



**HAL**  
open science

# Lipschitz regularity for local minimizers of some widely degenerate problems

Pierre Bousquet, Lorenzo Brasco, Vesa Julin

► **To cite this version:**

Pierre Bousquet, Lorenzo Brasco, Vesa Julin. Lipschitz regularity for local minimizers of some widely degenerate problems. *Annali della Scuola Normale Superiore di Pisa, Classe di Scienze*, 2016, 16, <10.2422/2036-2145.201410\_006>. <hal-01062508>

**HAL Id: hal-01062508**

**<https://hal.science/hal-01062508v1>**

Submitted on 9 Sep 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

# LIPSCHITZ REGULARITY FOR LOCAL MINIMIZERS OF SOME WIDELY DEGENERATE PROBLEMS

PIERRE BOUSQUET, LORENZO BRASCO, AND VESA JULIN

ABSTRACT. We consider local minimizers of the functional

$$\sum_{i=1}^N \int (|u_{x_i}| - \delta_i)_+^p dx + \int f u dx,$$

where  $\delta_1, \dots, \delta_N \geq 0$  and  $(\cdot)_+$  stands for the positive part. Under suitable assumptions on  $f$ , we prove that local minimizers are Lipschitz continuous functions if  $N = 2$  and  $p \geq 2$ , or if  $N \geq 2$  and  $p \geq 4$ .

## CONTENTS

1. Introduction	2
1.1. Overview	2
1.2. Main results	4
1.3. Plan of the paper	5
2. Preliminaries	5
2.1. Definitions and basic results	5
2.2. Approximation scheme	7
3. Local energy estimates for the approximating problem	9
3.1. Caccioppoli-type inequalities	10
3.2. A Sobolev estimate	12
3.3. Power-type subsolutions	14
4. Proof of Theorem A	15
5. Proof of Theorem B	22
Appendix A. Some properties of the functions $g_i$	27
Appendix B. An anisotropic Sobolev inequality in dimension 2	27
References	28

---

*Date:* September 9, 2014.

*2010 Mathematics Subject Classification.* 35J70, 35B65, 49K20.

*Key words and phrases.* Degenerate elliptic equations; Anisotropic problems; Lipschitz regularity.

## 1. INTRODUCTION

1.1. **Overview.** This paper is devoted to prove Lipschitz continuity for local minimizers of the anisotropic functional

$$(1.1) \quad \mathfrak{F}(u; \Omega') = \sum_{i=1}^N \int_{\Omega'} \frac{(|u_{x_i}| - \delta_i)_+^p}{p} dx + \int_{\Omega'} f u dx, \quad u \in W_{loc}^{1,p}(\Omega), \quad \Omega' \Subset \Omega.$$

Here  $\Omega \subset \mathbb{R}^N$  is an open set,  $2 \leq p < \infty$ ,  $\delta_i \geq 0$ ,  $(\cdot)_+$  stands for the positive part and  $f \in L_{loc}^{p'}(\Omega)$  where  $p' = p/(p-1)$ . This functional  $\mathfrak{F}$  stands for a model case of a more general class of problems, with specific growth and monotonicity assumptions. For the sake of clarity, the results in this paper are only stated for  $\mathfrak{F}$ . However, their proofs can be easily adapted to embrace general functionals having a similar structure.

The functional  $\mathfrak{F}$  naturally arises in problems of Optimal Transport with congestion and anisotropic effects, see for example [8, 9] for some motivations. These two papers contained among others some regularity results for local minimizers of (1.1). For instance [9, Main Theorem] proves that if  $f \in L_{loc}^\infty(\Omega)$ , then  $u$  is “almost Lipschitz”, i.e.  $u \in W_{loc}^{1,r}(\Omega)$  for every  $r \geq 1$ . On the other hand, in [8] it is proved that if  $f \in W_{loc}^{1,p'}(\Omega)$ , then

$$(1.2) \quad (|u_{x_i}| - \delta_i)_+^{\frac{p}{2}} \frac{u_{x_i}}{|u_{x_i}|} \in W_{loc}^{1,2}(\Omega), \quad i = 1, \dots, N.$$

However, it must be mentioned that to the best of our knowledge, *Lipschitz regularity of local minimizers is still unknown*. More surprisingly, even the case  $\delta_1 = \dots = \delta_N = 0$  does not seem to be fully understood. Only very recently some results have been obtained in this case, see [4, 14].

Observe that local minimizers of (1.1) are local weak solutions of the anisotropic degenerate equation

$$(1.3) \quad \sum_{i=1}^N \left( (|u_{x_i}| - \delta_i)_+^{p-1} \frac{u_{x_i}}{|u_{x_i}|} \right)_{x_i} = f,$$

which reduces to the Poisson equation for the so-called *pseudo  $p$ -Laplacian* when  $\delta_1 = \dots = \delta_N = 0$ , i.e.

$$(1.4) \quad \sum_{i=1}^N (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = f.$$

The terminology “pseudo  $p$ -Laplacian” appears in [1]. We just point out that such an operator already appeared in J.-L. Lions’s monograph [20], where existence issues for solutions to evolutions equations are tackled.

In order to neatly explain the difficulty of the problem, we now recall some class of functionals for which the Lipschitz property for local minimizers is known to be true. The first one is given by

$$(1.5) \quad \int G(\nabla u) dx,$$

with  $G$  enjoying a  *$p$ -Laplacian-type structure at infinity*. This means that there exist  $c, C > 0$  and  $m \geq 0$  such that  $G$  verifies the ellipticity condition

$$(1.6) \quad \langle D^2 G(z) \xi, \xi \rangle \geq c |z|^{p-2} |\xi|^2, \quad |z| \geq m,$$

and the growth condition

$$(1.7) \quad |DG(z)| \leq C |z|^{p-1}, \quad |z| \geq m.$$

We refer the reader to [7, 10, 11, 15] and [16] for example. For completeness, we also point out the papers [12, 13] and [22] for related regularity results on the term  $\nabla G(\nabla u)$ , when  $m > 0$ .

Another type of well-studied functionals having some similarities with  $\mathfrak{F}$  is given by (see for example [2, 3] and [17, Section 4])

$$(1.8) \quad \int \tilde{G}(\nabla u) dx, \quad \text{with} \quad \tilde{G}(z) = \sum_{i=1}^N (\mu + |z_i|^2)^{\frac{p_i}{2}}.$$

Here  $\mu > 0$  and  $1 < p_1 \leq p_2 \leq \dots \leq p_N$  are possibly different exponents. When the  $p_i$  are not equal, such a functional belongs to the class of *problems with non standard growth conditions*, whose systematic study started with the paper [21] by Marcellini. In this case we can infer local Lipschitz continuity if the exponents  $p_i$  are not “too far apart” (see the above mentioned references for more details).

However, our functional  $\mathfrak{F}$  *does not fall neither in the class of the functional (1.5) nor in that of (1.8)*. Indeed, observe that in our case

$$F(z) = \sum_{i=1}^N \frac{(|z_i| - \delta_i)_+^p}{p},$$

verifies (1.7), *but (1.6) crucially fails to hold*, since for every  $m > 0$ , there always exists  $z$  such that  $|z| = m$  and the least eigenvalue of  $D^2F(z)$  is 0. Observe that this phenomenon already occurs for the pseudo  $p$ -Laplacian, i.e. when  $\delta_1 = \dots = \delta_N = 0$ . Indeed, the main difficulty of the problem is that the region where ellipticity fails *is unbounded*.

For the same reason,  $\mathfrak{F}$  is not of the type (1.8), since already in the standard growth case  $2 \leq p_1 = p_2 = \dots = p_N$  we have

$$0 < \min_{|\xi|=1} \langle D^2\tilde{G}(z) \xi, \xi \rangle, \quad z \in \mathbb{R}^N.$$

When one allows  $\mu = 0$  in (1.8), the corresponding functional becomes degenerate along the axes  $z_i = 0$ , like in the case of the pseudo  $p$ -Laplacian. This case has been considered in the pioneering paper [25] by Uralt'seva and Urdaletova. There the Lipschitz character of minimizers has been shown under some restrictions on the exponents  $p_1, \dots, p_N$ , by using the so-called *Bernstein method*. Though the growth conditions considered are more general than ours, the type of degeneracy is again weaker than that admitted in  $\mathfrak{F}$  (see the next subsection for more comments on the result of [25]).

About the restriction  $p \geq 2$  considered in this paper, it is noteworthy to observe that for  $1 < p < 2$  our functional has a  $p$ -Laplacian-type structure. Indeed, in this case  $p - 2 < 0$  and thus (1.6) is satisfied with  $m = 0$ , i.e.

$$\langle D^2F(z) \xi, \xi \rangle = (p-1) \sum_{i=1}^N (|z_i| - \delta_i)_+^{p-2} |\xi_i|^2 \geq (p-1) |z|^{p-2} |\xi|^2, \quad \xi \in \mathbb{R}^N, z \in \mathbb{R}^N,$$

while of course

$$|DF(z)| \leq |z|^{p-1}, \quad z \in \mathbb{R}^N.$$

Then in this case local minimizers are locally Lipschitz continuous by<sup>1</sup> [16, Theorem 2.7].

**1.2. Main results.** In this paper, we prove the following results.

**Theorem A** (Two dimensional case). *Let  $N = 2$  and  $p \geq 2$ . Let  $f \in W_{loc}^{1,p'}(\Omega)$ , where  $p' = p/(p-1)$ . Then every local minimizer  $U \in W_{loc}^{1,p}(\Omega)$  of the functional  $\mathfrak{F}$  is a locally Lipschitz continuous function.*

**Theorem B** (Higher dimensional case). *Let  $N \geq 2$  and  $p \geq 4$ . Let  $f \in W_{loc}^{1,\infty}(\Omega)$ . Then every local minimizer  $U \in W_{loc}^{1,p}(\Omega)$  of the functional  $\mathfrak{F}$  is a locally Lipschitz continuous function.*

Let us now spend some words about the methods of proofs. The preliminary step in both cases is an approximation argument. Namely, the functional  $\mathfrak{F}$  is replaced by a regularized version  $\mathfrak{F}_\varepsilon$ , for a small parameter  $\varepsilon > 0$ . This permits to infer the necessary regularity on the solutions  $u_\varepsilon$  of the regularized problem, in order to justify the manipulations needed to obtain a priori Lipschitz estimates uniform in  $\varepsilon$ . Then one aims at taking these estimates to the limit as  $\varepsilon$  goes to 0. However, one should pay attention to the fact that  $\mathfrak{F}$  is not strictly convex when at least one  $\delta_i \neq 0$ . Thus a sequence of solutions  $u_\varepsilon$  may not necessarily converge to the selected local minimizer. In [9] a penalization argument was used to fix this issue. Here on the contrary, we use a simpler argument, based on the fact that the lack of strict convexity of  $t \mapsto (|t| - \delta_i)_+^p$  is “confined” (see Lemma 2.3).

The core of the proof of Theorem A is the a priori Lipschitz estimate of Proposition 4.1. Such an estimate is achieved by means of a Moser’s iteration technique applied to the equation solved by the partial derivatives  $u_{x_j}$  of the local minimizer. More precisely, we look at power-type subsolutions of this equation, i.e. quantities like  $|u_{x_j}|^s$  for  $s \geq 1$ . This is a standard strategy for equations having a  $p$ -Laplacian-type structure, but as already said our operator does not have such a structure and this entails several additional difficulties.

As explained in the introduction of [9], the main difficulty of this method is that the Caccioppoli inequality we get for  $|u_{x_j}|^s$  is quite involved. Indeed, due to the particular structure of  $D^2F$ , in principle we have a control only on a “weighted” norm of  $\nabla|u_{x_j}|^s$ , the weights being dependent on *all the other components*  $u_{x_i}$  of the gradient (see Lemma 3.6 below). Roughly speaking, what we control in the Caccioppoli inequality is a quantity like

$$\sum_{i=1}^N \int |u_{x_i}|^{p-2} \left| (|u_{x_j}|^{s+1})_{x_i} \right|^2.$$

For the diagonal term, i.e. when  $i = j$ , we can combine the  $x_j$ -derivative of  $u_{x_j}$  with the weight  $|u_{x_j}|^{p-2}$  and simply recognize the  $x_j$ -derivative of yet another power of  $u_{x_j}$ . Since we would like to have a control on the full gradient of such a power of  $u_{x_j}$ , we still miss all the  $x_i$ -derivatives ( $i \neq j$ ) of this function. To overcome this difficulty, we use in a crucial way the Sobolev property (1.2) together with Hölder inequality, in order to “cook-up” suitable Caccioppoli inequalities for all these *missing terms*. Surprisingly enough, even if the functional  $\mathfrak{F}$  has  $p$ -growth in every direction, we rely on the *anisotropic Sobolev inequality* due to Troisi (see [24]) in order to produce an iterative scheme of reverse Hölder inequalities. This procedure works for  $N = 2$ , but it seems to be limited just to the two dimensional case (see Remark 4.2 below).

<sup>1</sup>To be more precise, for  $1 < p < 2$  the function  $F$  is not  $C^2$ . However, this is not an issue, since the result of [16, Theorem 2.7] holds for convex functions satisfying a qualified form of uniform convexity for  $|z| \geq m$ . This coincides with (1.6) if the function is  $C^2$ , but it is otherwise more general.

In contrast Theorem B is valid in every dimension, but we need the restriction  $p \geq 4$ . This second result partially superposes with the already mentioned [25, Theorem 1] by Uralt'seva and Urdaletova. However, it should be noticed that the monotonicity assumptions on the operator<sup>2</sup> made in [25] does not allow for  $\delta_i > 0$ . Moreover, the result in [25] is stated for  $p > 3$ , but a careful inspection of the proof reveals that the same condition  $p \geq 4$  is needed there as well<sup>3</sup>.

Both the proofs of Theorem B and that of [25, Theorem 1] are based on a priori Lipschitz bounds, obtained by means of pointwise estimates in the vein of Bernstein method. However, computations are not the same and we believe ours to be slightly simpler. In [25] the first step is to look at the equation solved by a *concave* power of  $u$ , given by the function

$$w = (u + \|u\|_{L^\infty} + 1)^\gamma, \quad 0 < \gamma < 1.$$

Then they consider the equation solved by (some function of)  $\nabla w$ . There is an extra term in this new equation coming from the concave power which crucially leads to the result.

Here on the contrary we obtain the Lipschitz estimate by directly attacking equation (1.3). The main point is to consider the equation satisfied by the quantity

$$|\nabla u|^2 + \lambda u^2,$$

for a suitably large parameter  $\lambda$ . We notice that this is exactly the same test function used to prove classical gradient estimates for *linear* uniformly elliptic equations (see for example [19, Proposition 2.19]).

One of the drawbacks of these two strategies is the assumption on  $f$ , which does not seem to be optimal. Indeed, we expect the result to be true under the natural hypothesis  $f \in L^q_{loc}(\Omega)$  with  $q > N$ .

**1.3. Plan of the paper.** In Section 2 we set notations and preliminary results needed throughout the whole paper. In particular, we introduce there a regularized version of the problem which will be useful in order to get the desired Lipschitz estimate. Then Section 3 is devoted to prove some Caccioppoli-type inequalities for the gradient of the solution of the regularized problem. The proof of Theorem A is contained in 4, while Section 5 contains the proof of Theorem B. Two appendices containing some technical results complement the paper.

**Acknowledgements.** A quick but stimulating discussion with Nina Uralt'seva in June 2012 led to a better understanding of the paper [25], we thank her. Guillaume Carlier is warmly thanked for his interest in this work. Part of this paper has been written during the conferences “*Journées d'Analyse Appliquée Nice-Toulon-Marseille*” held in Porquerolles in May 2014, “*Nonlinear partial differential equations and stochastic methods*” held in Jyväskylä in June 2014 and “*Existence and Regularity for Nonlinear Systems of Partial Differential Equations*” held in Pisa in July 2014. Organizers and hosting institutions are gratefully acknowledged.

## 2. PRELIMINARIES

**2.1. Definitions and basic results.** Let  $\Omega \subset \mathbb{R}^N$  be an open set and  $p \geq 2$ . In what follows we set for simplicity

$$g_i(t) = \frac{1}{p} (|t| - \delta_i)_+^p, \quad t \in \mathbb{R}, \quad i = 1, \dots, N,$$

<sup>2</sup>See equation (8) of the paper [25].

<sup>3</sup>This comes from hypothesis (5) in [25]. Also observe that this condition contains a small typo,  $m_{i-2}$  should be replaced by  $m_i - 2$ .

where  $0 \leq \delta_1, \dots, \delta_N$  are given real numbers. We will also define

$$(2.1) \quad \delta = 1 + \max\{\delta_i : i = 1, \dots, N\}.$$

**Remark 2.1** (Smoothness of  $g_i$ ). When  $p$  is an integer and  $\delta_i > 0$ ,  $g_i$  is of class  $C^{p-1,1}$ . When  $p \notin \mathbb{N}$ , then  $g_i \in C^{[p], p-[p]}(\mathbb{R})$  where  $[\cdot]$  denotes the integer part.

**Remark 2.2** (The limit case  $p = 2$ ). Observe that for  $p = 2$  and  $\delta_i > 0$ , we have  $g_i \in C^{1,1}(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{1, -1\})$ , but  $g_i \notin C^2(\mathbb{R})$ . In this case, like in [9] a smoothing around  $|t| = \delta_i$  would be necessary, notably for the result of Lemma 2.8 below. However, in order not to overburden the presentation, for the sequel we will assume for simplicity  $p > 2$  (see [9, Section 2] for more details).

We are interested in local minimizers of the following variational integral

$$(2.2) \quad \mathfrak{F}(u; \Omega') = \sum_{i=1}^N \int_{\Omega'} g_i(u_{x_i}) dx + \int_{\Omega'} f u dx, \quad u \in W_{loc}^{1,p}(\Omega),$$

where  $f \in L_{loc}^{p'}(\Omega)$  and  $\Omega' \Subset \Omega$ . We recall that  $u \in W_{loc}^{1,p}(\Omega)$  is said to be a *local minimizer* of  $\mathfrak{F}$  if for every  $\Omega' \Subset \Omega$  we have

$$\mathfrak{F}(u; \Omega') \leq \mathfrak{F}(u + \varphi; \Omega'), \quad \text{for every } \varphi \in W_0^{1,p}(\Omega').$$

We first observe that  $\mathfrak{F}$  is not strictly convex, unless  $\delta = 1$ . Thus minimizers are not unique in general. The following result guarantees that it will be sufficient to prove the desired result for one minimizer.

**Lemma 2.3** (Propagation of regularity). *Let  $B \Subset \Omega$  be a ball and  $\varphi \in W^{1,p}(B)$ . Let  $u_1, u_2 \in W^{1,p}(\Omega)$  be two solutions of*

$$(2.3) \quad \min \left\{ \mathfrak{F}(v; B) : v - \varphi \in W_0^{1,p}(B) \right\}.$$

*Then it holds*

$$(2.4) \quad \left| |(u_1)_{x_i}| - |(u_2)_{x_i}| \right| \leq 2\delta_i, \quad \text{a. e. in } B, \quad i = 1, \dots, N.$$

*In particular, if a minimizer of (2.3) is (locally) Lipschitz, then this remains true for all the other minimizers.*

*Proof.* Let us suppose that (2.4) is not true. Then there exists  $i_0 \in \{1, \dots, N\}$  such that

$$E_{i_0} := \left\{ x \in B : \left| |(u_1)_{x_{i_0}}| - |(u_2)_{x_{i_0}}| \right| > 2\delta_{i_0} \right\},$$

has strictly positive measure. We then set  $u_s = (1-s)u_1 + su_2$  for some  $s \in (0, 1)$  and observe that this is admissible in (2.3). In view of Lemma A.1 in Appendix A,

$$g_{i_0} \left( (1-s)(u_1)_{x_{i_0}} + s(u_2)_{x_{i_0}} \right) < (1-s)g_{i_0}((u_1)_{x_{i_0}}) + sg_{i_0}((u_2)_{x_{i_0}}), \quad \text{a. e. in } E_{i_0}.$$

Thus we get

$$\mathfrak{F}(u_s) < (1-s)\mathfrak{F}(u_1) + s\mathfrak{F}(u_2) = \mathfrak{F}(u_1) = \mathfrak{F}(u_2),$$

which gives the desired contradiction.  $\square$

We will also need the following regularity result, which is reminiscent of [23]. A more general result of this type can be found in [5].

**Theorem 2.4** ([6]). *Let  $B \subset \mathbb{R}^N$  be a ball,  $\varphi \in C^2(\overline{B})$  and  $f \in L^\infty(B)$ . Let  $u \in W^{1,1}(B) \cap L^\infty(B)$  be a solution of*

$$\min \left\{ \int_B H(\nabla v) dx + \int_B f v dx : v - \varphi \in W_0^{1,1}(B) \right\},$$

where  $H : \mathbb{R}^N \rightarrow [0, \infty)$  is a  $C^2$  convex function such that for some  $\mu > 0$

$$(2.5) \quad \langle D^2 H(z) \xi, \xi \rangle \geq \mu |\xi|^2, \quad \xi, z \in \mathbb{R}^N.$$

Then  $u \in W_{loc}^{1,\infty}(B)$ .

**2.2. Approximation scheme.** We now introduce a regularized version of the original problem. We set

$$(2.6) \quad g_{i,\varepsilon}(t) = g_i(t) + \frac{\varepsilon}{2} t^2 = \frac{1}{p} (|t| - \delta_i)_+^p + \frac{\varepsilon}{2} |t|^2, \quad t \in \mathbb{R}.$$

Let  $U$  be a local minimizer of  $\mathfrak{F}$ . We also fix a ball

$$B \Subset \Omega \quad \text{such that} \quad 2B \Subset \Omega \text{ as well.}$$

Here  $2B$  denotes the ball having the same center as  $B$  scaled by a factor 2.

For every  $0 < \varepsilon \ll 1$  and every  $x \in \overline{B}$ , we set  $U_\varepsilon(x) = U * \varrho_\varepsilon(x)$ , where  $\varrho_\varepsilon$  is a smooth convolution kernel, supported in a ball of radius  $\varepsilon$  centered at the origin.

Then by definition of  $U_\varepsilon$  there exists  $0 < \varepsilon_0 < 1$  such that for every  $0 < \varepsilon \leq \varepsilon_0$

$$(2.7) \quad \|U_\varepsilon\|_{W^{1,p}(B)} = \|\nabla U_\varepsilon\|_{L^p(B)} + \|U_\varepsilon\|_{L^p(B)} \leq \|\nabla U\|_{L^p(2B)} + \|U\|_{L^p(2B)} =: C_1.$$

Finally, we define

$$\mathfrak{F}_\varepsilon(v; B) = \sum_{i=1}^N \int_B g_{i,\varepsilon}(v_{x_i}) dx + \int_B f_\varepsilon v dx,$$

where  $f_\varepsilon = f * \varrho_\varepsilon$ . The following preliminary result is standard.

**Lemma 2.5** (Basic energy estimate). *There exists a unique solution  $u_\varepsilon$  to the problem*

$$(2.8) \quad \min \left\{ \mathfrak{F}_\varepsilon(v; B) : v - U_\varepsilon \in W_0^{1,p}(B) \right\}.$$

The following uniform energy estimate holds

$$(2.9) \quad \int_B |\nabla u_\varepsilon|^p dx \leq C_2,$$

for some constant  $C_2 = C_2(N, \delta, p, |B|, C_1, \|f\|_{L^{p'}(2B)}) > 0$ .

*Proof.* We start by observing that a solution  $u_\varepsilon$  exists, by a standard application of the Direct Methods. Uniqueness then follows from strict convexity of the integrand

$$(2.10) \quad L_\varepsilon(x, u, z) = \sum_{i=1}^N g_i(z_i) + \frac{\varepsilon}{2} |z|^2 + f_\varepsilon(x) u,$$

in the gradient variable. In order to prove (2.9), we use the minimality of  $u_\varepsilon$ , which implies  $\mathfrak{F}_\varepsilon(u_\varepsilon; B) \leq \mathfrak{F}_\varepsilon(U_\varepsilon; B)$ . This gives

$$\sum_{i=1}^N \int_B g_{i,\varepsilon}((u_\varepsilon)_{x_i}) dx \leq \sum_{i=1}^N \int_B g_{i,\varepsilon}((U_\varepsilon)_{x_i}) dx + \int_B |f_\varepsilon| |u_\varepsilon - U_\varepsilon| dx.$$

We now use the fact that

$$(2.11) \quad \frac{1}{p} \left( \frac{1}{2^{p-1}} |t|^p - \delta^p \right) \leq g_{i,\varepsilon}(t) \leq \frac{2}{p} |t|^p + \frac{p-2}{2p}.$$

The lower bound in (2.11) follows from

$$|t|^p \leq 2^{p-1} ((|t| - \delta_i)_+^p + \delta_i^p),$$

while the upper bound is a consequence of Young inequality. This implies

$$\sum_{i=1}^N \int_B |(u_\varepsilon)_{x_i}|^p dx \leq C \sum_{i=1}^N \int_B |(U_\varepsilon)_{x_i}|^p + \int_B |f_\varepsilon| |u_\varepsilon - U_\varepsilon| dx + C$$

where  $C = C(N, p, \delta, |B|) > 0$  depends on  $N, p, \delta$  and  $|B|$  only. By using  $\|f_\varepsilon\|_{L^{p'}(B)} \leq \|f\|_{L^{p'}(2B)}$  and (2.7), standard computations lead to the desired conclusion.  $\square$

**Lemma 2.6** (Regularity of the minimizer I). *Let  $u_\varepsilon$  still denote the unique minimizer of (2.8). Then we have  $u_\varepsilon \in L^\infty(B)$ .*

*Moreover, if  $f \in L_{loc}^\infty(\Omega)$ , then there exists a constant  $M$  independent of  $\varepsilon$  such that*

$$(2.12) \quad \|u_\varepsilon\|_{L^\infty(B)} \leq M.$$

*Proof.* We use again (2.11). This implies that the integrand (2.10) satisfies

$$(2.13) \quad c |z|^p - \|f_\varepsilon\|_{L^\infty(B)} |u| - C' \leq L_\varepsilon(x, u, z) \leq \frac{1}{c} |z|^p + \|f_\varepsilon\|_{L^\infty(B)} |u| + C',$$

with  $c = c(N, p) > 0$ ,  $C = C(N, p) > 0$  and  $C' = C'(p, \delta) > 0$ . Thus  $u_\varepsilon \in L^\infty(B)$  by [18, Theorem 7.5 & Remark 7.6].

If  $f \in L_{loc}^\infty(\Omega)$ , we have  $\|f_\varepsilon\|_{L^\infty(B)} \leq \|f\|_{L^\infty(2B)}$ . By (2.13) we thus get that  $L_\varepsilon$  satisfies growth conditions independent of  $\varepsilon$ . Then by using again the a priori estimate of [18, Theorem 7.5] and (2.9) we get the desired conclusion.  $\square$

**Remark 2.7.** The previous  $L^\infty$  estimate uniform in  $\varepsilon$  will be needed in the proof of Theorem B.

The following result is not optimal, but it is suitable to our needs.

**Lemma 2.8** (Regularity of the minimizer II). *Let  $u_\varepsilon$  still denote the unique minimizer of (2.8). We have  $u_\varepsilon \in C_{loc}^k(B)$ , where*

$$k = \begin{cases} 2, & \text{if } 2 < p \leq 3, \\ 3, & \text{if } p > 3. \end{cases}$$

*Proof.* We divide the proof in two parts.

*Local Lipschitz regularity.* By Lemma 2.6, we know that  $u_\varepsilon$  is bounded. Then the local Lipschitz continuity is a plain consequence of Theorem 2.4, applied with

$$(2.14) \quad F_\varepsilon(z) = \sum_{i=1}^N g_i(z_i) + \frac{\varepsilon}{2} |z|^2, \quad z \in \mathbb{R}^N,$$

which verifies (2.5) with  $\mu = \varepsilon > 0$ .

*Local higher regularity.* Let  $\tilde{B} \Subset B$  be a ball and set  $\ell = \|\nabla u_\varepsilon\|_{L^\infty(\tilde{B})}$ , which is finite thanks to the previous step. By optimality, we have that  $u_\varepsilon$  solves the elliptic equation

$$(2.15) \quad \operatorname{div}(\nabla F_\varepsilon(\nabla u_\varepsilon)) = f_\varepsilon, \quad \text{in } \tilde{B},$$

where  $F_\varepsilon$  is as in (2.14). Since we have

$$\varepsilon |\xi|^2 \leq \langle D^2 F_\varepsilon(\nabla u_\varepsilon) \xi, \xi \rangle \leq (\varepsilon + (p-1) \ell^{p-2}) |\xi|^2, \quad \text{on } \tilde{B},$$

we can infer  $u_\varepsilon \in W_{loc}^{2,2}(\tilde{B})$  by a standard differential quotients argument (see for example [18, Theorem 8.1]). This in turn permits to find the equation locally solved by  $\nabla u_\varepsilon$ , by differentiating (2.15). Thus  $\nabla u_\varepsilon \in C_{loc}^{0,\sigma}(\tilde{B})$  by the celebrated De Giorgi–Moser–Nash Theorem, for some  $\sigma > 0$ . It remains to observe that  $F_\varepsilon \in C^{k,\alpha}$ , where  $k$  is as in the statement and

$$\alpha = \begin{cases} \min\{p-2, 1\}, & \text{if } 2 < p \leq 3, \\ \min\{p-3, 1\}, & \text{if } p > 3. \end{cases}$$

Then [18, Theorem 10.18] implies that  $u_\varepsilon$  has the claimed regularity properties.  $\square$

**Lemma 2.9** (Convergence to a minimizer). *With the same notation as before, we have*

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - \tilde{u}\|_{L^p(B)} = 0,$$

where  $\tilde{u}$  is a solution of

$$(2.16) \quad \min \left\{ \mathfrak{F}(\varphi; B) : \varphi - U \in W_0^{1,p}(B) \right\}.$$

*Proof.* By (2.9), there exists a sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  converging to 0 as  $k$  goes to  $\infty$  and a function  $\tilde{u} \in W^{1,p}(B)$  such that  $\{u_{\varepsilon_k}\}_{k \in \mathbb{N}}$  converges weakly to  $\tilde{u}$  in  $W^{1,p}(B)$  and strongly in  $L^p(B)$ . The function  $U_\varepsilon = U * \varrho_\varepsilon$  is of course admissible for the approximated problem and thus

$$\liminf_{k \rightarrow \infty} \mathfrak{F}_{\varepsilon_k}(U_{\varepsilon_k}; B) \geq \liminf_{k \rightarrow \infty} \mathfrak{F}_{\varepsilon_k}(u_{\varepsilon_k}; B) \geq \liminf_{k \rightarrow \infty} \mathfrak{F}(u_{\varepsilon_k}; B) \geq \mathfrak{F}(\tilde{u}; B),$$

where we used the weak lower semicontinuity of  $\mathfrak{F}$ . We then observe that by using the strong convergence of  $U_\varepsilon$  to  $U$  and inequality (A.2) in Appendix A, we get

$$\begin{aligned} \lim_{k \rightarrow \infty} |\mathfrak{F}_{\varepsilon_k}(U_{\varepsilon_k}; B) - \mathfrak{F}(U; B)| &\leq \lim_{k \rightarrow \infty} \sum_{i=1}^N \int_B |g_i((U_{\varepsilon_k})_{x_i}) - g_i(U_{x_i})| dx \\ &+ \lim_{k \rightarrow \infty} \frac{\varepsilon_k}{2} \int_B |\nabla U_{\varepsilon_k}|^2 + \lim_{k \rightarrow \infty} \int_B |f_{\varepsilon_k} U_{\varepsilon_k} - f U| dx = 0, \end{aligned}$$

and thus

$$\mathfrak{F}(U; B) = \lim_{k \rightarrow \infty} \mathfrak{F}_{\varepsilon_k}(U_{\varepsilon_k}; B) \geq \mathfrak{F}(\tilde{u}; B).$$

By definition of local minimizer, the function  $U$  itself is a solution of (2.16), then the previous inequality implies that  $\tilde{u}$  is a minimizer.  $\square$

### 3. LOCAL ENERGY ESTIMATES FOR THE APPROXIMATING PROBLEM

For the ball  $B \Subset \Omega$  we consider the regularized problem (2.8). We still denote by  $u_\varepsilon$  its unique solution, which verifies the Euler-Lagrange equation

$$(3.1) \quad \sum_{i=1}^N \int g'_{i,\varepsilon}((u_\varepsilon)_{x_i}) \varphi_{x_i} dx + \int f_\varepsilon \varphi dx = 0, \quad \varphi \in W_0^{1,p}(B).$$

From now on, in order to simplify the notation, we will systematically forget the subscript  $\varepsilon$  on  $u_\varepsilon$  and simply write  $u$ .

We now insert a test function of the form  $\varphi = \psi_{x_j} \in W_0^{1,p}(B)$  in (3.1), compactly supported in  $B$ . Then an integration by parts lead us to

$$(3.2) \quad \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) u_{x_i x_j} \psi_{x_i} dx - \int f_\varepsilon \psi_{x_j} dx = 0,$$

for  $j = 1, \dots, N$ . This is the equation solved by  $u_{x_j}$ .

**3.1. Caccioppoli-type inequalities.** In what follows we use the parameter  $\delta$  defined in (2.1). The general Caccioppoli inequality for an important class of subsolutions is given by the following result.

**Lemma 3.1.** *Let  $\Phi : \mathbb{R} \rightarrow \mathbb{R}^+$  be a  $C^2$  convex function such that*

$$(3.3) \quad \Phi'(t) \equiv 0 \quad \text{for } |t| \leq \delta.$$

*Then there exists a constant  $C_3 = C_3(p) > 0$  such that for every Lipschitz function  $\eta$  with compact support in  $B$ , we have*

$$(3.4) \quad \begin{aligned} & \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \eta^2 dx \\ & \leq C_3 \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) |\Phi(u_{x_j})|^2 |\eta_{x_i}|^2 dx \\ & \quad + C_3 \int_{A_j} |f_\varepsilon|^2 \left[ \Phi'(u_{x_j})^2 + \Phi''(u_{x_j}) \Phi(u_{x_j}) \right] \eta^2 dx + C_3 \int_{A_j} \Phi(u_{x_j})^2 |\eta_{x_j}|^2 dx, \end{aligned}$$

where we set  $A_j = \{x \in B : |u_{x_j}| \geq \delta\}$ .

*Proof.* In (3.2) we take the test function<sup>4</sup>  $\psi = \zeta \Phi'(u_{x_j})$ , with  $\Phi : \mathbb{R} \rightarrow \mathbb{R}^+$  as in the statement and  $\zeta$  nonnegative Lipschitz function with support in  $B$ . We thus obtain

$$\sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) (\Phi(u_{x_j}))_{x_i} \zeta_{x_i} dx + \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) u_{x_i x_j}^2 \Phi''(u_{x_j}) \zeta dx = \int_{A_j} f_\varepsilon (\zeta \Phi'(u_{x_j}))_{x_j} dx.$$

Finally, we test the previous equation against  $\zeta = \eta^2 \Phi(u_{x_j})$ , where  $\eta$  is again a Lipschitz function with support in  $B$ . Then we get

$$\begin{aligned} & \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \eta^2 dx + \mathcal{S}(\eta) \\ & \leq 2 \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right| \Phi(u_{x_j}) \eta |\eta_{x_i}| dx + \int_{A_j} |f_\varepsilon| \left| (\eta^2 \Phi(u_{x_j}) \Phi'(u_{x_j}))_{x_j} \right| dx. \end{aligned}$$

where we have introduced the *sponge term*

$$\mathcal{S}(\eta) = \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) u_{x_i x_j}^2 \Phi''(u_{x_j}) \Phi(u_{x_j}) \eta^2 dx.$$

---

<sup>4</sup>Observe that this is an admissible test function by Lemma 2.8.

From the previous inequality, by Young inequality in the first term on the right-hand side

$$(3.5) \quad \begin{aligned} & \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \eta^2 dx + 2\mathcal{S}(\eta) \\ & \leq 4 \sum_{i=1}^N \int_{A_j} g''_{i,\varepsilon}(u_{x_i}) \Phi(u_{x_j})^2 |\eta_{x_i}|^2 dx + 2 \int_{A_j} |f_\varepsilon| \left| (\eta^2 \Phi(u_{x_j}) \Phi'(u_{x_j}))_{x_j} \right| dx. \end{aligned}$$

We now estimate the term containing  $f_\varepsilon$ . We first observe

$$\begin{aligned} \int_{A_j} |f_\varepsilon| \left| (\eta^2 \Phi(u_{x_j}) \Phi'(u_{x_j}))_{x_j} \right| dx & \leq \int_{A_j} |f_\varepsilon| |\Phi'(u_{x_j})| \left| (\Phi(u_{x_j}))_{x_j} \right| \eta^2 dx \\ & \quad + 2 \int_{A_j} |f_\varepsilon| |\Phi'(u_{x_j})| \Phi(u_{x_j}) \eta |\eta_{x_j}| dx + \int_{A_j} |f_\varepsilon| \left| (\Phi'(u_{x_j}))_{x_j} \right| \Phi(u_{x_j}) \eta^2 dx. \end{aligned}$$

On the set  $A_j$  we have

$$(3.6) \quad g''_{j,\varepsilon}(u_{x_j}) \geq (p-1).$$

Let us consider the first term above containing  $f_\varepsilon$ :

$$\begin{aligned} \int_{A_j} |f_\varepsilon| |\Phi'(u_{x_j})| \left| (\Phi(u_{x_j}))_{x_j} \right| \eta^2 dx & \leq \frac{1}{2\tau} \int_{A_j} |f_\varepsilon|^2 |\Phi'(u_{x_j})|^2 \eta^2 dx \\ & \quad + \frac{\tau}{2} \int_{A_j} \left| (\Phi(u_{x_j}))_{x_j} \right|^2 \eta^2 dx \\ & \leq \frac{1}{2\tau} \int_{A_j} |f_\varepsilon|^2 |\Phi'(u_{x_j})|^2 \eta^2 dx \\ & \quad + \frac{\tau}{2(p-1)} \int_{A_j} g''_{j,\varepsilon}(u_{x_j}) \left| (\Phi(u_{x_j}))_{x_j} \right|^2 \eta^2 dx. \end{aligned}$$

The last term can be absorbed in the left-hand side of (3.5), by taking  $\tau = (p-1)/2$ . The second term containing  $f_\varepsilon$  is simply estimated by Young inequality

$$\int_{A_j} |f_\varepsilon| |\Phi'(u_{x_j})| \Phi(u_{x_j}) \eta |\eta_{x_j}| dx \leq \frac{1}{2} \int_{A_j} |f_\varepsilon|^2 |\Phi'(u_{x_j})|^2 \eta^2 dx + \frac{1}{2} \int_{A_j} \Phi(u_{x_j})^2 |\eta_{x_j}|^2 dx,$$

while for the last one we use the sponge term  $\mathcal{S}(\eta)$  to absorb the Hessian of  $u$ . Namely, we have

$$\begin{aligned} \int_{A_j} |f_\varepsilon| \left| (\Phi'(u_{x_j}))_{x_j} \right| \Phi(u_{x_j}) \eta^2 dx & = \int_{A_j} |f_\varepsilon| |u_{x_j x_j}| \Phi''(u_{x_j}) \Phi(u_{x_j}) \eta^2 dx \\ & \leq \tau \int_{A_j} u_{x_j x_j}^2 \Phi''(u_{x_j}) \Phi(u_{x_j}) \eta^2 dx \\ & \quad + \frac{1}{\tau} \int_{A_j} |f_\varepsilon|^2 \Phi''(u_{x_j}) \Phi(u_{x_j}) \eta^2 dx \\ & \leq \frac{\tau}{p-1} \mathcal{S}(\eta) + \frac{1}{\tau} \int_{A_j} |f_\varepsilon|^2 \Phi''(u_{x_j}) \Phi(u_{x_j}) \eta^2 dx. \end{aligned}$$

In the last estimate we used again (3.6). The term  $\tau/(p-1)\mathcal{S}(\eta)$  can then be absorbed in the left-hand side. This concludes the proof.  $\square$

If we allow for derivatives of  $f_\varepsilon$  on the right-hand side of (3.4), the previous estimate is simpler to get. In this case we can allow for more general subsolutions.

**Lemma 3.2** (Right-hand side in a Sobolev space). *Let  $\Phi : \mathbb{R} \rightarrow \mathbb{R}$  be a  $C^1$  convex function. Then there exists a constant  $C_3 = C_3(p) > 0$  such that for every Lipschitz function  $\eta$  with compact support in  $B$ , we have*

$$(3.7) \quad \begin{aligned} & \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \eta^2 dx \\ & \leq C_3 \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) |\Phi(u_{x_j})|^2 |\eta_{x_i}|^2 dx + C_3 \int |(f_\varepsilon)_{x_j}| |\Phi'(u_{x_j})| |\Phi(u_{x_j})| \eta^2 dx. \end{aligned}$$

*Proof.* Let us suppose for simplicity that  $\Phi \in C^2$ . If this were not the case, a standard smoothing argument will be needed, we leave the details to the reader.

We start observing that equation (3.2) can also be written as

$$(3.8) \quad \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) u_{x_i x_j} \psi_{x_i} dx + \int (f_\varepsilon)_{x_j} \psi dx = 0, \quad j = 1, \dots, N.$$

Then we take in (3.8) the test function  $\psi = \zeta \Phi'(u_{x_j})$  as before, with  $\Phi$  as in the statement and  $\zeta$  a nonnegative Lipschitz function supported in  $B$ . We obtain

$$\sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) (\Phi(u_{x_j}))_{x_i} \zeta_{x_i} dx \leq - \int (f_\varepsilon)_{x_j} \Phi'(u_{x_j}) \zeta dx.$$

Finally, we take again  $\zeta = \eta^2 \Phi(u_{x_j})$ , to get

$$\begin{aligned} \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \eta^2 dx & \leq 2 \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right| \Phi(u_{x_j}) \eta |\eta_{x_i}| dx \\ & \quad + \int |(f_\varepsilon)_{x_j}| |\Phi'(u_{x_j})| |\Phi(u_{x_j})| \eta^2 dx. \end{aligned}$$

By using Young inequality as before, we get

$$\begin{aligned} \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \eta^2 dx & \leq 4 \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \Phi(u_{x_j})^2 |\eta_{x_i}|^2 dx \\ & \quad + 2 \int |(f_\varepsilon)_{x_j}| |\Phi'(u_{x_j})| |\Phi(u_{x_j})| \eta^2 dx. \end{aligned}$$

This concludes the proof.  $\square$

**3.2. A Sobolev estimate.** In what follows we set

$$W_j = \delta^2 + (|u_{x_j}| - \delta)_+^2.$$

**Lemma 3.3.** *There exists a constant  $C_4 = C_4(p, \delta) > 0$  such that for every Lipschitz function  $\eta$  with compact support in  $B$ , we have*

$$(3.9) \quad \sum_{i=1}^N \int \left| \nabla W_i^{\frac{p}{4}} \right|^2 \eta^2 dx \leq C_4 \sum_{i,j=1}^N \int W_i^{\frac{p-2}{2}} W_j |\eta_{x_i}|^2 dx + C_4 \sum_{j=1}^N \int |(f_\varepsilon)_{x_j}| \sqrt{W_j} \eta^2 dx.$$

*Proof.* We choose the function  $\Phi(t) = t$  in (3.7). First observe that

$$\left| (\Phi(u_{x_j}))_{x_i} \right|^2 = u_{x_i x_j}^2,$$

and that by Lemma A.3 for  $|t| \geq \delta$

$$(3.10) \quad \begin{aligned} g_{i,\varepsilon}''(t) &\geq (p-1) \left( \frac{\delta - \delta_i}{\delta} \right)^{p-2} (\delta^2 + (|t| - \delta)_+^2)^{\frac{p-2}{2}} \\ &\geq \frac{1}{\delta^{p-2}} (\delta^2 + (|t| - \delta)_+^2)^{\frac{p-4}{2}} (|t| - \delta)_+^2. \end{aligned}$$

In the second inequality above we also used that  $p \geq 2$  and  $\delta - \delta_i \geq 1$ . Then, by (3.10) we have<sup>5</sup>

$$g_{i,\varepsilon}''(u_{x_i}) \left| (\Phi(u_{x_j}))_{x_i} \right|^2 \geq \frac{1}{\delta^{p-2}} W_i^{\frac{p-4}{2}} (|u_{x_i}| - \delta)_+^2 u_{x_i x_j}^2 = c \left| \partial_{x_j} W_i^{\frac{p}{4}} \right|^2,$$

where  $c = c(\delta, p) > 0$ . We further observe that

$$|\Phi(u_{x_j})| = |u_{x_j}| \leq \sqrt{2} \sqrt{W_j},$$

and<sup>6</sup>

$$(3.11) \quad g_{i,\varepsilon}''(u_{x_i}) = (p-1) (|u_{x_i}| - \delta_i)_+^{p-2} + \varepsilon \leq c W_i^{\frac{p-2}{2}},$$

where  $c = c(p) > 0$ . Then we get the desired result by summing (3.7) over  $j = 1, \dots, N$ .  $\square$

In what follows, we will use for simplicity the notation

$$\int_E \varphi \, dx := \frac{1}{|E|} \int_E \varphi \, dx.$$

**Corollary 3.4.** *There exists a constant  $C'_4 = C'_4(p, \delta, N) > 0$  such that for every pair of concentric balls  $B_{R_0} \Subset B_{\rho_0} \Subset B$ , we have*

$$(3.12) \quad \begin{aligned} \sum_{j=1}^N \frac{1}{R_0^{N-2}} \int_{B_{R_0}} \left| \nabla W_j^{\frac{p}{4}} \right|^2 \, dx &\leq C \left( \frac{\rho_0}{R_0} \right)^{N-2} \left( \frac{\rho_0}{\rho_0 - R_0} \right)^2 \sum_{j=1}^N \int_{B_{\rho_0}} W_j^{\frac{p}{2}} \, dx \\ &\quad + C \rho_0^{\frac{2}{p-1} - N + 2} \sum_{j=1}^N \int_{B_{\rho_0}} |(f_\varepsilon)_{x_j}|^{p'} \, dx. \end{aligned}$$

*Proof.* It is sufficient to insert the test function

$$\eta(x) = \min \left\{ 1, \frac{(\rho_0 - |x|)_+}{\rho_0 - R_0} \right\},$$

in (3.9) and then use Hölder and Young inequalities in the right-hand side. These give

$$\sum_{i,j=1}^N \int W_i^{\frac{p-2}{2}} W_j |\eta_{x_i}|^2 \, dx \leq \frac{1}{(\rho_0 - R_0)^2} \sum_{i,j=1}^N \left( \int_{B_{\rho_0}} W_i^{\frac{p}{2}} \, dx \right)^{\frac{p-2}{p}} \left( \int_{B_{\rho_0}} W_j^{\frac{p}{2}} \, dx \right)^{\frac{2}{p}},$$

<sup>5</sup>Observe that the inequality holds true everywhere, not only on  $A_i$ , since  $W_i$  is constant outside  $A_i$ .

<sup>6</sup>We use that  $t - \delta_i = (t - \delta) + (\delta - \delta_i) \leq (t - \delta)_+ + \delta$ , which implies

$$(t - \delta_i)_+ \leq (t - \delta)_+ + \delta.$$

and

$$\sum_{j=1}^N \int |(f_\varepsilon)_{x_j}| \sqrt{W_j} \eta^2 dx \leq C \sum_{j=1}^N \rho_0^{\frac{2}{p-1}} \int_{B_{\rho_0}} |(f_\varepsilon)_{x_j}|^{p'} dx + \frac{C}{\rho_0^2} \sum_{j=1}^N \int_{B_{\rho_0}} W_j^{\frac{p}{2}} dx,$$

which concludes the proof.  $\square$

**Remark 3.5** (Uniform Sobolev estimate). From the previous result, we obtain that if  $f \in W_{loc}^{1,p'}(\Omega)$ , then for every  $i = 1, \dots, N$  the function  $W_i^{p/4}$  enjoys a  $W_{loc}^{1,2}(B)$  estimate independent of  $\varepsilon$ , thanks to (2.9) and

$$\|f_\varepsilon\|_{W^{1,p'}(B_{\rho_0})} \leq \|f\|_{W^{1,p'}(2B)}.$$

**3.3. Power-type subsolutions.** We still use the notation

$$W_j = \delta^2 + (|u_{x_j}| - \delta)_+^2.$$

Then we have the following result.

**Lemma 3.6.** *There exists a constant  $C_5 = C_5(p) > 0$  such that for every  $s \geq 0$  and every Lipschitz function  $\eta$  with compact support in  $B$ , we have*

$$(3.13) \quad \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \left| \left( W_j^{\frac{s+1}{2}} \right)_{x_i} \right|^2 \eta^2 dx \leq C_5 \sum_{i=1}^N \int W_i^{\frac{p-2}{2}} W_j^{s+1} |\nabla \eta|^2 dx \\ + C_5 (s+1)^2 \int |f_\varepsilon|^2 W_j^s \eta^2 dx, \quad j = 1, \dots, N.$$

*Proof.* In equation (3.4) we make the choice<sup>7</sup>

$$\Phi(t) = \left( \delta^2 + (|t| - \delta)_+ \right)^{\frac{s+1}{2}},$$

for  $s \geq 0$  which satisfies hypothesis (3.3). Observe that by definition we have

$$\Phi(u_{x_j}) = W_j^{\frac{s+1}{2}},$$

so that

$$\left| (\Phi(u_{x_j}))_{x_i} \right|^2 = \left| \left( W_j^{\frac{s+1}{2}} \right)_{x_i} \right|^2.$$

Thus the left-hand side of (3.4) coincides with

$$\sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) \left| \left( W_j^{\frac{s+1}{2}} \right)_{x_i} \right|^2 \eta^2 dx.$$

We now come to the right-hand side:

$$\sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) |\Phi(u_{x_j})|^2 |\eta_{x_i}|^2 dx = \sum_{i=1}^N \int g''_{i,\varepsilon}(u_{x_i}) W_j^{s+1} |\eta_{x_i}|^2 dx \\ \leq C \sum_{i=1}^N \int W_i^{\frac{p-2}{2}} W_j^{s+1} |\eta_{x_i}|^2 dx,$$

<sup>7</sup>Observe that this function is not  $C^2$ , but only  $C^{1,1}$  near  $t = \delta$  or  $t = -\delta$ . This is not a big issue, since in any case  $\Phi''$  stays bounded as  $|t| \rightarrow \delta$ , thus we can use (3.4) for a regularization of  $\Phi$  and then pass to the limit at the end.

thanks to (3.11). For the other two terms, by using the definition of  $\Phi$  we simply have

$$\begin{aligned} \int_{A_j} |f_\varepsilon|^2 \left[ \Phi'(u_{x_j})^2 + \Phi''(u_{x_j}) \Phi(u_{x_j}) \right] \eta^2 dx + \int_{A_j} \Phi(u_{x_j})^2 |\eta_{x_j}|^2 dx \\ \leq C (s+1)^2 \int |f_\varepsilon|^2 W_j^s \eta^2 dx \\ + \sum_{i=1}^N \int W_i^{\frac{p-2}{2}} W_j^{s+1} |\eta_{x_j}|^2 dx, \end{aligned}$$

where we used that

$$\left[ \Phi'(t)^2 + \Phi''(t) \Phi(t) \right] \leq C (s+1)^2 \left( \delta^2 + (|t| - \delta)_+^2 \right)^s = C (1+s)^2 \Phi(t)^{\frac{2s}{s+1}},$$

and  $W_j^{s+1} \leq \sum_{i=1}^N W_i^{\frac{p-2}{2}} W_j^{s+1}$ , which follows from  $W_i \geq 1$ .  $\square$

In particular, we get an estimate for the *diagonal terms*, corresponding to  $i = j$ .

**Corollary 3.7.** *There exists a constant  $C_6 = C_6(p, \delta) > 0$  such that for every  $s \geq 0$  and every Lipschitz function  $\eta$  with compact support in  $\Omega$ , we have*

$$(3.14) \quad \begin{aligned} \int \left| \left( W_j^{\frac{p}{4} + \frac{s}{2}} \right)_{x_j} \right|^2 \eta^2 dx \leq C_6 \sum_{i=1}^N \int W_i^{\frac{p-2}{2}} W_j^{s+1} |\nabla \eta|^2 dx \\ + C_6 (s+1)^2 \int |f_\varepsilon|^2 W_j^s \eta^2 dx, \quad j = 1, \dots, N. \end{aligned}$$

*Proof.* We fix  $j$ , by keeping only the term  $i = j$  and dropping all the others in the left-hand side of (3.13), we get

$$\begin{aligned} \int_{A_j} g_{j,\varepsilon}''(u_{x_j}) \left| \left( W_j^{\frac{s+1}{2}} \right)_{x_j} \right|^2 \eta^2 dx \leq C_5 \sum_{i=1}^N \int W_i^{\frac{p-2}{2}} W_j^{s+1} |\nabla \eta|^2 dx \\ + C_5 (s+1)^2 \int |f_\varepsilon|^2 W_j^s \eta^2 dx. \end{aligned}$$

We now observe that again by Lemma A.3 on  $A_j$  we have

$$g_{j,\varepsilon}''(u_{x_j}) \geq \frac{p-1}{\delta^{p-2}} \left[ \delta^2 + (|u_{x_j}| - \delta)_+^2 \right]^{\frac{p-2}{2}} = \frac{p-1}{\delta^{p-2}} W_j^{\frac{p-2}{2}},$$

and that

$$W_j^{\frac{p-2}{2}} \left| \left( W_j^{\frac{s+1}{2}} \right)_{x_j} \right|^2 = \left( \frac{2+2s}{p+2s} \right)^2 \left| \left( W_j^{\frac{p}{4} + \frac{s}{2}} \right)_{x_j} \right|^2 \geq \left( \frac{2}{p} \right)^2 \left| \left( W_j^{\frac{p}{4} + \frac{s}{2}} \right)_{x_j} \right|^2,$$

so that the conclusion follows.  $\square$

#### 4. PROOF OF THEOREM A

The core of the proof of Theorem A is the a priori estimate of Proposition 4.1 below. We postpone its proof and proceed with that of Theorem A.

*Proof.* Let  $\Omega' \Subset \Omega$  and set  $d = \text{dist}(\Omega', \partial\Omega)$ . We take  $r_0 \leq d/10$ , then  $\Omega'$  can be covered by a finite number of balls centered at points in  $\Omega'$  and having radius  $r_0$ . Let  $B_{r_0} := B_{r_0}(x_0) \Subset \Omega$  be one of these balls, it is clearly sufficient to show that

$$\|\nabla U\|_{L^\infty(B_{r_0})} < +\infty.$$

To this aim we take the solution  $u_\varepsilon$  of the regularized problem (2.8) in the ball  $B := B_{4r_0}(x_0)$ . Observe that by construction we have  $2B = B_{8r_0}(x_0) \Subset \Omega$ . Then there exists  $\varepsilon_0 = \varepsilon_0(d) > 0$  such that for every  $0 < \varepsilon \leq \varepsilon_0$

$$\|f_\varepsilon\|_{W^{1,p'}(B)} \leq \|f\|_{W^{1,p'}(2B)}.$$

By using estimate (4.2) below with  $R_0 = 2r_0$  and  $\rho_0 = 3r_0$  we get

$$(4.1) \quad \|\nabla u_\varepsilon\|_{L^\infty(B_{r_0})} \leq C, \quad \text{for every } 0 < \varepsilon \leq \varepsilon_0,$$

where  $C > 0$  depends only on  $p, \delta, r_0, \|f\|_{W^{1,p'}(2B)}$  and the constant  $C_2$  in (2.9). We then observe that by Lemma 2.9, we can find a sequence  $\{\varepsilon_k\}_{k \in \mathbb{N}}$  converging to 0 and such that  $\{u_{\varepsilon_k}\}$  converges strongly in  $L^p(B)$  and weakly in  $W^{1,p}(B)$  to a solution  $\tilde{u}$  of

$$\min\{\mathfrak{F}(\varphi; B) : \varphi - U \in W^{1,p}(B)\}.$$

By lower semicontinuity we have that  $\tilde{u}$  still satisfies (4.1). It is now sufficient to use Lemma 2.3 in order to transfer this Lipschitz estimate from  $\tilde{u}$  to the original local minimizer  $U$ . This concludes the proof.  $\square$

**Proposition 4.1** (Uniform Lipschitz estimate,  $N = 2$ ). *Let  $N = 2$  and  $p \geq 2$ . Then for every triple of concentric balls  $B_{r_0} \Subset B_{R_0} \Subset B_{\rho_0} \Subset B$  and  $i = 1, 2$  we have*

$$(4.2) \quad \|(u_\varepsilon)_{x_i}\|_{L^\infty(B_{r_0})} \leq C_7 \left(\frac{R_0}{R_0 - r_0}\right)^4 \mathcal{J}(u_\varepsilon, f_\varepsilon; R_0, \rho_0)^2 \left[ \left( \int_{B_{R_0}} |(u_\varepsilon)_{x_i}|^p dx \right)^{\frac{1}{p}} + \delta \right],$$

where  $C_7 = C_7(p, \delta) > 0$  is a constant that only depends on  $p$  and  $\delta$  and

$$(4.3) \quad \begin{aligned} \mathcal{J}(u_\varepsilon, f_\varepsilon; R_0, \rho_0) &= \left(\frac{\rho_0}{\rho_0 - R_0}\right)^2 \left[ \int_{B_{\rho_0}} |\nabla u_\varepsilon|^p dx + \delta^p \right] \\ &+ \rho_0^{\frac{2}{p-1}} \int_{B_{\rho_0}} |\nabla f_\varepsilon|^{p'} dx + \rho_0^{\frac{2}{p}} \left( \int_{B_{\rho_0}} |f_\varepsilon|^{2p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

*Proof.* For notational simplicity, we write again  $u$  in place of  $u_\varepsilon$ . We still use the notation

$$W_j = \delta^2 + (|u_{x_j}| - \delta)_+^2, \quad j = 1, 2.$$

We give the proof for  $u_{x_1}$ , the one for  $u_{x_2}$  being exactly the same. By (3.14) we already know that

$$\int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)_{x_1} \right|^2 \eta^2 dx \leq C_6 \sum_{i=1}^2 \int W_i^{\frac{p-2}{2}} W_1^{s+1} |\nabla \eta|^2 dx + C_6 (s+1)^2 \int |f_\varepsilon|^2 W_1^s \eta^2 dx,$$

where  $\eta$  is any Lipschitz function supported on  $B$  and such that  $0 \leq \eta \leq 1$ . We add the term

$$\int |\eta_{x_1}|^2 W_1^{\frac{p}{2} + s} dx,$$

on both sides of the previous inequality and observe that

$$\int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)_{x_1} \right|^2 \eta^2 dx + \int W_1^{\frac{p}{2} + s} |\eta_{x_1}|^2 dx \geq \frac{1}{2} \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_1} \right|^2 dx.$$

We thus obtain

$$(4.4) \quad \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_1} \right|^2 dx \leq C \sum_{i=1}^2 \int W_i^{\frac{p-2}{2}} W_1^{s+1} |\nabla \eta|^2 dx + C(s+1)^2 \int |f_\varepsilon|^2 W_1^s \eta^2 dx,$$

with  $C = C(p, \delta) > 0$ .

The main problem of the Caccioppoli inequality (3.13) is that apparently we can not use it to control the *missing term*

$$\left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)_{x_2}.$$

Thus there is an obstruction to derive estimates for  $\nabla W_1^{\frac{p}{4} + \frac{s}{2}}$  which could lead to an iterative scheme of reverse Hölder inequalities. In order to overcome this problem, we observe that

$$\left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)_{x_2} \right| = \frac{p+2s}{p} \left| \left( W_1^{\frac{p}{4}} \right)_{x_2} \right| W_1^{\frac{s}{2}}.$$

Then if we fix  $1 < q < 2$ , by Hölder inequality with exponents  $2/q$  and  $2/(2-q)$ , we have

$$\left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)_{x_2} \right|^q \eta^q dx \right)^{\frac{2}{q}} \leq \left( \frac{p+2s}{p} \right)^2 \left( \int \left| \left( W_1^{\frac{p}{4}} \right)_{x_2} \right|^2 \eta^2 dx \right) \left( \int_{\text{spt}(\eta)} W_1^{\frac{q}{2-q} s} dx \right)^{\frac{2-q}{q}}.$$

The precise value of  $q$  will be specified later. We now add the term

$$\left( \int W_1^{\frac{pq}{4} + \frac{sq}{2}} |\eta_{x_2}|^q dx \right)^{\frac{2}{q}},$$

on both sides of the previous inequality and observe that by triangle inequality

$$\left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)_{x_2} \right|^q \eta^q dx \right)^{\frac{2}{q}} + \left( \int W_1^{\frac{pq}{4} + \frac{sq}{2}} |\eta_{x_2}|^q dx \right)^{\frac{2}{q}} \geq \frac{1}{2} \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_2} \right|^q dx \right)^{\frac{2}{q}}.$$

Thus we get

$$(4.5) \quad \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_2} \right|^q dx \right)^{\frac{2}{q}} \leq C(1+s)^2 \left( \int \left| \left( W_1^{\frac{p}{4}} \right)_{x_2} \right|^2 \eta^2 dx \right) \left( \int_{\text{spt}(\eta)} W_1^{\frac{q}{2-q} s} dx \right)^{\frac{2-q}{q}} + C \left( \int W_1^{\frac{pq}{4} + \frac{sq}{2}} |\eta_{x_2}|^q dx \right)^{\frac{2}{q}},$$

with  $C = C(p) > 0$ . For  $0 < r < R < R_0$ , we now take  $\eta \in W_0^{1,\infty}(B_R)$  to be the standard cut-off function

$$\eta(x) = \min \left\{ 1, \frac{(R - |x|)_+}{R - r} \right\},$$

then by multiplying (4.4) and (4.5) we get

$$\begin{aligned}
(4.6) \quad & \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_1} \right|^2 dx \right) \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_2} \right|^q dx \right)^{\frac{2}{q}} \\
& \leq C \left[ \frac{1}{(R-r)^2} \sum_{i=1}^2 \int_{B_R} W_i^{\frac{p-2}{2}} W_1^{s+1} dx + (s+1)^2 \int_{B_R} |f_\varepsilon|^2 W_1^s dx \right] \\
& \times \left[ (s+1)^2 \left( \int_{B_R} \left| \left( W_1^{\frac{p}{4}} \right)_{x_2} \right|^2 dx \right) \left( \int_{B_R} W_1^{\frac{q}{2-q} s} dx \right)^{\frac{2-q}{q}} \right. \\
& \left. + \frac{1}{(R-r)^2} \left( \int_{B_R} W_1^{\frac{pq}{4} + \frac{sq}{2}} dx \right)^{\frac{2}{q}} \right].
\end{aligned}$$

We now estimate the terms appearing in (4.6). To this aim, it will be useful to introduce the quantity

$$\begin{aligned}
(4.7) \quad \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) &= \sum_{i=1}^2 \left[ \int_{B_{R_0}} W_i^{\frac{p}{2}} dx + \int_{B_{R_0}} |\nabla W_i^{\frac{p}{4}}|^2 dx \right] \\
&+ R_0^{\frac{2}{p}} \left( \int_{B_{R_0}} |f_\varepsilon|^{2p'} dx \right)^{\frac{1}{p'}}.
\end{aligned}$$

Then we start with the first term on the right-hand side of (4.6). Observe that

$$\sum_{i=1}^2 \int_{B_R} W_i^{\frac{p-2}{2}} W_1^{s+1} dx = \int_{B_R} W_1^{\frac{p}{2}} W_1^s dx + \int_{B_R} W_2^{\frac{p-2}{2}} W_1 W_1^s dx.$$

We use Hölder inequality in conjunction with Sobolev-Poincaré inequality<sup>8</sup>, to get

$$\begin{aligned}
(4.8) \quad \int_{B_R} W_1^{\frac{p}{2}} W_1^s dx &\leq C \left[ \int_{B_{R_0}} W_1^{\frac{p}{2}} dx + \int_{B_{R_0}} |\nabla W_1^{\frac{p}{4}}|^2 dx \right] R_0^{\frac{2}{p'}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \\
&\leq C \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) R_0^{\frac{2}{p'}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}},
\end{aligned}$$

<sup>8</sup>Since we are in dimension  $N = 2$ , we have  $W^{1,2}(B_{R_0}) \hookrightarrow L^{2p'}(B_{R_0})$ . Then we have

$$\left( \int_{B_R} \left( W^{\frac{p}{4}} \right)^{2p'} dx \right)^{\frac{1}{p'}} \leq C R^{\frac{2}{p'}} \left[ \int_{B_R} \left( W^{\frac{p}{4}} \right)^2 dx + \int_{B_R} |\nabla W^{\frac{p}{4}}|^2 dx \right],$$

with a constant  $C = C(p) > 0$ .

and

$$\begin{aligned}
\int_{B_R} W_2^{\frac{p-2}{2}} W_1 W_1^s dx &\leq C \left[ \sum_{i=1}^2 \left( \int_{B_{R_0}} \left( W_i^{\frac{p}{4}} \right)^{2p'} dx \right)^{\frac{1}{p'}} \right] \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \\
(4.9) \quad &\leq C \left\{ \sum_{i=1}^2 \left[ \int_{B_{R_0}} W_i^{\frac{p}{2}} dx + \int_{B_{R_0}} |\nabla W_i^{\frac{p}{4}}|^2 dx \right] \right\} R_0^{\frac{2}{p'}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \\
&\leq C \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) R_0^{\frac{2}{p'}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}},
\end{aligned}$$

for some constant  $C = C(p) > 0$  depending only on  $p$ .

The term containing  $f_\varepsilon$  in (4.6) is estimated as follows. Observe that<sup>9</sup>

$$W^{1,p'}(B_{R_0}) \hookrightarrow L^{2p'}(B_{R_0}), \quad \text{since } 2p' < (p')^*,$$

then by Hölder's inequality and the definition of  $\mathcal{I}(W_1, W_2, f_\varepsilon; R_0)$

$$\begin{aligned}
(4.10) \quad \int_{B_R} |f_\varepsilon|^2 W_1^s dx &\leq \left( \int_{B_{R_0}} |f_\varepsilon|^{2p'} dx \right)^{\frac{1}{p'}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \\
&\leq \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) R_0^{-\frac{2}{p}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}}
\end{aligned}$$

For the last term on the right-hand side of (4.6), by Hölder inequality and estimate (4.8) we have

$$\begin{aligned}
(4.11) \quad \left( \int_{B_R} W_1^{\frac{pq}{4} + \frac{sq}{2}} dx \right)^{\frac{2}{q}} &\leq C R_0^{2\left(\frac{2}{q}-1\right)} \int_{B_R} W_1^{\frac{p}{2}} W_1^s dx \\
&\leq C R_0^{2\left(\frac{2}{q}-\frac{1}{p}\right)} \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}},
\end{aligned}$$

where  $C = C(q) > 0$ .

Finally, for the left-hand side of (4.6), we have

$$(4.12) \quad \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_1} \right|^2 dx \right) \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_2} \right|^q dx \right)^{\frac{2}{q}} \geq \mathcal{T}_q^2 \left( \int \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)^{\bar{q}^*} dx \right)^{\frac{4}{\bar{q}^*}}.$$

Here we used the *anisotropic Sobolev-Troisi inequality* (see Appendix B) for the compactly supported function  $W_1^{(p+2s)/4} \eta$ . The exponent  $\bar{q}^*$  is defined by

$$\bar{q}^* = \frac{2\bar{q}}{2-\bar{q}}, \quad \text{where} \quad \frac{1}{\bar{q}} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{q} \right),$$

so that

$$\bar{q} = \frac{4q}{2+q} \quad \text{and} \quad \bar{q}^* = \frac{4q}{2-q},$$

the constant  $\mathcal{T}_q$  only depends on  $q$  and it converges to 0 as  $q$  goes to 2.

<sup>9</sup>The exponent  $2p'$  is well-defined even in the case  $p = 2$ .

By using (4.8), (4.9), (4.10), (4.11) and (4.12) in (4.6), we then arrive at

$$\begin{aligned}
\left( \int_{B_r} \left( W_1^{\frac{p}{2}+s} \right)^{\frac{2q}{2-q}} dx \right)^{\frac{2-q}{q}} &\leq C \left[ \left( \frac{R_0}{R-r} \right)^2 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) R_0^{-\frac{2}{p}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \right. \\
&\quad \left. + (s+1)^2 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) R_0^{-\frac{2}{p}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \right] \\
(4.13) \quad &\times \left[ (s+1)^2 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) \left( \int_{B_R} W_1^{\frac{q}{2-q} s} dx \right)^{\frac{2-q}{q}} \right. \\
&\quad \left. + \left( \frac{R_0}{R-r} \right)^2 R_0^{2\left(\frac{2}{q}-\frac{1}{p}-1\right)} \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{1}{p}} \right],
\end{aligned}$$

for a constant  $C = C(p, q, \delta) > 0$ . We now choose  $1 < q < 2$  as follows

$$(4.14) \quad q = \frac{2p}{p+1}.$$

Observe that with such a choice, we have

$$\frac{q}{2-q} = p \quad \text{and} \quad \frac{2}{q} - \frac{1}{p} - 1 = 0.$$

We further observe that

$$\left( W_1^{\frac{p}{2}+s} \right)^{2p} \geq W_1^{2sp},$$

since  $W_i \geq 1$ . Then (4.13) becomes

$$\left( \int_{B_r} W_1^{2ps} dx \right)^{\frac{1}{p}} \leq C \mathcal{I}(W_1, W_2, f_\varepsilon; R_0)^2 \left[ \left( \frac{R_0}{R-r} \right)^2 + (s+1)^2 \right]^2 R_0^{-\frac{2}{p}} \left( \int_{B_R} W_1^{sp} dx \right)^{\frac{2}{p}}.$$

By using that  $R_0/(R-r) \geq 1$  and  $(s+1) \geq 1$  and introducing the notation  $\vartheta = ps$ , then the previous estimate finally gives

$$(4.15) \quad \|W_1\|_{L^{2\vartheta}(B_r)} \leq \left[ C \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) \left( \frac{R_0}{R-r} \right)^2 \left( \frac{\vartheta}{p} + 1 \right)^2 \right]^{\frac{p}{\vartheta}} R_0^{-\frac{1}{\vartheta}} \|W_1\|_{L^\vartheta(B_R)},$$

possibly for a different constant  $C = C(p, \delta) > 0$ . *This is the iterative scheme of reverse Hölder inequalities needed to launch a Moser's iteration.*

We then fix the two radii  $R_0 > r_0 > 0$  of the statement and consider the sequences

$$r_k = r_0 + \frac{R_0 - r_0}{2^k} \quad \text{and} \quad \vartheta_k = 2\vartheta_{k-1} = 2^k \vartheta_0 = 2^{k-1} p.$$

Then iterating (4.15) infinitely many times with  $R = r_k$  and  $r = r_{k+1}$ , we get

$$\|W_1\|_{L^\infty(B_{r_0})} \leq C \left( \frac{R_0}{R_0 - r_0} \right)^8 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0)^4 \left( \int_{B_{R_0}} W_1^{\frac{p}{2}} dx \right)^{\frac{2}{p}},$$

for some constant  $C = C(p, \delta) > 0$ . We notice that  $u_{x_1}^2 \leq W_1 \leq u_{x_1}^2 + \delta^2$ , by definition of  $W_1$ . Then we obtain with simple manipulations

$$\|u_{x_1}\|_{L^\infty(B_{r_0})} \leq C \left( \frac{R_0}{R_0 - r_0} \right)^4 \mathcal{I}(W_1, W_2, f_\varepsilon; R_0)^2 \left[ \left( \int_{B_{R_0}} |u_{x_1}|^p dx \right)^{\frac{1}{p}} + \delta \right],$$

for a possibly different constant  $C = C(p, \delta) > 0$ . Finally, by Corollary 3.4 the term  $\mathcal{I}(W_1, W_2, f_\varepsilon; R_0)$  defined in (4.7) can be estimated as follows

$$\begin{aligned} \mathcal{I}(W_1, W_2, f_\varepsilon; R_0) &\leq C \left( \frac{\rho_0}{\rho_0 - R_0} \right)^2 \left[ \sum_{j=1}^2 \int_{B_{\rho_0}} |u_{x_j}|^p dx + \delta^p \right] \\ &\quad + C \rho_0^{\frac{2}{p-1}} \int_{B_{\rho_0}} |\nabla f_\varepsilon|^{p'} dx + \rho_0^{\frac{2}{p}} \left( \int_{B_{\rho_0}} |f_\varepsilon|^{2p'} dx \right)^{\frac{1}{p'}}. \end{aligned}$$

This concludes the proof.  $\square$

**Remark 4.2.** Observe that the previous strategy does not seem to work for  $N \geq 3$ . Indeed, in this case we would have  $N - 1$  missing terms, i.e.

$$\partial_{x_i} W_1^{\frac{p}{4} + \frac{s}{2}}, \quad i = 2, \dots, N.$$

By proceeding as before for each of these terms, i.e. combining (3.9) and Hölder inequality, one would have on the left-hand side the term

$$\left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_1} \right|^2 dx \right) \prod_{i=2}^N \left( \int \left| \left( W_1^{\frac{p}{4} + \frac{s}{2}} \eta \right)_{x_i} \right|^q dx \right)^{\frac{2}{q}},$$

which in turn can be estimated from below by Sobolev-Troisi inequality by

$$\left( \int_{B_r} \left( W_1^{\frac{p}{4} + \frac{s}{2}} \right)^{\bar{q}^*} dx \right)^{\frac{2}{\bar{q}^*}}.$$

The right-hand side would still contain the term

$$\left( \int_{B_R} W_1^{\frac{q}{2-q} s} dx \right)^{\frac{2-q}{q}}.$$

The exponent  $\bar{q}^*$  is now defined by

$$\bar{q}^* = \frac{N \bar{q}}{N - \bar{q}}, \quad \text{where} \quad \frac{1}{\bar{q}} = \frac{1}{N} \left( \frac{1}{2} + \frac{N-1}{q} \right),$$

so that

$$\bar{q} = \frac{2Nq}{2N + q - 2} \quad \text{and} \quad \bar{q}^* = \frac{2Nq}{2N - q - 2}.$$

Then Moser's iteration would work if

$$\frac{\bar{q}^*}{2} s > \frac{q}{2-q} s \quad \iff \quad q < \frac{2}{N-1}.$$

Of course, when  $N \geq 3$  the last condition does not fit with the requirement  $q > 1$ .

## 5. PROOF OF THEOREM B

*Proof.* The proof is the same as that of Theorem A. The essential point is the uniform Lipschitz estimate of Proposition 5.1 below, which replaces that of Proposition 4.1.  $\square$

**Proposition 5.1** (Uniform Lipschitz estimate,  $p \geq 4$ ). *Let  $N \geq 3$  and  $p \geq 4$ . For every pair of concentric balls  $B_{r_0} \Subset B_{R_0} \Subset B$ , we have*

$$(5.1) \quad \|\nabla u_\varepsilon\|_{L^\infty(B_{r_0})} \leq C_8,$$

where  $C_8 = C_8(N, p, \delta, R_0 - r_0, M, \|f\|_{W^{1,\infty}(2B)}) > 0$  does not depend on  $\varepsilon$ . Here the constant  $M$  is the same appearing in (2.12).

*Proof.* As usual, for notational simplicity we simply write  $u$  in place of  $u_\varepsilon$ . By Lemma 2.8, we get that  $u$  is indeed a local  $C^3$  solution of the equation (3.1) in  $B$ , i.e. it verifies

$$(5.2) \quad \sum_{i=1}^N (g'_{i,\varepsilon}(u_{x_i}))_{x_i} = f_\varepsilon, \quad \text{in } B',$$

for every  $B' \Subset B$ . This means that pointwise we have

$$(5.3) \quad \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_i x_i} = f_\varepsilon \quad \text{in } B'.$$

We now derive the previous equation with respect to  $x_j$  and obtain

$$(5.4) \quad \sum_{i=1}^N \left[ g'''_{i,\varepsilon}(u_{x_i}) u_{x_i x_i} u_{x_i x_j} + g''_{i,\varepsilon}(u_{x_i}) u_{x_i x_i x_j} \right] = (f_\varepsilon)_{x_j}, \quad \text{in } B'.$$

We introduce the following linear differential operator

$$(5.5) \quad L[\psi] = \sum_{i=1}^N \left[ g'''_{i,\varepsilon}(u_{x_i}) u_{x_i x_i} \psi_{x_i} + g''_{i,\varepsilon}(u_{x_i}) \psi_{x_i x_i} \right],$$

then (5.4) can be simply written as  $L[u_{x_j}] = (f_\varepsilon)_{x_j}$ . Also observe that

$$L[\varphi \psi] = \varphi L[\psi] + \psi L[\varphi] + 2 \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) \varphi_{x_i} \psi_{x_i}.$$

Thus for  $u_{x_j}^2$  we obtain

$$L[u_{x_j}^2] = 2 u_{x_j} L[u_{x_j}] + 2 \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) \left[ (u_{x_j})_{x_i} \right]^2 = 2 u_{x_j} (f_\varepsilon)_{x_j} + 2 \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_j x_i}^2.$$

By linearity of  $L$  we thus get

$$L[|\nabla u|^2] = 2 \sum_{j=1}^N u_{x_j} (f_\varepsilon)_{x_j} + 2 \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_j x_i}^2.$$

We now fix a pair of concentric balls  $B_{r_0} \Subset B_{R_0} \Subset B$  as in the statement of Proposition 5.1. Let  $\zeta \in C_0^2(B_{R_0})$  be a function such that  $0 \leq \zeta \leq 1$  and

$$(5.6) \quad \zeta = 1 \text{ on } B_{r_0}, \quad |\nabla \zeta|^2 \leq \frac{C}{(R_0 - r_0)^2} \zeta \quad \text{and} \quad |D^2 \zeta| \leq \frac{C}{(R_0 - r_0)^2},$$

and consider in  $B_{R_0}$  the equation for the function  $\zeta |\nabla u|^2 + \lambda u^2$ . The crucial parameter  $\lambda$  will be chosen later. By using the product rule for  $L$  and its linearity, we get

$$\begin{aligned}
L[\zeta |\nabla u|^2 + \lambda u^2] &= \zeta L[|\nabla u|^2] + |\nabla u|^2 L[\zeta] + 2 \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})(u_{x_j}^2)_{x_i} \zeta_{x_i} + \lambda L[u^2] \\
&= 2 \sum_{j=1}^N u_{x_j} (f_\varepsilon)_{x_j} \zeta + 2 \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_j}^2 \zeta_{x_i} \\
&\quad + |\nabla u|^2 L[\zeta] + 2 \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i})(u_{x_j}^2)_{x_i} \zeta_{x_i} \\
&\quad + 2 u \lambda L[u] + 2 \lambda \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_i}^2.
\end{aligned}$$

By using the expression (5.5) of  $L$  and the equation (5.3), we can rewrite the previous identity as follows

$$(5.7) \quad L[\zeta |\nabla u|^2 + \lambda u^2] = 2\mathcal{I} + 2\zeta \mathcal{G}_1 + 2\mathcal{G}_2 + 2\lambda \mathcal{G}_3 + \mathcal{G}_4,$$

where we used the notation

$$\begin{aligned}
\mathcal{I} &= \lambda u f_\varepsilon + \zeta \sum_{j=1}^N u_{x_j} (f_\varepsilon)_{x_j}, & \mathcal{G}_1 &= \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i}) |u_{x_j} x_i|^2 \\
\mathcal{G}_2 &= \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i}) (|u_{x_j}|^2)_{x_i} \zeta_{x_i} + \frac{|\nabla u|^2}{2} \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) \zeta_{x_i} x_i, & \mathcal{G}_3 &= \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) |u_{x_i}|^2,
\end{aligned}$$

and

$$\mathcal{G}_4 = \sum_{i=1}^N g'''_{i,\varepsilon}(u_{x_i}) u_{x_i} x_i [2\lambda u u_{x_i} + |\nabla u|^2 \zeta_{x_i}].$$

We proceed to estimate separately each term on the right-hand side of (5.7).

#### The term $\mathcal{I}$ .

For this, by Young inequality we get

$$\begin{aligned}
(5.8) \quad \mathcal{I} &\geq -\lambda \|u\|_{L^\infty(B_{R_0})} \|f_\varepsilon\|_{L^\infty(B_{R_0})} - N \zeta |\nabla u| |\nabla f_\varepsilon| \\
&\geq -\lambda M \|f\|_{L^\infty(2B)} - \frac{N}{p} \zeta |\nabla u|^p - \frac{N}{p'} \zeta \|\nabla f\|_{L^\infty(2B)}^{p'},
\end{aligned}$$

where  $M$  is the constant appearing in (2.12). We also used that  $\|f_\varepsilon\|_{W^{1,\infty}(B_{R_0})} \leq \|f\|_{W^{1,\infty}(2B)}$ .

#### The term $\mathcal{G}_1$ .

This is a positive term and for the moment we simply keep it. It will act as a sponge term, in order to absorb (negative) terms containing the Hessian of  $u$ .

#### The term $\mathcal{G}_2$ .

This can be estimated by Young inequality and (5.6) as follows

$$\begin{aligned} \mathcal{G}_2 &\geq -\frac{\tau}{2} \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_j x_i}^2 \zeta_{x_i}^2 - \frac{1}{2\tau} \sum_{i,j=1}^N g''_{i,\varepsilon}(u_{x_i}) u_{x_j}^2 \\ &\quad - \frac{|\nabla u|^2}{2} \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) |\zeta_{x_i x_i}| \\ &\geq -\tau \frac{C}{2(R_0 - r_0)^2} \zeta \mathcal{G}_1 - \frac{|\nabla u|^2}{2} \left( \frac{C}{(R_0 - r_0)^2} + \frac{1}{\tau} \right) \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}), \end{aligned}$$

where  $\tau < 1$  is a small positive parameter. We then observe that the last term can be further estimated by using

$$(5.9) \quad g''_{i,\varepsilon}(u_{x_i}) \leq (p-1) |\nabla u|^{p-2} + 1,$$

so that

$$|\nabla u|^2 \sum_{j=1}^N g''_{i,\varepsilon}(u_{x_i}) \leq N((p-1) |\nabla u|^p + |\nabla u|^2) \leq N \left( p-1 + \frac{2}{p} \right) |\nabla u|^p + N \frac{p-2}{p}.$$

In the end we get

$$(5.10) \quad \mathcal{G}_2 \geq -\tau C'_1 \zeta \mathcal{G}_1 - \frac{C'_2}{2\tau} |\nabla u|^p - \frac{C'_2}{2\tau},$$

where  $C'_1 = C'_1(C, R_0 - r_0) > 0$  and  $C'_2 = C'_2(p, N, C, R_0 - r_0) > 0$ .

### The term $\mathcal{G}_3$ .

By using the form of  $g_{i,\varepsilon}$ , the convexity of the map  $m \mapsto m^{p-2}$  and recalling the definition (2.1) of  $\delta$ , we have

$$\mathcal{G}_3 = \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) |u_{x_i}|^2 \geq (p-1) \left( \frac{1}{2^{p-3}} \sum_{i=1}^N |u_{x_i}|^p - \delta^{p-2} |\nabla u|^2 \right).$$

By further applying Young inequality to estimate the term  $|\nabla u|^2$  and using that

$$\sum_{i=1}^N |u_{x_i}|^p \geq N^{\frac{2-p}{2}} |\nabla u|^p,$$

we end up with

$$(5.11) \quad \mathcal{G}_3 \geq C''_1 |\nabla u|^p - C''_2,$$

where  $C''_i = C''(p, N, \delta) > 0$ ,  $i = 1, 2$ .

### The term $\mathcal{G}_4$

This is the most delicate term and *it is precisely here that the condition  $p \geq 4$  becomes vital*. First we have

$$\mathcal{G}_4 \geq - \sum_{i=1}^N |g'''_{i,\varepsilon}(u_{x_i})| |u_{x_i x_i}| [2\lambda |u| |u_{x_i}| + |\nabla u|^2 |\zeta_{x_i}|].$$

Then we observe that by Cauchy-Schwarz inequality (recall the definition (2.6) of  $g_{i,\varepsilon}$ ) we have

$$\begin{aligned} |\nabla u|^2 \sum_{i=1}^N |g_{i,\varepsilon}'''(u_{x_i})| |u_{x_i x_i}| |\zeta_{x_i}| &\leq c |\nabla u|^2 \left( \sum_{i=1}^N (|u_{x_i}| - \delta_i)_+^{p-2} u_{x_i x_i}^2 |\zeta_{x_i}|^2 \right)^{\frac{1}{2}} \\ &\quad \times \left( \sum_{i=1}^N (|u_{x_i}| - \delta_i)_+^{p-4} \right)^{\frac{1}{2}} \\ &\leq c \left( \sum_{i=1}^N (|u_{x_i}| - \delta_i)_+^{p-2} u_{x_i x_i}^2 |\zeta_{x_i}|^2 \right)^{\frac{1}{2}} |\nabla u|^{\frac{p}{2}} \end{aligned}$$

for some constant  $c > 0$  depending on  $p$  and  $N$  only. In the last inequality we used that

$$(|u_{x_i}| - \delta_i)_+^{p-4} \leq |u_{x_i}|^{p-4},$$

since  $p \geq 4$ . By further using (5.6), the definition of  $g_{i,\varepsilon}$  and Young inequality, from the previous inequality we get

$$|\nabla u|^2 \sum_{i=1}^N |g_{i,\varepsilon}'''(u_{x_i})| |u_{x_i x_i}| |\zeta_{x_i}| \leq c (\zeta \mathcal{G}_1)^{\frac{1}{2}} |\nabla u|^{\frac{p}{2}} \leq \tau C_1''' \zeta \mathcal{G}_1 + \frac{C_1'''}{\tau} |\nabla u|^p,$$

for some constant  $C_1''' = C_1'''(C, N, p, R_0 - r_0) > 0$ . Similarly, we have

$$\begin{aligned} 2\lambda \sum_{i=1}^N |g_{i,\varepsilon}'''(u_{x_i})| |u_{x_i x_i}| |u| |u_{x_i}| &\leq c \lambda M \left( \sum_{i=1}^N g_{i,\varepsilon}''(u_{x_i}) u_{x_i x_i}^2 \right)^{\frac{1}{2}} \left( \sum_{i=1}^N (|u_{x_i}| - \delta_i)_+^{p-4} |u_{x_i}|^2 \right)^{\frac{1}{2}} \\ &\leq c \lambda \mathcal{G}_1^{\frac{1}{2}} |\nabla u|^{\frac{p-2}{2}} \leq C_2''' \lambda^{\frac{2p}{p+2}} \mathcal{G}_1^{\frac{p}{p+2}} + C_2''' |\nabla u|^p, \end{aligned}$$

for some constant  $C_2''' = C_2'''(N, p, M) > 0$ . By keeping everything together, we get

$$(5.12) \quad \mathcal{G}_4 \geq -\tau C_1''' \zeta \mathcal{G}_1 - \left( \frac{C_1'''}{\tau} + C_2''' \right) |\nabla u|^p - C_2''' \lambda^{\frac{2p}{p+2}} \mathcal{G}_1^{\frac{p}{p+2}}.$$

### Collecting all the estimates.

We now go back to (5.7) and use (5.8), (5.10), (5.11) and (5.12). Then we get

$$\begin{aligned} L[\zeta |\nabla u|^2 + \lambda u^2] &\geq \left[ 2 - \tau (2C_1' + C_1''') \right] \zeta \mathcal{G}_1 - C_2''' \lambda^{\frac{2p}{p+2}} \mathcal{G}_1^{\frac{p}{p+2}} \\ &\quad + \left( 2\lambda C_1'' - \frac{C_2'}{\tau} - 2\zeta \frac{N}{p} - \frac{C_1'''}{\tau} - C_2''' \right) |\nabla u|^p \\ &\quad - \left( 2M\lambda \|f\|_{L^\infty(2B)} + 2\zeta \frac{N}{p'} \|\nabla f\|_{L^\infty(2B)}^{p'} + \frac{C_2'}{\tau} + 2\lambda C_2'' \right). \end{aligned}$$

We now choose  $\tau$  small enough, in order to make the coefficient of  $\zeta \mathcal{G}_1$  strictly positive. Then we also choose  $\lambda \gg 1$  large enough, so that  $|\nabla u|^p$  as well has a strictly positive coefficient. Observe that the choices of  $\tau$  and  $\lambda$  only depend on the relevant data of the problem and are in particular independent of  $\varepsilon$ . By setting for simplicity  $\lambda^{2p/(p+2)} = \Lambda^2$ , this gives

$$(5.13) \quad L[\zeta |\nabla u|^2 + \lambda u^2] \geq \tilde{c}_1 \zeta \mathcal{G}_1 - \tilde{c}_2 \Lambda^2 \mathcal{G}_1^{\frac{p}{p+2}} + \tilde{c}_1 |\nabla u|^p - \tilde{c}_2,$$

where  $\tilde{c}_1 > 0$  and  $\tilde{c}_2 > 0$  are constants that depend only on  $\|f\|_{W^{1,\infty}(2B)}$ ,  $R_0 - r_0$ ,  $N$ ,  $p$ ,  $\delta$  and the constant  $M$  appearing in (2.12).

Let us now consider the maximum of the function  $\zeta |\nabla u|^2 + \lambda u^2$  in  $B_{R_0}$ . If this maximum is assumed at  $x_0 \in \partial B_{R_0}$ , then we get

$$\begin{aligned} \max_{B_{r_0}} |\nabla u|^2 &\leq \max_{B_{r_0}} \left[ \zeta |\nabla u|^2 + \lambda u^2 \right] \leq \max_{B_{R_0}} \left[ \zeta |\nabla u|^2 + \lambda u^2 \right] \\ &= \zeta(x_0) |\nabla u(x_0)|^2 + \lambda u(x_0)^2 \leq \lambda \|u\|_{L^\infty(B_{R_0})}^2 \leq \lambda M, \end{aligned}$$

thanks to the fact that  $\zeta = 1$  on  $B_{r_0}$  and  $\zeta \equiv 0$  on  $\partial B_{R_0}$ . This would prove the local Lipschitz estimate.

In order to conclude, let us now assume that  $x_0 \in B_{R_0}$ , then we get

$$\nabla (\zeta |\nabla u|^2 + \lambda u^2) = 0 \quad \text{at } x = x_0,$$

and

$$D^2 (\zeta |\nabla u|^2 + \lambda u^2) \leq 0 \quad \text{at } x = x_0.$$

Thus at the maximum point  $x_0$  we have

$$L[\zeta |\nabla u|^2 + \lambda u^2] = \sum_{i=1}^N g''_{i,\varepsilon}(u_{x_i}) (\zeta |\nabla u|^2 + \lambda u^2)_{x_i x_i} \leq 0.$$

By combining this with (5.13), we then get

$$\tilde{c}_2 \geq \tilde{c}_1 \zeta \mathcal{G}_1 - \tilde{c}_2 \Lambda^2 \mathcal{G}_1^{\frac{p}{p+2}} + \tilde{c}_1 |\nabla u|^p.$$

We multiply the previous by  $\zeta(x_0)^{p/2} > 0$ , then by Young inequality once again we get

$$\begin{aligned} \tilde{c}_2 \zeta(x_0)^{\frac{p}{2}} &\geq \tilde{c}_1 \zeta(x_0)^{\frac{p+2}{2}} \mathcal{G}_1 - \tilde{c}_2 \Lambda^2 \left( \zeta(x_0)^{\frac{p+2}{2}} \mathcal{G}_1 \right)^{\frac{p}{p+2}} + \tilde{c}_1 \zeta(x_0)^{\frac{p}{2}} |\nabla u(x_0)|^p \\ &\geq \left( \tilde{c}_1 - \tilde{c}_2 \frac{p}{p+2} \tau \Lambda^2 \right) \zeta(x_0)^{\frac{p+2}{2}} \mathcal{G}_1 - \frac{2\tau^{-\frac{p}{2}}}{p+2} \Lambda^2 + \tilde{c}_1 \zeta(x_0)^{\frac{p}{2}} |\nabla u(x_0)|^p. \end{aligned}$$

If we choose

$$\tau = \frac{p+2}{p} \frac{\tilde{c}_1}{\tilde{c}_2} \frac{1}{\Lambda^2},$$

and use that  $\zeta \leq 1$ , we finally get

$$(5.14) \quad \zeta(x_0) |\nabla u(x_0)|^2 \leq \tilde{C},$$

with  $\tilde{C} = \tilde{C}(N, p, \delta, \|f\|_{W^{1,\infty}(2B)}, M, R_0 - r_0) > 0$ . By using this bound we get

$$\begin{aligned} \max_{B_{r_0}} |\nabla u|^2 &\leq \max_{B_{r_0}} \left[ \zeta |\nabla u|^2 + \lambda u^2 \right] \leq \zeta(x_0) |\nabla u(x_0)|^2 + \lambda u(x_0)^2 \\ &\leq \tilde{C} + \lambda u(x_0)^2 \leq \tilde{C} + \lambda M^2, \end{aligned}$$

which gives the desired conclusion.  $\square$

APPENDIX A. SOME PROPERTIES OF THE FUNCTIONS  $g_i$ 

The functions  $g_i$  have the following convexity property.

**Lemma A.1.** *For every  $t_1, t_2 \in \mathbb{R}$  such that  $|t_1 - t_2| > 2\delta_i$  we have*

$$(A.1) \quad g_i((1-s)t_1 + st_2) < (1-s)g_i(t_1) + sg_i(t_2), \quad s \in (0, 1), \quad i = 1, \dots, N.$$

*Proof.* The function  $g_i$  is convex so that the inequality  $\leq$  holds true for every  $t_1, t_2$ . If  $g_i((1-s)t_1 + st_2) = (1-s)g_i(t_1) + sg_i(t_2)$ , then  $g_i$  is affine on the segment  $[t_1, t_2]$ . This can only happen when  $t_1, t_2 \in [-\delta_i, \delta_i]$ , in which case  $|t_1 - t_2| \leq 2\delta_i$ .  $\square$

They also satisfy the following Lipschitz-type estimate.

**Lemma A.2.** *Let  $p \geq 2$ . For every  $t_1, t_2 \in \mathbb{R}$  and  $i = 1, \dots, N$ , we have*

$$(A.2) \quad |g_i(t_1) - g_i(t_2)| \leq (|t_1|^{p-1} + |t_2|^{p-1}) |t_1 - t_2|.$$

*Proof.* By basic calculus we have

$$|g_i(t_1) - g_i(t_2)| = |g_i'((1-s)t_1 + st_2)| |t_1 - t_2|,$$

for some  $s \in [0, 1]$ . Since  $p \geq 2$ , the function  $t \mapsto |g_i'(t)|$  is convex and

$$|g_i'(t)| = (|t - \delta_i|_+^{p-1}) \leq |t|^{p-1}, \quad t \in \mathbb{R}.$$

Thus we get the conclusion.  $\square$

The following basic estimate has been used various times.

**Lemma A.3.** *Let  $p \geq 2$ . For every  $i = 1, \dots, N$  and every  $T \geq \delta_i$ , we have*

$$g_i''(t) \geq (p-1) \left( \frac{T - \delta_i}{T} \right)^{p-2} (T^2 + (|t - T|_+)^2)^{\frac{p-2}{2}}, \quad \text{for every } |t| \geq T.$$

*Proof.* For  $T = \delta_i$  there is nothing to prove, thus we can suppose that  $T > \delta_i$ . We use the elementary inequality

$$\frac{T}{T - \delta_i} (|t - \delta_i|) \geq |t|, \quad \text{for every } |t| \geq T.$$

This implies that for every  $|t| \geq T$ , we have

$$\frac{T}{T - \delta_i} (|t - \delta_i|_+ \geq T + (|t - T|_+)^2)^{\frac{1}{2}}.$$

By multiplying everything by  $(T - \delta_i)/T$  and raising to the power  $p - 2$ , we get the desired conclusion.  $\square$

## APPENDIX B. AN ANISOTROPIC SOBOLEV INEQUALITY IN DIMENSION 2

In the proof of Proposition 4.1 we used Sobolev-Troisi inequality. For the reader's convenience, we give a proof of the particular case we needed.

**Lemma B.1.** *Let  $1 < q < 2$ , then for every  $u \in C_0^\infty(\mathbb{R}^2)$  we have*

$$(B.1) \quad \mathcal{T}_q \left( \int_{\mathbb{R}^2} |u|^{\frac{4q}{2-q}} dx \right)^{\frac{2-q}{2q}} \leq \left( \int_{\mathbb{R}^2} |u_{x_1}|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |u_{x_2}|^q dx \right)^{\frac{1}{q}},$$

where the constant  $\mathcal{T}_q$  is given by

$$\mathcal{T}_q = \frac{(2-q)^2}{4q^2 - (2-q)^2} > 0.$$

*Proof.* We first observe that for every  $\alpha, \beta > 1$ , by basic calculus we have

$$|u(x_1, x_2)|^\alpha = \alpha \int_{-\infty}^{x_1} u_{x_1}(t, x_2) |u(t, x_2)|^{\alpha-2} u(t, x_2) dt,$$

and

$$|u(x_1, x_2)|^\beta = \beta \int_{-\infty}^{x_2} u_{x_2}(x_1, s) |u(x_1, s)|^{\beta-2} u(x_1, s) ds.$$

Thus

$$|u(x_1, x_2)|^{\alpha+\beta} \leq \alpha \beta \left( \int_{\mathbb{R}} |u_{x_1}(t, x_2)| |u(t, x_2)|^{\alpha-1} dt \right) \left( \int_{\mathbb{R}} |u_{x_2}(x_1, s)| |u(x_1, s)|^{\beta-1} ds \right).$$

If we now integrate over  $\mathbb{R}^2$  and use Fubini Theorem on the right-hand side, we get

$$(B.2) \quad \int_{\mathbb{R}^2} |u|^{\alpha+\beta} dx \leq \alpha \beta \left( \int_{\mathbb{R}^2} |u_{x_1}| |u|^{\alpha-1} dx \right) \left( \int_{\mathbb{R}^2} |u_{x_2}| |u|^{\beta-1} dx \right).$$

By Hölder inequality we then have

$$\begin{aligned} & \left( \int_{\mathbb{R}^2} |u_{x_1}| |u|^{\alpha-1} dx \right) \left( \int_{\mathbb{R}^2} |u_{x_2}| |u|^{\beta-1} dx \right) \\ & \leq \left( \int_{\mathbb{R}^2} |u_{x_1}|^2 dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |u_{x_2}|^q dx \right)^{\frac{1}{q}} \\ & \quad \times \left( \int_{\mathbb{R}^2} |u|^{2(\alpha-1)} dx \right)^{\frac{1}{2}} \left( \int_{\mathbb{R}^2} |u|^{\frac{q}{q-1}(\beta-1)} dx \right)^{\frac{q-1}{q}}. \end{aligned}$$

We now choose  $\alpha$  and  $\beta$  in such a way that

$$2(\alpha - 1) = \alpha + \beta \quad \text{and} \quad \frac{q}{q-1}(\beta - 1) = \alpha + \beta,$$

that is

$$\alpha = \frac{q+2}{2-q} \quad \text{and} \quad \beta = \frac{3q-2}{2-q}.$$

Observe that with these choices we have  $\alpha + \beta = 4q/(2-q)$ . Thus from (B.2) we get (B.1), with

$$\mathcal{T}_q = \frac{1}{\alpha} \frac{1}{\beta} = \frac{(2-q)^2}{4q^2 - (2-q)^2},$$

as desired. □

## REFERENCES

- [1] M. Belloni, B. Kawohl, The pseudo  $p$ -Laplace eigenvalue problem and viscosity solution as  $p \rightarrow \infty$ , ESAIM Control Optim. Calc. Var., **10** (2004), 28–52.
- [2] M. Bildhauer, M. Fuchs, X. Zhong, A regularity theory for scalar local minimizers of splitting-type variational integrals, Ann. Sc. Norm. Super. Pisa Cl. Sci., **6** (2007), 385–404.
- [3] M. Bildhauer, M. Fuchs, X. Zhong, Variational integrals with a wide range of anisotropy, St. Petersburg Math. J., **18** (2007), 717–736.

- [4] I. Birindelli, F. Demengel, Lipschitz regularity for solutions of the pseudo  $p$ -Laplace Poisson equation, in preparation.
- [5] P. Bousquet, L. Brasco, Global Lipschitz continuity for minima of degenerate problems, in preparation.
- [6] P. Bousquet, F. Clarke, Local Lipschitz continuity of solutions to a problem in the calculus of variations, *J. Differential Equations*, **243** (2007), 489–503.
- [7] L. Brasco, Global  $L^\infty$  gradient estimates for solutions to a certain degenerate elliptic equation, *Nonlinear Anal.*, **72** (2011), 516–531.
- [8] L. Brasco, G. Carlier, Congested traffic equilibria and degenerate anisotropic PDEs, *Dyn. Games Appl.*, **3** (2013), 508–522.
- [9] L. Brasco, G. Carlier, On certain anisotropic elliptic equations arising in congested optimal transport: local gradient bounds, *Adv. Calc. Var.*, **7** (2014), 379–407.
- [10] P. Celada, G. Cupini, M. Guidorzi, Existence and regularity of minimizers of nonconvex integrals with  $p - q$  growth, *ESAIM Control Optim. Calc. Var.*, **13** (2007), 343–358.
- [11] M. Chipot, L. C. Evans, Linearization at infinity and Lipschitz estimates for certain problems in the calculus of variations, *Proc. R. Soc. Edinb. Sect. A*, **102** (1986), 291–303.
- [12] M. Colombo, A. Figalli, An excess–decay result for a class of degenerate elliptic equations, *Discrete Contin. Dyn. Syst. Ser. S*, **7** (2014), 631–652.
- [13] M. Colombo, A. Figalli, Regularity results for very degenerate elliptic equations, *J. Math. Pures Appl.*, **101** (2014), 94–117.
- [14] F. Demengel, Lipschitz interior regularity for the viscosity and weak solutions of the pseudo  $p$ -Laplacian, preprint (2014), available at <http://arxiv.org/abs/1409.0810>
- [15] L. Esposito, G. Mingione, C. Trombetti, On the Lipschitz regularity for certain elliptic problems, *Forum Math.* **18** (2006), 263–292.
- [16] I. Fonseca, N. Fusco, P. Marcellini, An existence result for a nonconvex variational problem via regularity, *ESAIM Control Optim. Calc. Var.* **7** (2002), 69–95.
- [17] N. Fusco, C. Sbordone, Some remarks on the regularity of minima of anisotropic integrals, *Commun. Partial Differ. Equations*, **18** (1993), 153–167.
- [18] E. Giusti, *Metodi diretti nel calcolo delle variazioni*. (Italian) [Direct methods in the calculus of variations], Unione Matematica Italiana, Bologna, 1994.
- [19] Q. Han, F. Lin, *Elliptic partial differential equations*. Second edition. Courant Lecture Notes in Mathematics, **1**. Courant Institute of Mathematical Sciences, New York, AMS, Providence, RI, 2011.
- [20] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*. (French) Dunod; Gauthier-Villars, Paris 1969.
- [21] P. Marcellini, Regularity of minimizers of integrals of the Calculus of Variations under non standard growth conditions, *Arch. Rational Mech. Anal.*, **105** (1989), 267–284.
- [22] F. Santambrogio, V. Vespi, Continuity in two dimensions for a very degenerate elliptic equation, *Nonlinear Anal.*, **73** (2010), 3832–3841.
- [23] G. Stampacchia, On some regular multiple integral problems in the calculus of variations, *Comm. Pure Appl. Math.*, **16** (1963), 383–421.
- [24] M. Troisi, Teoremi di inclusione per spazi di Sobolev non isotropi, *Ricerche Mat.*, **18** (1969), 3–24.
- [25] N. Uralt'seva, N. Urdaletova, The boundedness of the gradients of generalized solutions of degenerate quasilinear nonuniformly elliptic equations, *Vest. Leningr. Univ. Math.*, **16** (1984), 263–270.

AIX-MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE  
*E-mail address:* pierre.bousquet@univ-amu.fr

AIX-MARSEILLE UNIVERSITÉ, CNRS, CENTRALE MARSEILLE, I2M, UMR 7373, 13453 MARSEILLE, FRANCE  
*E-mail address:* lorenzo.brasco@univ-amu.fr

DEPARTMENT OF MATHEMATICS AND STATISTICS, UNIVERSITY OF JYVÄSKYLÄ, P.O. BOX 35 (MAD), 40014 JYVÄSKYLÄ, FINLAND  
*E-mail address:* vesa.julin@jyu.fi