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► **To cite this version:**

Sofiene El Aoud, Frédéric Abergel. A stochastic control approach for options market making. 2014.
hal-01061852v1

HAL Id: hal-01061852

<https://hal.science/hal-01061852v1>

Preprint submitted on 8 Sep 2014 (v1), last revised 10 Nov 2015 (v3)

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A stochastic control approach for options market making

Sofiene El Aoud ^{†*}

Frédéric Abergel ^{†‡}

September 1, 2014

Abstract

In this paper, we establish a model for market making in options whose underlying is perfectly liquid. In our model framework, the stock price follows a generic stochastic volatility model under the real-world probability measure \mathcal{P} . Market participants price options on this stock under a risk-neutral pricing measure \mathcal{Q} , and they may misspecify the parameters controlling the dynamics of the volatility process. We consider that there is an agent who is willing to make markets in an option on the stock with the aim of maximizing his expected utility from terminal wealth at the maturity of this option. Since market impact is an important feature in the microscopic time scale and should be taken into account in high frequency trading, we study different forms of this function argued in the recent literature. Through the use of optimal stochastic control, we provide exact expressions of optimal bid and ask quotes of the market making strategy in the case where the agent is risk-neutral. Afterward, we suppose that the agent is risk-averse and wants to reduce the variance of the final wealth. In addition, this agent tries not to accumulate a large inventory in order not to have a significant exposure to market risk. For this purpose, we perturb the utility function by a penalty on the variance of final wealth and also on accumulated inventory. Using singular perturbation with respect to the penalty parameter, we provide analytic approximations of the optimal bid and ask quotes. In order to confirm our theoretical results, we perform Monte Carlo simulations of the optimal market making strategy in the case where the stock price process follows a Heston model. We show that the optimal strategy is more profitable than a zero-intelligence strategy. Besides, we highlight the effects of the misspecification of the parameters on the performance of the strategy.

*sofiene.elaoud@ecp.fr

[†]Ecole Centrale Paris, Laboratoire de Mathématiques Appliquées aux Systèmes, Grande Voie des Vignes, 92290 Châtenay Malabry, France

[‡]frederic.abergel@ecp.fr

JEL Classification: JEL: C51 Model construction and estimation, JEL: C52 Model evaluation and testing.

Keywords: Market making, stochastic control, utility maximization, inventory constraints.

1 Introduction

Market makers have a mandate to provide liquidity on some securities that they choose in advance. Their role consists of continuously setting bid and ask quotes on the instruments in which they make markets and are obligated to buy and sell at their displayed bid and offer prices. Market makers have a fundamental and important role in financial markets since they obviously increase the liquidity of their quotes instruments. Indeed, by setting their quotes in the order book, they increase its depth and they provide liquidity for impatient agents who are willing to cross the bid-ask spread to get their orders executed immediately. In periods of imbalance between buy and sell orders, market makers could hold a significant directional position for a while and accumulate a non negligible inventory. Thus, they may be required to bear a market risk which could cause significant losses. To be paid for such risks, market makers expect a positive profit since they set bid and ask quotes around the mid price, and then they offer to buy at a bid price which is lower than the mid price and they offer to sell at an ask price which is higher than the mid price.

This paper deals with the problem of options market making. There have not been many studies devoted to market making in options despite the importance of this subject. However, we can find, in the recent literature, several works focusing on the problem of market making and high frequency trading of stocks through the use of a stochastic optimization resolution (see [Avellaneda and Stoikov(2008)], [Carmona and Webster(2012)], [Cartea et al.(2013)Cartea, Donnelly, and Jaimungal], [Fodra and Labadie(2013)], [Guéant et al.(2013)Guéant, Lehalle, and Fernandez-Tapia], [Guilbaud and Pham(2013)], [Alfonsi et al.(2010)Alfonsi, Fruth, and Schied], [Schied(2013)]). Avellaneda and Stoikov addressed in [Avellaneda and Stoikov(2008)] the problem of high frequency market making of a stock. In their model framework, the stock price process diffuses as an arithmetic Brownian motion in the intraday time-scale, and the market maker aims to determine the optimal quotes in order to maximize her expected exponential utility from terminal wealth. The authors solved the optimization problem using a stochastic control approach and provided approximations of the optimal quotes using an expansion in the inventory parameter. Then, Fodra and Labadie extended in [Fodra and Labadie(2012)] the work of Avellaneda and Stoikov to the case where the stock price process has a drift term and a stochastic volatility. They provided the optimal bid and ask quotes in the case where the utility function is linear, but the solution is not valid under all stock models. Indeed, the value function is finite when the stock is an Ornstein-Uhlenbeck process for example but is not when the stock is a geometric Brownian motion. In [Fodra and Labadie(2013)], they extended their previous work by perturbing the utility function with a constraint on the accumulated inventory and they provided an approximation of the solution of the problem. In addition to that, Guéant, Lehalle and Fernandez-Tapia dealt in [Guéant et al.(2012)Guéant, Lehalle, and Fernandez-Tapia] with

the problematic of high frequency trading for the purpose of portfolio liquidation. The authors provided analytic expressions for optimal bid quotes (respectively ask quotes) in order to liquidate a short portfolio position (respectively a long portfolio position). In [Guéant et al.(2013)Guéant, Lehalle, and Fernandez-Tapia], a similar technique was used in order to provide expressions for the bid and ask quotes in the case where the market maker stops setting ask quotes (respectively bid quotes) if her inventory exceeds some negative level $-\mathcal{R}$ (respectively some positive level \mathcal{R}). This restriction can be considered as a risk limit and allows to solve the problem analytically.

The authors in [Cartea et al.(2013)Cartea, Donnelly, and Jaimungal] addressed the problem of model ambiguity and its effect on the optimal market making strategy. In their case of study, the ambiguity includes the choice of the model for the stock dynamics or also for the arrival rate of market orders. Using an entropic penalization, the authors solve the optimization problem and showed how the market maker adjusts her quotes to reduce adverse selection costs.

In [Stoikov and Sağlam(2009)], Stoikov and Sağlam proposed a mean variance framework to study the problem of optimal market making in options. At the beginning, the authors delivered the optimal bid and ask quotes on the option in the case of a complete market where the stock's volatility is constant and the option can be perfectly hedged by trading the stock. Afterward, they removed the hypothesis of constant volatility and they supposed that the stock and its instantaneous volatility remain constant during the trading session but may have large overnight moves. Using a dynamic programming method, they provide analytic expressions for the optimal quotes using a discrete time grid.

In this paper, the problem of optimal market making of options in a continuous time setting is addressed. The dynamics of the instantaneous volatility of the stock are specified in order to deliver a realistic framework for option pricing. Thus, the dimension of the market making problem increases as the instantaneous volatility represents an additional state variable. In addition, it is supposed here that the market maker hedges continuously her option inventory by trading the liquid stock which makes her hold an inventory in the stock. We aim, through this study, to determine the optimal market making strategy on options in the setting of a generic stochastic volatility model. In the second section, we define the joint dynamics of the spot and its instantaneous volatility under both the real-world probability measure and the pricing measure. In the third section, we present several forms of the market impact function in the option market. Conditional to this function, the intensity of arrival of market orders is determined analytically for the purpose of determination of the optimal strategy. In the fourth section, we consider that the market maker is risk neutral and aims to maximize the expectation of his wealth at the maturity date of the option. Thus, we determine analytic expressions for the optimal quotes through the use of a stochastic control approach. In the fifth section, the market maker is supposed to be risk averse, she aims to reduce the uncertainty of her final wealth and not to accumulate a large inventory. We propose, in this case, a mean-variance framework to determine the optimal quoting policy. Indeed, the utility function is penalized by an additional term which depends on the variance of final wealth. By means of a singular perturbation on the penalty parameter, we provide

approximations of the optimal quotes and show how they are affected by the penalty term. In the final section, we generate Monte Carlo simulations in order to test numerically the performance of the strategy. Through these simulations, we compare the performances of the optimal market making strategy and a zero-intelligence one. Next, we show the effects of the misspecification of the parameters on both the performance of the strategy and the accumulated inventory.

2 Model setup

Consider a financial market living on a stochastic basis $(\Omega, \mathcal{F}, \mathbb{F}, \mathcal{P})$, where the filtration $\mathbb{F} = \{\mathcal{F}_t\}$ satisfies the usual conditions, and where \mathcal{P} denotes the real-world probability measure. Under \mathcal{P} , the spot process S has the following dynamics:

$$\frac{dS_t}{S_t} = \mu dt + \sigma(y_t)dW_t^{(1)}, \quad (2.1)$$

$$dy_t = a_R(y_t)dt + b_R(y_t)dW_t^{(2)}, \quad (2.2)$$

where $W^{(1)}$ and $W^{(2)}$ are two $(\mathcal{P}, \mathcal{F})$ Brownian motions such that $d\langle W^{(1)}, W^{(2)} \rangle_t = \rho_R dt$.

Suppose that there is an European option with maturity T and payoff function $h(S_T)$ which is traded in the option market. Let the quantity $C_{\mathcal{P}}$ be defined as:

$$C_{\mathcal{P}}(t, S_t, y_t) = E^{\mathcal{P}}(h(S_T)|\mathcal{F}_t). \quad (2.3)$$

The quantity $C_{\mathcal{P}}$ doesn't represent the option price under \mathcal{P} since \mathcal{P} is not a risk-neutral probability measure. Nevertheless, this quantity will be useful in the sequel.

Market participants price options under a risk-neutral probability measure \mathcal{Q} different from \mathcal{P} . To make the model framework as general as possible, it is supposed here that market agents could have a misleading view of the dynamics of the process (y) , and then they may misspecify the parameters characterizing its distribution. This feature, discussed in [Jena and Tankov(2011)], accounts for the discrepancy between the physical measure and the pricing measure.

Let the interest rate be equal to r . It is supposed that, under the pricing measure \mathcal{Q} , the process S evolves as follows:

$$\frac{dS_t}{S_t} = r dt + \sigma(y_t)dW_t^{*,(1)}, \quad (2.4)$$

$$dy_t = a_I(y_t)dt + b_I(y_t)dW_t^{*,(2)}, \quad (2.5)$$

where $W^{*,(1)}$ and $W^{*,(2)}$ are two $(\mathcal{Q}, \mathcal{F})$ Brownian motions such that $d\langle W^{*,(1)}, W^{*,(2)} \rangle_t = \rho_I dt$. The option price $C_{\mathcal{Q}}$ under the risk-neutral probability measure \mathcal{Q} writes:

$$C_{\mathcal{Q}}(t, S_t, y_t) = e^{-r(T-t)} E^{\mathcal{Q}}(h(S_T)|\mathcal{F}_t). \quad (2.6)$$

Suppose that there is an agent who makes markets in this option and who trades with limit orders in the form of bid and ask quotes around the option's mid price $C_{\mathcal{Q}}$. In addition, the agent of interest trades continuously in the liquid stock in order to keep the option inventory delta-hedged. At a given time t , this market maker sets an ask price C_t^a and a bid price C_t^b such that:

$$\begin{aligned} C_t^a &= C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+, \\ C_t^b &= C_{\mathcal{Q}}(t, S_t, y_t) - \delta_t^-, \end{aligned}$$

where δ_t^- and δ_t^+ are positive quantities representing respectively the bid distance and the ask distance of the market maker. Those distances influence the option inventory held by the market maker since they affect the arrival rate of market buy and sell orders.

The notations defined here are going to be used in the rest of the study:

- $q_{1,t}$ denotes the option inventory held by the market maker at time t .
- $q_{2,t}$ is the stock inventory held by the market maker at time t .
- X_t is the cash held by the market maker at time t .
- $C_{\mathcal{Q}}(t, S_t, y_t)$ is the mid price of the option at time t .
- $\Delta_t = \Delta(t, S_t, y_t)$ is the option's delta at time t .

The arrival of liquidity-consuming orders to the order book are modeled by two independent Poisson processes: N^+ for buy orders consuming the ask quotes, and N^- for sell orders consuming the bid quotes. Therefore, the dynamics of the process q_1 can be described as following:

$$dq_{1,t} = dN_t^- - dN_t^+. \quad (2.7)$$

The market maker adjusts continuously her inventory in stock in order to delta-hedge her option position. This implies that $q_{2,t} = -q_{1,t}\Delta_t$ and then:

$$dq_{2,t} = -\Delta_t dq_{1,t} - q_{1,t} d\Delta_t - d\langle q_1, \Delta \rangle_t = -\Delta_t dN_t^- + \Delta_t dN_t^+ - q_{1,t} d\Delta_t. \quad (2.8)$$

The cash process (X) of the market maker is affected by three factors. Indeed, if her ask quote gets executed, she receives the ask price ($C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+$). On the other hand, if her bid quote gets executed, she pays the bid price ($C_{\mathcal{Q}}(t, S_t, y_t) - \delta_t^-$). In addition to that, the continuous trading in the stock, for the purpose of delta-hedging, induces a continuous variation in the cash process. Thus, the process (X) has the following dynamics:

$$dX_t = (C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+)dN_t^+ - (C_{\mathcal{Q}}(t, S_t, Y_t) - \delta_t^-)dN_t^- + q_{2,t}dS_t. \quad (2.9)$$

The market maker aims to provide liquidity optimally in order to maximize her expected utility from terminal wealth. She can control only the bid and ask distances δ^- and δ^+ which affect indirectly the arrival rate of market orders. Meanwhile, the dynamics of the state variables q_1, q_2 and X depend on market orders arrival. In order to formulate the optimization problem of the market maker, it is crucial to establish a model for the intensities of the Poisson processes N^+ and N^- representing market buy orders and market sell orders respectively.

3 Intensity of arrivals of market orders

This section is devoted to the modeling of the intensities of arrivals of market orders. Several works have been recently carried out to study the intensity of market-orders arrivals in the order book of a stock. These studies investigated the relation between the intensity λ^+ (respectively λ^-) of a buy market order (respectively a sell market-order) consuming liquidity of an ask quote (respectively a bid quote) at a distance δ^+ (respectively δ^-) from the mid price. It has been found that the functions λ^+ and λ^- are decreasing functions of δ^+ and δ^- respectively. In this study, we make the assumption that λ^+ and λ^- have the following forms: $\forall \delta^+, \delta^- \geq 0$:

$$\lambda^+(\delta^+) = \frac{A}{\left(B + (\delta^+)^{\frac{1}{\beta}}\right)^\gamma}, \quad \lambda^-(\delta^-) = \frac{A}{\left(B + (\delta^-)^{\frac{1}{\beta}}\right)^\gamma},$$

where $A, B > 0$, $\gamma > 1$ and β is a positive parameter characterizing the market impact function on the option market. This result is proven heuristically in (8.1).

4 Optimization problem for a risk-neutral market maker

The market maker aims to maximize her expected utility from terminal wealth at the option's maturity date T . This terminal wealth consists of the sum of the cash amount and the market value of the option inventory. The objective of the market maker is to determine, at a given time t , the optimal distances $(\delta_{*,t}^+, \delta_{*,t}^-)$ solution of the following problem:

$$(\delta_{L,*,t}^+, \delta_{L,*,t}^-) = \text{ArgMax}_{\{\delta^+, \delta^-\}} E^{\mathcal{P}}(U(X_T + q_{1,T}h(S_T)) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x).$$

where U is the utility function.

In this section, the market maker is considered to be risk-neutral, thus the utility function U is linear:

$$U(T, S_T, y_T, q_{1,T}, X_T) = X_T + q_{1,T}h(S_T), \quad (4.1)$$

where:

$$X_T = \int_0^T (C_{\mathcal{Q}}(t, S_t, y_t) + \delta_t^+) dN_t^+ - \int_0^T (C_{\mathcal{Q}}(t, S_t, Y_t) - \delta_t^-) dN_t^- + \int_0^T q_{2,t} dS_t.$$

The problem associated with the linear utility function is considered here. The cases corresponding to $\beta = \frac{1}{2}$ (square root market impact) and $\beta = 1$ (linear market impact) are studied separately and the optimal bid and ask quotes are provided analytically in each case.

In order to solve the optimization problem, a stochastic control approach is used. The value function of the market maker can be defined in the following way:

$$u(t, s, y, q_1, x) = \text{Max}_{\{(\delta^+, \delta^-) \in \mathcal{A}\}} E^{\mathcal{P}}(X_T + q_{1,T}h(S_T)) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x),$$

where \mathcal{A} denotes the set of admissible values for the controls (δ^-, δ^+) and is equal to $\mathcal{R}^+ \times \mathcal{R}^+$.

Let the functions J^+ and J^- be defined as follows:

$$\begin{aligned} J^+(\delta^+) &= \lambda^+(\delta^+) \left(u(t, s, y, q_{1,t^-} - 1, x_{t^-} + (C_{\mathcal{Q}} + \delta^+)) - u(t, s, y, q_{1,t^-}, x_{t^-}) \right), \\ J^-(\delta^-) &= \lambda^-(\delta^-) \left(u(t, s, y, q_{1,t^-} + 1, x_{t^-} - (C_{\mathcal{Q}} - \delta^-)) - u(t, s, y, q_{1,t^-}, x_{t^-}) \right), \end{aligned}$$

and the differential operators $\mathcal{L}_1, \mathcal{L}_2$ be introduced below:

$$\begin{aligned} \mathcal{L}_1 &= \mu S_t \frac{\partial}{\partial S} + \frac{1}{2} \sigma^2(y_t) S_t^2 \frac{\partial^2}{\partial S^2} + a_R(y_t) \frac{\partial}{\partial y} + \frac{1}{2} b_R^2(y_t) \frac{\partial^2}{\partial y^2} + \rho_R b_R(y_t) \sigma(y_t) S_t \frac{\partial^2}{\partial S \partial y}, \\ \mathcal{L}_2 &= \frac{\partial}{\partial x} q_{2,t} \mu S_t + \frac{1}{2} \frac{\partial^2}{\partial x^2} q_{2,t}^2 \sigma(y_t)^2 S_t^2 + \frac{\partial^2}{\partial x \partial S} q_{2,t} \sigma^2(y_t) S_t^2 + \frac{\partial^2}{\partial x \partial y} q_{2,t} \sigma(y_t) S_t b_R(y_t) \rho_R, \end{aligned}$$

The value function u is then the solution of the following Hamilton-Jacobi-Bellman equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u + \max_{\{(\delta^+, \delta^-) \in \mathcal{A}\}} (J^+(\delta^+) + J^-(\delta^-)) = 0, \quad (4.2)$$

with the final condition:

$$u(T, s, y, q_1, x) = x + q_1 h(s).$$

4.1 The case where $\beta = \frac{1}{2}$ (square root market impact)

The intensities of arrivals of market orders can be written as follows:

$$\forall \delta^+, \delta^- \geq 0, \lambda^+(\delta^+) = \frac{A}{(B + (\delta^+)^2)^\gamma} \text{ and } \lambda^-(\delta^-) = \frac{A}{(B + (\delta^-)^2)^\gamma},$$

where $A, B \geq 0$ and $\gamma > 1$.

4.1.1 Analytic resolution

Proposition 4.1. *Let $M_0(t, s, y) = C_{\mathcal{Q}}(t, s, y) - C_{\mathcal{P}}(t, s, y) + \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right)$.*

The optimal controls $(\delta_{L,,t}^+, \delta_{L,*,t}^-)$ of the market maker at time t are:*

$$\delta_{L,*,t}^+ = \frac{\sqrt{\gamma^2 M_0^2(t, s, y) + B(2\gamma - 1)} - \gamma M_0(t, s, y)}{2\gamma - 1}, \quad (4.3)$$

$$\delta_{L,*,t}^- = \frac{\gamma M_0(t, s, y) + \sqrt{\gamma^2 M_0^2(t, s, y) + B(2\gamma - 1)}}{2\gamma - 1}, \quad (4.4)$$

and the value function is:

$$u_0(t, s, y, q_1, x) = x + \theta_0(t, s, y) + q_1 \left(C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right) \right), \quad (4.5)$$

where:

$$\theta_0(t, s, y) = E_{t,s,y} \left(\int_t^T J_0(u, S_u, y_u) du \right).$$

and:

$$\begin{aligned} J_0(t, s, y) &= \lambda^+(\delta_{L,*,t}^+) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + M_0(\gamma - 1)}{2\gamma - 1} \right) \\ &+ \lambda^-(\delta_{L,*,t}^-) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + M_0(1 - \gamma)}{2\gamma - 1} \right). \end{aligned}$$

Proof. Since the utility function is linear as given in (4.1), we will look for a solution of the form:

$$u(t, s, y, q_1, x) = x + \theta_0(t, s, y) + q_1 \theta_1(t, s, y), \quad (4.6)$$

Let $f_0^+ = J^+$. Using (4.6), the function f_0^+ can be written explicitly as:

$$f_0^+(\delta^+) = \lambda^+(\delta^+) (\delta^+ + M_0(t, s, y)),$$

where $M_0(t, s, y) = C_{\mathcal{Q}}(t, s, y) - \theta_1(t, s, y)$.

In order to determine $\delta_{L,*,t}^+ = \text{ArgMax}_{\{x>0\}} f_0^+(x)$, the derivative of the function f_0^+ is computed as follows:

$$(f_0^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B + (\delta^+)^2} \left((\delta^+)^2 (1 - 2\gamma) - 2\gamma M_0(t, s, y) \delta^+ + B \right).$$

The function $(f_0^+)'$ vanishes at the points x_1^+ and x_2^+ :

$$x_1^+ = \frac{-\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1}, \quad x_2^+ = \frac{-\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1},$$

Since $\gamma \geq 1$ and $B > 0$:

$$\gamma |M_0| \leq \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)},$$

which implies that $x_1^+ < 0$ and $x_2^+ > 0$.

It can be recalled here that $1 - 2\gamma < 0$. Therefore, through the use of the table of variations of f_0^+ , it can be deduced that this function reaches its maximum on R^+ at x_2^+ . It follows that:

$$\delta_{L,*,t}^+ = \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} - \gamma M_0}{2\gamma - 1}$$

and:

$$f_0^+(\delta_{L,*,t}^+) = \lambda^+(\delta_{L,*,t}^+) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + M_0(\gamma - 1)}{2\gamma - 1} \right). \quad (4.7)$$

Let $f_0^- = J^-$. Using the form of the value function suggested in (4.6), the function f_0^- writes:

$$f_0^-(\delta^-) = \lambda^-(\delta^-)(\delta^- - M_0(t, s, y)).$$

In order to determine $\delta_{L,*,t}^- = \text{ArgMax}_{\{x>0\}} f_0^-(x)$, the derivative of the function f_0^- is computed:

$$(f_0^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + (\delta^-)^2} ((1 - 2\gamma)(\delta^-)^2 + 2\gamma M_0 \delta^- + B).$$

The function $(f_0^-)'$ changes sign and becomes null at the two following points:

$$x_1^- = \frac{\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1}, \quad x_2^- = \frac{\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1},$$

Using the same reasoning as before, it can be proved that $x_1^- < 0$ and $x_2^- > 0$ and that:

$$\delta_{L,*,t}^- = \frac{\gamma M_0 + \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1}.$$

The function f_0^- evaluated at its maximum gives:

$$f_0^-(\delta_{L,*,t}^-) = \lambda^-(\delta_{L,*,t}^-) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + M_0(1 - \gamma)}{2\gamma - 1} \right). \quad (4.8)$$

Finally, using (4.7) and (4.8), $J_0(t, s, y) = f_0^+(\delta_{L,*,t}^+) + f_0^-(\delta_{L,*,t}^-)$ can be written as follows:

$$\begin{aligned} J_0(t, s, y) &= \lambda^+(\delta_{L,*,t}^+) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + M_0(\gamma - 1)}{2\gamma - 1} \right) \\ &+ \lambda^-(\delta_{L,*,t}^-) \left(\frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + M_0(1 - \gamma)}{2\gamma - 1} \right). \end{aligned}$$

and it is straightforward to see that J_0 is independent of q_1 .

The equation (4.2) becomes:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)u + J_0(t, s, y) = 0.$$

For this equation to be solved, the terms are sorted by powers of q_1 . The functions θ_0, θ_1 and J_0 are independent of q_1 (they depend only on t, s, y). The terms obtained after the classification are either of order 0 or 1 in q_1 . By nullifying these terms, we obtain the following equations:

$$\begin{aligned} (0) &: (\partial_t + \mathcal{L}_1)\theta_0 + J_0(t, s, y) = 0, \\ (1) &: (\partial_t + \mathcal{L}_1)\theta_1 - \Delta_t \mu S_t = 0, \end{aligned}$$

The function θ_1 is the solution of the (1)-term (at order 1 in q_1):

$$(\partial_t + \mathcal{L}_1)\theta_1 - \Delta_t \mu S_t = 0,$$

and is subject to the final condition $\theta_1(T, s, y) = h(s)$. Using the Feynman-Kac formula, the following is obtained:

$$\theta_1(t, s, y) = E_{t,s,y}^{\mathcal{P}}(h(S_T)) - \mu E_{t,s,y}^{\mathcal{P}}\left(\int_t^T \Delta_u S_u du\right),$$

The quantity $M_0(t, s, y)$ can now be computed explicitly:

$$M_0(t, s, y) = C_{\mathcal{Q}}(t, s, y) - C_{\mathcal{P}}(t, s, y) + \mu E_{t,s,y}^{\mathcal{P}}\left(\int_t^T \Delta_u S_u du\right).$$

Finally, the function θ_0 is the solution of the following equation:

$$(\partial_t + \mathcal{L}_1)\theta_0 + J_0(t, s, y) = 0,$$

and satisfies also the final condition $\theta_0(T, s, y) = 0$. By the use of the Feynman-Kac formula:

$$\theta_0(t, s, y) = E_{t,s,y}\left(\int_t^T J_0(u, S_u, y_u) du\right),$$

As a conclusion, $u_0(t, s, y, q_1, x) = x + \theta_0(t, s, y) + q_1 \left(C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}}\left(\int_t^T \Delta_u S_u du\right)\right)$ is the solution of the HJB equation (4.2).

It should be proven now that $|\theta_0(t, s, y)| < +\infty$ in order to insure the validity of the verification theorem.

Using the concavity of the square root function:

$$|J_0(t, s, y)| \leq \frac{2A(2\gamma - 1)|M_0| + \sqrt{B(2\gamma - 1)}}{2\gamma - 1},$$

If the option is a call or a put, then its delta is bounded which implies that:

$\exists M_1, M_2 > 0, \forall u < T, |M_0(u, S_u, y_u)| \leq M_1 + M_2 S_u$. This implies that:

$$\begin{aligned} |E_{t,s,y}(J_0(u, S_u, y_u))| &\leq E_{t,s,y}|J_0(u, S_u, y_u)|, \\ &\leq \frac{2A}{B\gamma(2\gamma - 1)} \left((2\gamma - 1)(M_1 + M_2 S_t e^{\mu(u-t)}) + \sqrt{B(2\gamma - 1)} \right), \end{aligned}$$

and:

$$\int_t^T |E_{t,s,y}(J_0(u, S_u, y_u))| du \leq \frac{2A}{B\gamma} \left(\left(M_1 + \sqrt{\frac{B}{2\gamma - 1}} \right) (T - t) + M_2 S_t \frac{e^{\mu(T-t)} - 1}{\mu} \right).$$

Consequently, $\int_t^T |E_{t,s,y}(J_0(u, S_u, y_u))| du$ is finite.

Besides, u is regular (twice differentiable with respect to S, y, x) since it is a combination of regular functions. Then, it coincides with the value function. \square

4.1.2 Interpretation of the strategy

The quantity $C_{\mathcal{P}}(t, s, y)$ represents the expected payoff of the option under the probability measure \mathcal{P} . In order to delta hedge his position, the buyer of this option holds a quantity $-\Delta_t$ of the stock at every date $t \leq T$. If the trend of the stock is positive ($\mu > 0$), then the portfolio of delta-hedging will have in average a negative return (respectively a positive return) if the delta of the option is positive (respectively negative). Thus, the indifference price for the option is $C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right)$. It can be deduced that $M_0(t, s, y) = C_{\mathcal{Q}}(t, s, y) - \left(C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right) \right)$ represents the difference between the price of the option under the risk-neutral probability \mathcal{Q} and its indifference price under the historic probability \mathcal{P} .

Let f, g be the functions defined as:

$$f(x) = \frac{\gamma x + \sqrt{\gamma^2 x^2 + B(2\gamma - 1)}}{2\gamma - 1}, \quad g(x) = \frac{\sqrt{\gamma^2 x^2 + B(2\gamma - 1)} - \gamma x}{2\gamma - 1}.$$

By simple derivation, it can be shown that:

$$\begin{aligned} \frac{\partial \delta_{L,*,t}^-}{\partial M_0} &= f'(M_0) = \frac{\gamma}{2\gamma - 1} \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)} + \gamma M_0}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \geq 0, \\ \frac{\partial \delta_{L,*,t}^+}{\partial M_0} &= g'(M_0) = \frac{\gamma}{2\gamma - 1} \frac{\gamma M_0 - \sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \leq 0, \end{aligned}$$

It follows then that the bid distance $\delta_{L,*,t}^- = f(M_0)$ is an increasing function of M_0 and the ask distance $\delta_{L,*,t}^+ = g(M_0)$ is a decreasing function of M_0 . Indeed, if M_0 increases, it becomes more profitable for the market maker to sell the option. The market maker lowers the bid quote ($\delta_{L,*,t}^-$ increases) and also lowers the ask quote ($\delta_{L,*,t}^+$ decreases). In this way, the probability to get the ask quote executed augments and the probability of execution of the bid quote is lowered.

If $M_0 = 0$, the market price of the option is equal to its indifference price under \mathcal{P} . In this case $\delta_{L,*,t}^- = \delta_{L,*,t}^+ = \frac{B}{\sqrt{2\gamma - 1}}$, which means that the bid quote C_t^b and the ask quote C_t^a are symmetric around the mid price $C_{\mathcal{Q}}$.

The expressions of the optimal quotes $\delta_{L,*,t}^-$ and $\delta_{L,*,t}^+$ show that, in the presence of mispricing on the option, the market maker adjusts her bid and ask quotes by being more aggressive in the side that makes her profit from the mispricing term M_0 and more conservative in the opposite side. The MM insures then the role of a liquidity provider while trying to profit from the mispricing induced by the misspecification of the parameters.

The bid-ask spread $\mathcal{S}_{L,*,t}$ of the market maker can be defined as the distance between the bid and ask quotes meaning that $\mathcal{S}_{L,*,t} = \delta_{L,*,t}^- + \delta_{L,*,t}^+$. It follows that $\mathcal{S}_{L,*,t} = 2 \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1}$

and:

$$\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} = -\frac{2(B(2\gamma-1) + \gamma M_0^2)}{(2\gamma-1)^2 \sqrt{\gamma^2 M_0^2 + B(2\gamma-1)}} < 0.$$

This means that the bid-ask spread $\mathcal{S}_{L,*,t}$ is a decreasing function of the parameter γ . Indeed when γ increases, the intensity of arrivals of market orders decreases and the probability of execution of a quote at a distance δ from the mid price decreases. Consequently, the market maker contracts her bid-ask spread and places the quotes closer to the mid price.

4.2 The case where $\beta = 1$ (linear market impact)

The intensities of arrivals of market orders can be written as follows:

$\forall \delta^+, \delta^- \geq 0$, $\lambda^+(\delta^+) = \frac{A}{(B+\delta^+)^\gamma}$ and $\lambda^-(\delta^-) = \frac{A}{(B+\delta^-)^\gamma}$. The optimization problem can be solved similarly.

4.2.1 Analytic resolution

Proposition 4.2. Let $\mathcal{S} = \frac{B}{\gamma}$, and $M_0(t, s, y) = C_{\mathcal{Q}}(t, s, y) - C_{\mathcal{P}}(t, s, y) + \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta_u S_u du \right)$.

The optimal controls $(\delta_{L,*}^+, \delta_{L,*}^-)$ of the market maker are:

$$\delta_{L,*,t}^+ = \left(\frac{B - \gamma M_0(t, s, y)}{\gamma - 1} \right)^+, \quad (4.9)$$

$$\delta_{L,*,t}^- = \left(\frac{B + \gamma M_0(t, s, y)}{\gamma - 1} \right)^+, \quad (4.10)$$

and the value function is:

$$u_0(t, s, y, q_1, x) = x + \theta_0(t, s, y) + q_1 \left(C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta_u S_u du \right) \right), \quad (4.11)$$

where:

$$\theta_0(t, s, y) = E_{t,s,y} \left(\int_t^T J_0(u, S_u, y_u) du \right),$$

and:

$$J_0(t, s, y) = \begin{cases} \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-M_0(t,s,y))^{\gamma-1}} - \frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma (B+M_0(t,s,y))^{\gamma-1}} & \text{if } M_0(t, s, y) \in [-\mathcal{S}, \mathcal{S}] \\ \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-M_0(t,s,y))^{\gamma-1}} + \frac{A}{(B-C)^\gamma} (-M_0(t, s, y)) & \text{if } M_0(t, s, y) \in]-\infty, -\mathcal{S}] \\ -\frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma (B+M_0(t,s,y))^{\gamma-1}} + \frac{A}{B^\gamma} (M_0(t, s, y)) & \text{if } M_0(t, s, y) \in [\mathcal{S}, +\infty[\end{cases}$$

The proof is given in Appendix (8.2).

4.2.2 Interpretation of the strategy

Based on the results stated in (4.2), the following derivatives can be computed:

$$\begin{aligned}\frac{\partial \delta_{L,*,t}^+}{\partial M_0} &= -\frac{\gamma}{\gamma-1} 1_{\{M_0 \leq \mathcal{S}\}} \leq 0, \\ \frac{\partial \delta_{L,*,t}^-}{\partial M_0} &= \frac{\gamma}{\gamma-1} 1_{\{M_0 \geq -\mathcal{S}\}} \geq 0,\end{aligned}$$

It can then be noticed that:

1. The distance $\delta_{L,*,t}^+$ of the ask-quote to the mid price is a decreasing function of the mispricing term M_0 (as it is the case for $\beta = \frac{1}{2}$).
2. The distance $\delta_{L,*,t}^-$ of the bid quote to the mid price is an increasing function of the mispricing term M_0 (same finding as for $\beta = \frac{1}{2}$).

The same conclusions drawn in the case where $\beta = \frac{1}{2}$ concerning the effect of the mispricing term M_0 on the optimal bid and ask quotes are conserved when $\beta = 1$. This is normal since the market impact function will affect the values of the optimal distances but not their monotonicity with respect to the model parameters. In other words, the market impact function does not change the tactic of the market maker, but just the amplitude of the quotes.

The spread $\mathcal{S}_{L,*,t} = \delta_{L,*,t}^+ + \delta_{L,*,t}^-$ writes:

$$\mathcal{S}_{L,*,t} = \begin{cases} \frac{2B}{\gamma-1} & \text{if } M_0(t, s, y) \in [-\mathcal{S}, \mathcal{S}], \\ \frac{B-\gamma M_0(t, s, y)}{\gamma-1} & \text{if } M_0(t, s, y) \leq -\mathcal{S}, \\ \frac{B+\gamma M_0(t, s, y)}{\gamma-1} & \text{if } M_0(t, s, y) \geq \mathcal{S}, \end{cases}$$

By differentiating the last expression with respect to the variable γ , it follows that:

$$\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} = \begin{cases} -\frac{2B}{(\gamma-1)^2} & \text{if } M_0(t, s, y) \in [-\mathcal{S}, \mathcal{S}], \\ \frac{M_0(t, s, y) - B}{(\gamma-1)^2} & \text{if } M_0(t, s, y) \leq -\mathcal{S}, \\ -\frac{M_0(t, s, y) + B}{(\gamma-1)^2} & \text{if } M_0(t, s, y) \geq \mathcal{S}, \end{cases}$$

It can be seen that $\frac{\partial \mathcal{S}_{L,*,t}}{\partial \gamma} \leq 0$ and then $\mathcal{S}_{L,*,t}$ is a decreasing function of the parameter γ (same finding for $\beta = \frac{1}{2}$).

5 Optimization problem for a risk-averse market maker

The profit of the market maker comes from the bid-ask spread received when market orders hits her quotes and trades occur. Since the times of arrivals of market orders are uncertain, there is a risk on the value of future wealth and also on future inventory. In order to risk-manage her strategy, the market maker may solve an optimization problem to determine

the optimal distances (δ_*^-, δ_*^+) which allows her to maximize the expectation of her future wealth while minimizing its variance.

In the following, to make the HJB equation possible to solve, the variance of final wealth is going to be approximated. Indeed, under the assumption that $\delta^+, \delta^- \ll C_{\mathcal{Q}}$, we can write:

$$X_T \sim \int_t^T C_{\mathcal{Q}}(u, S_u, y_u) dN_u^+ - \int_t^T C_{\mathcal{Q}}(u, S_u, Y_u) dN_u^- + \int_0^T q_{2,u} dS_u.$$

Supposing that N^+ and N^- are independent Poisson processes, it can be deduced:

$$V(X_T | \mathcal{F}_t) \sim E^{\mathcal{P}} \left(\int_t^T C_{\mathcal{Q}}^2(u, S_u, y_u) (\lambda_u^+ + \lambda_u^-) du + \int_t^T q_{1,u}^2 \Delta_u^2 \sigma^2(y_u) S_u^2 du \right),$$

Besides:

$$V(q_{1,T} h(S_T) | \mathcal{F}_t) \sim E^{\mathcal{P}} \left(\int_t^T E^{\mathcal{P}} (h^2(S_T) | \mathcal{F}_u) (\lambda_u^+ + \lambda_u^-) du \right),$$

and:

$$\text{Cov}(X_T, q_{1,T} h(S_T)) \sim -E^{\mathcal{P}} \left(\int_t^T C_{\mathcal{Q}}(u, S_u, y_u) C_{\mathcal{P}}(u, S_u, y_u) (\lambda_u^+ + \lambda_u^-) du \right)$$

Then:

$$\begin{aligned} \text{Var}(X_T + q_{1,T} h(S_T)) &= E^{\mathcal{P}} \left(\int_t^T (C_{\mathcal{Q}}^2 + E^{\mathcal{P}} (h^2(S_T) | \mathcal{F}_u) - 2C_{\mathcal{Q}} C_{\mathcal{P}}) (\lambda_u^+ + \lambda_u^-) du \right) \\ &\quad + E^{\mathcal{P}} \left(\int_t^T q_{1,u}^2 \Delta_u^2 \sigma^2(y_u) S_u^2 du \right). \end{aligned}$$

The quantities λ_u^+ and λ_u^- , which depend on δ_u^+ and δ_u^- respectively, are replaced by $\lambda_{L,u}^+ = \lambda^+(\delta_{L,*}^+)$ and $\lambda_{L,u}^- = \lambda^-(\delta_{L,*}^-)$ in order to simplify the resolution of the problem. Indeed, the value function u^ϵ is defined as follows:

$$u^\epsilon(t, s, y, q_1, x) = \text{Max}_{\{(\delta^-, \delta^+) \in \mathcal{A}\}} E_{t,s,y,q_1,x}^{\mathcal{P}} (H^\epsilon(t, T, S_T, y_T, q_{1,T}, X_T)),$$

where the quantity H^ϵ writes:

$$H^\epsilon(t, T, S_T, y_T, q_{1,T}, X_T) = X_T + q_{1,T} h(S_T) - \epsilon \eta \mathcal{I} - \epsilon \eta \int_t^T q_{1,u}^2 \Delta_u^2 \sigma^2(y_u) S_u^2 du,$$

and:

$$\mathcal{I} = \int_t^T (C_{\mathcal{Q}}^2 + E^{\mathcal{P}} (h^2(S_T) | \mathcal{F}_u) - 2C_{\mathcal{Q}} C_{\mathcal{P}}) (\lambda_{L,u}^+ + \lambda_{L,u}^-) du.$$

Using a stochastic control approach, u^ϵ is the solution of the following HJB equation:

$$(\partial_t + \mathcal{L}) u^\epsilon + \max_{\{(\delta^-, \delta^+) \in \mathcal{A}\}} J^\epsilon(\delta^-, \delta^+) = \epsilon \eta (q_{1,t}^2 V_t + T_t), \quad (5.1)$$

where:

$$\begin{aligned} T_t &= (C_{\mathcal{Q}}^2 + E^{\mathcal{P}}(h^2(S_T)|\mathcal{F}_t) - 2C_{\mathcal{Q}}C_{\mathcal{P}})(\lambda_{L,t}^+ + \lambda_{L,t}^-), \\ V_t &= \Delta_t^2 \sigma^2(y_t) S_t^2. \end{aligned}$$

The operator \mathcal{L} is defined such that $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, and the term J^ϵ satisfies $J^\epsilon = J^{-,\epsilon} + J^{+,\epsilon}$ where the functions $J^{+,\epsilon}, J^{-,\epsilon}$ are defined in the following way:

$$\begin{aligned} J^{+,\epsilon}(\delta^+) &= \lambda^+(\delta^+)(u^\epsilon(t, s, y, q_{1,t^-} - 1, x_{t^-} + (C_{\mathcal{Q}} + \delta^+)) - u^\epsilon(t, s, y, q_{1,t^-}, x_{t^-})), \\ J^{-,\epsilon}(\delta^-) &= \lambda^-(\delta^-)(u^\epsilon(t, s, y, q_{1,t^-} + 1, x_{t^-} - (C_{\mathcal{Q}} - \delta^-)) - u^\epsilon(t, s, y, q_{1,t^-}, x_{t^-})), \end{aligned}$$

The objective of the market maker, at a given date t , is to determine $(\delta_{*,t}^+, \delta_{*,t}^-)$ solution of the following problem:

$$(\delta_{*,t}^+, \delta_{*,t}^-) = \text{ArgMax}_{\{\delta^+, \delta^-\}} E^{\mathcal{P}}(H^\epsilon(t, T, S_T, y_T, q_{1,T}, X_T) | S_t = s, y_t = y, q_{1,t} = q_1, X_t = x).$$

in order to set optimal bid and ask quotes in the option order book.

The problem is going to be solved in the cases where $\beta = \frac{1}{2}$ and $\beta = 1$.

5.1 The case where $\beta = \frac{1}{2}$ (square-root market impact)

5.1.1 Analytic approximation

Proposition 5.1. *The optimal controls $(\delta_{*,t}^+, \delta_{*,t}^-)$ of the market maker can be approximated at order 1 in ϵ by $(\hat{\delta}_{*,t}^-, \hat{\delta}_{*,t}^+)$ which are defined as:*

$$\hat{\delta}_{*,t}^+ = \frac{-\gamma M^+ + \sqrt{\gamma^2 (M^+)^2 + B(2\gamma - 1)}}{2\gamma - 1}, \quad (5.2)$$

$$\hat{\delta}_{*,t}^- = \frac{\gamma M^- + \sqrt{\gamma^2 (M^-)^2 + B(2\gamma - 1)}}{2\gamma - 1}, \quad (5.3)$$

where the quantities $M^+(t, s, y, q_1)$ and $M^-(t, s, y, q_1)$ are defined as:

$$\begin{aligned} M^+(t, s, y, q_1) &= M_0(t, s, y) + \epsilon M_1(t, s, y, q_1), \\ M^-(t, s, y, q_1) &= M_0(t, s, y) + \epsilon M_2(t, s, y, q_1), \end{aligned}$$

and $M_1(t, s, y, q_1)$ and $M_2(t, s, y, q_1)$ are given explicitly:

$$\begin{aligned} M_1(t, s, y, q_1) &= -\theta_3(t, s, y) + (1 - 2q_1)\theta_4(t, s, y), \\ M_2(t, s, y, q_1) &= -\theta_3(t, s, y) - (1 + 2q_1)\theta_4(t, s, y), \end{aligned}$$

with:

$$\begin{aligned} \theta_4(t, s, y) &= -\eta E_{t,s,y}^{\mathcal{P}} \left(\int_t^T V_u du \right), \\ \theta_3(t, s, y) &= -2E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \theta_4(u, s_u, y_u) (\lambda^+(\delta_{L,*}^+) - \lambda^-(\delta_{L,*}^-)) du \right), \end{aligned}$$

The approximation error is at order 2 in ϵ :

$$\begin{aligned} |\delta_{*,t}^+ - \hat{\delta}_{*,t}^+| &= O(\epsilon^2), \\ |\delta_{*,t}^- - \hat{\delta}_{*,t}^-| &= O(\epsilon^2). \end{aligned}$$

Proof. Let u^ϵ be the solution of the HJB equation (5.1). Under the assumption that $\epsilon \sim 0$, a singular perturbation technique can be performed with respect to the parameter ϵ :

$$u^\epsilon(t, s, y, q_1, x) = x + \sum_{k=0}^{+\infty} \epsilon^k v_k(t, s, y, q_1). \quad (5.4)$$

If $\epsilon = 0$, then the case of a linear utility function is recovered. This implies that $u^0(t, s, y, q_1, x) = x + v_0(t, s, y, q_1) = u_0(t, s, y, q_1, x)$ where u_0 is the value function defined in (4.5). Therefore, it is assumed that v_0 has the following form:

$$v_0(t, s, y, q_1) = \theta_0(t, s, y) + q_1 \theta_1(t, s, y).$$

Since the utility function contains a constraint on the square of the inventory q_1 in the traded option, it is assumed that v_1 has the following form:

$$v_1(t, s, y, q_1) = \theta_2(t, s, y) + q_1 \theta_3(t, s, y) + q_1^2 \theta_4(t, s, y),$$

In order to solve the HJB equation, the jump terms $J^{+,\epsilon}$ and $J^{-,\epsilon}$ have to be calculated.

Let $f^+ = J^{+,\epsilon}$, the function f^+ writes:

$$\begin{aligned} f^+(\delta^+) &= \lambda^+(\delta^+) (u(t, s, y, q_1 - 1, x + (c + \delta^+)) - u(t, s, y, q_1, x)), \\ &= \lambda^+(\delta^+) (\delta^+ + M_0(t, s, y) + \epsilon M_1(t, s, y, q_1) + \epsilon^2 R^+(t, s, y, q_1)), \end{aligned}$$

where:

$$\begin{aligned} M_1(t, s, y, q_1) &= v_1(t, s, y, q_1 - 1) - v_1(t, s, y, q_1), \\ &= -\theta_3(t, s, y) + (1 - 2q_1)\theta_4(t, s, y), \end{aligned}$$

and:

$$R^+(t, s, y, q_1) = \sum_{k=2}^{+\infty} \epsilon^{k-2} (v_k(t, s, y, q_1 - 1) - v_k(t, s, y, q_1))$$

Let $M^+(t, s, y, q_1) = M_0(t, s, y) + \epsilon M_1(t, s, y, q_1)$. In order to determine $\delta_{*,t}^+ = \text{ArgMax}_{(x \geq 0)} f^+(x)$, the derivative $(f^+)'$ is computed:

$$(f^+)'(\delta^+) = \frac{\lambda^+(\delta^+)}{B + (\delta^+)^2} \left((\delta^+)^2 (1 - 2\gamma) - 2\gamma (M^+(t, s, y) + \epsilon^2 R^+(t, s, y, q_1)) \delta^+ + B \right).$$

The derivative function $(f^+)'$ has the two following zeros x_1^+ and x_2^+ :

$$\begin{aligned} x_1^+ &= \frac{-\gamma(M^+ + \epsilon^2 R^+) - \sqrt{\gamma^2(M^+ + \epsilon^2 R^+)^2 + B(2\gamma - 1)}}{2\gamma - 1} < 0, \\ x_2^+ &= \frac{-\gamma(M^+ + \epsilon^2 R^+) + \sqrt{\gamma^2(M^+ + \epsilon^2 R^+)^2 + B(2\gamma - 1)}}{2\gamma - 1} > 0. \end{aligned}$$

Through the use of the table of variations of $(f^+)'$, it can be deduced that $\delta_{*,t}^+ = x_2^+$, which implies:

$$\delta_{*,t}^+ = \frac{-\gamma M^+ + \sqrt{\gamma^2(M^+)^2 + B(2\gamma - 1)}}{2\gamma - 1} + O(\epsilon^2).$$

Performing Taylor expansion with respect to the parameter ϵ yields:

$$\delta_{*,t}^+ = \delta_{L,*,t}^+ + \epsilon \left(-\frac{\gamma}{2\gamma - 1} M_1 + \frac{\gamma^2 M_0 M_1}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \right) + O(\epsilon^2).$$

It is then useful to write $f^+(\delta_{*,t}^+)$ as the sum of $f_0^+(\delta_{L,*,t}^+)$ plus a perturbation term due to the parameter ϵ . This will be of great use in the resolution of the HJB equation.

Noticing that $f^+(x) = f_0^+(x) + \epsilon \lambda^+(x) M_1(t, s, y, q_1) + O(\epsilon^2)$ and using Taylor's expansion, enables to obtain the following result:

$$\begin{aligned} f^+(\delta_{*,t}^+) &= f^+(\delta_{L,*,t}^+) + (f^+)'\!(\delta_{L,*,t}^+)(\delta_{*,t}^+ - \delta_{L,*,t}^+) + O((\delta_{*,t}^+ - \delta_{L,*,t}^+)^2), \\ &= f_0^+(\delta_{L,*,t}^+) + \epsilon \lambda^+(\delta_{L,*,t}^+) M_1 + \epsilon \left(-\frac{\gamma}{2\gamma - 1} M_1 + \frac{\gamma^2 M_0 M_1}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \right) (f^+)'\!(\delta_{L,*,t}^+) + O(\epsilon^2). \end{aligned}$$

Thus, since $(f^+)'\!(x) = (f_0^+)'\!(x) + O(\epsilon)$ and $(f_0^+)'\!(\delta_{L,*,t}^+) = 0$, it can be deduced:

$$f^+(\delta_{*,t}^+) = f_0^+(\delta_{L,*,t}^+) + \epsilon \lambda^+(\delta_{L,*,t}^+) M_1 + O(\epsilon^2),$$

Let $f^- = J^{-\cdot\epsilon}$. Using the form of the value function suggested in (4.6), the function f^- writes:

$$\begin{aligned} f^-(\delta^-) &= \lambda^-(\delta^-)(u(t, s, y, q_1 + 1, x - (c - \delta^-)) - u(t, s, y, q_1, x)), \\ &= \lambda^-(\delta^-) (\delta^- - (M_0(t, s, y) + \epsilon M_2(t, s, y, q_1) + \epsilon^2 R^-(t, s, y, q_1))) \end{aligned}$$

where:

$$\begin{aligned} M_2(t, s, y, q_1) &= -(v_1(t, s, y, q_1 + 1) - v_1(t, s, y, q_1)), \\ &= -\theta_3(t, s, y) - (1 + 2q_1)\theta_4(t, s, y), \end{aligned}$$

and:

$$R^-(t, s, y, q_1) = -\sum_{k=2}^{+\infty} \epsilon^{k-2} (v_k(t, s, y, q_1 + 1) - v_k(t, s, y, q_1)).$$

The quantity $M^-(t, s, y, q_1) = M_0(t, s, y) + \epsilon M_2(t, s, y, q_1)$ is introduced to simplify the notations. By differentiating f^- , it can be shown that:

$$(f^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + (\delta^-)^2} \left((1 - 2\gamma)(\delta^-)^2 + 2\gamma(M^- + \epsilon^2 R^-)\delta^- + B \right).$$

Thus, $(f^-)'$ gets null in the two following points:

$$\begin{aligned} x_1^- &= \frac{\gamma(M^- + \epsilon^2 R^-) - \sqrt{\gamma^2(M^- + \epsilon^2 R^-)^2 + B(2\gamma - 1)}}{2\gamma - 1}, \\ x_2^- &= \frac{\gamma(M^- + \epsilon^2 R^-) + \sqrt{\gamma^2(M^- + \epsilon^2 R^-)^2 + B(2\gamma - 1)}}{2\gamma - 1}. \end{aligned}$$

Since $x_1^- < 0$, $x_2^- > 0$ and $\gamma > 1$, then $\delta_{*,t}^- = \text{ArgMax}_{\{x \geq 0\}} f^-(x) = x_2^-$. It follows that:

$$\delta_{*,t}^- = \frac{\gamma M^- + \sqrt{\gamma^2(M^-)^2 + B(2\gamma - 1)}}{2\gamma - 1} + O(\epsilon^2).$$

Through the use of Taylor expansion with respect to the parameter ϵ , the last relation yields:

$$\delta_{*,t}^- = \delta_{L,*,t}^- + \epsilon \left(\frac{\gamma}{2\gamma - 1} M_2 + \frac{\gamma^2 M_0 M_2}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \right) + O(\epsilon^2).$$

The quantity $f^-(\delta_{*,t}^-)$ can be written as the sum of $f_0^-(\delta_{L,*,t}^-)$ plus a perturbation term due to the parameter ϵ . Indeed, since:

$$f^-(x) = f_0^-(x) - \epsilon \lambda^-(x) M_2(t, s, y, q_1) + O(\epsilon^2),$$

then, by using Taylor's expansion, it follows:

$$\begin{aligned} f^-(\delta_{*,t}^-) &= f^-(\delta_{L,*,t}^-) + (f^-)'(\delta_{L,*,t}^-)(\delta_{*,t}^- - \delta_{L,*,t}^-) + O((\delta_{*,t}^- - \delta_{L,*,t}^-)^2), \\ &= f_0^-(\delta_{L,*,t}^-) - \epsilon \lambda^-(\delta_{L,*,t}^-) M_2 + \epsilon \left(\frac{\gamma}{2\gamma - 1} M_2 + \frac{\gamma^2 M_0 M_2}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} \right) (f^-)'(\delta_{L,*,t}^-) + O(\epsilon^2). \end{aligned}$$

It can be recalled at this stage that $(f^-)'(x) = (f_0^-)'(x) + O(\epsilon)$, and $(f_0^-)'(\delta_{L,*,t}^-) = 0$. Thus, the last equation becomes:

$$f^-(\delta_{*,t}^-) = f_0^-(\delta_{L,*,t}^-) - \epsilon \lambda^-(\delta_{L,*,t}^-) M_2 + O(\epsilon^2).$$

Now that the terms $f^+(\delta_{*,t}^+)$ and $f^-(\delta_{*,t}^-)$ are approximated separately, the quantity $J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) = f^-(\delta_{*,t}^-) + f^+(\delta_{*,t}^+)$ can be written as below:

$$\begin{aligned} J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) &= f_0^+(\delta_{L,*,t}^+) + \epsilon M_1 \lambda^+(\delta_{L,*,t}^+) + f_0^-(\delta_{L,*,t}^-) - \epsilon M_2 \lambda^-(\delta_{L,*,t}^-) + O(\epsilon^2), \\ &= J_0(\delta_{L,*,t}^-, \delta_{L,*,t}^+) + \epsilon (M_1 \lambda^+(\delta_{L,*,t}^+) - M_2 \lambda^-(\delta_{L,*,t}^-)) + O(\epsilon^2). \end{aligned}$$

By rearranging the terms of $J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+)$ by powers of ϵ , it can be obtained:

$$J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) = J_0(t, s, y) + \epsilon J_1(t, s, y, q_1) + O(\epsilon^2),$$

where $J_1(t, s, y, q_1) = J_{1,0}(t, s, y) + q_1 J_{1,1}(t, s, y)$ and:

$$\begin{aligned} J_{1,0}(t, s, y) &= \lambda^+(\delta_{L,*,t}^+)(-\theta_3 + \theta_4) - \lambda^-(\delta_{L,*,t}^-)(-\theta_3 - \theta_4), \\ J_{1,1}(t, s, y) &= -2\theta_4 (\lambda^+(\delta_{L,*,t}^+) - \lambda^-(\delta_{L,*,t}^-)). \end{aligned}$$

The HJB equation (5.1) can be regrouped by terms according to their order in ϵ . The term at order 0 in ϵ leads to the equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0 + q_1\theta_1) + J_0(t, s, y) = 0,$$

with the final conditions:

$$\theta_0(T, s, y) = 0, \quad \theta_1(T, s, y) = h(s).$$

The functions θ_0, θ_1 are then deduced:

$$\begin{aligned} \theta_1(t, s, y) &= C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right), \\ \theta_0(t, s, y) &= E_{t,s,y} \left(\int_t^T J_0(u, S_u, y_u) du \right). \end{aligned}$$

The term at order 1 in ϵ , gives the following equation:

$$(\partial_t + \mathcal{L})(\theta_2 + q_1\theta_3 + q_1^2\theta_4) + J_1(t, s, y) = \eta q_{1,t}^2 V_t + \eta T_t,$$

The terms of the last equation can be sorted by their orders in q_1 . The term of order 2 in q_1 gives the following equation for θ_4 :

$$(\partial_t + \mathcal{L}_1)\theta_4(t, s, y) = \eta V_t.$$

Using the final condition $\theta_4(T, s, y) = 0$, it can be deduced that:

$$\theta_4(t, s, y) = -\eta E_{t,s,y}^{\mathcal{P}} \left(\int_t^T V_u du \right).$$

The function θ_3 is the solution of the equation resulting from the term of order 1 in q_1 :

$$(\partial_t + \mathcal{L}_1)\theta_3(t, s, y) + J_{1,1}(t, s, y) = 0,$$

and it also satisfies the final condition $\theta_3(T, s, y) = 0$, it follows that:

$$\theta_3(t, s, y) = E_{t,s,y}^{\mathcal{P}} \left(\int_t^T J_{1,1}(u, s_u, y_u) du \right).$$

Finally, the function θ_2 is the solution of the equation related to the term at order 0 in q_1 :

$$(\partial_t + \mathcal{L}_1)\theta_2(t, s, y) + J_{1,0}(t, s, y) = \eta T_t,$$

and is subject to the final condition $\theta_2(T, s, y) = 0$. Using again the Feynman-Kac formula, the following is obtained:

$$\theta_2(t, s, y) = E_{t,s,y}^{\mathcal{P}} \left(\int_t^T (J_{1,0} - \eta T)(u, s_u, y_u) du \right).$$

□

5.1.2 Interpretation of the strategy

The effect of the mispricing term M_0 on the optimal bid and ask distances remains the same as in the case without inventory constraints ($\epsilon = 0$), indeed:

$$\begin{aligned}\frac{\partial \hat{\delta}_{*,t}^+}{\partial M_0} &= \frac{\partial \hat{\delta}_{*,t}^+}{\partial M^+} \frac{\partial M^+}{\partial M_0} = g'(M^+) < 0, \\ \frac{\partial \hat{\delta}_{*,t}^-}{\partial M_0} &= \frac{\partial \hat{\delta}_{*,t}^-}{\partial M^-} \frac{\partial M^-}{\partial M_0} = f'(M^-) > 0,\end{aligned}$$

Consequently, the market maker still adjusts the bid and ask quotes depending on the term M_0 .

The new feature here is that the distances $\hat{\delta}_{*,t}^-$ and $\hat{\delta}_{*,t}^+$ depend on the inventories q_1 . Indeed, it can be seen that:

$$\begin{aligned}\frac{\partial \hat{\delta}_{*,t}^+}{\partial q_1} &= -2\epsilon\theta_4(t, s, y) \frac{\gamma}{2\gamma - 1} g'(M^+) < 0, \\ \frac{\partial \hat{\delta}_{*,t}^-}{\partial q_1} &= -2\epsilon\theta_4(t, s, y) \frac{\gamma}{2\gamma - 1} f'(M^-) > 0,\end{aligned}$$

This means that when the inventory in options q_1 of the market maker increases, more aggressive ask quotes and more conservative bid quotes are posted. The market maker uses this quoting policy in order to reduce her inventory in options.

The bid-ask spread of the market maker writes $\mathcal{S}_{*,t} = \hat{\delta}_{*,t}^- + \hat{\delta}_{*,t}^+$. Using Taylor expansion, $\mathcal{S}_{*,t}$ can be approximated as following:

$$\begin{aligned}\mathcal{S}_{*,t} &= 2 \frac{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}}{2\gamma - 1} + \epsilon \frac{\gamma}{2\gamma - 1} \left[(M_2 - M_1) + \frac{M_0 \gamma}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} (M_1 + M_2) \right], \\ &= \mathcal{S}_{L,*,t} + \epsilon \frac{\gamma}{2\gamma - 1} \left[-2\theta_4 + 2 \frac{M_0 \gamma}{\sqrt{\gamma^2 M_0^2 + B(2\gamma - 1)}} (-\theta_3 - 2q_1\theta_4) \right],\end{aligned}$$

The term $(M_2 - M_1)$ is a positive quantity that increases the bid-ask spread. Indeed, in presence of inventory constraints, the market maker wants to have a bigger margin in order to be paid for the risk she bears. Therefore, the bid-ask spread is widened. The second term depends on q_1 and M_0 , its effect on $\mathcal{S}_{*,t}$ is not straightforward. It is easier to see the effects of q_1 directly on $\hat{\delta}_{*,t}^-$ and $\hat{\delta}_{*,t}^+$.

5.2 The case where $\beta = 1$ (linear market impact)

5.2.1 Analytic approximation

Proposition 5.2. *The optimal controls $(\delta_{*,t}^+, \delta_{*,t}^-)$ of the market maker can be approximated at order 1 in ϵ by $(\hat{\delta}_{*,t}^-, \hat{\delta}_{*,t}^+)$ which are defined as:*

$$\hat{\delta}_{*,t}^+ = \begin{cases} \frac{B-\gamma M^+(t,s,y,q_1)}{\gamma-1} & \text{if } M^+(t,s,y) \leq \mathcal{S}, \\ 0 & \text{if } M^+(t,s,y) \geq \mathcal{S}, \end{cases} \quad (5.5)$$

$$\hat{\delta}_{*,t}^- = \begin{cases} \frac{B+\gamma M^-(t,s,y,q_1)}{\gamma-1} & \text{if } M^-(t,s,y) \geq -\mathcal{S}, \\ 0 & \text{if } M^-(t,s,y) \leq -\mathcal{S}, \end{cases} \quad (5.6)$$

and the error of approximation is at order 2 in ϵ :

$$\begin{aligned} |\hat{\delta}_{*,t}^+ - \delta_{*,t}^+| &= O(\epsilon^2), \\ |\hat{\delta}_{*,t}^- - \delta_{*,t}^-| &= O(\epsilon^2). \end{aligned}$$

The proof of this Proposition and the value function are given in Appendix (8.3).

5.2.2 Interpretation of the strategy

The effect of the mispricing term M_0 on the optimal bid and ask distances remains the same as in the case without inventory constraints ($\epsilon = 0$), indeed:

$$\begin{aligned} \frac{\partial \hat{\delta}_{*,t}^+}{\partial M_0} &= -\frac{\gamma}{\gamma-1} 1_{\{M^+ \leq \mathcal{S}\}} \leq 0, \\ \frac{\partial \hat{\delta}_{*,t}^-}{\partial M_0} &= \frac{\gamma}{\gamma-1} 1_{\{M^- \geq -\mathcal{S}\}} \geq 0, \end{aligned}$$

The proxies $\hat{\delta}_{*,t}^-$ and $\hat{\delta}_{*,t}^+$ of the optimal distances depend on the inventory q_1 . In order to understand the effect of q_1 on $\hat{\delta}_{*,t}^-$ and $\hat{\delta}_{*,t}^+$, the first derivatives with respect to the variable q_1 can be computed explicitly:

$$\begin{aligned} \frac{\partial \hat{\delta}_{*,t}^+}{\partial q_1} &= 2\epsilon\theta_4(t,s,y) \frac{\gamma}{\gamma-1} 1_{\{M^+ \leq \mathcal{S}\}} \leq 0, \\ \frac{\partial \hat{\delta}_{*,t}^-}{\partial q_1} &= -2\epsilon\theta_4(t,s,y) \frac{\gamma}{\gamma-1} 1_{\{M^- \geq -\mathcal{S}\}} \geq 0, \end{aligned}$$

The first derivatives highlights the effect of the option inventory q_1 on the distances $\hat{\delta}_{*,t}^-$ and $\hat{\delta}_{*,t}^+$. Indeed, if q_1 increases, the market maker lowers the ask and bid quotes with the aim to cut down the option inventory.

6 Numerical Simulations

In this section, Monte Carlo simulations are performed in order to test the performance of the market making strategies stated previously. It is supposed in this section that the spot process follows a Heston model which means that under the real-world probability measure \mathcal{P} , the spot process S has the following dynamics:

$$\begin{aligned}\frac{dS_t}{S_t} &= \mu dt + \sqrt{y_t} dW_t^{(1)} \\ dy_t &= k_R(\theta_R - y_t)dt + \eta_R \sqrt{y_t} dW_t^{(2)}\end{aligned}$$

where $d\langle W^{(1)}, W^{(2)} \rangle_t = \rho_R dt$.

Market participant price options on the stock under a risk-neutral pricing measure \mathcal{Q} , under which the spot process S has the following dynamics:

$$\begin{aligned}\frac{dS_t}{S_t} &= r dt + \sqrt{y_t} dW_t^{*,(1)} \\ dy_t &= k_I(\theta_I - y_t)dt + \eta_I \sqrt{y_t} dW_t^{*,(2)}\end{aligned}$$

where $d\langle W^{*,(1)}, W^{*,(2)} \rangle_t = \rho_I dt$.

The functions a_R , b_R and σ are:

$$\begin{aligned}a_R(y_t) &= k_R(\theta_R - y_t), \\ b_R(y_t) &= \eta_R \sqrt{y_t}, \\ \sigma(y_t) &= \sqrt{y_t},\end{aligned}$$

The term $C_{\mathcal{Q}}(t, s, y)$ is the option price in the Heston model (see [Heston(1993)]). The term $C_{\mathcal{P}} = E^{\mathcal{P}}((S_T - K)^+ | \mathcal{F}_t)$ can be computed explicitly as explained in Appendix (8.4).

The first part of the numerical study is devoted to the comparison between the optimal strategy in the case of a linear utility function, and a zero-intelligence strategy. In the second part, the effect of the misspecification of the parameters is studied. It can be seen how that could affect the performance of the strategy as well as the inventory of the market maker.

The numerical simulations are performed as following. We consider here that the traded option has a maturity equal to 3 Months ($T = 0.25$) and a strike equal to 100 ($K = 100$). For each of the following cases, we fix the set of parameters $(r, k_I, \theta_I, \eta_I, \rho_I)$ which characterizes the stock dynamics under the risk-neutral probability \mathcal{Q} as well as the set of the parameters $(\mu, k_R, \theta_R, \eta_R, \rho_R)$ characterizing the dynamics under the real-world probability measure \mathcal{P} . We simulate 1000 paths of the spot and instantaneous variance processes $(S_t, v_t)_{\{0 \leq t \leq T\}}$ starting from $S_0 = 100$ and $v_0 = 0.04$. It is considered here that there is 6 trading hours per day and that the market maker refreshes the quotes every 5 minutes. This means that there is $12 \times 6 = 72$ points per day. Since there is approximately 64 business days in a 3 months period, this amounts to $64 \times 72 = 4608$ points per simulated path. At each point, the quantities $C_{\mathcal{Q}}(t, s, y)$ and $C_{\mathcal{P}}(t, s, y)$ are computed using a Fast Fourier Transform method. This

simulation task is numerically consuming and was performed using the computing cluster at Ecole Centrale Paris.

For each simulated path, the optimal market making strategy is used from the inception date ($t = 0$) until the maturity date of the option ($t = T$). We then performed a statistical study on the results obtained upon the test of the optimal strategy on 1000 independent paths.

6.1 Comparison with a zero-intelligence agent

Let the implied volatility $\sigma_{t,I}(K, T)$ of the call option be defined such that $C_{\mathcal{Q}}(t, S_t, y_t) = P_{BS}(t, S_t, \sigma_{t,I}(K, T))$, where P_{BS} denotes the Black-Scholes price formula. In addition, let $\vartheta_{BS} = \frac{\partial P_{BS}(t, S_t, \sigma_{t,I})}{\partial \sigma_{t,I}}$ be the Black-Scholes Vega of the option. It is supposed here that the zero-intelligence agent places equidistant bid and ask quotes around the mid-price and tries to earn $0.05 \times \vartheta_{BS}$ per trade. For this reason, this agent places the bid price $C_{ZI,t}^b$ and the ask price $C_{ZI,t}^a$ as follows:

$$\begin{aligned} C_{ZI,t}^a &= C_{\mathcal{Q}}(t, S_t, y_t) + \delta_{ZI,t}, \\ C_{ZI,t}^b &= C_{\mathcal{Q}}(t, S_t, y_t) - \delta_{ZI,t}, \end{aligned}$$

where $\delta_{ZI,t} = 0.05 \times \vartheta_{BS}$.

It is supposed here that there is no misspecification of the model parameters which means that $(k_R, \theta_R, \eta_R, \rho_R) = (k_I, \theta_I, \eta_I, \rho_I) = (4, 0.04, 0.5, -0.4)$. Besides, the stock doesn't have a drift under the real world-probability measure ($\mu = 0$) and the risk-free rate is null ($r = 0$).

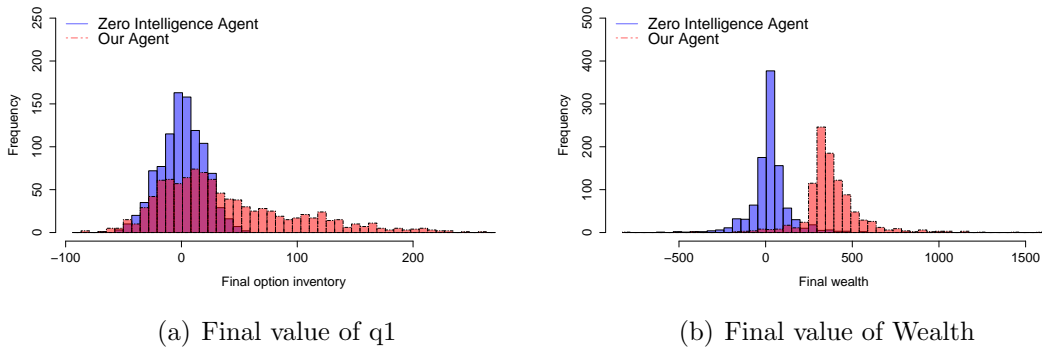


Figure 1: Statistics for $\beta = 0.5$

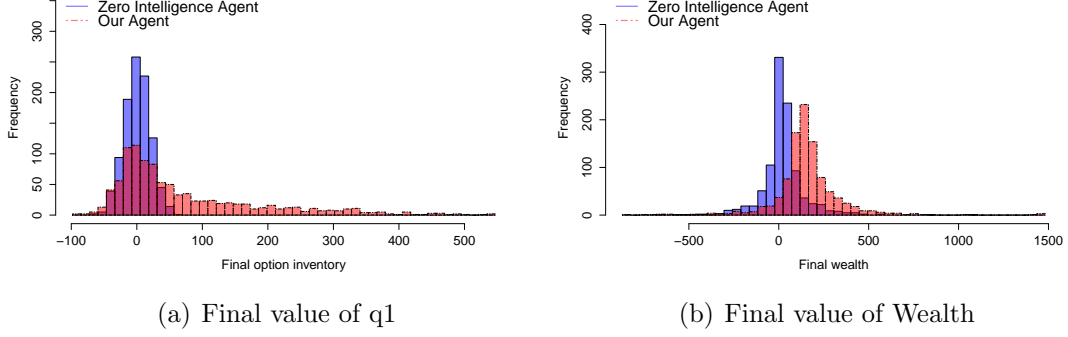


Figure 2: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and with optimal quotes	23.00	38.42	58.66	0.98	0.63
$\beta = 1$ and with optimal quotes	29.00	72.94	114.14	1.52	1.94
$\beta = \frac{1}{2}$ and with zero-intelligence	1.00	0.80	19.05	-0.14	-0.02
$\beta = 1$ and with zero-intelligence	1.00	1.13	19.90	-0.07	0.00

Table 1: Statistics on final inventory q_1

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and with optimal quotes	356.89	379.64	176.13	0.72	9.38
$\beta = 1$ and with optimal quotes	144.54	154.90	187.97	0.75	12.24
$\beta = \frac{1}{2}$ and with zero-intelligence	21.34	25.01	116.44	0.82	11.05
$\beta = 1$ and with zero-intelligence	17.60	23.32	121.75	0.86	11.51

Table 2: Statistics on final wealth

The statistical study shows that the strategy using the optimal quotes performs better than the zero-intelligence strategy. This finding is logical because the optimal strategy has the advantage of using the information on the liquidity of the option and the arrival rate of market orders.

6.2 Effect of the misspecification of parameters (The case of a linear utility function)

6.2.1 Misspecification of the parameter ρ

Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.04, 0.5, -0.9)$.

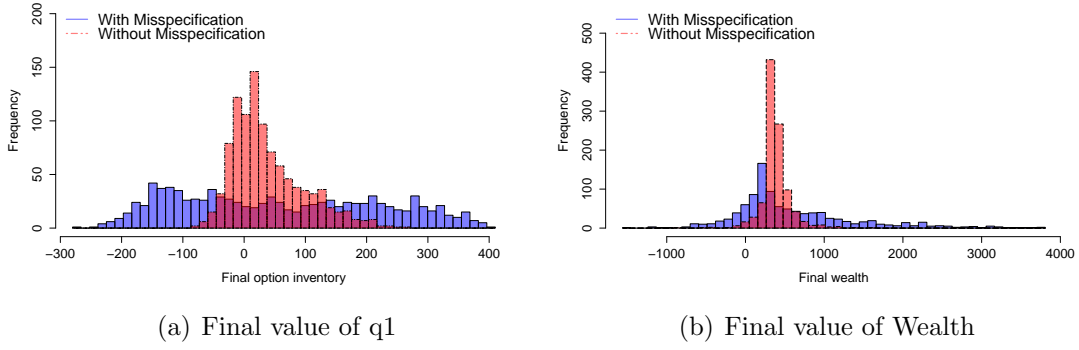


Figure 3: Statistics for $\beta = 0.5$

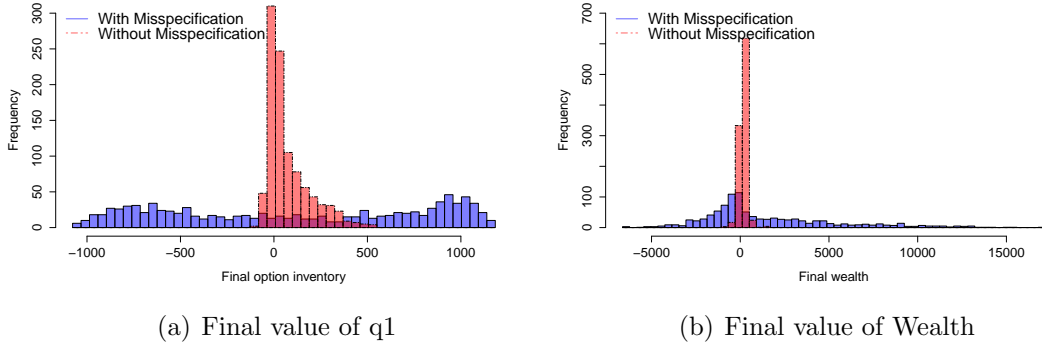


Figure 4: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\rho_R = \rho_I$	23.00	38.41	58.66	0.97	0.62
$\beta = 1$ and $\rho_R = \rho_I$	29.00	72.942	114.13	1.51	1.93
$\beta = \frac{1}{2}$ and $\rho_R \neq \rho_I$	42.50	57.30	165.14	0.19	-1.16
$\beta = 1$ and $\rho_R \neq \rho_I$	105.50	112.38	699.43	-0.049	-1.48

Table 3: Statistics on final q_1

The distribution of the option inventory $q_1(T)$ at the maturity date T is more dispersed in the case where $\rho_R \neq \rho_I$. The final option inventory $q_1(T)$ is path dependent. Indeed, in each path, the evolution of the moneyness of the option affects the quantity $C_P(t, s, y) - C_Q(t, s, y)$ and influences consequently the aggressivity of the quotes (on the bid or the ask side).

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\rho_R = \rho_I$	356.89	379.63	176.13	0.72	9.38
$\beta = 1$ and $\rho_R = \rho_I$	144.54	154.90	187.97	0.74	12.24
$\beta = \frac{1}{2}$ and $\rho_R \neq \rho_I$	331.58	584.06	772.01	1.29	2.01
$\beta = 1$ and $\rho_R \neq \rho_I$	102.22	1485.31	3780.57	1.20	1.26

Table 4: Statistics on final wealth.

The statistics show that the final wealth in the case where $\rho_R = -0.4$ and $\rho_I = -0.9$ is on average higher than in the case where $\rho_R = \rho_I = -0.4$. Indeed, the market maker adapts her strategy in order to profit from the misspecification of the parameter ρ .

6.2.2 Misspecification of the parameter θ

Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.0625, 0.5, -0.4)$. The statistics of 1000 simulations are given below:

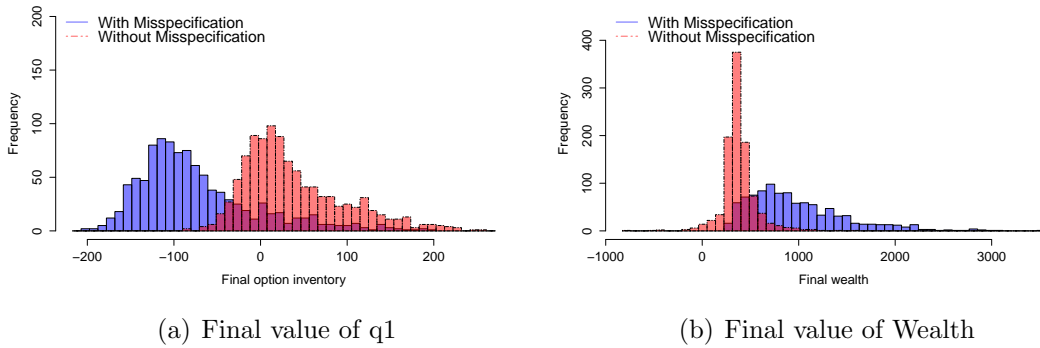


Figure 5: Statistics for $\beta = 0.5$

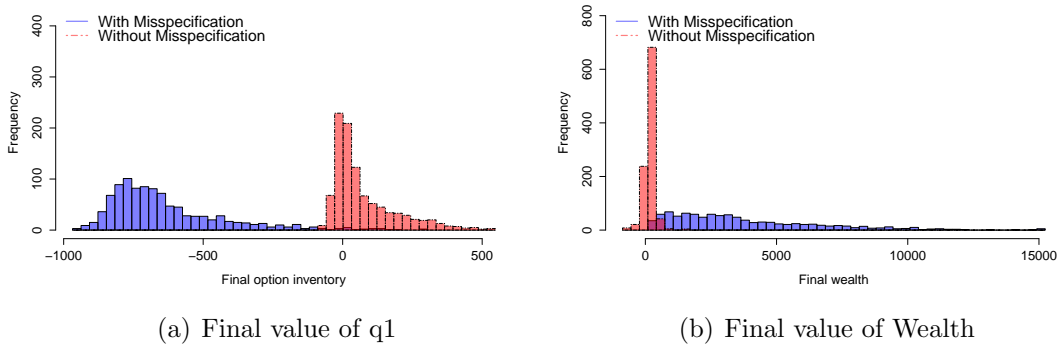


Figure 6: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\theta_R = \theta_I$	23.00	38.42	58.66	0.98	0.63
$\beta = 1$ and $\theta_R = \theta_I$	29.00	72.94	114.14	1.52	1.94
$\beta = \frac{1}{2}$ and $\theta_R \neq \theta_I$	-90.50	-73.81	68.76	1.22	1.45
$\beta = 1$ and $\theta_R \neq \theta_I$	-693.00	-636.17	204.57	1.51	2.52

Table 5: Statistics on final q_1 .

The distribution of the final option inventory $q_1(T)$ corresponding to the case where $\theta_R = 0.04$ and $\theta_I = 0.0625$ is shifted to the left compared to the case where $\theta_R = \theta_I = 0.04$. Indeed, since $\theta_R < \theta_I$, we have $C_{\mathcal{P}}(t, S_t, y_t) < C_{\mathcal{Q}}(t, S_t, y_t)$. Consequently, the optimal strategy is such that the ask quotes are aggressive and the bid quotes are conservative compared to the case without parameters misspecification.

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\theta_R = \theta_I$	356.89	379.64	176.13	0.72	9.38
$\beta = 1$ and $\theta_R = \theta_I$	144.54	154.90	187.97	0.75	12.24
$\beta = \frac{1}{2}$ and $\theta_R \neq \theta_I$	856.03	972.96	504.59	1.26	1.91
$\beta = 1$ and $\theta_R \neq \theta_I$	3029.61	3737.93	2834.59	1.18	1.26

Table 6: Statistics on final wealth

The final wealth in the case with parameters misspecification ($\theta_R = 0.04, \theta_I = 0.0625$) is on average higher than in the case without parameters misspecification ($\theta_R = \theta_I = 0.04$). The simulations show well how the optimal strategy enables the market maker to take advantage of the misspecification of the parameter θ .

6.2.3 Effect of the spot trend μ

Monte Carlo simulations are performed using the following parameters: $(\mu, k_R, \theta_R, \eta_R, \rho_R) = (0.05, 4, 0.04, 0.5, -0.4)$ and $(r, k_I, \theta_I, \eta_I, \rho_I) = (0, 4, 0.04, 0.5, -0.4)$.

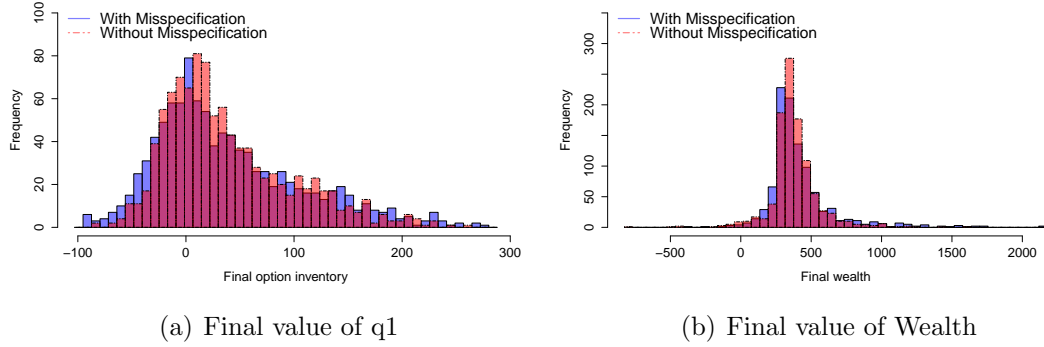


Figure 7: Statistics for $\beta = 0.5$

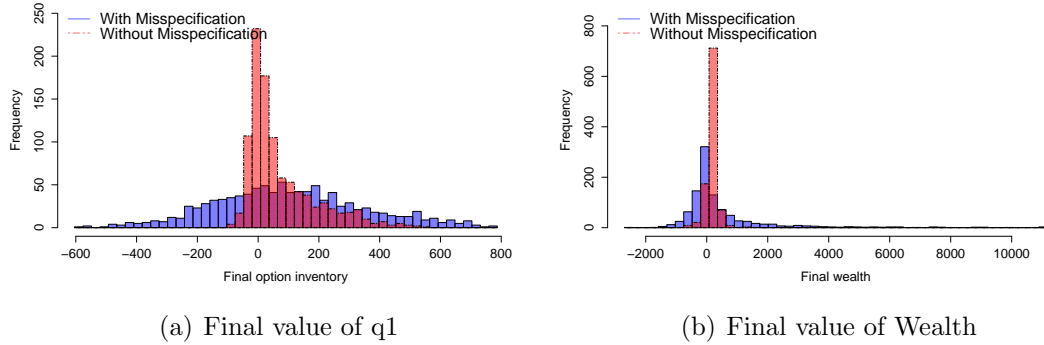


Figure 8: Statistics for $\beta = 1$

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\mu_R = r$	23.00	38.42	58.66	0.98	0.63
$\beta = 1$ and $\mu_R = r$	29.00	72.94	114.14	1.52	1.94
$\beta = \frac{1}{2}$ and $\mu_R \neq r$	22.50	39.45	68.27	0.86	0.34
$\beta = 1$ and $\mu_R \neq r$	86.00	103.12	248.24	0.20	-0.17

Table 7: Statistics on final option inventory q_1

The final option inventory $q_1(T)$ in the case where $\mu_R = 0.05$ and $r = 0$ is higher than in the case where $\mu_R = r = 0$. Indeed, as proved in (8.5), in the case where $r = 0$, a positive

trend ($\mu_R > 0$) leads to $C_{\mathcal{P}}(t, S_t, y_t) > C_{\mathcal{Q}}(t, S_t, y_t)$ whereas a negative trend ($\mu_R < 0$) leads to $C_{\mathcal{P}}(t, S_t, y_t) < C_{\mathcal{Q}}(t, S_t, y_t)$. Consequently, when $r = 0$ and $\mu_R = 0.05$, the optimal bid quote is more aggressive and the optimal ask quote is more conservative than in the case where $\mu_R = r = 0$.

Cases	Median	Mean	Std	Skewness	Kurtosis
$\beta = \frac{1}{2}$ and $\mu_R = r$	356.89	379.64	176.13	0.72	9.38
$\beta = 1$ and $\mu_R = r$	144.54	154.90	187.97	0.75	12.24
$\beta = \frac{1}{2}$ and $\mu_R \neq r$	351.68	414.31	245.09	2.64	11.05
$\beta = 1$ and $\mu_R \neq r$	15.82	467.42	533.84	3.40	14.75

Table 8: Statistics on final wealth

The presence of a trend on the spot leads to a difference between $C_{\mathcal{P}}(t, S_t, y_t)$ and $C_{\mathcal{Q}}(t, S_t, y_t)$. This difference is taken into consideration in the strategy and explains the difference between the distributions of final wealth between the cases $\mu_R = 0$ and $\mu_R = 0.05$.

7 Conclusion

In this paper, a framework for options market making was established. Using a stochastic control approach, analytic expressions for optimal bid and ask quotes on the option are obtained. Next to that, a mean variance framework is considered in order to take into account the risk adversity of the market maker. Using a singular perturbation technique, approximations for the optimal quotes are obtained. Through the use of Monte Carlo simulations, the impact of parameters misspecification on the final wealth and inventory of the market maker was shown. A deeper study of the market making strategy with inventory constraints is kept for a future work.

8 Appendices

8.1 Appendix 1: Intensity of arrivals of market orders

In order to determine the form of the functions λ^+ and λ^- , we need to specify the distribution function of the size of market orders and also the market impact following the execution of a market order.

Let $f_{\mathcal{V}}$ denote the density distribution of the size of market orders in absolute value of their cash amount. Several studies proved that this density decays as a power law (see [Avellaneda and Stoikov(2008)]). We will suppose here that $f_{\mathcal{V}}$ can be fitted by a power-law density:

$$\forall x > 0, f_{\mathcal{V}}(x) = \frac{\gamma L^\gamma}{(L+x)^{\gamma+1}}. \quad (8.1)$$

From a practical point of view, there is a strictly positive lower bound for x which corresponds to the option price x_{Min} . Nevertheless, it is supposed here that the density $f_{\mathcal{V}}$ is positive for $0 \leq x \leq x_{Min}$.

In the other hand, the market impact has been studied by different authors in the econophysics literature and it is widely accepted that the change in price ΔP following a market order of size \mathcal{V} can be written as follows:

$$\Delta P = K\mathcal{V}^\beta. \quad (8.2)$$

There are two values of β which are supported by different researchers: $\beta = 1$ which corresponds to a linear market impact and $\beta = \frac{1}{2}$ which corresponds to a square root market impact.

The probability that a bid quote (respectively ask quote) placed at a distance δ^- (respectively δ^+) from the mid price gets executed is equal to the probability that a sell market order (respectively buy market order) triggers a market impact which is higher or equal to δ^- (respectively δ^+). Suppose as in [Avellaneda and Stoikov(2008)] that the frequency of market orders is constant and equal to F . Then, for $\delta \geq 0$:

$$\begin{aligned} \lambda(\delta) &= F \times P(\Delta P \geq \delta), \\ &= F \times P(\mathcal{V}^\beta \geq \frac{\delta}{K}), \\ &= F \times P\left(\mathcal{V} \geq \left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}\right), \\ &= F \int_{\left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}}^{+\infty} f_{\mathcal{V}}(x) dx, \\ &= F \frac{L^\gamma}{\left(L + \left(\frac{\delta}{K}\right)^{\frac{1}{\beta}}\right)^\gamma}. \end{aligned}$$

Therefore, the intensity of order arrivals can be written as $\lambda(\delta) = \frac{A}{(B+\delta^C)^\gamma}$ where : $A = FK^{\frac{\gamma}{\beta}}L^\gamma$, $B = LK^{\frac{1}{\beta}}$ and $C = \frac{1}{\beta}$.

For $A = 1 \text{ Tick}^2 s^{-1}$, $B = 5 \text{ Tick}$, $C = 1$, $\gamma = 2$ and $\text{Tick} = 0.01$, the following intensities λ^- and λ^+ are obtained:

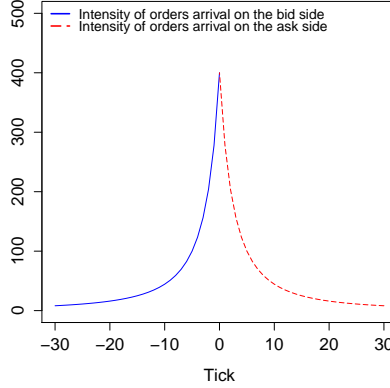


Figure 9: Shape of the functions λ^+ (red line) and λ^- (blue line)

Remark: Several empirical studies proved that $\gamma > 1$. Indeed, for Gopikrishnan et al, $\gamma = 1.53$. For Maslow and Mills, $\gamma = 1.4$ and for Gabaix et al $\gamma = 1.5$.

8.2 Appendix 2: Resolution in the case of a linear utility function ($\beta = 1$)

The utility function U given in (4.1) is linear. Therefore, we make the following Ansatz for u :

$$u(t, s, y, q_1, x) = x + \theta_0(t, s, y) + q_1 \theta_1(t, s, y), \quad (8.3)$$

Let $f_0^+ = J^+$, this function represents the jump part due to the execution of the ask quote. Using (8.3), the function f_0^+ can be written explicitly as:

$$f_0^+(\delta^+) = \lambda^+(\delta^+) (\delta^+ + M_0(t, s, y)),$$

where $M_0(t, s, y) = C_Q(t, s, y) - \theta_1(t, s, y)$.

In order to determine $\delta_{L,*,t}^+ = \text{ArgMax}_{\{x \geq 0\}} f_0^+(x)$, the derivative of the function f_0^+ is computed:

$$(f_0^+) '(\delta^+) = \frac{\lambda^+(\delta^+)}{B + \delta^+} (\delta^+ (1 - \gamma) + B - \gamma M_0(t, s, y)),$$

If $M_0(t, s, y) \geq \frac{B}{\gamma}$, then $\forall \delta^+ \geq 0$, $(f_0^+)'\delta^+ \leq -\frac{(\gamma-1)\delta^+}{B+\delta^+}\lambda^+(\delta^+) < 0$. It follows that f_0^+ is decreasing on $[0, +\infty[$ and then $\delta_{L,*,t}^+ = 0$.

If $M_0(t, s, y) \leq \frac{B}{\gamma}$, the function $(f_0^+)'$ changes sign on the interval $[0, +\infty[$ and becomes null at $x^+ = \frac{B-\gamma M_0(t,s,y)}{\gamma-1}$. The table of variations yields that $\delta_{L,*,t}^+ = x^+$.

In conclusion, $\delta_{L,*,t}^+$ can be determined as follows:

$$\delta_{L,*,t}^+ = \left(\frac{B - \gamma M_0(t, s, y)}{\gamma - 1} \right)^+,$$

and:

$$f_0^+(\delta_{L,*,t}^+) = \begin{cases} \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma(B-M_0(t,s,y))^{\gamma-1}} & \text{if } M_0(t, s, y) \leq \frac{B}{\gamma} \\ \frac{A}{B^\gamma} M_0(t, s, y) & \text{if } M_0(t, s, y) \geq \frac{B}{\gamma} \end{cases} \quad (8.4)$$

The same approach can be applied to the function $f_0^- = J^-$. Indeed, using the form of the value function suggested in (8.3), the function f_0^- can be written as follows:

$$f_0^-(\delta^-) = \lambda^-(\delta^-)(\delta^- - M_0(t, s, y)).$$

In order to determine $\delta_{L,*,t}^- = \text{ArgMax}_x f_0^-(x)$, the derivative of the function f_0^- is computed:

$$(f_0^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + \delta^-} ((1 - \gamma)\delta^- + (B + \gamma M_0(t, s, y))).$$

If $M_0(t, s, y) \leq -\frac{B}{\gamma}$, then $\forall \delta^- \geq 0$, $(f_0^-)'(\delta^-) < 0$. Consequently, the function (f_0^-) is decreasing on $[0, +\infty[$ and $\delta_{L,*,t}^- = 0$. On the other hand, if $M_0(t, s, y) \geq -\frac{B}{\gamma}$, the function $(f_0^-)'$ changes sign in the interval $[0, +\infty[$ and becomes null at $x^- = \frac{B+\gamma M_0(t,s,y)}{\gamma-1}$. Since the function f_0^- is increasing on $[0, x^-]$ and decreasing on $[x^-, +\infty[$, then $\delta_{L,*,t}^- = x^-$.

Consequently, $\delta_{L,*,t}^-$ writes:

$$\delta_{L,*,t}^- = \left(\frac{B + \gamma M_0(t, s, y)}{\gamma - 1} \right)^+,$$

and:

$$f_0^-(\delta_{L,*,t}^-) = \begin{cases} -\frac{A(1-\gamma)^{\gamma-1}}{(-\gamma)^\gamma(B+M_0(t,s,y))^{\gamma-1}} & \text{if } M_0(t, s, y) \geq -\frac{B}{\gamma} \\ -\frac{A}{B^\gamma} M_0(t, s, y) & \text{if } M_0(t, s, y) \leq -\frac{B}{\gamma} \end{cases} \quad (8.5)$$

In order to simplify the notations, the following quantities are introduced: $\mathcal{S} = \frac{B}{\gamma}$ and $J_0(t, s, y) = f_0^+(\delta_{L,*,t}^+) + f_0^-(\delta_{L,*,t}^-)$.

The equation (4.2) becomes:

$$\mathcal{H} : (\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0(t, s, y) + q_1\theta_1(t, s, y)) + J_0(t, s, y) = 0$$

The terms of the HJB equation are sorted by powers of q_1 :

$$\begin{aligned} (0) & : (\partial_t + \mathcal{L}_1)\theta_0 + J_0(t, s, y) = 0, \\ (1) & : (\partial_t + \mathcal{L}_1)\theta_1 - \mu\Delta_t S_t = 0, \end{aligned}$$

Using the final conditions and applying the Feynman-Kac formula yields:

$$\begin{aligned}\theta_1(t, s, y) &= C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right), \\ \theta_0(t, s, y) &= E_{t,s,y} \left(\int_t^T J_0(u, S_u, y_u) du \right).\end{aligned}$$

Based on these results, the quantity $M_0(t, s, y)$ can be deduced:

$$M_0(t, s, y) = C_{\mathcal{Q}}(t, s, y) - C_{\mathcal{P}}(t, s, y) + \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right),$$

and $u_0(t, s, y, q_1, x) = x + \theta_0(t, s, y) + q_1 \left(C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y}^{\mathcal{P}} \left(\int_t^T \Delta(u, S_u, y_u) S_u du \right) \right)$ is the solution of the HJB equation (4.2).

The function u_0 coincides with the value function if it is finite and regular. In order to check the finiteness of u_0 , we prove first that $E \left(\left| \int_t^T J_0(u, S_u, y_u) du \right| \right)$ is finite:

1. If $M_0(t, s, y) \in [-\mathcal{S}, \mathcal{S}]$:

$$\begin{aligned}\left(\frac{\gamma+1}{\gamma} B \right)^{-(\gamma-1)} &\leq (B - M_0(t, s, y))^{-(\gamma-1)} \leq \left(\frac{\gamma-1}{\gamma} B \right)^{-(\gamma-1)}, \\ \left(\frac{\gamma+1}{\gamma} B \right)^{-(\gamma-1)} &\leq (B + M_0(t, s, y))^{-(\gamma-1)} \leq \left(\frac{\gamma-1}{\gamma} B \right)^{-(\gamma-1)},\end{aligned}$$

Then $\exists M > 0$ such that $|J_0(t, s, y)| \leq M$.

2. If $M_0(t, s, y) \geq \mathcal{S}$:

$$E_{t,s,y}^{\mathcal{P}} (|J_0(u, S_u, y_u)|) \leq \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B+\mathcal{S})^{\gamma-1}} + \frac{A}{B^\gamma} E_{t,s,y}^{\mathcal{P}} (|M_0(u, S_u, y_u)|),$$

Since $E(|M_0(u, S_u, y_u)|)$ is finite, then $E(|J_0(u, S_u, y_u)|)$ is also finite.

3. If $M_0(t, s, y) \leq -\mathcal{S}$:

Then :

$$E_{t,s,y}^{\mathcal{P}} (|J_0(u, S_u, y_u)|) \leq \frac{A(\gamma-1)^{\gamma-1}}{\gamma^\gamma (B-\mathcal{S})^{\gamma-1}} + \frac{A}{B^\gamma} E(|M_0(u, S_u, y_u)|),$$

and the rest of the proof is similar to the previous case.

This establishes the proof that $\left| \int_t^T E(J_0(u, S_u, y_u)) du \right|$ is finite. Thus, it can be deduced that $|\theta_0(t, s, y)| < +\infty$. On the other hand, the function θ_0 is regular since J_0 is continuous (J_0 is at least $C^{0,0,0}$), then the function θ is also regular. Consequently, θ coincides with the value function.

8.3 Appendix 3: Resolution in the mean-variance framework ($\beta = 1$)

Let u^ϵ be the solution of the HJB equation (5.1). Under the assumption that $\epsilon \sim 0$, a singular perturbation technique is performed with respect to the parameter ϵ :

$$u^\epsilon(t, s, y, q_1, x) = x + \sum_{k=0}^{+\infty} \epsilon^k v_k(t, s, y, q_1),$$

Given the form of the utility function, the following Ansatz on v_0 and v_1 is made :

$$\begin{aligned} v_0(t, s, y, q_1) &= \theta_0(t, s, y) + q_1 \theta_1(t, s, y), \\ v_1(t, s, y, q_1) &= \theta_2(t, s, y) + q_1 \theta_3(t, s, y) + q_1^2 \theta_4(t, s, y), \end{aligned}$$

In order to solve the HJB equation, the Jump terms $J^{+,\epsilon}$ and $J^{-,\epsilon}$ have to be calculated. Let $f^+ = J^{+,\epsilon}$, the function f^+ writes:

$$\begin{aligned} f^+(\delta^+) &= \lambda^+(\delta^+)(u(t, s, y, q_1 - 1, x + (c + \delta^+)) - u(t, s, y, q_1, x)), \\ &= \lambda^+(\delta^+) (\delta^+ + M_0(t, s, y) + \epsilon M_1(t, s, y, q_1) + \epsilon^2 R^+(t, s, y, q_1)), \end{aligned}$$

Let $M^+(t, s, y, q_1) = M_0(t, s, y, q_1) + \epsilon M_1(t, s, y, q_1)$. By differentiating f^+ , it can be shown that:

$$(f^+) '(\delta^+) = \frac{\lambda^+(\delta^+)}{B + \delta^+} (\delta^+ (1 - \gamma) + B - \gamma M^+(t, s, y, q_1) - \gamma \epsilon^2 R^+(t, s, y, q_1)),$$

In order to determine $\delta_{*,t}^+ = \text{ArgMax}_{\{x \geq 0\}} f^+(x)$, two cases should be distinguished:

- $M^+(t, s, y, q_1) + \epsilon^2 R^+(t, s, y, q_1) \geq \mathcal{S}$: In this case, $\forall \delta^+ \geq 0$, $(f^+) '(\delta^+) \leq 0$, then the function f^+ is decreasing on the interval $[0, +\infty[$ and $\delta_{*,t}^+ = 0$.
- $M^+(t, s, y, q_1) + \epsilon^2 R^+(t, s, y, q_1) \leq \mathcal{S}$: The function $(f^+) '$ changes its sign on $[0, +\infty[$ and gets null in x^+ :

$$x^+ = \frac{B - \gamma (M^+(t, s, y, q_1) + \epsilon^2 R^+(t, s, y, q_1))}{\gamma - 1}.$$

Since $\gamma > 1$ then $\delta_{*,t}^+ = x^+$.

In conclusion, $\delta_{*,t}^+$ writes:

$$\delta_{*,t}^+ = \left(\frac{B - \gamma (M^+(t, s, y, q_1) + \epsilon^2 R^+(t, s, y, q_1))}{\gamma - 1} \right)^+,$$

and, using Taylor expansion, it follows:

$$\delta_{*,t}^+ = \delta_{L,*,t}^+ - \epsilon \frac{\gamma}{\gamma - 1} M_1(t, s, y, q_1) 1_{\{M^+ + \epsilon^2 R^+ \leq \mathcal{S}\}} + O(\epsilon^2).$$

In order to solve the HJB equation, it is useful to write $f^+(\delta_{*,t}^+)$ as the sum of $f_0^+(\delta_{L,*,t}^+)$ plus a correction that depends on the parameter ϵ .

We have:

$$f^+(x) = f_0^+(x) + \epsilon \lambda^+(x) M_1(t, s, y, q_1) + O(\epsilon^2),$$

then, through the use of Taylor expansion, one can write:

$$\begin{aligned} f^+(\delta_{*,t}^+) &= f^+(\delta_{L,*,t}^+) + (f^+)'(\delta_{L,*,t}^+)(\delta_{*,t}^+ - \delta_{L,*,t}^+) + O((\delta_{*,t}^+ - \delta_{L,*,t}^+)^2), \\ &= f_0^+(\delta_{L,*,t}^+) + \epsilon \lambda^+(\delta_{L,*,t}^+) M_1 - \epsilon (f^+)'(\delta_{L,*,t}^+) \frac{\gamma}{\gamma - 1} M_1 1_{\{M^+ + \epsilon^2 R^+ \leq \mathcal{S}\}} + O(\epsilon^2). \end{aligned}$$

Since $(f^+)'(x) = (f_0^+)'(x) + O(\epsilon)$, the last equation becomes:

$$f^+(\delta_{*,t}^+) = f_0^+(\delta_{L,*,t}^+) + \epsilon M_1 \left(\lambda^+(\delta_{L,*,t}^+) - \frac{\gamma}{\gamma - 1} (f_0^+)'(\delta_{L,*,t}^+) 1_{\{M^+ + \epsilon^2 R^+ \leq \mathcal{S}\}} \right) + O(\epsilon^2),$$

It can be recalled at this stage that $(f_0^+)'(\delta_{L,*,t}^+) = \frac{B - \gamma M_0}{B} \lambda^+(0) 1_{\{M_0 > \mathcal{S}\}}$.

Notice that if $M_0 \in [\min(\mathcal{S}, \mathcal{S} - \epsilon M_1 - \epsilon^2 \mathcal{R}^+), \max(\mathcal{S}, \mathcal{S} - \epsilon M_1 - \epsilon^2 \mathcal{R}^+)]$, then $|\frac{B - \gamma M_0}{B} \lambda^+(0)| = O(\epsilon)$. This means:

$$(f_0^+)'(\delta_{L,*,t}^+) = \frac{B - \gamma M_0}{B} \lambda^+(0) 1_{\{M^+ + \epsilon^2 R^+ > \mathcal{S}\}} + O(\epsilon),$$

and then:

$$f^+(\delta_{*,t}^+) = f_0^+(\delta_{L,*,t}^+) + \epsilon M_1(t, s, y, q_1) \lambda^+(\delta_{L,*,t}^+) + O(\epsilon^2).$$

The optimal bid distance $\delta_{*,t}^-$ can be determined using the same method. Indeed, let $f^- = J^{-,\epsilon}$, the function f^- writes:

$$\begin{aligned} f^-(\delta^-) &= \lambda^-(\delta^-)(u(t, s, y, q_1 + 1, x - (c - \delta^-)) - u(t, s, y, q_1, x)), \\ &= \lambda^-(\delta^-) (\delta^- - (M_0(t, s, y) + \epsilon M_2(t, s, y, q_1) + \epsilon^2 R^-(t, s, y, q_1))). \end{aligned}$$

Let $M^-(t, s, y, q_1) = M_0(t, s, y) + \epsilon M_2(t, s, y, q_1)$. Differentiating f^- yields:

$$(f^-)'(\delta^-) = \frac{\lambda^-(\delta^-)}{B + \delta^-} (\delta^- (1 - \gamma) + B + \gamma M^-(t, s, y, q_1) + \gamma \epsilon^2 R^-(t, s, y, q_1)).$$

Afterward, it can be seen that if $M^- + \epsilon^2 R^- < -\mathcal{S}$, then $\delta_{*,t}^- = 0$, whereas if $M^- + \epsilon^2 R^- \geq -\mathcal{S}$, then $(f^-)'$ changes its sign on $[0, +\infty[$ and gets null in $x^- = \frac{B + \gamma M^-(t, s, y, q_1) + \gamma \epsilon^2 R^-(t, s, y, q_1)}{\gamma - 1} = \delta_{*,t}^-$.

So, the quantity $\delta_{*,t}^-$ writes:

$$\delta_{*,t}^- = \left(\frac{B + \gamma (M^-(t, s, y, q_1) + \epsilon^2 R^-(t, s, y, q_1))}{\gamma - 1} \right)^+$$

which implies that:

$$\delta_{*,t}^- = \delta_{L,*,t}^- + \epsilon \frac{\gamma}{\gamma - 1} M_2(t, s, y, q_1) 1_{\{M^- + \epsilon^2 R^- \geq -\mathcal{S}\}} + O(\epsilon^2).$$

The quantity $f^-(\delta_{*,t}^-)$ can be written as the sum of $f_0^-(\delta_{L,*,t}^-)$ plus a correction term due to the parameter ϵ . Indeed:

$$f^-(x) = f_0^-(x) - \epsilon \lambda^-(x) M_2(t, s, y, q_1) + O(\epsilon^2),$$

Then, using Taylor's expansion:

$$\begin{aligned} f^-(\delta_{*,t}^-) &= f^-(\delta_{L,*,t}^-) + (f^-)'(\delta_{L,*,t}^-)(\delta_{*,t}^- - \delta_{L,*,t}^-) + O((\delta_{*,t}^- - \delta_{L,*,t}^-)^2), \\ &= f_0^-(\delta_{L,*,t}^-) - \epsilon \lambda^-(\delta_{L,*,t}^-) M_2 + \epsilon (f^-)'(\delta_{L,*,t}^-) \frac{\gamma}{\gamma - 1} M_2 1_{\{M^- + \epsilon^2 R^- \geq -S\}} + O(\epsilon^2). \end{aligned}$$

Using the relation $(f^-)'(x) = (f_0^-)'(x) + O(\epsilon)$ implies:

$$f^-(\delta_{*,t}^-) = f_0^-(\delta_{L,*,t}^-) + \epsilon M_2(t, s, y, q_1) \left(-\lambda^-(\delta_{L,*,t}^-) + \frac{\gamma}{\gamma - 1} (f_0^-)'(\delta_{L,*,t}^-) 1_{\{M^- + \epsilon^2 R^- \geq -S\}} \right) + O(\epsilon^2).$$

We have also $(f_0^-)'(\delta_{L,*,t}^-) = \left(\frac{B + \gamma M_0}{B} \lambda^-(0) \right) 1_{\{M_0 < -S\}}$. Following the same method, it can be shown that if $M_0 \in [\min(-S, -S - \epsilon M_2 - \epsilon^2 \mathcal{R}^-), \max(-S, -S - \epsilon M_2 - \epsilon^2 \mathcal{R}^-)]$, then $|\frac{B + \gamma M_0}{B} \lambda^-(0)| = O(\epsilon)$. Therefore, it can be deduced that:

$$(f_0^-)'(\delta_{L,*,t}^-) = \frac{B + \gamma M_0}{B} \lambda^-(0) 1_{\{M^- + \epsilon^2 R^- < -S\}} + O(\epsilon).$$

and:

$$f^-(\delta_{*,t}^-) = f_0^-(\delta_{L,*,t}^-) - \epsilon M_2(t, s, y, q_1) \lambda^-(\delta_{L,*,t}^-) + O(\epsilon^2),$$

Now that the terms $f^+(\delta_{*,t}^+)$ and $f^-(\delta_{*,t}^-)$ are computed separately, the term $J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) = f^+(\delta_{*,t}^+) + f^-(\delta_{*,t}^-)$ is deduced:

$$\begin{aligned} J(\delta_{*,t}^-, \delta_{*,t}^+) &= f_0^+(\delta_{L,*,t}^+) + \epsilon M_1 \lambda^+(\delta_{L,*,t}^+) + f_0^-(\delta_{L,*,t}^-) - \epsilon M_2 \lambda^-(\delta_{L,*,t}^-) + O(\epsilon^2), \\ &= J_0(\delta_{L,*,t}^-, \delta_{L,*,t}^+) + \epsilon M_1(t, s, y, q_1) \lambda^+(\delta_{L,*,t}^+) - \epsilon M_2(t, s, y, q_1) \lambda^-(\delta_{L,*,t}^-) + O(\epsilon^2), \end{aligned}$$

The terms of $J(\delta_{*,t}^-, \delta_{*,t}^+)$ are classified according to their power in ϵ :

$$J(\delta_{*,t}^-, \delta_{*,t}^+) = J_0(t, s, y) + \epsilon J_1(t, s, y, q_1) + O(\epsilon^2),$$

where $J_1(t, s, y, q_1) = J_{1,0}(t, s, y) + q_1 J_{1,1}(t, s, y)$ and:

$$\begin{aligned} J_{1,0}(t, s, y) &= \lambda^+(\delta_{L,*,t}^+) (-\theta_3 + \theta_4) - \lambda^-(\delta_{L,*,t}^-) (-\theta_3 - \theta_4), \\ J_{1,1}(t, s, y) &= -2\theta_4 (\lambda^+(\delta_{L,*,t}^+) - \lambda^-(\delta_{L,*,t}^-)), \end{aligned}$$

The HJB equation can be separated into several terms according to the order of the parameter ϵ . By nullifying the term of order 0 in ϵ , it can be obtained that:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(x + \theta_0 + q_1 \theta_1) + J_0 = 0,$$

with the final conditions:

$$\theta_0(T, s, y) = 0, \quad \theta_1(T, s, y) = h(s).$$

The functions θ_0 and θ_1 are equivalent to those found in the case of a linear utility function without inventory constraints, thus:

$$\begin{aligned}\theta_1(t, s, y) &= C_{\mathcal{P}}(t, s, y) - \mu E_{t,s,y} \left(\int_t^T \Delta_u S_u du \right) \\ \theta_0(t, s, y) &= E_{t,s,y} \left(\int_t^T J_0(u, S_u, y_u) du \right),\end{aligned}$$

The term of order 1 in ϵ leads to the following equation:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2)(\theta_2 + q_1\theta_3 + q_1^2\theta_4) + J_1(t, s, y) = \eta q_1^2 V + \eta T,$$

with the final conditions:

$$\theta_2(T, s, y) = 0, \quad \theta_3(T, s, y) = 0, \quad \theta_4(T, s, y) = 0.$$

The functions θ_2 , θ_3 and θ_4 are:

$$\begin{aligned}\theta_2(t, s, y) &= E_{t,s,y}^{\mathcal{P}} \left(\int_t^T (J_{1,0} - \eta T)(u, s_u, y_u) du \right), \\ \theta_3(t, s, y) &= E_{t,s,y}^{\mathcal{P}} \left(\int_t^T J_{1,1}(u, s_u, y_u) du \right), \\ \theta_4(t, s, y) &= -\eta E_{t,s,y}^{\mathcal{P}} \left(\int_t^T V_u du \right).\end{aligned}$$

Remark:

The optimal ask quote $\delta_{*,t}^+$ can be approximated at order 1 in ϵ by $\hat{\delta}_{*,t}^+$:

$$\hat{\delta}_{*,t}^+ = \left(\frac{B - \gamma M^+(t, s, y, q_1)}{\gamma - 1} \right)^+.$$

Indeed, if $M^+(t, s, y) \in [\text{Min}(\mathcal{S} - \epsilon^2 R^+, \mathcal{S}), \text{Max}(\mathcal{S} - \epsilon^2 R^+, \mathcal{S})]$, then $|\frac{B - \gamma M^+(t, s, y, q_1)}{\gamma - 1}| = O(\epsilon^2)$ and consequently:

$$|\hat{\delta}_{*,t}^+ - \delta_{*,t}^+| = O(\epsilon^2),$$

Similarly, it can be seen that if $M^- \in [\text{Min}(-\mathcal{S}, -\mathcal{S} - \epsilon^2 R^-), \text{Max}(-\mathcal{S}, -\mathcal{S} - \epsilon^2 R^-)]$, then $|\frac{B + \gamma M^-}{\gamma - 1}| = O(\epsilon^2)$. Thus, the optimal bid quote $\delta_{*,t}^-$ can be approximated at order 1 in ϵ by $\hat{\delta}_{*,t}^-$:

$$\hat{\delta}_{*,t}^- = \left(\frac{B + \gamma M^-}{\gamma - 1} \right)^+,$$

and the approximation error is at order 2 in ϵ :

$$|\hat{\delta}_{*,t}^- - \delta_{*,t}^-| = O(\epsilon^2)$$

8.4 Appendix 4: Solving the Heston PDE with a trend on the underlying

We aim to determine here an analytic expression for the quantity $C(t, S_t, y_t) = E^{\mathcal{P}}((S_T - K)^+)$ in the Heston model. In this model, $C(t, S_t, y_t)$ is the solution of the following equation:

$$\frac{\partial C}{\partial t} + \frac{\partial C}{\partial S} \mu S_t + \frac{1}{2} \frac{\partial^2 C}{\partial S^2} y_t S_t^2 + \frac{\partial C}{\partial y} k(\theta - y_t) + \frac{1}{2} \frac{\partial^2 C}{\partial y^2} \eta^2 y_t + \frac{\partial^2 C}{\partial S \partial y} \rho S_t y_t \eta = 0,$$

with the final condition: $C(T, S_T, y_T) = (S_T - K)^+$.

We are going to use for this purpose the same approach as in [Heston(1993)] and [Gatheral(2006)].

Let $x_t = \log(\frac{S_t}{K})$ and $P(t, x_t, y_t) = C(t, S_t, y_t)$, then P is the solution of the following equation:

$$\frac{\partial P}{\partial t} + \frac{\partial P}{\partial x} (\mu - \frac{y_t}{2}) + \frac{1}{2} \frac{\partial^2 P}{\partial x^2} y_t + \frac{\partial P}{\partial y} k(\theta - y_t) + \frac{1}{2} \frac{\partial^2 P}{\partial y^2} \eta^2 y_t + \frac{\partial^2 P}{\partial x \partial y} \rho y_t \eta = 0, \quad (8.6)$$

By analogy with the Black-Scholes formula, the guessed solution of the previous equation has the form:

$$P(t, x_t, y_t) = K (e^{x_t} P_1 - e^{-\mu(T-t)} P_2), \quad (8.7)$$

By inserting (8.7) in (8.6), it can be obtained that:

$$K e^{x_t} T_1(t, x, y) - K e^{-\mu(T-t)} T_2(t, x, y) = 0,$$

where:

$$T_1(t, x, y) = \frac{\partial P_1}{\partial t} + (\mu - \frac{y_t}{2})(P_1 + \frac{\partial P_1}{\partial x}) + \frac{y_t}{2} (2 \frac{\partial P_1}{\partial x} + P_1 + \frac{\partial^2 P_1}{\partial x^2}) + k(\theta - y_t) \frac{\partial P_1}{\partial y} + \frac{\eta^2 y_t}{2} \frac{\partial^2 P_1}{\partial y_t^2} + \rho \eta y_t (\frac{\partial^2 P_1}{\partial x \partial y} + \frac{\partial P_1}{\partial y}),$$

and:

$$T_2(t, x, y) = \mu P_2 + \frac{\partial P_2}{\partial t} + (\mu - \frac{y_t}{2}) \frac{\partial P_2}{\partial x} + \frac{y_t}{2} \frac{\partial^2 P_2}{\partial x^2} + k(\theta - y_t) \frac{\partial P_2}{\partial y} + \frac{\eta^2 y_t}{2} \frac{\partial^2 P_2}{\partial y_t^2} + \rho \eta y_t \frac{\partial^2 P_2}{\partial x \partial y}$$

Let $\tau = T - t$, it follows that for $j \in \{1, 2\}$, P_j satisfies the following equation:

$$-\frac{\partial P_j}{\partial \tau} + \mu P_j + (\mu + u_j y_t) \frac{\partial P_j}{\partial x} + \frac{y_t}{2} \frac{\partial^2 P_j}{\partial x^2} + (a - b_j y_t) \frac{\partial P_j}{\partial y} + \frac{\eta^2 y_t}{2} \frac{\partial^2 P_j}{\partial y_t^2} + \rho \eta y_t \frac{\partial^2 P_j}{\partial x \partial y} = 0$$

where $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = k\theta$, $b_1 = k - \rho\eta$, $b_2 = k$.

Using a Fourier transform method, let $\tilde{P}_j(\tau, u, y)$ be defined as following:

$$\tilde{P}_j(\tau, u, y) = \int_{-\infty}^{+\infty} e^{-iux} P_j(\tau, x, y) dx$$

Evaluating $\tilde{P}_j(\tau, u, y)$ at $\tau = 0$ gives:

$$\begin{aligned}\tilde{P}_j(0, u, y) &= \int_{-\infty}^{+\infty} e^{-iux} P_j(0, x, y) dx, \\ &= \int_{-\infty}^{+\infty} e^{-iux} \mathbf{1}_{\{x>0\}} dx, \\ &= \frac{1}{iu}\end{aligned}$$

By doing necessary calculations, it can be found that:

$$-\frac{\partial \tilde{P}_j}{\partial \tau} + \mu \tilde{P}_j + (\mu + u_j y_t) i u \tilde{P}_j - u^2 \frac{y_t}{2} \tilde{P}_j + (a - b_j y_t) \frac{\partial \tilde{P}_j}{\partial y} + \frac{\eta^2 y_t}{2} \frac{\partial^2 \tilde{P}_j}{\partial y_t^2} + \rho \eta y_t i u \frac{\partial \tilde{P}_j}{\partial y} = 0$$

Let:

$$\beta' = b_j - \rho \eta i u, \quad \gamma' = \frac{\eta^2}{2}, \quad \alpha' = -\frac{u^2}{2} + i u u_j,$$

Using the new notations, \tilde{P}_j satisfies the following equation:

$$-\frac{\partial \tilde{P}_j}{\partial \tau} + a \frac{\partial \tilde{P}_j}{\partial y} + y_t (\alpha' \tilde{P}_j - \beta' \frac{\partial \tilde{P}_j}{\partial y} + \gamma' \frac{\partial^2 \tilde{P}_j}{\partial y_t^2}) + (1 + i u) \mu \tilde{P}_j = 0, \quad (8.8)$$

Now by substituting $\tilde{P}_j(\tau, u, y)$ by:

$$\begin{aligned}\tilde{P}_j(\tau, u, y) &= \exp(C(u, \tau)\theta + D(u, \tau)y_t) \tilde{P}_j(0, u, y), \\ &= \frac{1}{iu} \exp(C(u, \tau)\theta + D(u, \tau)y_t)\end{aligned}$$

and by using the relations:

$$\frac{\partial \tilde{P}_j}{\partial \tau} = \left(\theta \frac{\partial C}{\partial \tau} + y_t \frac{\partial D}{\partial \tau}\right) \tilde{P}_j, \quad \frac{\partial \tilde{P}_j}{\partial y} = D(u, \tau) \tilde{P}_j, \quad \frac{\partial^2 \tilde{P}_j}{\partial y^2} = D(u, \tau)^2 \tilde{P}_j,$$

the equation (8.8) becomes:

$$\left[-\theta \frac{\partial C}{\partial \tau} + a D + (1 + i u) \mu\right] + y_t \left[\alpha' - \beta' D + \gamma' D^2 - \frac{\partial D}{\partial \tau}\right] = 0$$

The two terms at order 0 and 1 in y_t should then be null, which gives the two following equations:

$$\begin{aligned}\frac{\partial C}{\partial \tau} &= \frac{a}{\theta} D + \frac{1 + i u}{\theta} \mu, \\ \frac{\partial D}{\partial \tau} &= \alpha' - \beta' D + \gamma' D^2,\end{aligned}$$

Let:

$$r_+ = \frac{\beta' + \sqrt{\beta'^2 - 4\alpha'\gamma'}}{2\gamma'}, \quad r_- = \frac{\beta' - \sqrt{\beta'^2 - 4\alpha'\gamma'}}{2\gamma'}, \quad g = \frac{r_-}{r_+}.$$

Integrating with the terminal conditions $C(u, 0) = 0$ and $D(u, 0) = 0$ gives:

$$\begin{aligned} D(u, \tau) &= r - \frac{1 - e^{-d\tau}}{1 - ge^{-d\tau}}, \\ C(u, \tau) &= \frac{1 + iu}{\theta} \mu \tau + k \left[r - \tau - \frac{2}{\eta^2} \ln\left(\frac{1 - ge^{-d\tau}}{1 - g}\right) \right] \end{aligned}$$

and finally:

$$P_j(\tau, x, y_t) = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \operatorname{Re}(\tilde{P}_j(\tau, u, y_t) e^{iux}) du,$$

8.5 Appendix 5: Effect of the trend

Let C_μ be the solution of the equation:

$$\frac{\partial C_\mu}{\partial t} + \frac{\partial C_\mu}{\partial S} \mu S_t + \frac{1}{2} \frac{\partial^2 C_\mu}{\partial S^2} y_t S_t^2 + \frac{\partial C_\mu}{\partial y} k(\theta - y_t) + \frac{1}{2} \frac{\partial^2 C_\mu}{\partial y^2} \eta^2 y_t + \frac{\partial^2 C_\mu}{\partial S \partial y} \rho S_t y_t \eta = 0,$$

with the final condition: $C_\mu(T, S_T, y_T) = (S_T - K)^+$, and C_0 be the solution of the equation:

$$\frac{\partial C_0}{\partial t} + \frac{1}{2} \frac{\partial^2 C_0}{\partial S^2} y_t S_t^2 + \frac{\partial C_0}{\partial y} k(\theta - y_t) + \frac{1}{2} \frac{\partial^2 C_0}{\partial y^2} \eta^2 y_t + \frac{\partial^2 C_0}{\partial S \partial y} \rho S_t y_t \eta = 0,$$

with the final condition: $C_0(T, S_T, y_T) = (S_T - K)^+$.

Let $\tilde{\mathcal{P}}$ be the probability measure defined as $\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}|_{\mathcal{F}_t}} = e^{-\int_0^t \frac{\mu}{\sigma(y_s)} dW_s^{(1)} - \frac{1}{2} \int_0^t \frac{\mu^2}{\sigma^2(y_s)} ds}$. Therefore, under $\tilde{\mathcal{P}}$:

$$\begin{aligned} \frac{dS_t}{S_t} &= \sigma(y_t) d\tilde{W}_t^{(1)}, \\ dy_t &= a_R(y_t) dt + b_R(y_t) d\tilde{W}_t^{(2)}, \end{aligned}$$

where $\tilde{W}^{(1)}$ and $\tilde{W}^{(2)}$ are two $(\tilde{\mathcal{P}}, \mathcal{F})$ Brownian motions defined as $\tilde{W}_t^{(2)} = W_t^{(2)}$ and $\tilde{W}_t^{(1)} = W_t^{(1)} + \int_0^t \frac{\mu}{\sigma(y_s)} ds$. Using the Feynman-Kac formula, it can be deduced that $C_0(t, S_t, y_t) = E_{t, S_t, y_t}^{\tilde{\mathcal{P}}}((S_T - K)^+)$.

Let the quantity $R(t, S_t, y_t) = C_\mu(t, S_t, y_t) - C_0(t, S_t, y_t)$, it can be seen that R is solution of the equation:

$$\frac{\partial R}{\partial t} + \frac{\partial R}{\partial S} \mu S_t + \frac{1}{2} \frac{\partial^2 R}{\partial S^2} y_t S_t^2 + \frac{\partial R}{\partial y} k(\theta - y_t) + \frac{1}{2} \frac{\partial^2 C_\mu}{\partial y^2} \eta^2 y_t + \frac{\partial^2 C_\mu}{\partial S \partial y} \rho S_t y_t \eta + \frac{\partial C_0}{\partial S} \mu S_t = 0,$$

with the final condition $R(T, S_T, y_T) = 0$. Using the Feynman-Kac formula, it follows that:

$$R(t, S_t, y_t) = E_{t, S_t, y_t}^{\mathcal{P}} \left(\int_t^T \mu S_u \frac{\partial C_0}{\partial S}(u, S_u, y_u) du \right),$$

Since $\frac{\partial C_0}{\partial S} > 0$ (C_0 is a call price under $\tilde{\mathcal{P}}$), then $R(t, S_t, y_t) \geq 0$.

8.6 Appendix 6: Accuracy of the approximation

The value function u^ϵ , solution of the HJB equation (5.1), writes:

$$u^\epsilon(t, s, y, q_1, x) = x + v_0(t, s, y, q_1) + \epsilon v_1(t, s, y, q_1) + \mathcal{R}(t, s, y, q_1),$$

with $\mathcal{R}(t, s, y, q_1) = \sum_{k=2}^{+\infty} \epsilon^k v_k(t, s, y, q_1)$.

On the other hand, through the use of Taylor expansion of the terms $f^+(\delta_{*,t}^+)$ and $f^-(\delta_{*,t}^-)$, it has been shown that:

$$J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) = J_0(t, s, y) + \epsilon J_1(t, s, y, q_1) + O(\epsilon^2),$$

Let $g_\epsilon(t, s, y, q_1) = J^\epsilon(\delta_{*,t}^-, \delta_{*,t}^+) - (J_0(t, s, y) + \epsilon J_1(t, s, y, q_1))$, then (5.1) writes:

$$\begin{aligned} & (\partial_t + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3)(x + v_0(t, s, y, q_1) + \epsilon v_1(t, s, y, q_1) + \mathcal{R}(t, s, y, q_1)) \\ & + (J_0(t, s, y) + \epsilon J_1(t, s, y, q_1) + g_\epsilon(t, s, y, q_1)) = \epsilon (\eta_3 q_{1,t}^2 V_t + q_{2,t}^2 \sigma^2(y_t) S_t^2 + C_{\mathcal{Q}}^2 (\lambda_{L,t}^- + \lambda_{L,t}^+)), \end{aligned}$$

it follows that:

$$(\partial_t + \mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3)\mathcal{R}(t, s, y, q_1) + g_\epsilon(t, s, y, q_1) = 0,$$

Since $\mathcal{R}(T, S_T, y_T, q_{1,T}, q_{2,T}) = 0$, then using Feynman-Kac formula:

$$\mathcal{R}(t, s, y, q_1) = E_{t,s,y,q_1}^P \left(\int_t^T g_\epsilon(u, S_u, y_u, q_{1,u}, q_{2,u}) du \right),$$

which implies that $\mathcal{R}(t, s, y, q_1) = O(\epsilon^2)$ and then $|u^\epsilon(t, s, y, q_1, x) - (x + v_0(t, s, y, q_1) + \epsilon v_1(t, s, y, q_1))| = O(\epsilon^2)$.

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