



HAL
open science

Continuous-Discrete Observer for State Affine Systems With Sampled and Delayed Measurements

Tarek Ahmed-Ali, Vincent van Assche, Jean-François Massieu, Philippe
Dorléans

► **To cite this version:**

Tarek Ahmed-Ali, Vincent van Assche, Jean-François Massieu, Philippe Dorléans. Continuous-Discrete Observer for State Affine Systems With Sampled and Delayed Measurements. IEEE Transactions on Automatic Control, 2013, 58 (4), pp.1085-1091. 10.1109/TAC.2012.2225555 . hal-01062210

HAL Id: hal-01062210

<https://hal.science/hal-01062210>

Submitted on 10 Jul 2022

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Continuous-Discrete Observer for State Affine Systems With Sampled and Delayed Measurements

Tarek Ahmed-Ali, Vincent Van Assche, Jean-François Massieu, and
Philippe Dorléans

Abstract—The observation of a class of multi-input multi-output (MIMO) state affine systems with both sampled and delayed output measurements is addressed. These two constraints disturb simultaneously the convergence of the observer. Assuming some persistent excitation conditions to hold, and by using Lyapunov tools adapted to impulsive systems, two classes of global exponential observers are proposed. Some explicit relations between maximum allowable delay and maximum allowable sampling period are given. An extension to some classes of nonlinear systems is also given.

Index Terms—Continuous-discrete observer, delayed measurements, impulsive systems.

I. INTRODUCTION

This work is devoted to observer design for a class of uniformly observable systems with sampled and delayed measurements. In the last decades, the design of nonlinear observers for continuous systems with sampled measurements has received a great attention. This interest is motivated by many engineering applications, such as network control systems (NCSs) in which the output is transmitted over a shared digital communication network, and is only available at discrete-time instants. For linear systems it is usually possible to design observers by using the discrete time model of the continuous time system. This is

The authors are with the Laboratoire GREYC, UMR CNRS 6072, UFR Sciences, Dépt. Physique EEA, Université de Caen Basse-Normandie/ENSICAEN, Caen, France (e-mail: tarek.ahmed-ali@ensicaen.fr; vincent.vanassche@unicaen.fr; jean-francois.massieu@unicaen.fr; philippe.dorleans@unicaen.fr).

not always possible for nonlinear systems because the exact discrete time model is generally not available. In this case, there exist two main approaches dealing with this problem. The first one, is based on the design of a discrete observer by using a consistent approximation of the exact discretized model. This approach provides a semi-global practical stability of the observation error. More details on this method can be found in [1] and its references. The second one is based on a mixed continuous and discrete design. This approach has been inspired by Jazwinski [2] who introduced the continuous-discrete Kalman filter to solve a filtering problem for stochastic continuous-discrete time systems. It consists on two steps. In the first one (which is called the prediction step), the observer is a copy of the model system, whereas in the second step, the value of the state estimate is updated using the newly available sampled measure. The exponential convergence of the observation error is then ensured under some sufficient conditions on the sampling period through the stability analysis of impulsive systems. In [3] the authors use this approach to write a discrete-continuous version of the well known high gain observer [4].

In [5], observers for a MIMO class of state affine systems where the dynamical matrix depends on the inputs have been designed when the inputs are regularly persistent. This work was extended to adaptive observers in [6]. In [7], a similar method has been used for a larger class of systems and applied to the observation of an emulsion copolymerization process. The observation of a class of systems with output injection has been treated in [8] and recently, in [9], a high gain continuous-discrete observer has been developed by using constant observation gains. In [10], the authors extend the work of [11] to the discrete-time measurements case. Recently, a hybrid sampled-data observer dedicated to a class of nonlinear systems has been presented in [12]. This scheme is based on an inter-sample time predictor which estimates the output between two sampling instants. The advantage of this algorithm is in the fact that the estimates remain continuous and only the predictor is re-initialized at each sampling instant. This algorithm has been extended to some networked control systems in [13] by using a *Lyapunov Krasovskii* approach. On the other hand, the design of observers for linear detectable systems with sampled and delayed measurements was treated in [14] by using a descriptor system approach and a *Lyapunov Krasovskii* functional. The authors have proposed a hybrid observer for a class of linear systems and derive sufficient conditions based on *linear matrix inequalities* to guarantee exponential convergence of the observation error. This idea has also been used in [15] and [16] for some classes of nonlinear systems with nonuniformly sampled measurements.

In this technical note, we present two classes of observers. The first one is an extension to delayed measurements of the continuous-discrete observer developed in [5]. The second one is based on the introduction of an estimator of the delayed output between two updating instants. It can be viewed as an extension of the work of [17] to sampled-data case by using a prediction of the output between two sampling instants following the ideas developed in [12]. The main advantage of the second class compared to the first one is in the fact that the states of the observer remain continuous and only the estimator is re-initialized. This property will simplify the implementation compared to the first class. For both classes, we give explicit sufficient conditions on the delay which is considered unknown but bounded and the sampling period, to ensure exponential stability of the observation error. This second observer structure is also extended to a wide class of nonlinear globally Lipschitz systems. We also emphasize the fact that the result presented in this technical note can be easily extended to the case of scheduling protocols [13]. The present note is organized as follows: in Sections II–VI, lies some notations are introduced, then the problem is stated; in the fourth section we present our main results and in Section V we extend the second observer structure to a wide class of nonlinear systems.

II. NOMENCLATURE

First, some mathematical notations are introduced. Let $\mathbb{R} \stackrel{\text{def}}{=} (-\infty, \infty)$, $\mathbb{R}^+ \stackrel{\text{def}}{=} (0, \infty)$, $\mathbb{R}_0^+ \stackrel{\text{def}}{=} [0, \infty)$, and let $\|\cdot\|$ be the Euclidean norm. For $p, q, n, m \in \mathbb{N}$, $\mathbb{R}^{p \times q}$ represents the set of real matrices of order $p \times q$ and $\mathbb{I}_p \in \mathbb{R}^{p \times p}$ stands for the identity matrix of order $p \times p$. If $X \subset \mathbb{R}^{p \times q}$ and $Y \subset \mathbb{R}^{n \times m}$, $C(\mathcal{X}, \mathbf{Y})$ denotes the space of all continuous functions mapping $\text{mathscr{X}} \rightarrow \mathcal{Y}$. The notation $\|P\|$, for $P \in \mathbb{R}^{p \times q}$, represents the L_2 -norm of P and X' represent the transposed vector of X . We say that $\alpha \mathbb{I}_n \leq S \leq \beta \mathbb{I}_n$ where $S \in \mathbb{R}^{n \times n}$ if $\lambda_{\min}(S) \geq \alpha$ and $\lambda_{\max}(S) \leq \beta$ where $\lambda_{\min}(S)$ and $\lambda_{\max}(S)$ denote respectively the smallest and the biggest eigenvalues of the matrix S . For $f : \mathbb{R} \rightarrow \mathbb{R}^{p \times q}$ and $t \in \mathbb{R}$, the notation $f^-(t)$ denotes the left limit of f at instant t , if it exists. In all this study, the initial time is called $t_0 \in \mathbb{R}$.

III. PROBLEM STATEMENT

The following class of systems is considered:

$$\begin{cases} \dot{x}(t) = A(u)x(t) + b(u) \\ y(t) = Cx(t) \end{cases} \quad (1)$$

where $x \in \mathbb{R}^n$ is the instantaneous *state* vector, $u \in D \subset \mathbb{R}^m$ the *input* vector (D is a compact set) and $y \in \mathbb{R}^p$ the *output* vector. The matrix $A \in C(\mathbb{R}^m, \mathbb{R}^{n \times n})$, $b \in C(\mathbb{R}^m, \mathbb{R}^n)$, $C \in \mathbb{R}^{p \times n}$, $\Psi_u \in C(\mathbb{R}^m, \mathbb{R}^{n \times l})$ are known, with $n, m, p, l \in \mathbb{N}$ and $x_0 = x(t_0)$. In this work, we suppose that the measures of y are sampled at instants t_k and available for the observer only at instants $t_k + \tau_k$. The notation $(t_k)_{k \geq 0}$ represents a strictly increasing sequence such that $\lim_{k \rightarrow \infty} t_k = \infty$, and $\tau_k > 0$ represents the transmission delay. The sampling intervals are bounded with $0 \leq T_{\min} \leq t_k - t_{k-1} \leq T_{\max}$ for all $k = 1, 2, \dots, \infty$. The transmission delays τ_k are unknown, only an upper bound $\tau \geq \max(\tau_k)$ is known for all $k = 1, 2, \dots, \infty$. We also assume that $T_{\min} \geq \tau$. This assumption means that the measures sampled at t_k are available for the observer before the next measures sampled at t_{k+1} .

We will design two different classes of observers for systems (1), and provide for each of them upper bounds of the maximum allowable sampling period and the maximum allowable delay, so that the observation error converges globally exponentially towards zero.

IV. OBSERVERS DESIGN AND STABILITY STUDY

A. First Observation Structure

The following continuous-discrete observer is proposed. For $k \geq 1$, $t \in [t_{k-1} + \tau, t_k + \tau)$

$$\begin{cases} \dot{\hat{x}}(t) = A(u)\hat{x}(t) + b(u) \\ \dot{S}(t) = -A'(u)S(t) - S(t)A(u) - \mu S(t) \end{cases} \quad (2)$$

For $t = t_k + \tau$, with $T_k = t_k - t_{k-1}$

$$\begin{cases} S(t_k + \tau) = S^-(t_k + \tau) + T_k C' C \\ \hat{x}(t_k + \tau) = \hat{x}^-(t_k + \tau) \\ \quad + \rho T_k S^{-1}(t_k + \tau) C' (y(t_k) - C \hat{x}(t_k)) \end{cases} \quad (3)$$

The notation $\hat{x} \in \mathbb{R}^n$ denotes the estimate of the state x , $S(t) \in \mathbb{R}^{n \times n}$ is the so-called observation gain, and the parameters $\mu \in \mathbb{R}^+$ and $\rho \in [1, \infty)$ are some design parameters. Note that the existence of the inverse of matrix S will be ensured in the following. The initial conditions are denoted $\hat{x}_0 \in \mathbb{R}^n$ and S_0 is symmetric positive definite.

The observer (2) and (3) is composed of a predictor part (2) and a correction one (3). During $[t_{k-1} + \tau, t_k + \tau)$, the state estimate \hat{x} has the same dynamics as system (1). The matrix S is defined as the solu-

tion of the third differential equation in (2) and is updated at instants $t_k + \tau$ to ensure that it remains positive definite (see for example in [5]). At each instant $t_k + \tau$, observer trajectory is corrected taking into account the measure obtained at instant t_k . Thanks to (3), conditions on the inputs and restriction on T_{\max} , it will be shown that the matrix S remains positive definite. The dynamical equation satisfied by the state observation error, $e = \hat{x} - x$, is, in view of (1), (2) and (3), for $k \geq 1$

$$\begin{cases} \dot{e}(t) = A(u)e(t) & t \in [t_{k-1} + \tau, t_k + \tau) \\ e(t_k + \tau) = e^-(t_k + \tau) - \rho T_k S^{-1}(t_k + \tau) C' C e(t_k) \end{cases} \quad (4)$$

Before stating the main results, some preliminary definitions and results are required.

Definition 1: Consider the following system, for $t \in [t_0, \infty)$:

$$\begin{cases} \dot{x}(t) = A(u)x(t) \\ y(t) = Cx(t) \end{cases} \quad (5)$$

and the transition matrix $\Psi_u(\cdot, t_0)$ associated to system (5), such that $\Psi_u(t_0, t_0) = \mathbb{I}_n$. The bounded input u is said to be *regularly persistent* if there exist $t_a, t_b \in [t_0, \infty)$, $\delta \in \mathbb{R}^+$, such that, for all $t \geq t_b$

$$\int_t^{t+t_a} \Psi_u'(s, t_0) C' C \Psi_u(s, t_0) ds \geq \delta \mathbb{I}_n. \quad (6)$$

Regularly persistent inputs guarantee the system to be observable. For more details, see [18]–[20]. In the sequel, we will assume that the following hypothesis is fulfilled.

Hypothesis 1: The input u is regularly persistent and the constants t_a and δ are known.

Remark 1: The constants t_a and δ are preponderant for determining a bound on the admissible sampling period. Due to the difficulty to verify the condition on the observability Grammian, in practice, numerical tools can be used for this purpose.

1) *Technical Results:* The following proposition guarantees that the matrix S remains positive definite for any $t \in [t_0, \infty)$, under conditions on the types of inputs applied to the system and on the sampling period $T_k = t_k - t_{k-1}$.

Proposition 1: Let u be a regularly persistent input for system (1), for all $\mu \geq 2\xi$ where $\xi = \sup_{t \geq t_0} \|A(u(t))\|$, if $T_{\max} \leq \bar{T}$, where \bar{T} is the unique positive term such that $\bar{T} = (\delta e^{-2\xi(t_a + \bar{T})} / 4\xi \|C^T C\| (t_a + \bar{T}))$, then, for all $S(t_0)$ symmetric positive definite, there exist constants $\alpha, \beta \in \mathbb{R}^+$ such that, for $t \in [t_0, \infty)$

$$\alpha \mathbb{I}_n \leq S(t) \leq \beta \mathbb{I}_n. \quad (7)$$

The proof of Proposition 1 can be found in [5].

- Since considered inputs belong to a compact set \mathcal{S} , and because A is continuous, ξ is well defined.
- There always exists such a \bar{T} ; indeed writing condition $\bar{T} = \delta e^{-2\xi(t_a + \bar{T})} / 4\xi \|C^T C\| (t_a + \bar{T})$ as $4\xi \|C^T C\| (t_a + \bar{T}) \bar{T} = \delta e^{-2\xi(t_a + \bar{T})}$, loosely speaking it is clear that, for positive values of \bar{T} , the polynomial term on the left-hand side will ‘cross’ the exponential on the right-hand side at a unique point.
- Considering regularly persistent inputs, the positive definition of the solution S of equations (2) and (3) can only be ensured under condition on the sampling period, otherwise the solution of (2) may become non-positive definite [5].
- Notice that α, β implicitly depend on the maximum allowable sampling period T_{\max} .

2) *Stability Analysis:* Based on the above hypotheses, we are now able to state our main result.

Theorem 1: Let us consider system (1). Assume that hypothesis 1 holds and that $\mu > 2\xi$ as in Proposition 1. Then for all

$$T_{\max} \leq \min \left\{ \frac{\alpha (\frac{3}{2}\rho - 1)}{\|C\|^2 \rho^2}, \bar{T} \right\} \quad (8)$$

and

$$\tau < \min \left\{ T_{\min}, \frac{1}{\xi} \sqrt{\frac{\alpha (1 - e^{-\mu T_{\min}})}{\psi}} \right\} \quad (9)$$

with

$\psi = \rho T_{\max} \|C\|^2 \left[2 (\rho T_{\max} \|C\|^2 / \alpha)^2 + 5 (\rho T_{\max} \|C\|^2 / \alpha) + 2 \right]$, system (2) and (3) is a global exponential observer for system (1).

Proof: Note that condition (8) satisfies Proposition 1. This means that (7) is verified for all $t \geq t_0$.

Now, let us consider the following Lyapunov candidate function:

$$V(t) = e(t)' S(t) e(t). \quad (10)$$

As in [5], let us compute the time derivative of $V(t)$ for $t \in [t_{k-1} + \tau, t_k + \tau)$, then we have

$$\dot{V}(t) = -\mu V(t). \quad (11)$$

For $t = t_k + \tau$, first, by using the Leibniz integration formula

$$e^-(t_k + \tau) = e(t_k) + \int_{t_k}^{t_k + \tau} \dot{e}(s) ds$$

then we can rewrite (4) as follows:

$$\begin{aligned} e(t_k + \tau) &= e^-(t_k + \tau) - \rho T_k S^{-1}(t_k + \tau) C' C e^-(t_k + \tau) \\ &\quad + \rho T_k S^{-1}(t_k + \tau) C' C \int_{t_k}^{t_k + \tau} \dot{e}(s) ds \end{aligned} \quad (12)$$

which leads to

$$\begin{aligned} e(t_k + \tau) &= e^-(t_k + \tau) - \rho T_k S^{-1}(t_k + \tau) C' C e^-(t_k + \tau) \\ &\quad + \rho T_k S^{-1}(t_k + \tau) C' C \int_{t_k}^{t_k + \tau} A(u(s)) e(s) ds. \end{aligned} \quad (13)$$

Now, let us compute $V(t_k + \tau)$, then

$$\begin{aligned} V(t_k + \tau) &= e(t_k + \tau)' S(t_k + \tau) e(t_k + \tau) \\ &= e^-(t_k + \tau)' \Delta_1(t_k + \tau) e^-(t_k + \tau) \\ &\quad + e^-(t_k + \tau)' \Delta_2(t_k + \tau) I_0 \\ &\quad + I_0' \Delta_3(t_k + \tau) I_0 \end{aligned} \quad (14)$$

where

$$I_0 = \int_{t_k}^{t_k + \tau} A(u(s)) e(s) ds \quad (15)$$

and

$$\begin{aligned} \Delta_1(t_k + \tau) &= S(t_k + \tau) - 2\rho T_k C' C \\ &\quad + \rho^2 T_k^2 C' C S^{-1}(t_k + \tau) C' C \\ \Delta_2(t_k + \tau) &= 2\rho T_k (\mathbb{I}_n - \rho T_k C' C S^{-1}(t_k + \tau)) C' C I_0 \\ \Delta_3(t_k + \tau) &= \rho^2 T_k^2 C' C S^{-1}(t_k + \tau) C' C \end{aligned}$$

In the sequel, $e^-(t_k + \tau)$, $V(t_k + \tau)$, $V^-(t_k + \tau)$, $S(t_k + \tau)$, and $S^-(t_k + \tau)$ will be noted respectively e^- , V , V^- , S , and S^- .

Now let us consider the terms $e^{-'} \Delta_1 e^{-}$ and $e^{-'} \Delta_2 I_0$. After some computations we have

$$\begin{aligned} e^{-'} \Delta_1 e^{-} &= V^{-} + \rho^2 T_k^2 e^{-} C' C S^{-1} C' C e^{-} \\ &\quad - (2\rho - 1) T_k \|e_y^{-}\|^2 \\ &\leq V^{-} + \rho^2 T_k^2 \alpha^{-1} \|C\|^2 \|e_y^{-}\|^2 \\ &\quad - (2\rho - 1) T_k \|e_y^{-}\|^2 \end{aligned} \quad (16)$$

where

$$e_y^{-} = C \hat{x}^{-}(t_k + \tau) - y(t_k + \tau) = C e^{-} \quad (17)$$

and

$$\begin{aligned} e^{-'} \Delta_2 I_0 &= 2\rho T_k e^{-'} C' (1 - \rho T_k C S^{-1} C') C I_0 \\ &\leq 2\rho T_k \left[1 + \frac{\rho T_k}{\alpha} \|C\|^2 \right] \|C I_0\| \|e_y^{-}\| \\ &\leq 2\rho T_k \left[1 + \frac{\rho T_k}{\alpha} \|C\|^2 \right]^2 \|C I_0\|^2 \\ &\quad + \frac{1}{2} \rho T_k \|e_y^{-}\|^2. \end{aligned} \quad (18)$$

From equations (14)–(18), by gathering the terms in $\|e_y^{-}\|$ on the one hand and the terms in $\|C I_0\|$ on the other, one can write

$$V \leq V^{-} + T_k \Delta_4 \|e_y^{-}\|^2 + \Delta_5 \|C I_0\|^2$$

with

$$\Delta_4 = \frac{\rho^2 T_k \|C\|^2}{\alpha} - \left(\frac{3}{2} \rho - 1 \right)$$

and

$$\Delta_5 = \rho T_k \left(2 \left[1 + \frac{\rho T_k}{\alpha} \|C\|^2 \right]^2 + \frac{\rho T_k}{\alpha} \|C\|^2 \right). \quad (19)$$

One can check that $\Delta_4 \leq 0$ if the following inequality holds:

$$T_k \leq \frac{\alpha}{\rho^2 \|C\|^2} \left(\frac{3}{2} \rho - 1 \right). \quad (20)$$

On the other hand, from (11), we can write

$$V(t) = V(t_{k-1} + \tau) e^{-\mu(t-t_{k-1}-\tau)} \leq V(t_{k-1} + \tau) \quad (21)$$

on $[t_{k-1} + \tau, t_k + \tau)$. Using the property (7), then we have

$$\alpha \|e(t)\|^2 \leq V(t) \leq V(t_{k-1} + \tau). \quad (22)$$

Using the definition of I_0 and the fact that $\dot{e}(t) = A(u)e(t)$ on $[t_{k-1} + \tau, t_k + \tau)$, then we can write

$$\|C I_0\| = \left\| C \int_{t_k}^{t_k + \tau} A(u(s)) e(s) ds \right\| \quad (23)$$

and from this

$$\|C I_0\| \leq \|C\| \int_{t_k}^{t_k + \tau} \|A(u(s))\| \|e(s)\| ds \quad (24)$$

from the fact that $\|A(u)\| \leq \xi$ and (22), we deduce

$$\|C I_0\|^2 \leq \frac{\xi^2 \|C\|^2}{\alpha} \tau^2 V(t_{k-1} + \tau) \quad (25)$$

and

$$V(t_k + \tau) \leq \left[e^{-\mu T_k} + \Delta_5 \frac{\xi^2 \|C\|^2}{\alpha} \tau^2 \right] V(t_{k-1} + \tau). \quad (26)$$

Then, to ensure exponential stability, it is sufficient to have

$$e^{-\mu T_k} + \Delta_5 \frac{\xi^2 \|C\|^2}{\alpha} \tau^2 < 1 \quad (27)$$

which gives the following upper bound on the delay τ :

$$\tau < \min \left\{ T_{\min}, \frac{1}{\xi} \sqrt{\frac{\alpha(1 - e^{-\mu T_{\min}})}{\psi}} \right\} \quad (28)$$

with

$$\psi = \rho T_{\max} \|C\|^2 \left[2(\rho T_{\max} \|C\|^2 / \alpha)^2 + 5(\rho T_{\max} \|C\|^2 / \alpha) + 2 \right]$$

i.e., the second condition of the theorem. \blacksquare

B. Second Observation Structure

In this section, another class of observers structure is proposed. The main difference compared to the first class is in the introduction of an output delayed estimator which is re-initialized at each updating instant $t_k + \tau$

$$\forall t \geq 0 \begin{cases} \dot{\hat{x}}(t) = A(u)\hat{x}(t) - S_1^{-1} C' (C\hat{x}(t - \tau) - \omega(t)) \\ \quad + b(u), \\ \dot{S}_1(t) = -\mu S_1(t) - S_1(t)A(u) - A' S_1(t) + C' C, \end{cases} \quad (29a)$$

$$\forall k \in \mathbb{N}^* \begin{cases} \dot{\omega}(t) = CA(u(t - \tau))\hat{x}(t - \tau) + Cb(u(t - \tau)), \\ \forall t \in [t_{k-1} + \tau, t_k + \tau), \\ \omega(t_k + \tau) = y(t_k). \end{cases} \quad (29b)$$

The observer (29) is composed of a classical observer (29a) and an output estimator (58b). Note that \hat{x} and S_1 are continuous on \mathbb{R}^+ (the set of strictly positive real numbers) and only ω is re-initialized at each updating instant.

Theorem 2: Let us consider system (1). Assume that hypothesis 1 holds then for sufficiently large values of μ , there exist bounds on the delays and on the sampling period, such that system (29a)–(58b) is a global exponential observer for system (1).

Proof: In view of (29a)

$$\dot{e}(t) = A(u(t - \tau))e(t) - S_1^{-1} C' (C\hat{x}(t - \tau) - \omega(t)). \quad (30)$$

If we introduce the error due to the maximum allowable delay τ

$$e_\omega(t) \stackrel{\text{def}}{=} \omega(t) - y(t - \tau) \quad (31)$$

then (30) can be rewritten

$$\dot{e}(t) = A(u)e(t) - S_1^{-1} C' C e(t - \tau) + S_1^{-1} C' e_\omega(t) \quad (32)$$

$$\begin{aligned} &= [A(u) - S_1^{-1} C' C] e(t) + S_1^{-1} C' C \int_{t-\tau}^t \dot{e}(\theta) d\theta \\ &\quad + S_1^{-1} C' e_\omega(t). \end{aligned} \quad (33)$$

Since $x(t)$ and $\hat{x}(t)$ are both continuous on \mathbb{R}^+ , the error $e(t)$ is continuous on \mathbb{R}^+ . Before the study of the convergence of the observation error $e = \hat{x} - x$, let us recall the following technical result [21]

Proposition 2: Let u be a regularly persistent input for system (1). Then for all $\mu \geq 2\xi$ where $\xi = \sup_{t \geq t_0} \|A(u(t))\|$, and for all $S_1(t_0)$ symmetric positive definite, there exist constants $\alpha_1, \beta_1 \in \mathbb{R}^+$ such that, for $t \in [t_0, \infty)$

$$\alpha_1 \mathbb{1}_n \leq S_1(t) \leq \beta_1 \mathbb{1}_n \quad (34)$$

where $\alpha_1 = \delta e^{-2\mu t_a}$ and β_1 does not depend on μ .

Now, let us consider the following Lyapunov–Krasovskii candidate which is inspired from [14]:

$$U = e'(t)S_1 e(t) + \int_{t-\tau}^t \int_{\zeta}^t \|\dot{e}(\theta)\|^2 d\theta d\zeta + \gamma \phi(t) \|e_\omega(t)\|^2 \quad (35)$$

where γ is a positive design parameter and $\phi(t)$ is a piecewise differentiable function such that $\forall t > 0, \phi(t) > 0, \forall k \in \mathbb{N}, \phi(t_k) = \phi_{\max} \in \mathbb{R}^+$, and $\forall t \in [t_k + \tau, t_{k+1} + \tau), \phi(t) < 0$. In order to prove the exponential convergence of the observation error, it is sufficient to find conditions on the maximum allowable delay and the maximum sampling allowable period ensuring

$$\dot{U} \leq -\varepsilon U \quad (36)$$

$$\forall t \in [t_{k-1} + \tau, t_k + \tau) \quad \varepsilon > 0 \quad (36)$$

$$U(t_k + \tau) \leq U^-(t_k + \tau) \quad \forall k \in \mathbf{N}^*. \quad (37)$$

From the fact that the observation error e is continuous and the error $e_\omega(t_k + \tau) = 0$, then it is obvious that the inequality (37) is fulfilled. Now let us consider the derivative of U on $t \in [t_{k-1} + \tau, t_k + \tau)$

$$\begin{aligned} \dot{U} &= e'(t)\dot{S}_1 e(t) + 2e'(t)S_1 \dot{e}(t) + \tau \|\dot{e}(t)\|^2 - \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta \\ &\quad + 2\gamma \phi(t) \|e_\omega(t)\| \cdot \|\dot{e}_\omega(t)\| + \gamma \dot{\phi}(t) \|e_\omega(t)\|^2. \end{aligned} \quad (38)$$

Since

$$\begin{aligned} 2e' C' \left(C \int_{t-\tau}^t \dot{e}(\theta) d\theta + e_\omega \right) &\leq e' C' C e \\ &\quad + \left\| C \int_{t-\tau}^t \dot{e}(\theta) d\theta + e_\omega \right\|^2 \end{aligned}$$

by replacing $\dot{S}_1(t)$ and $\dot{e}(t)$ by their expressions from (29a) and (33), we get

$$\begin{aligned} Z &= e' \dot{S}_1 e + 2e' S_1 \dot{e} \\ &= -\mu e' S_1 e - e' C' C e + 2e' C' \left(C \int_{t-\tau}^t \dot{e}(\theta) d\theta + e_\omega \right) \\ &\leq -\mu e' S_1 e + \left\| C \int_{t-\tau}^t \dot{e}(\theta) d\theta + e_\omega(t) \right\|^2 \\ &\leq -\mu e' S_1 e + 2\|C\|^2 \cdot \left\| \int_{t-\tau}^t \dot{e}(\theta) d\theta \right\|^2 + 2\|e_\omega(t)\|^2. \end{aligned}$$

Applying the Jensen's inequality, then we have

$$\left\| \int_{t-\tau}^t \dot{e}(\theta) d\theta \right\|^2 \leq \tau \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta \quad (39)$$

leads to

$$Z \leq -\mu e' S_1 e + 2\tau \|C\|^2 \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta + 2\|e_\omega(t)\|^2. \quad (40)$$

Using (40) into (38) gives

$$\begin{aligned} \dot{U} &\leq -\mu e'(t)S_1 e(t) + 2\tau \|C\|^2 \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta + 2\|e_\omega(t)\|^2 \\ &\quad + \tau \|\dot{e}(t)\|^2 - \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta \\ &\quad + 2\gamma \phi(t) \|e_\omega(t)\| \cdot \|\dot{e}_\omega(t)\| + \gamma \dot{\phi}(t) \|e_\omega(t)\|^2. \end{aligned} \quad (41)$$

Now, to handle the terms in e_ω , one can differentiate its expression (31) and get, through a direct application of the observer definition (29)

$$\begin{aligned} \dot{e}_\omega(t) &= \dot{\omega}(t) - \dot{y}(t - \tau) \\ &= CA(u(t - \tau))e(t - \tau) \\ &= CA(u(t - \tau))e(t) \\ &\quad - CA(u(t - \tau)) \int_{t-\tau}^t \dot{e}(\theta) d\theta. \end{aligned} \quad (42)$$

Hence one has

$$\|\dot{e}_\omega(t)\| \leq \|CA(u)\| \cdot \left[\|e(t)\| + \left\| \int_{t-\tau}^t \dot{e}(\theta) d\theta \right\| \right] \quad (43)$$

and therefore

$$\begin{aligned} 2\gamma \phi(t) \|e_\omega(t)\| \cdot \|\dot{e}_\omega(t)\| &\leq \gamma \phi(t) \|CA(u)\| \\ &\quad \times \left[\|e_\omega(t)\|^2 + \|e(t)\|^2 + \left\| \int_{t-\tau}^t \dot{e}(\theta) d\theta \right\|^2 \right]. \end{aligned}$$

Using once again Jensen's inequality (39)

$$\begin{aligned} 2\gamma \phi(t) \|e_\omega(t)\| \cdot \|\dot{e}_\omega(t)\| &\leq \gamma \phi(t) \|CA(u)\| \\ &\quad \times \left[\|e_\omega(t)\|^2 + 2\|e(t)\|^2 + 2\tau \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta \right]. \end{aligned} \quad (44)$$

Using (44) to rewrite the upper bound on \dot{U} from (41), one gets

$$\begin{aligned} \dot{U} &\leq -\mu e'(t)S_1 e(t) + 2\|C\|^2 \tau \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta + 2\|e_\omega(t)\|^2 \\ &\quad + \tau \|\dot{e}(t)\|^2 - \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta + \gamma \dot{\phi}(t) \|e_\omega(t)\|^2 \\ &\quad + \gamma \phi(t) \|CA(u)\| \left[\|e_\omega(t)\|^2 + 2\|e(t)\|^2 + 2\tau \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta \right] \end{aligned} \quad (45)$$

On the other hand from (32) we can easily deduce that

$$\begin{aligned} \|\dot{e}(t)\|^2 &\leq k_1 \left[\|e(t)\|^2 + \|e_\omega(t)\|^2 + \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta \right], \\ k_1 &\in \mathbb{R}_0^+ \end{aligned} \quad (46)$$

and from the proposition 2, we have

$$-\mu e'(t)S_1 e(t) \leq -\mu \alpha_1 \|e(t)\|^2. \quad (47)$$

Then

$$\begin{aligned} \dot{U} \leq & -[\mu\alpha_1 - 2\gamma\phi(t)\|CA(u)\| - \tau k_1] \|e(t)\|^2 \\ & + \left[\gamma\phi(t)\|CA(u)\| + \tau k_1 + \gamma\dot{\phi}(t) + 2 \right] \|e_\omega(t)\|^2 \\ & + \left[2\tau\gamma\phi(t)\|CA(u)\| + \tau k_1 + 2\|C\|^2\tau - 1 \right] \\ & \times \int_{t-\tau}^t \|\dot{e}(\theta)\|^2 d\theta. \end{aligned} \quad (48)$$

Then we can say that the exponential convergence of the observation error is ensured if following conditions hold:

$$\mu\alpha_1 - 2\gamma\phi(t)\|CA(u)\| - \tau k_1 - \varepsilon_1\beta_1 \geq 0 \quad (49a)$$

$$\gamma\phi(t)\|CA(u)\| + \tau k_1 + \gamma\dot{\phi}(t) + 2 + \varepsilon_1\gamma\phi_{\max} \leq 0 \quad (49b)$$

$$2\tau\gamma\phi(t)\|CA(u)\| + \tau k_1 + 2\|C\|^2\tau - 1 + \varepsilon_1\tau \leq 0. \quad (49c)$$

In order to derive the maximum allowable sampling period and the maximum allowable delay, we consider that $\varepsilon_1 \rightarrow 0^+$. Assume that $\tau k_1 \leq \alpha_1$ and $\gamma = \alpha_1$, then (49a) becomes

$$\mu > 2\phi_{\max}\|C\|\xi + 1 \quad (50)$$

where $\xi = \sup_{t \geq t_0} \|A(u(t))\|$. Furthermore one can choose $\phi(t)$ as a sawtooth function with $\forall k > 0$, $\phi(t_k) = \phi_{\max} \in \mathbb{R}$ and $\forall t \geq 0$, $\dot{\phi}(t) < -k_\phi \in \mathbb{R}^+$. Thus, condition (49b) is verified when

$$k_\phi = \frac{2}{\alpha_1} + \phi_{\max}\|C\|\xi + 1. \quad (51)$$

But one must also have $\forall t \geq 0$, $\phi(t) > 0$. This implies that the sampling period must be smaller than $T_{\max} = \phi_{\max}/k_\phi$. Together with (51), this leads to the following condition on the maximal allowable sampling period:

$$T_{\max} < \frac{\phi_{\max}}{\frac{2}{\alpha_1} + \phi_{\max}\|C\|\xi + 1}. \quad (52)$$

From this, (49c) leads to a bound on τ

$$\tau < \min \left\{ T_{\min}, \frac{\alpha_1}{k_1}, \frac{1}{2\delta\phi_{\max}\|C\|\xi + k_1 + 2\|C\|^2} \right\}. \quad (53)$$

If conditions (50), (52), and (53) are fulfilled, then the three conditions (49) are fulfilled and this means that the observation error converges exponentially towards zero. ■

Remark 2: Notice that the free strictly positive value ϕ_{\max} can be used to tune the lower bound of μ following (50) and the corresponding bounds of the sampling period and delay following (52) and (53).

V. EXTENSION TO GLOBALLY LIPSCHITZ NONLINEAR SYSTEMS

In this section we extend the result of Theorem 2 to the following class of nonlinear system:

$$\begin{cases} \dot{x}(t) = f(x, u) \\ y(t) = h(x) \end{cases} \quad (54)$$

where $f : \mathbf{R}^n \times \mathbf{R}^k \rightarrow \mathbf{R}^n$ and $h : \mathbf{R}^n \rightarrow \mathbf{R}^k$ are smooth mapping, and $u \in U$ where $U \in \mathbf{R}^m$ is a non empty set. We assume that the following hypotheses hold:

Hypothesis 2: There exists a constant $L > 0$ such that

$$\|f(x, u) - f(z, u)\| \leq L\|x - z\|, \forall x, z \in \mathbf{R}^n, \forall u \in U \quad (55)$$

Hypothesis 3: There exist a symmetric, positive definite matrix $P \in \mathbf{R}^{n \times n}$, a constant $q > 0$ and matrices $K \in \mathbf{R}^{n \times k}$, $H \in \mathbf{R}^{k \times n}$ such that

$$h(x) = Hx, \quad \forall x \in \mathbf{R}^n \quad (56a)$$

$$\begin{aligned} (z-x)'P(f(z, u) - f(x, u)) + (z-x)'PKH(z-x) \\ \leq -q\|z-x\|^2 \quad \forall x, z \in \mathbf{R}^n, \forall u \in U. \end{aligned} \quad (56b)$$

This hypothesis means that the following system:

$$\dot{z} = f(z, u) - K(Hz - y) \quad (57)$$

is a global exponential observer for systems (54). It has also to be noticed that hypotheses 2 and 3 are automatically satisfied for triangular systems (see [4]) and the class of nonlinear systems defined in [22].

Now we are able to give the result of this extension.

Theorem 3: Let us consider systems (54) and suppose that hypotheses 2 and 3 hold. Then there exist bounds on the delay and sampling period such that the following system:

$$\forall t \geq 0 \dot{\hat{x}}(t) = f(u(t), \hat{x}(t)) - K \left(H\hat{x}(t-\tau) - \omega(t) \right). \quad (58a)$$

$$\forall k \in \mathbb{N}^* \begin{cases} \dot{\omega}(t) = Hf(\hat{x}(t-\tau), u(t-\tau)) \\ \forall t \in [t_{k-1} + \tau, t_k + \tau), \\ \omega(t_k + \tau) = y(t_k) \end{cases} \quad (58b)$$

is a global exponential observer for systems (54).

1) Sketch of Proof: The proof of this Theorem is similar to the proof of Theorem 2 by using the same Lyapunov–Krasovskii candidate function which is inspired from [14]

$$U = e'(t)Pe(t) + \int_{t-\tau}^t \int_{\zeta}^t \|\dot{e}(\theta)\|^2 d\theta d\zeta + \gamma\phi(t)\|e_\omega(t)\|^2 \quad (59)$$

where $e = \hat{x} - x$ and P is defined in hypothesis 3.

Remark 3: Following the same methodology, it has to be noticed that the second structure used in the results of Theorems 2 and 3 can be easily extended to classes of observers contained in [23].

Remark 4: The main contribution of this note is that we give several results concerning the observers design where both delay and sampling phenomena appear on the output measurements. This work must be compared to the result of [24] where the authors consider this case for linear systems and without using predictor. The introduction of predictor will improve the bound of sampling period because the model of the output is used to predict it between two updating instants.

VI. CONCLUSION

In this technical note, the observation of a class of continuous MIMO systems with sampled and delayed measurements has been studied. Under some sufficient conditions, some classes of global exponential observers have been developed. This contribution can be obviously extended to the case of scheduling protocols in order to include a maximum number of communication constraints which often appear in the networked control systems. But for the class of finite-time observers developed in [25] and [26], this problem remains still open because the corrector term of the observer is not locally or globally Lipschitz.

REFERENCES

- [1] M. Arcak and D. Nešić, "A framework for nonlinear sampled-data observer design via approximate discrete-time models and emulation," *Automatica*, vol. 40, pp. 1931–1938, 2004.

- [2] A. Jazwinski, *Stochastic Processes and Filtering Theory. Mathematics in Science and Engineering*. New York: Academic Press, 1970.
- [3] F. Deza, E. Busvelle, J. Gauthier, and D. Rakotopora, "High gain estimation for nonlinear systems," *Sys. Control Lett.*, vol. 18, pp. 295–299, 1992.
- [4] J. Gauthier, H. Hammouri, and S. Othman, "A simple observer for nonlinear systems applications to bioreactors," *IEEE Trans. Autom. Control*, vol. 37, no. 6, pp. 875–880, Jun. 1992.
- [5] M. Nadri and H. Hammouri, "Design of a continuous-discrete observer for state affine systems," *Appl. Math. Lett.*, vol. 16, pp. 967–974, 2003.
- [6] T. Ahmed-Ali, R. Postoyan, and F. Lamnabhi-Lagarriague, "Continuous-discrete adaptive observers for state affine systems," *Automatica*, vol. 45, pp. 2986–2990, 2009.
- [7] C.-M. Astorga, N. Othman, S. Othman, H. Hammouri, and T.-F. McKenna, "Nonlinear continuous-discrete observers: Applications to emulsion polymerization reactors," *Control Eng. Practice*, vol. 10, pp. 3–13, 2002.
- [8] M. Nadri, H. Hammouri, and C.-M. Astorga, "Observer design for continuous-discrete time state affine systems up to output injection," *Eur. J. Control*, vol. 10, pp. 252–263, 2004.
- [9] H. Hammouri, M. Nadri, and R. Mota, "Constant gain observer for continuous-discrete time uniformly observable systems," in *Proc. 45th IEEE Conf. Dec. Control*, San Diego, CA, 2006, pp. 6240–6244.
- [10] V. Andrieu and M. Nadri, "Observer design for Lipschitz systems with discrete-time measurements," in *Proc. Conf. Decision Control (CDC'10)*, Atlanta, GA, 2010, pp. 6522–6527.
- [11] A. Zemouche, M. Boutayeb, and G. I. Bara, "Observers for a class of Lipschitz systems with extension to \mathcal{H}_∞ performance analysis," *Syst. Control Lett.*, vol. 57, no. 1, pp. 18–27, 2008.
- [12] I. Karafyllis and C. Kravaris, "From continuous-time design to sampled-data design of nonlinear observers," *IEEE Trans. Autom. Control*, vol. 54, pp. 2169–2174, 2009.
- [13] T. Ahmed-Ali and F. Lamnabhi-Lagarriague, "High gain observer design for some networked control systems," *IEEE Trans. Autom. Control*, vol. 57, pp. 995–1000, 2012.
- [14] J. Hespanha, P. Naghshtabrizi, and Y. Xu, "A survey of recent results in networked control systems," *IEEE Special Issue Technol. Netw. Control Syst.*, vol. 95, no. 1, pp. 138–162, 2007.
- [15] T. Raff, M. Kögel, and F. Allgöwer, "Observer with sample-and-hold updating for Lipschitz nonlinear systems with nonuniformly sampled measurements," in *Proc. Amer. Control Conf. (ACC'08)*, Seattle, Washington, 2008, pp. 5254–5257.
- [16] V. Van Assche, T. Ahmed-Ali, C. Hann, and F. Lamnabhi-Lagarriague, "High gain observer design for nonlinear systems with time varying delayed measurements," in *Proc. 18th IFAC World Congress*, Milan, Italy, 2011, pp. 692–696.
- [17] T. Ahmed-Ali, E. Cherrier, and F. Lamnabhi-Lagarriague, "Cascade high predictors for a class of nonlinear systems," *IEEE Trans. Autom. Control*, vol. 57, pp. 224–229, 2012.
- [18] G. Bornard, F. Celle, and N. Couenne, "Regularly persistent observers for bilinear systems," in *Proc. 29th Int. Conf. Nonlinear Syst., New Trends Nonlinear Syst. Theory*, Nantes, France, 1988, vol. 122, pp. 130–140.
- [19] F. Celle, J. Gauthier, D. Kazakos, and G. Sallet, "Synthesis of nonlinear observers: A harmonic-analysis approach," *Math. Syst. Theory*, vol. 22, pp. 291–322, 1989.
- [20] H. Hammouri and J. D. Morales, "Chapter: Topological properties of observer's inputs," in *Progress in Syst. and Control Theory*, Birkhauser. Boston, MA: , 2002.
- [21] G. Besançon, G. Bornard, and H. Hammouri, "Observer synthesis for a class of nonlinear control systems," *Eur. J. Control*, pp. 176–192, 1996.
- [22] R. Rajamani, "Observers for Lipschitz nonlinear systems," *IEEE Trans. Autom. Control*, vol. 43, pp. 397–401, 1998.
- [23] I. Karafyllis and C. Kravaris, "Global exponential observers for two classes of nonlinear systems," *Syst. Control Lett.*, vol. 61, pp. 797–806, 2012.
- [24] P. N. J. Hespanha and Y. Xu, "A survey of recent results in networked control systems," *IEEE Special Issue Technol. Netw. Control Syst.*, vol. 95, no. 1, pp. 138–162, 2007.
- [25] Y. Shen and X. Xia, "Semi-global finite-time observers for nonlinear systems," *Automatica*, vol. 44, pp. 3152–3156, 2008.
- [26] Y. Shen, W. Shen, M. Jiang, and Y. Huang, "Semi-global finite-time observers for multi-output," *Int. J. Robust Nonlin. Control*, vol. 7, no. 20, pp. 789–801, 2010.