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The Mézard-Parisi equation for matchings in pseudo-dimension $d > 1$

Justin Salez

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Abstract

We establish existence and uniqueness of the solution to the cavity equation for the random assignment problem in pseudo-dimension $d > 1$, as conjectured by Aldous and Bandyopadhyay (Annals of Applied Probability, 2005) and Wästlund (Annals of Mathematics, 2012). This fills the last remaining gap in the proof of the original Mézard-Parisi prediction for this problem (Journal de Physique Lettres, 1985).

Keywords: recursive distributional equation; random assignment problem; mean-field combinatorial optimization; cavity method.

2010 MSC: 60C05, 82B44, 90C35.

1 Introduction

The random assignment problem is a now classical problem in probabilistic combinatorial optimization. Given an $n \times n$ array $\{X_{i,j}\}_{1 \leq i,j \leq n}$ of IID non-negative random variables, it asks about the statistics of

$$M_n := \min_{\sigma} \sum_{i=1}^{n} X_{i,\sigma(i)},$$

where the minimum runs over all permutations $\sigma$ of $\{1, \ldots, n\}$. This corresponds to finding a minimum-length perfect matching on the complete bipartite graph $K_{n,n}$ with edge-lengths $\{X_{i,j}\}_{1 \leq i,j \leq n}$. Using the celebrated replica symmetry ansatz from statistical physics, Mézard and Parisi [10, 11, 12] made a remarkably precise prediction concerning the regime where $n$ tends to infinity while the distribution of $X_{i,j}$ is kept fixed and satisfies

$$\mathbb{P}(X_{i,j} \leq x) \sim x^d \quad \text{as} \quad x \to 0^+,$$
for some exponent $0 < d < \infty$. Specifically, they conjectured that
\[
\frac{M_n}{n^{1-1/d}} \xrightarrow{p}{\frac{\lim}{n \to \infty}} -d \int_{\mathbb{R}} f(x) \ln f(x) dx,
\]
(1)
where the function $f: \mathbb{R} \to [0,1]$ solves the so-called cavity equation:
\[
f(x) = \exp \left( - \int_{-\infty}^{+\infty} d(x+y)^{d-1} f(y) dy \right).
\]
(2)
Aldous \cite{1, 3} proved this conjecture in the special case $d = 1$, where the term $(x+y)^{d-1}$ simplifies and makes the cavity equation exactly solvable, yielding
\[
f(x) = \frac{1}{1 + e^x} \quad \text{and} \quad -d \int_{\mathbb{R}} f(x) \ln f(x) dx = \frac{\pi^2}{6}.
\]
Since then, several alternative proofs have been found \cite{9, 13, 15}. This stands in sharp contrast with the case $d \neq 1$, where showing that the Mézard-Parisi equation (2) admits a unique solution has until now remained an open problem \cite[Open Problem 63]{4}. Wästlund \cite{16} circumvented this issue by considering instead the truncated equation
\[
f_\lambda(x) = \exp \left( - \int_{-\lambda}^{\lambda} d(x+y)^{d-1} f_\lambda(y) dy \right), \quad 0 < \lambda < \infty.
\]
(3)
Using an ingenious game-theoretical interpretation of this equation, he showed the existence of a unique, global attractive solution $f_\lambda: [-\lambda, \lambda] \to [0,1]$ for every $0 < \lambda < \infty$, provided $d \geq 1$. He then used this fact to establish that
\[
\frac{M_n}{n^{1-1/d}} \xrightarrow{p}{\frac{\lim}{n \to \infty}} \lim_{\lambda \to +\infty} -d \int_{-\lambda}^{\lambda} f_\lambda(x) \ln f_\lambda(x) dx.
\]
(4)
Wästlund \cite{16} explicitly left open the problem of completing the proof of the original Mézard-Parisi prediction by showing (i) that the untruncated cavity equation admits a unique solution $f$ and (ii) that $f_\lambda \to f$ as $\lambda \to \infty$. The purpose of this short paper is to establish this conjecture.

**Theorem 1.** For $d > 1$, the Mézard-Parisi equation (2) admits a unique solution $f: \mathbb{R} \to [0,1]$. Moreover, $f_\lambda \to f$ pointwise as $\lambda \to +\infty$, and
\[
\int_{-\lambda}^{\lambda} f_\lambda(x) \ln f_\lambda(x) dx \xrightarrow{\lambda \to +\infty} \int_{\mathbb{R}} f(x) \ln f(x) dx.
\]
Consequently, the two limits in (1) and (4) coincide.

In addition, we provide a short alternative proof of the crucial result of \cite{16} that the truncated equation (3) admits a unique, attractive solution.
Remark 1. Very recently, a proof of uniqueness for the truncated equation (3) has been announced [8] for the case $0 < d < 1$. It would be interesting to see if the result of the present paper can be extended to this regime.

Remark 2. For a random variable $Z$ with $P(Z > x) = f(x)$, the cavity equation (2) simply expresses the fact that $Z$ solves the distributional identity

$$Z \overset{d}{=} \min_{i \geq 1} \{ \xi_i - Z_i \},$$

where $\{\xi_i\}_{i \geq 1}$ is a Poisson point process with intensity $dx^{d-1} \partial x$ on $[0, \infty)$, and $\{Z_i\}_{i \geq 1}$ are IID with the same distribution as $Z$, independent of $\{\xi_i\}_{i \geq 1}$. Such recursive distributional equations arise naturally in a variety of models from statistical physics, and the question of existence and uniqueness of solutions plays a crucial role for the rigorous understanding of those models. We refer the interested reader to the comprehensive surveys [2, 4] for more details. In particular, [4, Section 7.4] contains a detailed discussion on equation (5), and [4, Open Problem 63] raises explicitly the uniqueness issue. We note that the refined question of endogeneity remains a challenging open problem. Recursive distributional equations for other mean-field combinatorial optimization problems have been analysed in e.g. [5, 14, 6].

The remainder of the paper is organized as follows. Section 2 deals with the truncated equation (3) for fixed $0 < \lambda < \infty$ and is devoted to the alternative analytical proof that there is a unique, globally attractive solution $f_\lambda$. Section 3 prepares the $\lambda \to \infty$ limit by providing uniform controls on the family $\{f_\lambda : 0 < \lambda < \infty\}$ and by characterizing the possible limit points. This reduces the proof of Theorem 1 to establishing uniqueness in the un-truncated Mézard-Parisi equation ($\lambda = \infty$), which is done in Section 4.

## 2 The truncated cavity equation ($\lambda < \infty$)

Fix a parameter $0 < \lambda < \infty$. On the set $\mathcal{F}$ of non-increasing functions $f : [-\lambda, \lambda] \to [0, 1]$, define an operator $T$ by

$$(Tf)(x) = \exp \left( -d \int_{-x}^\lambda (x + y)^{d-1} f(y) dy \right).$$

The purpose of this section is to give a short and purely analytical proof of the following result, which was the main technical ingredient in [16] and was therein established using an ingenious game-theoretical framework.
Proposition 1. $T$ admits a unique fixed point $f_\lambda$ and it is attractive in the sense that $|T^n f(x) - f_\lambda(x)| \rightarrow 0$, uniformly in both $x \in [-\lambda, \lambda]$ and $f \in \mathcal{F}$.

Proof. Write $f \leq g$ to mean $f(x) \leq g(x)$ for all $x \in [-\lambda, \lambda]$. In particular, $0 \leq f \leq T0$

for every $f \in \mathcal{F}$, where 0 denotes the constant-zero function. Note also that the operator $T$ is non-increasing, in the sense that $f \leq g \Rightarrow T f \geq T g$.

Those two observations imply that the sequences $\{T^{2n}0\}_{n \geq 0}$ and $\{T^{2n+1}0\}_{n \geq 0}$ are respectively non-decreasing and non-increasing, and that their respective pointwise limits $f^-$ and $f^+$ satisfy

$$f^- \leq \liminf_{n \to \infty} T^n f \leq \limsup_{n \to \infty} T^n f \leq f^+,$$

for any $f \in \mathcal{F}$. Moreover, the dominated convergence Theorem ensures that $T$ is continuous with respect to pointwise convergence, allowing to pass to the limit in the identity $T^{n+1}0 = T(T^n0)$ to deduce that

$$T f^- = f^+ \quad \text{and} \quad T f^+ = f^-.$$  \hfill (7)

Therefore, the proof boils down to the identity $f^- = f^+$, which we now establish. By definition, we have for any $f \in \mathcal{F}$,

$$(T f)(x) = \exp \left( -d \int_{-\lambda}^{\lambda} (x+y)^{d-1} 1_{(x+y \geq 0)} f(y) dy \right).$$

Since $d > 1$, we may differentiate under the integral sign to obtain

$$(T f)'(x) = -d(d-1)(T f)(x) \int_{-\lambda}^{\lambda} (x+y)^{d-2} 1_{(x+y \geq 0)} f(y) dy.$$ Integrating over $[-\lambda, \lambda]$ and noting that $(T f)(-\lambda) = 1$, we conclude that

$$1 - (T f)(\lambda) = d(d-1) \int_{[-\lambda,\lambda]^2} (x+y)^{d-2} 1_{(x+y \geq 0)} (T f)(x) f(y) dx dy.$$ Let us now specialize to $f = f^\pm$. In both cases, the right-hand side is

$$d(d-1) \int_{[-\lambda,\lambda]^2} (x+y)^{d-2} 1_{(x+y \geq 0)} f^+(x) f^-(y) dx dy.$$
by (7). Therefore, we have \((Tf^+)(\lambda) = (Tf^-)(\lambda)\), i.e.

\[
\int_{-\lambda}^{\lambda} d(\lambda + y)^{d-1} f^+(y)dy = \int_{-\lambda}^{\lambda} d(\lambda + y)^{d-1} f^-(y)dy.
\]

Since we already know that \(f^- \leq f^+\), this forces \(f^- = f^+\) almost-everywhere on \([-\lambda, \lambda]\), and hence everywhere by continuity. Finally, the convergence \(T^nf \to f = f^\pm\) is automatically uniform on \([-\lambda, \lambda]\), by Dini’s Theorem.

3 Relative compactness of solutions \((\lambda \to \infty)\)

In order to study properties of the family \(\{f_\lambda: 0 < \lambda < \infty\}\), we extend the domain of \(f_\lambda\) to \(\mathbb{R}\) by setting \(f_\lambda(x) = 1\) for \(x \leq -\lambda\) and \(f_\lambda(x) = 0\) for \(x > \lambda\).

**Proposition 2** (Uniform bounds). For all \(0 < \lambda < \infty\) and \(x \geq 0\),

\[
\begin{align*}
    f_\lambda(x) & \leq \exp \left( -\frac{xd}{e} \right) \\
    1 - f_\lambda(-x) & \leq \exp \left( -\frac{xd}{e} \right) \\
    f_\lambda(-x) \ln \frac{1}{f_\lambda(-x)} & \leq \exp \left( -\frac{xd}{e} \right) \\
    f_\lambda(x) \ln \frac{1}{f_\lambda(x)} & \leq \left( 1 + \frac{xd}{e} \right) \exp \left( -\frac{xd}{e} \right).
\end{align*}
\]

**Proof.** Let \(0 < \lambda < \infty\). We may assume that \(x \in [0, \lambda]\), otherwise the above bounds are trivial. By definition, we have

\[
f_\lambda(x) = \exp \left( -\int_{-x}^{\lambda} d(x + y)^{d-1} f_\lambda(y)dy \right).
\]

Now, since \(x \geq 0\) and \(f_\lambda\) is non-increasing, we have

\[
\int_{-x}^{\lambda} (x + y)^{d-1} f_\lambda(y)dy = \int_{-x}^{0} (x + y)^{d-1} f_\lambda(y)dy + \int_{0}^{\lambda} (x + y)^{d-1} f_\lambda(y)dy \\
\geq f_\lambda(0) \frac{x^d}{d} + \int_{0}^{\lambda} y^{d-1} f_\lambda(y)dy.
\]

Applying \(u \mapsto \exp(-du)\) to both sides and using (8), we obtain

\[
f_\lambda(x) \leq f_\lambda(0) \exp(-f_\lambda(0)x^d).
\]
In turn, this inequality implies that for all \( x \geq 0 \),
\[
\int_{x}^{+\infty} d(y - x)^{d-1} f_\lambda(y)dy \leq f_\lambda(0) \int_{x}^{+\infty} dy^{d-1} e^{-f_\lambda(0)y^d} = \exp(-f_\lambda(0)x^d).
\]
Applying \( u \mapsto \exp(-u) \) to both sides, we conclude that
\[
f_\lambda(-x) \geq \exp\left(-e^{-f_\lambda(0)x^d}\right).
\] (10)
In particular, taking \( x = 0 \) yields \( f_\lambda(0) \geq e^{-1} \), and reinjecting this into (9) and (10) easily yields the first three claims. For the last one, observe that \( u \mapsto u \ln \frac{1}{u} \) increases on \([0, e^{-1}]\) and decreases on \([e^{-1}, 1]\), with the value at \( u = e^{-1} \) being precisely \( e^{-1} \). Therefore, if \( \exp(-x^d/e) \leq e^{-1} \), we may use the bound \( f_\lambda(x) \leq \exp(-x^d/e) \) to deduce that
\[
f_\lambda(x) \ln \frac{1}{f_\lambda(x)} \leq \frac{x^d}{e} \exp\left(-\frac{x^d}{e}\right).
\]
On the other hand, if \( \exp(-x^d/e) \geq e^{-1} \), then
\[
f_\lambda(x) \ln \frac{1}{f_\lambda(x)} \leq e^{-1} \leq \exp\left(-\frac{x^d}{e}\right).
\]
In both cases, the last inequality holds, and the proof is complete. \( \square \)

**Proposition 3.** The family \( \{f_\lambda : 0 < \lambda < \infty\} \) is relatively compact with respect to the topology of uniform convergence on \( \mathbb{R} \), and any sub-sequential limit as \( \lambda \to \infty \) must solve the cavity equation (2).

**Proof.** Let \( \{\lambda_n\}_{n \geq 0} \) be any sequence of positive numbers such that \( \lambda_n \to \infty \) as \( n \to \infty \). By Helly’s compactness principle for uniformly bounded monotone functions (see e.g. [7, Theorem 36.5]), there exists an increasing sequence \( \{n_k\}_{k \geq 0} \) in \( \mathbb{N} \) and a non-increasing function \( f : \mathbb{R} \to [0, 1] \) such that
\[
f_{\lambda_{n_k}}(x) \xrightarrow{k \to \infty} f(x),
\] (11)
for all \( x \in \mathbb{R} \). Thanks to the first inequality in Proposition 2, we may invoke dominated convergence to deduce that for each \( x \in \mathbb{R} \),
\[
\int_{-x}^{\lambda_{n_k}} f_{\lambda_{n_k}}(y)(x + y)^{d-1}dy \xrightarrow{k \to \infty} \int_{-x}^{+\infty} f(y)(x + y)^{d-1}dy.
\]
Applying \( u \mapsto \exp(-du) \) and recalling (8), we see that
\[
f(x) = \exp\left(-d \int_{-x}^{+\infty} f(y)(x + y)^{d-1}dy\right),
\]
which shows that \( f \) must solve the cavity equation (2). This identity easily implies that \( f \) is continuous. Consequently, the convergence (11) is uniform in \( x \in \mathbb{R} \), by Dini’s Theorem. \( \square \)
4 The un-truncated cavity equation (\( \lambda = \infty \))

To conclude the proof of Theorem 1, it now remains to show that the un-truncated equation
\[
f(x) = \exp \left( -d \int_{-x}^{+\infty} (x + y)^{d-1} f(y)dy \right).
\]
(12)

admits at most one fixed point \( f: \mathbb{R} \to [0,1] \). Proposition 3 will then guarantee the convergence \( f_{\lambda} \xrightarrow{\lambda \to \infty} f \), which will in turn imply
\[
\int_{-\lambda}^{\lambda} f_{\lambda}(x) \ln f_{\lambda}(x)dx \xrightarrow{\lambda \to +\infty} \int_{\mathbb{R}} f(x) \ln f(x)dx,
\]
by dominated convergence, thanks to the last inequalities in Proposition 2.

A quick inspection of the proof of Proposition 2 reveals that it remains valid when \( \lambda = \infty \). In particular, any solution \( f \) to (12) must satisfy
\[
\max(f(x), 1 - f(-x)) \leq \exp \left( -\frac{x^d}{e} \right),
\]
(13)

for all \( x \geq 0 \). It also clear from (12) that \( f \) must be \((0,1)\)-valued and continuous. We will use those properties in the proofs below.

**Lemma 1.** If \( f, g \) solve (12), then there exists \( t \geq 0 \) such that for all \( x \in \mathbb{R} \),
\[
f(x + t) \leq g(x) \leq f(x - t).
\]

**Proof.** (13) ensures that for any \( t \in \mathbb{R} \), \( y \mapsto (1 + |y|)(f(y - t) - g(y)) \) is integrable on \( \mathbb{R} \), so that by dominated convergence,
\[
\frac{1}{x^{d-1}} \int_{-x}^{+\infty} (y + x)^{d-1} (f(y - t) - g(y))dy \xrightarrow{x \to +\infty} \Delta(t),
\]
(14)

where
\[
\Delta(t) := \int_{\mathbb{R}} (f(y - t) - g(y))dy.
\]
(15)

Observe that \( t \mapsto \Delta(t) \) increases continuously from \(-\infty\) to \(+\infty\), as can be seen from the decomposition
\[
\Delta(t) = \int_{0}^{+\infty} (1 - g(y) - g(y))dy + \int_{-t}^{+\infty} f(y)dy - \int_{t}^{+\infty} (1 - f(-y))dy.
\]
In particular, we can find \( t_0 \geq 0 \) such that \( \Delta(-t_0) < 0 < \Delta(t_0) \). In view of (14), we deduce the existence of \( a \geq 0 \) such that for all \( x \geq a \),
\[
\int_{-x}^{+\infty} (y + x)^{d-1} g(y) dy \geq \int_{-x}^{+\infty} (y + x)^{d-1} f(y + t_0) dy \quad (16)
\]
\[
\int_{-x}^{+\infty} (y + x)^{d-1} g(y) dy \leq \int_{-x}^{+\infty} (y + x)^{d-1} f(y - t_0) dy. \quad (17)
\]
Applying \( u \mapsto \exp(-du) \), we conclude that for all \( x \geq a \),
\[
f(x + t_0) \leq g(x) \leq f(x - t_0). \quad (18)
\]
In turn, this implies that (16)-(17) also hold when \( x \leq -a \), so that (18) actually holds for all \( x \) outside \((-a, a)\). On the other hand, since \( g \) is \((0, 1)\)-valued and \( f \) has limits \(0, 1\) at \(\pm\infty\), we can choose \( t_1 \geq 0 \) large enough so that
\[
f(-a + t_1) \leq g(a) \leq g(-a) \leq f(-a - t_1).
\]
Since \( f, g \) are non-increasing, this inequality implies that for all \( x \in [-a, a] \),
\[
f(x + t_1) \leq g(x) \leq f(x - t_1). \quad (19)
\]
In view of (18)-(19), taking \( t := \max(t_0, t_1) \) concludes the proof.

**Proof of Proposition 3.** Let \( f, g \) solve equation (12) and let \( t \) be the smallest non-negative number satisfying for all \( x \in \mathbb{R} \),
\[
f(x + t) \leq g(x) \leq f(x - t). \quad (20)
\]
Note that \( t \) exists by Lemma 1 and the continuity of \( f \). Now assume for a contradiction that \( t > 0 \). Clearly, each of the two inequalities in (20) must be strict at some point \( x \in \mathbb{R} \) (and hence on some open interval by continuity), otherwise we would have \( g \geq f \) or \( g \leq f \) and (12) would then force \( g = f \), contradicting the assumption that \( t > 0 \). Consequently, the function \( \Delta \) defined in (15) must satisfy \( \Delta(-t) < 0 < \Delta(t) \). By continuity of \( \Delta \), there must exists \( t_0 < t \) such that \( \Delta(-t_0) < 0 < \Delta(t_0) \). As we have already seen, this inequality implies
\[
f(x + t_0) \leq g(x) \leq f(x - t_0), \quad (21)
\]
for all \( x \) outside some compact \([-a, a]\). In particular, we now see that the inequalities in (20) must be strict for all large enough \( x \). Thus, for all \( x \in \mathbb{R} \),
\[
\int_{-x}^{+\infty} (y + x)^{d-1} g(y) dy > \int_{-x}^{+\infty} (y + x)^{d-1} f(y + t) dy
\]
\[
\int_{-x}^{+\infty} (y + x)^{d-1} g(y) dy < \int_{-x}^{+\infty} (y + x)^{d-1} f(y - t) dy.
\]
Applying \( u \mapsto \exp(-du) \) now shows that the inequalities in (20) must actually be strict everywhere on \( \mathbb{R} \), hence in particular on the compact \([-a,a]\). By uniform continuity, there must exist \( t_1 < t \) such that

\[
f(x + t_1) \leq g(x) \leq f(x - t_1),
\]

for all \( x \in [-a,a] \). In view of (21)-(22), the number \( t' := \max(t_0, t_1) \) now contradicts the minimality of \( t \).

References


