}\right) \quad (36)$$

where a_θ and b_θ are given by (29). This completes the proof of theorem 3.1.

Remark 3.1: In order to emphasize the relationship between the rate of the exponential decreasing to zero and the ultimate bound of the observation error given by the continuous case (see [4]) and the theorem 3.1 for the continuous-discrete time case, we shall consider the case where the sampling period is constant and investigate the behaviour of the exponential decay rate and the ultimate bound. Indeed, one has

$$\begin{aligned} \eta_\theta(T_s) &\triangleq (a_\theta - b_\theta T_s) e^{-a_\theta T_s} \\ N_\theta(T_s) &\triangleq T_s \sqrt{\sigma} \left(1 + \frac{1}{1 - e^{-\eta_\theta(T_s)T_s}}\right) \end{aligned}$$

It is easy to check that for all $T_s \geq 0$ satisfying (30), the function $\eta_\theta(T_s)$ is a decreasing function of T_s and $N_\theta(T_s)$ is an increasing function of T_s . Now, let us take the limits of these functions when T_s tends to zero. Indeed, one has:

$$\begin{aligned} \lim_{T_s \rightarrow 0} \eta_\theta(T_s) &= a_\theta = \frac{\mu\theta}{2\lambda_M} = \mu\theta \\ \lim_{T_s \rightarrow 0} N_\theta(T_s) &= \lim_{T_s \rightarrow 0} T_s \sqrt{\sigma} \left(1 + \frac{1}{1 - e^{-\eta_\theta(T_s)T_s}}\right) \\ &= \lim_{T_s \rightarrow 0} T_s \sqrt{\sigma} \left(1 + \frac{1}{T_s \eta_\theta(T_s)}\right) = \frac{\sqrt{\sigma}}{\mu\theta} = M_\theta \end{aligned}$$

where μ_θ and M_θ are respectively the exponential decay rate and the function intervening in the ultimate bound expression obtained in the continuous output case. So, the results obtained in the sampled output case are quite coherent with those derived in the continuous output case: the decreasing to zero of the observation error is inversely proportional to the magnitude of the sampling period while the value of the ultimate bound is proportional to this magnitude. Moreover, when the sampling period tends to zero, we meet the same expressions for the decay rate and ultimate bound as those derived in the continuous output case.

IV. EXAMPLE

We consider a simple microbial culture which involves a single biomass X growing on a single substrate S . The bioprocess is operating in a continuous mode with a time-varying dilution rate $D(t)$ and a constant input substrate concentration S_{in} . The underlying mathematical dynamical model of the process is

$$\begin{cases} \dot{X} &= r - DX \\ \dot{S} &= -kr + D(S_{in} - S) \end{cases} \quad (37)$$

where X and S are the respective concentrations of the considered species, r is the reaction rate and k is a yield coefficient. The reaction rate r is generally a very complex function of the operating conditions and the state of the process. The analytical modelling of this function is often cumbersome and still constitutes a challenging problem. To tackle this problem, this function is treated as a time-varying parameter that needs to be estimated from the biomass concentration. Such an estimation shall be provided by an impulsive continuous-discrete time observer of the form (20) designed by considering the following rate estimation model

$$\begin{cases} \dot{X}(t) &= r(t) - DX \\ \hat{r}(t) &= \varepsilon(t) \\ y(kT_s) &= X(kT_s) \quad k \in \mathbb{N} \end{cases} \quad (38)$$

The equations of the impulsive continuous-discrete time observer specialize as follows:

$$\begin{cases} \dot{\hat{X}}(t) &= \hat{r}(t) - D(t)\hat{X}(t) \\ &\quad - 2\theta e^{-2(t-kT_s)}(\hat{X}(kT_s) - X(kT_s)) \\ \hat{r}(t) &= -\theta^2 e^{-2(t-kT_s)}(\hat{X}(kT_s) - X(kT_s)) \end{cases}$$

The gain K of the observer (20) is chosen as $K = [2 \ 1]^T$ in order to assign all the modes of the matrix $A - KC$ at (-1) .

The simulation experiments have been performed using the following model for the reaction rate

$$r = \frac{\mu^* SX}{k_C X + S} \quad (39)$$

where μ^* and K_C are constant kinetic parameters. The values of the kinetic parameters and the initial conditions have been respectively given by

$$\mu^* = 1/6 \text{ h}^{-1}, \quad k = 1 \text{ g.g}^{-1}, \quad K_C = 1 \text{ g.g}^{-1}, \quad S_{in} = 1.3158 \text{ g.l}^{-1}$$

and

$$X(0) = \hat{X}(0) = 1 \text{ g.l}^{-1}, \quad S(0) = 0.3158 \text{ g.l}^{-1}, \quad \hat{r}(0) = 0 \text{ g.(l.h)}^{-1}$$

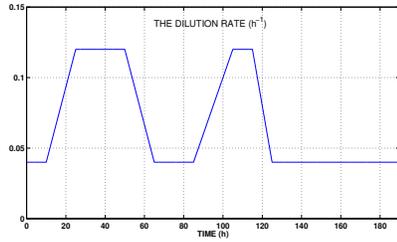
The dilution rate that has been considered is shown in figure 1a, i.e. a trapezoidal signal varying from 0.04 to 0.12 h^{-1} . In order to simulate practical situations, all the simulation experiments have been carried out with noisy data measurements: the measurements of X issued from the simulation of system (37)-(39) are corrupted by an additive gaussian noise with zero mean value and a standard deviation equal to 0.0316.

The simulation study has been performed as follows. First, several simulation experiments have been carried out assuming continuous time output measurements. This allows to obtain a lower bound on the design parameter θ for which the underlying continuous time observer performs well i.e. a good compromise between an admissible tracking of the kinetic parameter variation and a good behavior of the observer with respect to noise measurements. The value of θ ensuring such a compromise was about 1.5. Then, an other set of simulation experiments has been performed with this fixed design parameter value ($\theta = 1.5$) and different values for the sampling period. From an experiment to the other, the value of the sampling period was constant, $\tau_M = T_s$, and has been increased until reaching the maximum allowable value τ_M for which the performance of the continuous-discrete time observer still be quite satisfactory. The obtained value was about $\tau_M = 1.1 \text{ h}$. We have reproduced in figure 1e three sets of results corresponding to the estimation of the reaction rate r obtained with the same value of the design parameter $\theta = 1.5$. The first set provides the estimate of r in the case of continuous time measurements and it is obtained by simulating an observer given in [4]. This estimate can serve as a reference (the ideal case) for the estimates obtained with the continuous-discrete time observer (39). The second and third sets are obtained with $T_s = 0.05 \text{ h}$ and $T_s = 1 \text{ h}$, respectively. As expected from the theoretical analysis, the best estimates are obtained with smaller values of the sampling period. Moreover, for relatively low values of T_s (e.g. $T_s = 0.05 \text{ h}$), the obtained estimates are quite similar to those obtained with continuous measurements.

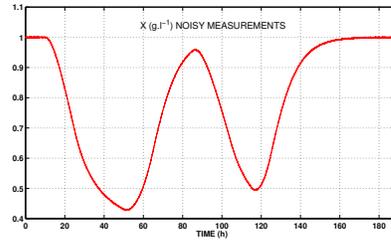
It is worth noticing that the expressions of the reactions rates, introduced for simulation purposes and in order to obtain the pseudo-measurements of X and P , are not known by observer (39).

V. CONCLUSION

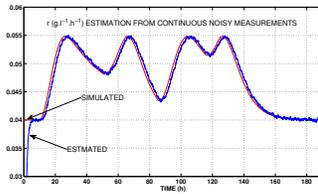
A high gain observer has been derived for a class of nonlinear systems with sampled output measurements involving specific uncertainties. This observer provides continuous time estimates of the whole system state from the available output samples. The underlying observation error convergence has been established under a well defined condition on the observer, the system parameter and the maximum allowable value of the sampling partition diameter. More specifically, it



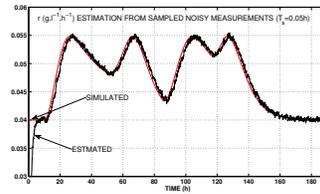
(a) Time evolution of the dilution rate



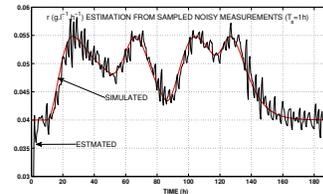
(b) Continuous noisy measurements of X



(c) Estimation of r (i)



(d) Estimation of r (ii)



(e) Estimation of r (iii)

has been shown that the observation error remains confined in a ball centered at the origin with a radius proportional to the magnitudes of the uncertainties and the maximum of the sampling partition diameter. A particular emphasis has been put on the observation error exponential convergence to zero in the uncertainties free case. Of practical interest, it has been shown that the proposed observer can be used to perform an on-line relatively accurate estimation of the reaction rate in biochemical reactors.

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