



**HAL**  
open science

# The Mezard-Parisi equation for matchings in pseudo-dimension $d > 1$

Justin Salez

► **To cite this version:**

Justin Salez. The Mezard-Parisi equation for matchings in pseudo-dimension  $d > 1$ . *Electronic Communications in Probability*, 2015, 20 (13), pp.1-7. 10.1214/ECP.v20-3791 . hal-01062106

**HAL Id: hal-01062106**

**<https://hal.science/hal-01062106>**

Submitted on 9 Sep 2014

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# The Mézard-Parisi equation for matchings in pseudo-dimension $d > 1$

Justin Salez

September 9, 2014

## Abstract

We establish existence and uniqueness of the solution to the cavity equation for the random assignment problem in pseudo-dimension  $d > 1$ , as conjectured by Aldous and Bandyopadhyay (Annals of Applied Probability, 2005) and Wästlund (Annals of Mathematics, 2012). This fills the last remaining gap in the proof of the original Mézard-Parisi prediction for this problem (Journal de Physique Lettres, 1985).

**Keywords:** recursive distributional equation; random assignment problem; mean-field combinatorial optimization; cavity method.

**2010 MSC:** 60C05, 82B44, 90C35.

## 1 Introduction

The *random assignment problem* is a now classical problem in probabilistic combinatorial optimization. Given an  $n \times n$  array  $\{X_{i,j}\}_{1 \leq i,j \leq n}$  of IID non-negative random variables, it asks about the statistics of

$$M_n := \min_{\sigma} \sum_{i=1}^n X_{i,\sigma(i)},$$

where the minimum runs over all permutations  $\sigma$  of  $\{1, \dots, n\}$ . This corresponds to finding a minimum-length perfect matching on the complete bipartite graph  $K_{n,n}$  with edge-lengths  $\{X_{i,j}\}_{1 \leq i,j \leq n}$ . Using the celebrated *replica symmetry ansatz* from statistical physics, Mézard and Parisi [10, 11, 12] made a remarkably precise prediction concerning the regime where  $n$  tends to infinity while the distribution of  $X_{i,j}$  is kept fixed and satisfies

$$\mathbb{P}(X_{i,j} \leq x) \sim x^d \quad \text{as } x \rightarrow 0^+,$$

for some exponent  $0 < d < \infty$ . Specifically, they conjectured that

$$\frac{M_n}{n^{1-1/d}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} -d \int_{\mathbb{R}} f(x) \ln f(x) dx, \quad (1)$$

where the function  $f: \mathbb{R} \rightarrow [0, 1]$  solves the so-called *cavity equation*:

$$f(x) = \exp \left( - \int_{-x}^{+\infty} d(x+y)^{d-1} f(y) dy \right). \quad (2)$$

Aldous [1, 3] proved this conjecture in the special case  $d = 1$ , where the term  $(x+y)^{d-1}$  simplifies and makes the cavity equation exactly solvable, yielding

$$f(x) = \frac{1}{1+e^x} \quad \text{and} \quad -d \int_{\mathbb{R}} f(x) \ln f(x) dx = \frac{\pi^2}{6}.$$

Since then, several alternative proofs have been found [9, 13, 15]. This stands in sharp contrast with the case  $d \neq 1$ , where showing that the Mézard-Parisi equation (2) admits a unique solution has until now remained an open problem [4, Open Problem 63]. Wästlund [16] circumvented this issue by considering instead the truncated equation

$$f_\lambda(x) = \exp \left( - \int_{-x}^{\lambda} d(x+y)^{d-1} f_\lambda(y) dy \right), \quad 0 < \lambda < \infty. \quad (3)$$

Using an ingenious game-theoretical interpretation of this equation, he showed the existence of a unique, global attractive solution  $f_\lambda: [-\lambda, \lambda] \rightarrow [0, 1]$  for every  $0 < \lambda < \infty$ , provided  $d \geq 1$ . He then used this fact to establish that

$$\frac{M_n}{n^{1-1/d}} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \lim_{\lambda \rightarrow +\infty} \uparrow -d \int_{-\lambda}^{\lambda} f_\lambda(x) \ln f_\lambda(x) dx. \quad (4)$$

Wästlund [16] explicitly left open the problem of completing the proof of the original Mézard-Parisi prediction by showing (i) that the untruncated cavity equation admits a unique solution  $f$  and (ii) that  $f_\lambda \rightarrow f$  as  $\lambda \rightarrow \infty$ . The purpose of this short paper is to establish this conjecture.

**Theorem 1.** *For  $d > 1$ , the Mézard-Parisi equation (2) admits a unique solution  $f: \mathbb{R} \rightarrow [0, 1]$ . Moreover,  $f_\lambda \rightarrow f$  pointwise as  $\lambda \rightarrow +\infty$ , and*

$$\int_{-\lambda}^{\lambda} f_\lambda(x) \ln f_\lambda(x) dx \xrightarrow[\lambda \rightarrow +\infty]{} \int_{\mathbb{R}} f(x) \ln f(x) dx.$$

*Consequently, the two limits in (1) and (4) coincide.*

In addition, we provide a short alternative proof of the crucial result of [16] that the truncated equation (3) admits a unique, attractive solution.

**Remark 1.** Very recently, a proof of uniqueness for the truncated equation (3) has been announced [8] for the case  $0 < d < 1$ . It would be interesting to see if the result of the present paper can be extended to this regime.

**Remark 2.** For a random variable  $Z$  with  $\mathbb{P}(Z > x) = f(x)$ , the cavity equation (2) simply expresses the fact that  $Z$  solves the distributional identity

$$Z \stackrel{d}{=} \min_{i \geq 1} \{\xi_i - Z_i\}, \quad (5)$$

where  $\{\xi_i\}_{i \geq 1}$  is a Poisson point process with intensity  $dx^{d-1}\partial x$  on  $[0, \infty)$ , and  $\{Z_i\}_{i \geq 1}$  are IID with the same distribution as  $Z$ , independent of  $\{\xi_i\}_{i \geq 1}$ . Such *recursive distributional equations* arise naturally in a variety of models from statistical physics, and the question of existence and uniqueness of solutions plays a crucial role for the rigorous understanding of those models. We refer the interested reader to the comprehensive surveys [2, 4] for more details. In particular, [4, Section 7.4] contains a detailed discussion on equation (5), and [4, Open Problem 63] raises explicitly the uniqueness issue. We note that the refined question of *endogeny* remains a challenging open problem. Recursive distributional equations for other mean-field combinatorial optimization problems have been analysed in e.g. [5, 14, 6].

The remainder of the paper is organized as follows. Section 2 deals with the truncated equation (3) for fixed  $0 < \lambda < \infty$  and is devoted to the alternative analytical proof that there is a unique, globally attractive solution  $f_\lambda$ . Section 3 prepares the  $\lambda \rightarrow \infty$  limit by providing uniform controls on the family  $\{f_\lambda : 0 < \lambda < \infty\}$  and by characterizing the possible limit points. This reduces the proof of Theorem 1 to establishing uniqueness in the untruncated Mézard-Parisi equation ( $\lambda = \infty$ ), which is done in Section 4.

## 2 The truncated cavity equation ( $\lambda < \infty$ )

Fix a parameter  $0 < \lambda < \infty$ . On the set  $\mathfrak{F}$  of non-increasing functions  $f: [-\lambda, \lambda] \rightarrow [0, 1]$ , define an operator  $T$  by

$$(Tf)(x) = \exp\left(-d \int_{-x}^{\lambda} (x+y)^{d-1} f(y) dy\right). \quad (6)$$

The purpose of this section is to give a short and purely analytical proof of the following result, which was the main technical ingredient in [16] and was therein established using an ingenious game-theoretical framework.

**Proposition 1.**  $T$  admits a unique fixed point  $f_\lambda$  and it is attractive in the sense that  $|T^n f(x) - f_\lambda(x)| \xrightarrow[n \rightarrow \infty]{} 0$ , uniformly in both  $x \in [-\lambda, \lambda]$  and  $f \in \mathfrak{F}$ .

*Proof.* Write  $f \leq g$  to mean  $f(x) \leq g(x)$  for all  $x \in [-\lambda, \lambda]$ . In particular,

$$\mathbf{0} \leq f \leq T\mathbf{0}$$

for every  $f \in \mathfrak{F}$ , where  $\mathbf{0}$  denotes the constant-zero function. Note also that the operator  $T$  is non-increasing, in the sense that

$$f \leq g \implies Tf \geq Tg.$$

Those two observations imply that the sequences  $\{T^{2n}\mathbf{0}\}_{n \geq 0}$  and  $\{T^{2n+1}\mathbf{0}\}_{n \geq 0}$  are respectively non-decreasing and non-increasing, and that their respective pointwise limits  $f^-$  and  $f^+$  satisfy

$$f^- \leq \liminf_{n \rightarrow \infty} T^n f \leq \limsup_{n \rightarrow \infty} T^n f \leq f^+,$$

for any  $f \in \mathfrak{F}$ . Moreover, the dominated convergence Theorem ensures that  $T$  is continuous with respect to pointwise convergence, allowing to pass to the limit in the identity  $T^{n+1}\mathbf{0} = T(T^n\mathbf{0})$  to deduce that

$$Tf^- = f^+ \quad \text{and} \quad Tf^+ = f^-. \quad (7)$$

Therefore, the proof boils down to the identity  $f^- = f^+$ , which we now establish. By definition, we have for any  $f \in \mathfrak{F}$ ,

$$(Tf)(x) = \exp\left(-d \int_{-\lambda}^{\lambda} (x+y)^{d-1} \mathbf{1}_{(x+y \geq 0)} f(y) dy\right).$$

Since  $d > 1$ , we may differentiate under the integral sign to obtain

$$(Tf)'(x) = -d(d-1)(Tf)(x) \int_{-\lambda}^{\lambda} (x+y)^{d-2} \mathbf{1}_{(x+y \geq 0)} f(y) dy.$$

Integrating over  $[-\lambda, \lambda]$  and noting that  $(Tf)(-\lambda) = 1$ , we conclude that

$$1 - (Tf)(\lambda) = d(d-1) \iint_{[-\lambda, \lambda]^2} (x+y)^{d-2} \mathbf{1}_{(x+y \geq 0)} (Tf)(x) f(y) dx dy.$$

Let us now specialize to  $f = f^\pm$ . In both cases, the right-hand side is

$$d(d-1) \iint_{[-\lambda, \lambda]^2} (x+y)^{d-2} \mathbf{1}_{(x+y \geq 0)} f^+(x) f^-(y) dx dy,$$

by (7). Therefore, we have  $(Tf^+)(\lambda) = (Tf^-)(\lambda)$ , i.e.

$$\int_{-\lambda}^{\lambda} d(\lambda + y)^{d-1} f^+(y) dy = \int_{-\lambda}^{\lambda} d(\lambda + y)^{d-1} f^-(y) dy.$$

Since we already know that  $f^- \leq f^+$ , this forces  $f^- = f^+$  almost-everywhere on  $[-\lambda, \lambda]$ , and hence everywhere by continuity. Finally, the convergence  $T^n \mathbf{0} \rightarrow f_\lambda = f^\pm$  is automatically uniform on  $[-\lambda, \lambda]$ , by Dini's Theorem.  $\square$

### 3 Relative compactness of solutions ( $\lambda \rightarrow \infty$ )

In order to study properties of the family  $\{f_\lambda: 0 < \lambda < \infty\}$ , we extend the domain of  $f_\lambda$  to  $\mathbb{R}$  by setting  $f_\lambda(x) = 1$  for  $x \leq -\lambda$  and  $f_\lambda(x) = 0$  for  $x > \lambda$ .

**Proposition 2** (Uniform bounds). *For all  $0 < \lambda < \infty$  and  $x \geq 0$ ,*

$$\begin{aligned} f_\lambda(x) &\leq \exp\left(-\frac{x^d}{e}\right) \\ 1 - f_\lambda(-x) &\leq \exp\left(-\frac{x^d}{e}\right) \\ f_\lambda(-x) \ln \frac{1}{f_\lambda(-x)} &\leq \exp\left(-\frac{x^d}{e}\right) \\ f_\lambda(x) \ln \frac{1}{f_\lambda(x)} &\leq \left(1 + \frac{x^d}{e}\right) \exp\left(-\frac{x^d}{e}\right). \end{aligned}$$

*Proof.* Let  $0 < \lambda < \infty$ . We may assume that  $x \in [0, \lambda]$ , otherwise the above bounds are trivial. By definition, we have

$$f_\lambda(x) = \exp\left(-\int_{-x}^{\lambda} d(x+y)^{d-1} f_\lambda(y) dy\right). \quad (8)$$

Now, since  $x \geq 0$  and  $f_\lambda$  is non-increasing, we have

$$\begin{aligned} \int_{-x}^{\lambda} (x+y)^{d-1} f_\lambda(y) dy &= \int_{-x}^0 (x+y)^{d-1} f_\lambda(y) dy + \int_0^{\lambda} (x+y)^{d-1} f_\lambda(y) dy \\ &\geq f_\lambda(0) \frac{x^d}{d} + \int_0^{\lambda} y^{d-1} f_\lambda(y) dy. \end{aligned}$$

Applying  $u \mapsto \exp(-du)$  to both sides and using (8), we obtain

$$f_\lambda(x) \leq f_\lambda(0) \exp(-f_\lambda(0)x^d). \quad (9)$$

In turn, this inequality implies that for all  $x \geq 0$ ,

$$\int_x^\lambda d(y-x)^{d-1} f_\lambda(y) dy \leq f_\lambda(0) \int_x^{+\infty} dy^{d-1} e^{-f_\lambda(0)y^d} dy = \exp(-f_\lambda(0)x^d).$$

Applying  $u \mapsto \exp(-u)$  to both sides, we conclude that

$$f_\lambda(-x) \geq \exp\left(-e^{-f_\lambda(0)x^d}\right). \quad (10)$$

In particular, taking  $x = 0$  yields  $f_\lambda(0) \geq e^{-1}$ , and reinjecting this into (9) and (10) easily yields the first three claims. For the last one, observe that  $u \mapsto u \ln \frac{1}{u}$  increases on  $[0, e^{-1}]$  and decreases on  $[e^{-1}, 1]$ , with the value at  $u = e^{-1}$  being precisely  $e^{-1}$ . Therefore, if  $\exp(-x^d/e) \leq e^{-1}$ , we may use the bound  $f_\lambda(x) \leq \exp(-x^d/e)$  to deduce that

$$f_\lambda(x) \ln \frac{1}{f_\lambda(x)} \leq \frac{x^d}{e} \exp\left(-\frac{x^d}{e}\right).$$

On the other hand, if  $\exp(-x^d/e) \geq e^{-1}$ , then

$$f_\lambda(x) \ln \frac{1}{f_\lambda(x)} \leq e^{-1} \leq \exp\left(-\frac{x^d}{e}\right).$$

In both cases, the last inequality holds, and the proof is complete.  $\square$

**Proposition 3.** *The family  $\{f_\lambda: 0 < \lambda < \infty\}$  is relatively compact with respect to the topology of uniform convergence on  $\mathbb{R}$ , and any sub-sequential limit as  $\lambda \rightarrow \infty$  must solve the cavity equation (2).*

*Proof.* Let  $\{\lambda_n\}_{n \geq 0}$  be any sequence of positive numbers such that  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$ . By Helly's compactness principle for uniformly bounded monotone functions (see e.g. [7, Theorem 36.5]), there exists an increasing sequence  $\{n_k\}_{k \geq 0}$  in  $\mathbb{N}$  and a non-increasing function  $f: \mathbb{R} \rightarrow [0, 1]$  such that

$$f_{\lambda_{n_k}}(x) \xrightarrow[k \rightarrow \infty]{} f(x), \quad (11)$$

for all  $x \in \mathbb{R}$ . Thanks to the first inequality in Proposition 2, we may invoke dominated convergence to deduce that for each  $x \in \mathbb{R}$ ,

$$\int_{-x}^{\lambda_{n_k}} f_{\lambda_{n_k}}(y)(x+y)^{d-1} dy \xrightarrow[k \rightarrow \infty]{} \int_{-x}^{+\infty} f(y)(x+y)^{d-1} dy.$$

Applying  $u \mapsto \exp(-du)$  and recalling (8), we see that

$$f(x) = \exp\left(-d \int_{-x}^{+\infty} f(y)(x+y)^{d-1} dy\right),$$

which shows that  $f$  must solve the cavity equation (2). This identity easily implies that  $f$  is continuous. Consequently, the convergence (11) is uniform in  $x \in \mathbb{R}$ , by Dini's Theorem.  $\square$

## 4 The un-truncated cavity equation ( $\lambda = \infty$ )

To conclude the proof of Theorem 1, it now remains to show that the un-truncated equation

$$f(x) = \exp\left(-d \int_{-x}^{+\infty} (x+y)^{d-1} f(y) dy\right). \quad (12)$$

admits at most one fixed point  $f: \mathbb{R} \rightarrow [0, 1]$ . Proposition 3 will then guarantee the convergence  $f_\lambda \xrightarrow{\lambda \rightarrow \infty} f$ , which will in turn imply

$$\int_{-\lambda}^{\lambda} f_\lambda(x) \ln f_\lambda(x) dx \xrightarrow{\lambda \rightarrow +\infty} \int_{\mathbb{R}} f(x) \ln f(x) dx,$$

by dominated convergence, thanks to the last inequalities in Proposition 2.

A quick inspection of the proof of Proposition 2 reveals that it remains valid when  $\lambda = \infty$ . In particular, any solution  $f$  to (12) must satisfy

$$\max(f(x), 1 - f(-x)) \leq \exp\left(-\frac{x^d}{e}\right), \quad (13)$$

for all  $x \geq 0$ . It is also clear from (12) that  $f$  must be  $(0, 1)$ -valued and continuous. We will use those properties in the proofs below.

**Lemma 1.** *If  $f, g$  solve (12), then there exists  $t \geq 0$  such that for all  $x \in \mathbb{R}$ ,*

$$f(x+t) \leq g(x) \leq f(x-t).$$

*Proof.* (13) ensures that for any  $t \in \mathbb{R}$ ,  $y \mapsto (1 + |y|)(f(y-t) - g(y))$  is integrable on  $\mathbb{R}$ , so that by dominated convergence,

$$\frac{1}{x^{d-1}} \int_{-x}^{+\infty} (y+x)^{d-1} (f(y-t) - g(y)) dy \xrightarrow{x \rightarrow +\infty} \Delta(t), \quad (14)$$

where

$$\Delta(t) := \int_{\mathbb{R}} (f(y-t) - g(y)) dy. \quad (15)$$

Observe that  $t \mapsto \Delta(t)$  increases continuously from  $-\infty$  to  $+\infty$ , as can be seen from the decomposition

$$\Delta(t) = \int_0^{+\infty} (1 - g(-y) - g(y)) dy + \int_{-t}^{+\infty} f(y) dy - \int_t^{+\infty} (1 - f(-y)) dy.$$



In particular, we can find  $t_0 \geq 0$  such that  $\Delta(-t_0) < 0 < \Delta(t_0)$ . In view of (14), we deduce the existence of  $a \geq 0$  such that for all  $x \geq a$ ,

$$\int_{-x}^{+\infty} (y+x)^{d-1} g(y) dy \geq \int_{-x}^{+\infty} (y+x)^{d-1} f(y+t_0) dy \quad (16)$$

$$\int_{-x}^{+\infty} (y+x)^{d-1} g(y) dy \leq \int_{-x}^{+\infty} (y+x)^{d-1} f(y-t_0) dy. \quad (17)$$

Applying  $u \mapsto \exp(-du)$ , we conclude that for all  $x \geq a$ ,

$$f(x+t_0) \leq g(x) \leq f(x-t_0). \quad (18)$$

In turn, this implies that (16)-(17) also hold when  $x \leq -a$ , so that (18) actually holds for all  $x$  outside  $(-a, a)$ . On the other hand, since  $g$  is  $(0, 1)$ -valued and  $f$  has limits  $0, 1$  at  $\pm\infty$ , we can choose  $t_1 \geq 0$  large enough so that

$$f(-a+t_1) \leq g(a) \leq g(-a) \leq f(a-t_1).$$

Since  $f, g$  are non-increasing, this inequality implies that for all  $x \in [-a, a]$ ,

$$f(x+t_1) \leq g(x) \leq f(x-t_1). \quad (19)$$

In view of (18)-(19), taking  $t := \max(t_0, t_1)$  concludes the proof.  $\square$

*Proof of Proposition 3.* Let  $f, g$  solve equation (12) and let  $t$  be the smallest non-negative number satisfying for all  $x \in \mathbb{R}$ ,

$$f(x+t) \leq g(x) \leq f(x-t). \quad (20)$$

Note that  $t$  exists by Lemma 1 and the continuity of  $f$ . Now assume for a contradiction that  $t > 0$ . Clearly, each of the two inequalities in (20) must be strict at some point  $x \in \mathbb{R}$  (and hence on some open interval by continuity), otherwise we would have  $g \geq f$  or  $g \leq f$  and (12) would then force  $g = f$ , contradicting the assumption that  $t > 0$ . Consequently, the function  $\Delta$  defined in (15) must satisfy  $\Delta(-t) < 0 < \Delta(t)$ . By continuity of  $\Delta$ , there must exist  $t_0 < t$  such that  $\Delta(-t_0) < 0 < \Delta(t_0)$ . As we have already seen, this inequality implies

$$f(x+t_0) \leq g(x) \leq f(x-t_0), \quad (21)$$

for all  $x$  outside some compact  $[-a, a]$ . In particular, we now see that the inequalities in (20) must be strict for all large enough  $x$ . Thus, for all  $x \in \mathbb{R}$ ,

$$\begin{aligned} \int_{-x}^{+\infty} (y+x)^{d-1} g(y) dy &> \int_{-x}^{+\infty} (y+x)^{d-1} f(y+t) dy \\ \int_{-x}^{+\infty} (y+x)^{d-1} g(y) dy &< \int_{-x}^{+\infty} (y+x)^{d-1} f(y-t) dy. \end{aligned}$$

Applying  $u \mapsto \exp(-du)$  now shows that the inequalities in (20) must actually be strict everywhere on  $\mathbb{R}$ , hence in particular on the compact  $[-a, a]$ . By uniform continuity, there must exist  $t_1 < t$  such that

$$f(x + t_1) \leq g(x) \leq f(x - t_1), \quad (22)$$

for all  $x \in [-a, a]$ . In view of (21)-(22), the number  $t' := \max(t_0, t_1)$  now contradicts the minimality of  $t$ .  $\square$

## References

- [1] David Aldous. Asymptotics in the random assignment problem. *Probab. Theory Related Fields*, 93(4):507–534, 1992.
- [2] David Aldous and J. Michael Steele. The objective method: probabilistic combinatorial optimization and local weak convergence. In *Probability on discrete structures*, volume 110 of *Encyclopaedia Math. Sci.*, pages 1–72. Springer, Berlin, 2004.
- [3] David J. Aldous. The  $\zeta(2)$  limit in the random assignment problem. *Random Structures Algorithms*, 18(4):381–418, 2001.
- [4] David J. Aldous and Antar Bandyopadhyay. A survey of max-type recursive distributional equations. *Ann. Appl. Probab.*, 15(2):1047–1110, 2005.
- [5] David Gamarnik, Tomasz Nowicki, and Grzegorz Swirszcz. Maximum weight independent sets and matchings in sparse random graphs. Exact results using the local weak convergence method. *Random Structures Algorithms*, 28(1):76–106, 2006.
- [6] M. Khandwawala. Solutions to recursive distributional equations for the mean-field TSP and related problems. *ArXiv e-prints*, May 2014.
- [7] A. N. Kolmogorov and S. V. Fomīn. *Introductory real analysis*. Dover Publications, Inc., New York, 1975. Translated from the second Russian edition and edited by Richard A. Silverman, Corrected reprinting.
- [8] J. Larsson. The Minimum Perfect Matching in Pseudo-dimension  $0 < q < 1$ . *ArXiv e-prints*, March 2014.
- [9] Svante Linusson and Johan Wästlund. A proof of Parisi’s conjecture on the random assignment problem. *Probab. Theory Related Fields*, 128(3):419–440, 2004.

- [10] M. Mézard and G. Parisi. Replicas and optimization. *J. Physique Lett.*, 46(17):771–778, 1985.
- [11] M. Mézard and G. Parisi. Mean-field equations for the matching and the travelling salesman problems. *EPL (Europhysics Letters)*, 2(12):913, 1986.
- [12] M. Mézard and G. Parisi. On the solution of the random link matching problems. *J. Phys. France*, 48(9):1451–1459, 1987.
- [13] Chandra Nair, Balaji Prabhakar, and Mayank Sharma. Proofs of the Parisi and Coppersmith-Sorkin random assignment conjectures. *Random Structures Algorithms*, 27(4):413–444, 2005.
- [14] G. Parisi and J. Wästlund. Mean field matching and traveling salesman problems in pseudo-dimension 1. 2012.
- [15] Johan Wästlund. An easy proof of the  $\zeta(2)$  limit in the random assignment problem. *Electron. Commun. Probab.*, 14:261–269, 2009.
- [16] Johan Wästlund. Replica symmetry of the minimum matching. *Ann. of Math. (2)*, 175(3):1061–1091, 2012.