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# HARMONIC FUNCTIONS ON MULTIPLICATIVE GRAPHS AND INVERSE PITMAN TRANSFORM ON INFINITE RANDOM PATHS

CÉDRIC LECOUCVEY, EMMANUEL LESIGNE AND MARC PEIGNÉ

ABSTRACT. We introduce central probability distributions on Littelmann paths and show they coincide with the distributions on such paths already appearing in our previous works. Next we establish a law of large numbers and a central limit theorem for the generalized Pitman transform. We then study harmonic functions on multiplicative graphs defined from the tensor powers of finite-dimensional Lie algebras representations. Finally, we show there exists an inverse of the generalized Pitman transform defined almost surely on the set of infinite paths remaining in the Weyl chamber.

## 1. INTRODUCTION

The present paper is a sequel of our previous works [7], [8] and [9] where we study the random Littelmann path defined from a simple module  $V$  of a Kac-Moody algebra  $\mathfrak{g}$  and use the generalized Pitman transform  $\mathcal{P}$  introduced by Biane, Bougerol and O'Connell [1] to obtain its conditioning to stay in the dominant Weyl chamber of  $\mathfrak{g}$ . Roughly speaking, this random path is obtained by concatenation of elementary paths randomly chosen among the vertices of the crystal graph  $B$  associated to  $V$  following a distribution depending on the graph structure of  $B$ . It is worth noticing that for  $\mathfrak{g} = \mathfrak{sl}_2$ , this random path reduces to the random walk on  $\mathbb{Z}$  with steps  $\{\pm 1\}$  and the transform  $\mathcal{P}$  is the usual Pitman transform [16]. Also when  $V$  is the defining representation of  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ , the vertices of  $B$  are simply the paths linking 0 to each vector of the standard basis of  $\mathbb{R}^{n+1}$  and we notably recover some results by O'Connell exposed in [14].

We will assume here that  $\mathfrak{g}$  is a simple (finite-dimensional) Lie algebra over  $\mathbb{C}$  of rank  $n$ . The irreducible finite-dimensional representations of  $\mathfrak{g}$  are then parametrized by the dominant weights of  $\mathfrak{g}$  which are the elements of the set  $P_+ = P \cap \mathcal{C}$  where  $P$  and  $\mathcal{C}$  are the weight lattice and the dominant Weyl chamber of  $\mathfrak{g}$ , respectively. The random path  $\mathcal{W}$  we considered in [9] is defined from the crystal  $B(\kappa)$  of the irreducible  $\mathfrak{g}$ -module  $V(\kappa)$  with highest weight  $\kappa \in P_+$  ( $\kappa$  is fixed for each  $\mathcal{W}$ ). The crystal  $B(\kappa)$  is an oriented graph graded by the weights of  $\mathfrak{g}$  whose vertices are Littelmann paths of length 1. The vertices and the arrows of  $B(\kappa)$  are obtained by simple combinatorial rules from a path  $\pi_\kappa$  connecting 0 to  $\kappa$  and remaining in  $\mathcal{C}$  (highest weight path). We endowed it with a probability distribution  $p$  compatible with the weight graduation defined from the choice of a  $n$ -tuple  $\tau$  of positive reals (a positive real for each simple root of  $\mathfrak{g}$ ). The probability distribution considered on the successive tensor powers  $B(\kappa)^{\otimes \ell}$  is the product distribution  $p^{\otimes \ell}$ . It has the crucial property to be central: two paths in  $B(\kappa)^{\otimes \ell}$  with the same ends have the same probability. We can then define, following the classical construction of a Bernoulli process, a random path  $\mathcal{W}$  with underlying probability space  $(B(\kappa)^{\otimes \mathbb{Z}_{\geq 0}}, p^{\otimes \mathbb{Z}_{\geq 0}})$  as the direct limit of the spaces  $(B(\kappa)^{\otimes \ell}, p^{\otimes \ell})$ . The trajectories of  $\mathcal{W}$  are the concatenations of the Littelmann paths appearing in  $B(\kappa)$ . It makes sense to consider the image of  $\mathcal{W}$  by the generalized Pitman transform  $\mathcal{P}$ . This yields a Markov process  $\mathcal{H} = \mathcal{P}(\mathcal{W})$  whose trajectories are the concatenations of the paths appearing in  $B(\kappa)$  which remain in the dominant Weyl

chamber  $\mathcal{C}$ . When the drift of  $\mathcal{W}$  belongs to the interior of  $\mathcal{C}$ , we establish in [9] that the law of  $\mathcal{H}$  coincides with the law of  $\mathcal{W}$  conditioned to stay in  $\mathcal{C}$ .

The processes  $\mathcal{W}$  and  $\mathcal{H}$  are continuous in time. By considering their values at integer times  $t = \ell, \ell \in \mathbb{Z}_{\geq 0}$ , we obtain a random walk  $W$  on  $P$  and a Markov chain  $H$  on  $P_+$ , respectively. Alternatively, we can consider  $H$  as a Markov chain on the multiplicative graph  $\mathcal{G}$  with vertices the pairs  $(\lambda, \ell) \in P_+ \times \mathbb{Z}_{\geq 0}$  and weighted arrows  $(\lambda, \ell) \xrightarrow{m_{\lambda, \kappa}^\Lambda} (\Lambda, \ell + 1)$  where  $m_{\lambda, \kappa}^\Lambda$  is the multiplicity of the representation  $V(\Lambda)$  in the tensor product  $V(\lambda) \otimes V(\kappa)$ . This yields an harmonic function on  $\mathcal{G}$  associating to each vertex  $(\lambda, \ell)$ , the probability of the cylinder starting at  $(\lambda, \ell)$  (i.e. the probability that  $H$  starting at  $\lambda$  remains in  $\mathcal{C}$ ).

When  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ , the elements of  $P_+$  can be regarded as the partitions  $\lambda = (\lambda_1 \geq \dots \geq \lambda_n \geq 0) \in \mathbb{Z}^n$ . Moreover, if we choose  $V(\kappa) = V$ , the defining representation of  $\mathfrak{g} = \mathfrak{sl}_{n+1}$ , we have  $m_{\lambda, \kappa}^\Lambda \neq 0$  if and only if the Young diagram of  $\Lambda$  is obtained by adding one box to that of  $\lambda$ . The connected component of  $\mathcal{G}$  obtained from  $(\emptyset, 0)$  thus coincides with the Young lattice on partitions with at most  $n$  parts (one can obtain the whole Young lattice by working with  $\mathfrak{g} = \mathfrak{sl}_\infty$  for which the result of [14] can easily be extended).

It was also observed by Kerov and Vershik (see [6]) that, conversely to the previous construction where we obtain an harmonic function on  $\mathcal{G}$  defined from  $\mathcal{H}$  (thus from the conditioning of  $\mathcal{W}$ ), each harmonic function on a multiplicative graph  $\mathcal{G}$  completely determines a Markov chain on the vertices of  $\mathcal{G}$ . In the case of the Young lattice, they showed that these harmonic functions have nice expressions in terms of generalized Schur functions.

It was established in [16] that the usual (one-dimensional) Pitman transform is almost surely invertible on infinite trajectories (i.e. reversible on a space of trajectories of probability 1). It is then a natural question to ask whether its generalized version  $\mathcal{P}$  shares the same invertibility property. Observe that in the case of the defining representation of  $\mathfrak{sl}_{n+1}$  (or  $\mathfrak{sl}_\infty$ ), the generalized Pitman transform can be expressed in terms of a Robinson-Schensted-Knuth (RSK) type correspondence. Such an invertibility property was obtained by O'Connell in [14] (usual RSK related to ordinary Schur functions) and very recently extended by Sniady [17] to the generalized version of RSK used by Kerov and Vershik (related to the generalized Schur functions). Our result shows that this invertibility property survives beyond type  $A$  and for random paths constructed from any irreducible representation.

In what follows, we first prove that the probability distributions  $p$  on  $B(\kappa)$  we introduced in [7], [8] and [9] are precisely all the possible distributions yielding central distributions on  $B(\kappa)^{\otimes \ell}$ . We believe this will make the restriction we did in these papers more natural. We also establish a law of large numbers and a central limit theorem for the Markov process  $\mathcal{H}$ . Here we need our assumption that  $\mathfrak{g}$  is finite-dimensional since in this case  $\mathcal{P}$  has a particular simple expression as a composition of (ordinary) Pitman transforms. Then we determine the harmonic functions on the multiplicative graph  $\mathcal{G}$  for which the associated Markov chain verifies a law of large numbers. We establish in fact that these Markov chains are exactly the processes  $H$  defined in [7] and have simple expressions in terms of the Weyl characters of  $\mathfrak{g}$ . This can be regarded as an analogue of the result of Kerov and Vershik determining the harmonic functions on the Young lattice. Finally, we prove that the generalized Pitman transform  $\mathcal{P}$  is almost surely invertible and explain how the inverse  $\mathcal{P}^{-1}$  can be computed. Here we will extend the approach developed by Sniady in [17] for the generalized RSK to our context. In particular,  $\mathcal{P}^{-1}$  is defined by using the dynamical system on the trajectories that remain in  $\mathcal{C}$  defined as the shift of the first elementary path of the trajectory (i.e. its deletion) composed with the transform  $\mathcal{P}$ . Nevertheless, we cannot use the stabilization property of Jeu de Taquin trajectories which is central in [17] because it is only

relevant for the combinatorics of tableaux. Instead, we prove a stabilization phenomenon for the composition of the Pitman transform with a convenient involution on crystal defined from the Lusztig involution.

The paper is organized as follows. In Section 2, we recall some background on continuous time Markov processes. Section 3 is a recollection of results on representation theory of Lie algebras and the Littelmann path model. We state in Section 4 the main results of [9] and prove that the probability distributions  $p$  introduced in [7] are in fact the only possible yielding central measures on trajectories. The law of large numbers and the central limit theorem for  $\mathcal{H}$  are established in Section 5. We study the harmonic functions of the graphs  $\mathcal{G}$  in Section 6. In Section 7 we show that the spaces of trajectories for  $\mathcal{W}$  and  $\mathcal{H}$  both have the structure of dynamical systems coming from the shift operation. We then prove that these dynamical systems are intertwined by  $\mathcal{P}$ . Finally, we establish the existence of  $\mathcal{P}^{-1}$  in Section 7 by using a natural involution on the crystals  $B(\kappa)^{\otimes \ell}$ , a stabilization property of the transformation  $\mathcal{P}$  and the previous dynamical system on the trajectories that remain in  $\mathcal{C}$ .

**MSC classification:** 05E05, 05E10, 60G50, 60J10, 60J22.

## 2. RANDOM PATHS

**2.1. Background on Markov chains.** Consider a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and a countable set  $M$ . A sequence  $Y = (Y_\ell)_{\ell \geq 0}$  of random variables defined on  $\Omega$  with values in  $M$  is a *Markov chain* when

$$\mathbb{P}(Y_{\ell+1} = \mu_{\ell+1} \mid Y_\ell = \mu_\ell, \dots, Y_0 = \mu_0) = \mathbb{P}(Y_{\ell+1} = \mu_{\ell+1} \mid Y_\ell = \mu_\ell)$$

for any  $\ell \geq 0$  and any  $\mu_0, \dots, \mu_\ell, \mu_{\ell+1} \in M$ . The Markov chains considered in the sequel will also be assumed time homogeneous, that is  $\mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu) = \mathbb{P}(Y_\ell = \lambda \mid Y_{\ell-1} = \mu)$  for any  $\ell \geq 1$  and  $\mu, \lambda \in M$ . For all  $\mu, \lambda$  in  $M$ , the transition probability from  $\mu$  to  $\lambda$  is then defined by

$$\Pi(\mu, \lambda) = \mathbb{P}(Y_{\ell+1} = \lambda \mid Y_\ell = \mu)$$

and we refer to  $\Pi$  as the transition matrix of the Markov chain  $Y$ . The distribution of  $Y_0$  is called the initial distribution of the chain  $Y$ .

A *continuous time Markov process*  $\mathcal{Y} = (\mathcal{Y}(t))_{t \geq 0}$  on  $(\Omega, \mathcal{F}, \mathbb{P})$  with values in  $\mathbb{R}^n$  is a measurable family of random variables defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  such that, for any integer  $k \geq 1$  and any  $0 \leq t_1 < \dots < t_{k+1}$  the conditional distribution<sup>(1)</sup> of  $\mathcal{Y}(t_{k+1})$  given  $(\mathcal{Y}(t_1), \dots, \mathcal{Y}(t_k))$  is equal to the conditional distribution of  $\mathcal{Y}(t_{k+1})$  given  $\mathcal{Y}(t_k)$ ; in other words, for almost all  $(y_1, \dots, y_k)$  with respect to the distribution of the random vector  $(\mathcal{Y}(t_1), \dots, \mathcal{Y}(t_k))$  and for all Borelian set  $B \subset \mathbb{R}^n$

$$\mathbb{P}(\mathcal{Y}(t_{k+1}) \in B \mid \mathcal{Y}(t_1) = y_1, \dots, \mathcal{Y}(t_k) = y_k) = \mathbb{P}(\mathcal{Y}(t_{k+1}) \in B \mid \mathcal{Y}(t_k) = y_k).$$

We refer to the book [3], chapter 3, for a description of such processes.

<sup>1</sup>Let us recall briefly the definition of the conditional distribution of a random variable given another one. Let  $X$  and  $Y$  be random variables defined on some probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with values respectively in  $\mathbb{R}^n$  and  $\mathbb{R}^m$ ,  $n, m \geq 1$ . Denote by  $\mu_X$  the distribution of  $X$ , it is a probability measure on  $\mathbb{R}^n$ . The conditional distribution of  $Y$  given  $X$  is defined by the following ‘‘disintegration’’ formula: for any Borelian sets  $A \subset \mathbb{R}^n$  and  $B \subset \mathbb{R}^m$

$$\mathbb{P}\left((X \in A) \cap (Y \in B)\right) = \int_A \mathbb{P}(Y \in B \mid X = x) d\mu_X(x).$$

Notice that the function  $x \mapsto \mathbb{P}(Y \in B \mid X = x)$  is a Radon-Nicodym derivative with respect to  $\mu_X$  and is thus just defined modulo the measure  $\mu_X$ . The measure  $B \mapsto \mathbb{P}(Y \in B \mid X = x)$  is called the **conditional distribution of  $Y$  given  $X = x$** .

From now on, we consider a  $\mathbb{R}^n$ -valued Markov process  $(\mathcal{Y}(t))_{t \geq 0}$  defined on  $(\Omega, \mathcal{F}, \mathbb{P})$  and we assume the following conditions:

- (i)  $M \subset \mathbb{R}^n$
- (ii) for any integer  $\ell \geq 0$

$$(1) \quad Y_\ell := \mathcal{Y}(\ell) \in M \quad \mathbb{P}\text{-almost surely.}$$

It readily follows that the sequence  $Y = (Y_\ell)_{\ell \geq 0}$  is a  $M$ -valued Markov chain.

- (iii) for any integer  $\ell \geq 0$ , the conditional distribution of  $(\mathcal{Y}(t))_{t \geq \ell}$  given  $Y_\ell$  is equal to the one of  $(\mathcal{Y}(t))_{t \geq 0}$  given  $Y_0$ ; in other words, for any Borel set  $B \subset (\mathbb{R}^n)^{\otimes [0, +\infty[}$  and any  $\lambda \in M$ , one gets

$$\mathbb{P}((\mathcal{Y}(t))_{t \geq \ell} \in B \mid Y_\ell = \lambda) = \mathbb{P}((\mathcal{Y}(t))_{t \geq 0} \in B \mid Y_0 = \lambda).$$

In the following, we will assume that the initial distribution of the Markov process  $(\mathcal{Y}(t))_{t \geq 0}$  has full support, i.e.  $\mathbb{P}(\mathcal{Y}(0) = \lambda) > 0$  for any  $\lambda \in M$ .

**2.2. Elementary random paths.** Consider a  $\mathbb{Z}$ -lattice  $P \subset \mathbb{R}^n$  with rank  $n$ . An *elementary Littelmann path* is a piecewise continuous linear map  $\pi : [0, 1] \rightarrow P_{\mathbb{R}}$  such that  $\pi(0) = 0$  and  $\pi(1) \in P$ . Two paths which coincide up to reparametrization are considered as identical.

The set  $\mathcal{F}$  of continuous functions from  $[0, 1]$  to  $\mathbb{R}^n$  is equipped with the norm  $\|\cdot\|_\infty$  of uniform convergence : for any  $\pi \in \mathcal{F}$ , on has  $\|\pi\|_\infty := \sup_{t \in [0, 1]} \|\pi(t)\|$  where  $\|\cdot\|$  denotes the euclidean norm on  $P \subset \mathbb{R}^n$ . Let  $B$  be a *finite set of elementary paths* and fix a probability distribution  $p = (p_\pi)_{\pi \in B}$  on  $B$  such that  $p_\pi > 0$  for any  $\pi \in B$ . Let  $X$  be a random variable with values in  $B$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  and with distribution  $p$  (in other words  $\mathbb{P}(X = \pi) = p_\pi$  for any  $\pi \in B$ ). The variable  $X$  admits a moment of order 1 defined by

$$m := \mathbb{E}(X) = \sum_{\pi \in B} p_\pi \pi.$$

The concatenation  $\pi_1 * \pi_2$  of two elementary paths  $\pi_1$  and  $\pi_2$  is defined by

$$\pi_1 * \pi_2(t) = \begin{cases} \pi_1(2t) & \text{for } t \in [0, \frac{1}{2}], \\ \pi_1(1) + \pi_2(2t - 1) & \text{for } t \in [\frac{1}{2}, 1]. \end{cases}$$

In the sequel,  $\mathcal{C}$  is a closed convex cone in  $P \subset \mathbb{R}^n$ .

Let  $B$  be a set of elementary paths and  $(X_\ell)_{\ell \geq 1}$  a sequence of i.i.d. random variables with same law as  $X$  where  $X$  is the random variable with values in  $B$  introduced just above. We define a random process  $\mathcal{W}$  as follows: for any  $\ell \in \mathbb{Z}_{>0}$  and  $t \in [\ell, \ell + 1]$

$$\mathcal{W}(t) := X_1(1) + X_2(1) + \cdots + X_{\ell-1}(1) + X_\ell(t - \ell).$$

The sequence of random variables  $W = (W_\ell)_{\ell \in \mathbb{Z}_{\geq 0}} := (\mathcal{W}(\ell))_{\ell \geq 0}$  is a random walk with set of increments  $I := \{\pi(1) \mid \pi \in B\}$ .

### 3. LITTELMANN PATHS

**3.1. Background on representation theory of Lie algebras.** Let  $\mathfrak{g}$  be a simple finite-dimensional Lie algebra over  $\mathbb{C}$  of rank  $n$  and  $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{h} \oplus \mathfrak{g}_-$  a triangular decomposition. We shall follow the notation and convention of [2]. According to the Cartan-Killing classification,  $\mathfrak{g}$  is characterized (up to isomorphism) by its root system  $R$ . This root system is determined by the previous triangular decomposition and realized in the euclidean space  $\mathbb{R}^n$ . We denote by  $\Delta_+ = \{\alpha_i \mid i \in I\}$  the set of simple roots of  $\mathfrak{g}$ , by  $R_+$  the (finite) set of positive roots. We then have  $n = \text{card}(\Delta_+)$  and  $R = R_+ \cup R_-$  with  $R_- = -R_+$ . The root lattice of  $\mathfrak{g}$  is the integral lattice  $Q = \bigoplus_{i=1}^n \mathbb{Z}\alpha_i$ . Write  $\omega_i, i = 1, \dots, n$  for the fundamental weights associated to  $\mathfrak{g}$ . The

weight lattice associated to  $\mathfrak{g}$  is the integral lattice  $P = \bigoplus_{i=1}^n \mathbb{Z}\omega_i$ . It can be regarded as an integral sublattice of  $\mathfrak{h}_{\mathbb{R}}^*$  (the real form of the dual  $\mathfrak{h}^*$  of  $\mathfrak{h}$ ). We have  $\dim(P) = \dim(Q) = n$  and  $Q \subset P$ .

The cone of dominant weights for  $\mathfrak{g}$  is obtained by considering the positive integral linear combinations of the fundamental weights, that is  $P_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\omega_i$ . The corresponding open Weyl chamber is the cone  $\overset{\circ}{C} = \bigoplus_{i=1}^n \mathbb{R}_{>0}\omega_i$ . We also introduce its closure  $C = \bigoplus_{i=1}^n \mathbb{R}_{\geq 0}\omega_i$ . In type  $A$ , we shall use the weight lattice of  $\mathfrak{gl}_n$  rather than that of  $\mathfrak{sl}_n$  for simplicity. We also introduce the Weyl group  $W$  of  $\mathfrak{g}$  which is the group generated by the orthogonal reflections  $s_i$  through the hyperplanes perpendicular to the simple root  $\alpha_i, i = 1, \dots, n$ . Each  $w \in W$  may be decomposed as a product of the  $s_i, i = 1, \dots, n$ . All the minimal length decompositions of  $w$  have the same length  $l(w)$ . The group  $W$  contains a unique element  $w_0$  of maximal length  $l(w_0)$  equal to the number of positive roots of  $\mathfrak{g}$ , this  $w_0$  is an involution and if  $s_{i_1} \cdots s_{i_r}$  is a minimal length decomposition of  $w_0$ , we have

$$(2) \quad R_+ = \{\alpha_{i_1}, s_{i_1} \cdots s_{i_a}(\alpha_{i_{a+1}}) \text{ with } a = 1, \dots, r-1\}.$$

**Example 3.1.** *The root system of  $\mathfrak{g} = \mathfrak{sp}_4$  has rank 2. In the standard basis  $(e_1, e_2)$  of the euclidean space  $\mathbb{R}^2$ , we have  $\omega_1 = (1, 0)$  and  $\omega_2 = (1, 1)$ . So  $P = \mathbb{Z}^2$  and  $C = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq x_2 \geq 0\}$ . The simple roots are  $\alpha_1 = e_1 - e_2$  and  $\alpha_2 = 2e_2$ . We also have  $R_+ = \{\alpha_1, \alpha_2, \alpha_1 + \alpha_2, 2\alpha_1 + \alpha_2\}$ . The Weyl group  $W$  is the octahedral group with 8 elements. It acts on  $\mathbb{R}^2$  by permuting the coordinates of the vectors and flipping their sign. More precisely, for any  $\beta = (\beta_1, \beta_2) \in \mathbb{R}^2$ , we have  $s_1(\beta) = (\beta_2, \beta_1)$  and  $s_2(\beta) = (\beta_1, -\beta_2)$ . The longest element is  $w_0 = -id = s_1s_2s_1s_2$ . One easily verifies we indeed have*

$$R_+ = \{\alpha_1, s_1s_2s_1(\alpha_2) = \alpha_2, s_1s_2(\alpha_1) = \alpha_1 + \alpha_2, s_1(\alpha_2) = 2\alpha_1 + \alpha_2\}.$$

We now summarize some properties of the action of  $W$  on the weight lattice  $P$ . For any weight  $\beta$ , the orbit  $W \cdot \beta$  of  $\beta$  under the action of  $W$  intersects  $P_+$  in a unique point. We define a partial order on  $P$  by setting  $\mu \leq \lambda$  if  $\lambda - \mu$  belongs to  $Q_+ = \bigoplus_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i$ .

Let  $U(\mathfrak{g})$  be the enveloping algebra associated to  $\mathfrak{g}$ . Each finite dimensional  $\mathfrak{g}$  (or  $U(\mathfrak{g})$ )-module  $M$  admits a decomposition in weight spaces  $M = \bigoplus_{\mu \in P} M_\mu$  where

$$M_\mu := \{v \in M \mid h(v) = \mu(h)v \text{ for any } h \in \mathfrak{h} \text{ and some } \mu(h) \in \mathbb{C}\}.$$

This means that the action of any  $h \in \mathfrak{h}$  on the weight space  $M_\mu$  is diagonal with eigenvalue  $\mu(h)$ . In particular,  $(M \oplus M')_\mu = M_\mu \oplus M'_\mu$ . The Weyl group  $W$  acts on the weights of  $M$  and for any  $\sigma \in W$ , we have  $\dim M_\mu = \dim M_{\sigma \cdot \mu}$ . For any  $\gamma \in P$ , let  $e^\gamma$  be the generator of the group algebra  $\mathbb{C}[P]$  associated to  $\gamma$ . By definition, we have  $e^\gamma e^{\gamma'} = e^{\gamma+\gamma'}$  for any  $\gamma, \gamma' \in P$  and the group  $W$  acts on  $\mathbb{C}[P]$  as follows:  $w(e^\gamma) = e^{w(\gamma)}$  for any  $w \in W$  and any  $\gamma \in P$ .

The character of  $M$  is the Laurent polynomial in  $\mathbb{C}[P]$   $\text{char}(M)(x) := \sum_{\mu \in P} \dim(M_\mu)e^\mu$  where  $\dim(M_\mu)$  is the dimension of the weight space  $M_\mu$ .

The irreducible finite dimensional representations of  $\mathfrak{g}$  are labelled by the dominant weights. For each dominant weight  $\lambda \in P_+$ , let  $V(\lambda)$  be the irreducible representation of  $\mathfrak{g}$  associated to  $\lambda$ . The category  $\mathcal{C}$  of finite dimensional representations of  $\mathfrak{g}$  over  $\mathbb{C}$  is semisimple: each module decomposes into irreducible components. The category  $\mathcal{C}$  is equivariant to the (semisimple) category of finite dimensional  $U(\mathfrak{g})$ -modules (over  $\mathbb{C}$ ). Roughly speaking, this means that the representation theory of  $\mathfrak{g}$  is essentially identical to the representation theory of the *associative* algebra  $U(\mathfrak{g})$ . Any finite dimensional  $U(\mathfrak{g})$ -module  $M$  decomposes as a direct sum of irreducible  $M = \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus m_{M,\lambda}}$  where  $m_{M,\lambda}$  is the multiplicity of  $V(\lambda)$  in  $M$ . Here we slightly abuse the notation and also denote by  $V(\lambda)$  the irreducible f.d.  $U(\mathfrak{g})$ -module associated to  $\lambda$ .



When  $M = V(\lambda)$  is irreducible, we set  $s_\lambda := \text{char}(M) = \sum_{\mu \in P} K_{\lambda, \mu} e^\mu$  with  $\dim(M_\mu) = K_{\lambda, \mu}$ . Then  $K_{\lambda, \mu} \neq 0$  only if  $\mu \leq \lambda$ . Recall also that the characters can be computed from the Weyl character formula but we do not need this approach in the sequel.

Given  $\kappa, \mu$  in  $P_+$  and a nonnegative integer  $\ell$ , we define the tensor multiplicities  $f_{\lambda/\mu, \kappa}^\ell$  by

$$(3) \quad V(\mu) \otimes V(\kappa)^{\otimes \ell} \simeq \bigoplus_{\lambda \in P_+} V(\lambda)^{\oplus f_{\lambda/\mu, \kappa}^\ell}.$$

For  $\mu = 0$ , we set  $f_{\lambda, \kappa}^\ell = f_{\lambda/0, \kappa}^\ell$ . When there is no risk of confusion, we write simply  $f_{\lambda/\mu}^\ell$  (resp.  $f_\lambda^\ell$ ) instead of  $f_{\lambda/\mu, \kappa}^\ell$  (resp.  $f_{\lambda, \kappa}^\ell$ ). We also define the multiplicities  $m_{\mu, \kappa}^\lambda$  by

$$(4) \quad V(\mu) \otimes V(\kappa) \simeq \bigoplus_{\mu \rightsquigarrow \lambda} V(\lambda)^{\oplus m_{\mu, \kappa}^\lambda}$$

where the notation  $\mu \rightsquigarrow \lambda$  means that  $\lambda \in P_+$  and  $V(\lambda)$  appears as an irreducible component of  $V(\mu) \otimes V(\kappa)$ . We have in particular  $m_{\mu, \kappa}^\lambda = f_{\lambda/\mu, \kappa}^1$ .

**3.2. Littelmann path model.** We now give a brief overview of the Littelmann path model. We refer to [11], [12], [13] and [5] for examples and a detailed exposition. Consider a Lie algebra  $\mathfrak{g}$  and its root system realized in the euclidean space  $P_{\mathbb{R}} = \mathbb{R}^n$ . We fix a scalar product  $\langle \cdot, \cdot \rangle$  on  $P_{\mathbb{R}}$  invariant under  $W$ . For any root  $\alpha$ , we set  $\alpha^\vee = \frac{2\alpha}{\langle \alpha, \alpha \rangle}$ . We define the notion of elementary continuous piecewise linear paths in  $P_{\mathbb{R}}$  as we did in § 2.2. Let  $\mathcal{L}$  be the set of elementary paths having *only rational turning points* (i.e. whose inflexion points have rational coordinates) and ending in  $P$  i.e. such that  $\pi(1) \in P$ . We then define the weight of the path  $\pi$  by  $\text{wt}(\pi) = \pi(1)$ .

Littelmann associated to each simple root  $\alpha_i, i = 1, \dots, n$ , some root operators  $\tilde{e}_i$  and  $\tilde{f}_i$  acting on  $\mathcal{L} \cup \{\mathbf{0}\}$ . We do not need their complete definition in the sequel and refer to the above mentioned papers for a complete review. Recall nevertheless that roots operators  $\tilde{e}_i$  and  $\tilde{f}_i$  essentially act on a path  $\eta$  by applying the symmetry  $s_\alpha$  on parts of  $\eta$ . It therefore preserve the length of the paths since the elements of  $W$  are isometries. Also if  $\tilde{f}_i(\eta) = \eta' \neq \mathbf{0}$ , we have

$$(5) \quad \tilde{e}_i(\eta') = \eta \text{ and } \text{wt}(\tilde{f}_i(\eta)) = \text{wt}(\eta) - \alpha_i.$$

By drawing an arrow  $\eta \xrightarrow{i} \eta'$  between the two paths  $\eta, \eta'$  of  $\mathcal{L}$  as soon as  $\tilde{f}_i(\eta) = \eta'$  (or equivalently  $\eta = \tilde{e}_i(\eta')$ ), we obtain a Kashiwara crystal graph with set of vertices  $\mathcal{L}$ . By abuse of notation, we yet denote it by  $\mathcal{L}$  which so becomes a colored oriented graph. For any  $\eta \in \mathcal{L}$ , we denote by  $B(\eta)$  the connected component of  $\eta$  i.e. the subgraph of  $\mathcal{L}$  generated by  $\eta$  by applying operators  $\tilde{e}_i$  and  $\tilde{f}_i, i = 1, \dots, n$ . For any path  $\eta \in \mathcal{L}$  and  $i = 1, \dots, n$ , set  $\varepsilon_i(\eta) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{e}_i^k(\eta) = \mathbf{0}\}$  and  $\varphi_i(\eta) = \max\{k \in \mathbb{Z}_{\geq 0} \mid \tilde{f}_i^k(\eta) = \mathbf{0}\}$ .

The set  $\mathcal{L}_{\min \mathbb{Z}}$  of *integral paths* is the set of paths  $\eta$  such that  $m_\eta(i) = \min_{t \in [0, 1]} \{\langle \eta(t), \alpha_i^\vee \rangle\}$  belongs to  $\mathbb{Z}$  for any  $i = 1, \dots, n$ . We also recall that we have

$$\mathcal{C} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha_i^\vee \rangle \geq 0\} \text{ and } \mathring{\mathcal{C}} = \{x \in \mathfrak{h}_{\mathbb{R}}^* \mid \langle x, \alpha_i^\vee \rangle > 0\}.$$

Any path  $\eta$  such that  $\text{Im } \eta \subset \mathcal{C}$  verifies  $m_\eta(i) = 0$  so belongs to  $\mathcal{L}_{\min \mathbb{Z}}$ . One gets the

**Proposition 3.2.** *Let  $\eta$  and  $\pi$  two paths in  $\mathcal{L}_{\min \mathbb{Z}}$ . Then*

- (i) *the concatenation  $\pi * \eta$  belongs to  $\mathcal{L}_{\min \mathbb{Z}}$ ,*
- (ii) *for any  $i = 1, \dots, n$  we have*

$$(6) \quad \tilde{e}_i(\eta * \pi) = \begin{cases} \eta * \tilde{e}_i(\pi) & \text{if } \varepsilon_i(\pi) > \varphi_i(\eta) \\ \tilde{e}_i(\eta) * \pi & \text{otherwise,} \end{cases} \quad \text{and} \quad \tilde{f}_i(\eta * \pi) = \begin{cases} \tilde{f}_i(\eta) * \pi & \text{if } \varphi_i(\eta) > \varepsilon_i(\pi) \\ \eta * \tilde{f}_i(\pi) & \text{otherwise.} \end{cases}$$

- In particular,  $\tilde{\varepsilon}_i(\eta * \pi) = \mathbf{0}$  if and only if  $\tilde{\varepsilon}_i(\eta) = \mathbf{0}$  and  $\varepsilon_i(\pi) \leq \varphi_i(\eta)$  for any  $i = 1, \dots, n$ .
- (iii)  $\tilde{\varepsilon}_i(\eta) = \mathbf{0}$  for any  $i = 1, \dots, n$  if and only if  $\text{Im } \eta$  is contained in  $\mathcal{C}$ .

The following theorem summarizes crucial results by Littelmann (see [11], [12] and [13]).

**Theorem 3.3.** Consider  $\lambda, \mu$  and  $\kappa$  dominant weights and choose arbitrarily elementary paths  $\eta_\lambda, \eta_\mu$  and  $\eta_\kappa$  in  $\mathcal{L}$  such that  $\text{Im } \eta_\lambda \subset \mathcal{C}$ ,  $\text{Im } \eta_\mu \subset \mathcal{C}$  and  $\text{Im } \eta_\kappa \subset \mathcal{C}$  and joining respectively 0 to  $\lambda$ , 0 to  $\mu$  and 0 to  $\kappa$ .

- (i) We have  $B(\eta_\lambda) := \{\tilde{f}_{i_1} \cdots \tilde{f}_{i_k} \eta_\lambda \mid k \in \mathbb{Z}_{\geq 0} \text{ and } 1 \leq i_1, \dots, i_k \leq n\} \setminus \{\mathbf{0}\}$ .  
In particular  $\text{wt}(\eta) - \text{wt}(\eta_\lambda) \in Q_+$  for any  $\eta \in B(\eta_\lambda)$ .
- (ii) All the paths in  $B(\eta_\lambda)$  have the same length than  $\eta_\lambda$ .
- (iii) The paths on  $B(\eta_\lambda)$  belong to  $\mathcal{L}_{\min \mathbb{Z}}$ .
- (iv) If  $\eta'_\lambda$  is another elementary path from 0 to  $\lambda$  such that  $\text{Im } \eta'_\lambda$  is contained in  $\mathcal{C}$ , then  $B(\eta_\lambda)$  and  $B(\eta'_\lambda)$  are isomorphic as oriented graphs i.e. there exists a bijection  $\theta : B(\eta_\lambda) \rightarrow B(\eta'_\lambda)$  which commutes with the action of the operators  $\tilde{\varepsilon}_i$  and  $\tilde{f}_i$ ,  $i = 1, \dots, n$ .
- (v) We have

$$(7) \quad s_\lambda = \sum_{\eta \in B(\eta_\lambda)} e^{\eta(1)}.$$

- (vi) For any  $b \in B(\eta_\lambda)$  we have  $\text{wt}(b) = \sum_{i=1}^n (\varphi_i(b) - \varepsilon_i(b)) \omega_i$ .
- (vii) For any  $i = 1, \dots, n$  and any  $b \in B(\eta_\lambda)$ , let  $s_i(b)$  be the unique path in  $B(\eta_\lambda)$  such that

$$\varphi_i(s_i(b)) = \varepsilon_i(b) \text{ and } \varepsilon_i(s_i(b)) = \varphi_i(b)$$

(in other words,  $s_i$  acts on each  $i$ -chain  $\mathcal{C}_i$  as the symmetry with respect to the center of  $\mathcal{C}_i$ ). The actions of the  $s_i$ 's extend to an action of  $\mathbb{W}$  on  $\mathcal{L}$  which stabilizes  $B(\eta_\lambda)$ . In particular, for any  $w \in \mathbb{W}$  and any  $b \in B(\eta_\lambda)$ , we have  $w(b) \in B(\eta_\lambda)$  and  $\text{wt}(w(b)) = w(\text{wt}(b))$ .

- (viii) Given any integer  $\ell \geq 0$ , set

$$(8) \quad B(\eta_\mu) * B(\eta_\kappa)^{* \ell} = \{\pi = \eta * \eta_1 * \cdots * \eta_\ell \in \mathcal{L} \mid \eta \in B(\eta_\mu) \text{ and } \eta_k \in B(\eta_\kappa) \text{ for any } k = 1, \dots, \ell\}.$$

The graph  $B(\eta_\mu) * B(\eta_\kappa)^{* \ell}$  is contained in  $\mathcal{L}_{\min \mathbb{Z}}$ .

- (ix) The multiplicity  $m_{\mu, \kappa}^\lambda$  defined in (4) is equal to the number of paths of the form  $\mu * \eta$  with  $\eta \in B(\eta_\kappa)$  contained in  $\mathcal{C}$ .
- (x) The multiplicity  $f_{\lambda/\mu}^\ell$  defined in (3) is equal to cardinality of the set

$$H_{\lambda/\mu}^\ell := \{\pi \in B(\eta_\mu) * B(\eta_\kappa)^{* \ell} \mid \tilde{\varepsilon}_i(\pi) = 0 \text{ for any } i = 1, \dots, n \text{ and } \pi(1) = \lambda\}.$$

Each path  $\pi = \eta * \eta_1 * \cdots * \eta_\ell \in H_{\lambda/\mu}^\ell$  verifies  $\text{Im } \pi \subset \mathcal{C}$  and  $\eta = \eta_\mu$ .

**Remarks 3.4.** (i) Combining (5) with assertions (i) and (v) of Theorem 3.3, one may check that the function  $e^{-\lambda} s_\lambda$  is in fact a polynomial in the variables  $T_i = e^{-\alpha_i}$ , namely

$$(9) \quad s_\lambda = e^\lambda S_\lambda(T_1, \dots, T_n)$$

where  $S_\lambda \in \mathbb{C}[X_1, \dots, X_n]$ .

- (ii) Using assertion (i) of Theorem 3.3, we obtain  $m_{\mu, \kappa}^\lambda \neq 0$  only if  $\mu + \kappa - \lambda \in Q_+$ . Similarly, when  $f_{\lambda/\mu}^{\kappa, \ell} \neq 0$  one necessarily has  $\mu + \ell\kappa - \lambda \in Q_+$ .



## 4. RANDOM PATHS FROM LITTELMANN PATHS

In this Section we recall some results of [9]. We also introduce the notion of central probability distribution on elementary Littelmann paths and show these distributions coincide with those used in the seminal works [1], [14] and also in our previous papers [7], [8],[9].

**4.1. Central probability measure on trajectories.** Consider  $\kappa \in P_+$  and a path  $\pi_\kappa \in \mathcal{L}$  from 0 to  $\kappa$  such that  $\text{Im } \pi_\kappa$  is contained in  $\mathcal{C}$ . Let  $B(\pi_\kappa)$  be the connected component of  $\mathcal{L}$  containing  $\pi_\kappa$ . Assume that  $\{\pi_1, \dots, \pi_\ell\}$  is a family of elementary paths in  $B(\pi_\kappa)$ ; the path  $\pi_1 \otimes \dots \otimes \pi_\ell$  of length  $\ell$  is defined by: for all  $k \in \{1, \dots, \ell - 1\}$  and  $t \in [k, k + 1]$

$$(10) \quad \pi_1 \otimes \dots \otimes \pi_\ell(t) = \pi_1(1) + \dots + \pi_k(1) + \pi_{k+1}(t - k).$$

Let  $B(\pi_\kappa)^{\otimes \ell}$  be the set of paths of the form  $b = \pi_1 \otimes \dots \otimes \pi_\ell$  where  $\pi_1, \dots, \pi_\ell$  are elementary paths in  $B(\pi_\kappa)$ ; there exists a bijection  $\Delta$  between  $B(\pi_\kappa)^{\otimes \ell}$  and the set  $B(\pi_\kappa)^{* \ell}$  of paths in  $\mathcal{L}$  obtained by concatenations of  $\ell$  paths of  $B(\pi_\kappa)$ :

$$(11) \quad \Delta : \begin{cases} B(\pi_\kappa)^{\otimes \ell} & \longrightarrow & B(\pi_\kappa)^{* \ell} \\ \pi_1 \otimes \dots \otimes \pi_\ell & \longmapsto & \pi_1 * \dots * \pi_\ell \end{cases} .$$

In fact  $\pi_1 \otimes \dots \otimes \pi_\ell$  and  $\pi_1 * \dots * \pi_\ell$  coincide up to a reparametrization and we define the weight of  $b = \pi_1 \otimes \dots \otimes \pi_\ell$  setting

$$\text{wt}(b) := \text{wt}(\pi_1) + \dots + \text{wt}(\pi_\ell) = \pi_1(1) + \dots + \pi_\ell(1).$$

Consider  $p$  a probability distribution on  $B(\pi_\kappa)$  such that  $p_\pi > 0$  for any  $\pi \in B(\pi_\kappa)$ . For any integer  $\ell \geq 1$ , we endow  $B(\pi_\kappa)^{\otimes \ell}$  with the product density  $p^{\otimes \ell}$ . That is we set  $p_\pi^{\otimes \ell} = p_{\pi_1} \times \dots \times p_{\pi_\ell}$  for any  $\pi = \pi_1 \otimes \dots \otimes \pi_\ell \in B(\pi_\kappa)^{\otimes \ell}$ . Here, we follow the classical construction of a Bernoulli process. Write  $\Pi_\ell : B(\pi_\kappa)^{\otimes \ell} \rightarrow B(\pi_\kappa)^{\otimes \ell - 1}$  the projection defined by  $\Pi_\ell(\pi_1 \otimes \dots \otimes \pi_{\ell-1} \otimes \pi_\ell) = \pi_1 \otimes \dots \otimes \pi_{\ell-1}$ ; the sequence  $(B(\pi_\kappa)^{\otimes \ell}, \Pi_\ell, p^{\otimes \ell})_{\ell \geq 1}$  is a projective system of probability spaces. We denote by  $\Omega = (B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}}, p^{\otimes \mathbb{Z}_{\geq 0}})$  its projective limit. The elements of  $B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}}$  are infinite sequences  $\omega = (\pi_\ell)_{\ell \geq 1}$  we call trajectories. By a slight abuse of notation, we will write  $\Pi_\ell(\omega) = \pi_1 \otimes \dots \otimes \pi_\ell$ . We also write  $\mathbb{P} = p^{\otimes \mathbb{Z}_{\geq 0}}$  for short. For any  $b \in B(\pi_\kappa)^{\otimes \ell}$ , we denote by  $U_b = \{\omega \in \Omega \mid \Pi_\ell(\omega) = b\}$  the cylinder defined by  $\pi$  in  $\Omega$ .

**Definition 4.1.** *The probability distribution  $\mathbb{P} = p^{\otimes \mathbb{Z}_{\geq 0}}$  is central on  $\Omega$  when for any  $\ell \geq 1$  and any vertices  $b$  and  $b'$  in  $B(\pi_\kappa)^{\otimes \ell}$  such that  $\text{wt}(b) = \text{wt}(b')$  we have  $\mathbb{P}(U_b) = \mathbb{P}(U_{b'})$ .*

**Remark 4.2.** *The probability distribution  $\mathbb{P}$  is central when for any integer  $\ell \geq 1$  and any vertices  $b, b'$  in  $B(\pi_\kappa)^{\otimes \ell}$  such that  $\text{wt}(b) = \text{wt}(b')$ , we have  $p_b^{\otimes \ell} = p_{b'}^{\otimes \ell}$ . We indeed have  $U_b = b \otimes \Omega$  and  $U_{b'} = b' \otimes \Omega$ . Hence  $\mathbb{P}(U_b) = p_b^{\otimes \ell}$  and  $\mathbb{P}(U_{b'}) = p_{b'}^{\otimes \ell}$ .*

The following proposition shows that  $\mathbb{P}$  can only be central when the probability distribution  $p$  on  $B(\pi_\kappa)$  is compatible with the graduation of  $B(\pi_\kappa)$  by the set of simple roots. This justifies the restriction we did in [7] and [9] on the probability distributions we have considered on  $B(\pi_\kappa)$ . This restriction will also be relevant in the remaining of this paper.

**Proposition 4.3.** *The following assertions are equivalent*

- (i) *The probability distribution  $\mathbb{P}$  is central.*
- (ii) *There exists an  $n$ -tuple  $\tau = (\tau_1, \dots, \tau_n) \in ]0, +\infty[^n$  such that for each arrow  $\pi \xrightarrow{i} \pi'$  in  $B(\pi_\kappa)$ , we have the relation  $p_{\pi'} = p_\pi \times \tau_i$ .*

*Proof.* Assume probability distribution  $\mathbb{P}$  is central. For any path  $\pi \in B(\pi_\kappa)$ , we define the depth  $d(\pi)$  as the number of simple roots appearing in the decomposition of  $\kappa - \text{wt}(\pi)$  on the basis of simple roots (see assertion (i) of Theorem 3.3). This is also the length of any path joining  $\pi_\kappa$  to  $\pi$  in the crystal graph  $B(\pi_\kappa)$ . We have to prove that  $\frac{p_\pi}{p_\pi}$  is a constant depending only on  $i$  as soon as we have an arrow  $\pi \xrightarrow{i} \pi'$  in  $B(\pi_\kappa)$ . For any  $k \geq 1$ , we set  $B(\pi_\kappa)_k = \{\pi \in B(\pi_\kappa) \mid d(\pi) \leq k\}$ . We will proceed by induction and prove that  $\frac{p_{\pi'}}{p_\pi}$  is a constant depending only on  $i$  as soon as there is an arrow  $\pi \xrightarrow{i} \pi'$  in  $B(\pi_\kappa)_k$ . This is clearly true in  $B(\pi_\kappa)_1$  since there is at most one arrow  $i$  starting from  $\pi_\kappa$ . Assume, the property is true in  $B(\pi_\kappa)_k$  with  $k \geq 1$ . Consider  $\pi'$  in  $B(\pi_\kappa)_{k+1}$  and an arrow  $\pi \xrightarrow{i} \pi'$  in  $B(\pi_\kappa)_{k+1}$ . We must have  $\pi \in B(\pi_\kappa)_k$ . If  $B(\pi_\kappa)_k$  does not contains any arrow  $\xrightarrow{i}$ , there is nothing to verify. So assume there is at least an arrow  $\pi_1 \xrightarrow{i} \pi_2$  in  $B(\pi_\kappa)_k$ . In  $B(\pi_\kappa)^{\otimes 2}$ , we have  $\text{wt}(\pi_1 \otimes \pi') = \text{wt}(\pi_1) + \text{wt}(\pi) - \alpha_i$  since  $\text{wt}(\pi') = \text{wt}(\pi) - \alpha_i$ . Similarly, we have  $\text{wt}(\pi_2 \otimes \pi) = \text{wt}(\pi_1) - \alpha_i + \text{wt}(\pi)$  since  $\text{wt}(\pi_2) = \text{wt}(\pi_1) - \alpha_i$ . Thus  $\text{wt}(\pi_1 \otimes \pi') = \text{wt}(\pi_2 \otimes \pi)$ . Since  $\mathbb{P}$  is central, we deduce from the above remark the equality  $p^{\otimes 2}(\pi_1 \otimes \pi') = p^{\otimes 2}(\pi_2 \otimes \pi)$ . This yields  $p_{\pi_1} p_{\pi'} = p_{\pi_2} p_\pi$ . Hence  $\frac{p_{\pi'}}{p_\pi} = \frac{p_{\pi_2}}{p_{\pi_1}}$ . So by our induction hypothesis,  $\frac{p_{\pi'}}{p_\pi}$  is equal to a constant which only depends on  $i$ .

Conversely, assume there exists an  $n$ -tuple  $\tau = (\tau_1, \dots, \tau_n) \in ]0, +\infty[^n$  such that for each arrow  $\pi \xrightarrow{i} \pi'$  in  $B(\pi_\kappa)$ , we have the relation  $p_{\pi'} = p_\pi \times \tau_i$ . Consider vertices  $b, b'$  in  $B(\pi_\kappa)^{\otimes \ell}$  such that  $\text{wt}(b) = \text{wt}(b')$ . Since  $b$  and  $b'$  have the same weight, we derive from (5) that the paths from  $\pi_\kappa$  to  $b$  and the paths from  $\pi_\kappa$  to  $b'$  contain the same number (says  $a_i$ ) of arrows  $\xrightarrow{i}$  for any  $i = 1, \dots, n$ . We therefore have  $p_b = p_{b'} = p_{\pi_\kappa} \tau_1^{a_1} \dots \tau_n^{a_n}$  and the probability distribution  $\mathbb{P}$  is central.  $\square$

**4.2. Central probability distribution on elementary paths.** In the remaining of the paper, we fix the  $n$ -tuple  $\tau = (\tau_1, \dots, \tau_n) \in ]0, +\infty[^n$  and assume that  $\mathbb{P}$  is a central distribution on  $\Omega$  defined from  $\tau$  (in the sense of Definition 4.1. For any  $u = u_1 \alpha_1 + \dots + u_n \alpha_n \in Q$ , we set  $\tau^u = \tau_1^{u_1} \dots \tau_n^{u_n}$ . Since the root and weight lattices have both rank  $n$ , any weight  $\beta \in P$  also decomposes on the form  $\beta = \beta_1 \alpha_1 + \dots + \beta_n \alpha_n$  with possibly non integral coordinates  $\beta_i$ . The transition matrix between the bases  $\{\omega_i, i = 1, \dots, n\}$  and  $\{\alpha_i, i = 1, \dots, n\}$  (regarded as bases of  $P_{\mathbb{R}}$ ) being the Cartan matrix of  $\mathfrak{g}$  whose entries are integers, the coordinates  $\beta_i$  are rational. We will also set  $\tau^\beta = \tau_1^{\beta_1} \dots \tau_n^{\beta_n}$ .

Let  $\pi \in B(\pi_\kappa)$ : by assertion (i) of Theorem 3.3, one gets

$$\pi(1) = \text{wt}(\pi) = \kappa - \sum_{i=1}^n u_i(\pi) \alpha_i$$

where  $u_i(\pi) \in \mathbb{Z}_{\geq 0}$  for any  $i = 1, \dots, n$ . We define  $S_\kappa(\tau) := S_\kappa(\tau_1, \dots, \tau_n) = \sum_{\pi \in B(\pi_\kappa)} \tau^{\kappa - \text{wt}(\pi)}$ .

**Definition 4.4.** We define the probability distribution  $p = (p_\pi)_{\pi \in B(\pi_\kappa)}$  on  $B(\pi_\kappa)$  associated to  $\tau$  by setting  $p_\pi = \frac{\tau^{\kappa - \text{wt}(\pi)}}{S_\kappa(\tau)}$ .

**Remark 4.5.** By assertion (iii) of Theorem 3.3, for  $\pi'_\kappa$  another elementary path from 0 to  $\kappa$  such that  $\text{Im } \pi'_\kappa$  is contained in  $\mathcal{C}$ , there exists an isomorphism  $\Theta$  between the crystals  $B(\pi_\kappa)$  and  $B(\pi'_\kappa)$ . For  $p'$  the central probability distribution defined from  $\tau$  on  $B(\pi'_\kappa)$ , one gets  $p_\pi = p'_{\Theta(\pi)}$  for any  $\pi \in B(\pi_\kappa)$ . Therefore, the probability distributions we use on the graph  $B(\pi_\kappa)$  are invariant by crystal isomorphisms and also the probabilistic results we will establish in the paper.

The following proposition gathers results of [7] (Lemma 7.2.1) and [9] (Proposition 5.4). Recall that  $m = \sum_{\pi \in B(\pi_\kappa)} p_\pi \pi$ . We set  $\bar{m} = m(1)$ .

**Proposition 4.6.**

- (i) We have  $\bar{m} \in \mathring{\mathcal{C}}$  if and only if  $\tau_i \in ]0, 1[$  for any  $i = 1, \dots, n$ .
- (ii) Denote by  $L$  the common length of the paths in  $B(\pi_\kappa)$ . Then, the length of  $m$  is less or equal to  $L$ .

Set  $\mathcal{M}_\kappa = \{\bar{m} \mid \tau = (\tau_1, \dots, \tau_n) \in ]0, +\infty[ \}$  be the set of all vectors  $\bar{m}$  obtained from the central distributions defined on  $B(\pi_\kappa)$ . Observe that  $\mathcal{M}_\kappa$  only depends on  $\kappa$  and not of the choice of the highest path  $\pi_\kappa$ . This is the set of possible mean obtained from central probability distributions defined on  $B(\pi_\kappa)$ . We will also need the set

$$(12) \quad \mathcal{D}_\kappa = \mathcal{M}_\kappa \cap \mathring{\mathcal{C}} = \{\bar{m} \in \mathcal{M}_\kappa \mid \tau_i \in ]0, 1[, i = 1, \dots, n\}$$

of drifts in  $\mathring{\mathcal{C}}$ .

**Example 4.7.** We resume Example 3.1 and consider the Lie algebra  $\mathfrak{g} = \mathfrak{sp}_4$  of type  $C_2$  for which  $P = \mathbb{Z}^2$  and  $\mathcal{C} = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq x_2 \geq 0\}$ .

For  $\kappa = \omega_1$  and  $\pi_\kappa$  the line between 0 and  $\varepsilon_1$ , we get  $B(\pi_\kappa) = \{\pi_1, \pi_2, \pi_{\bar{2}}, \pi_{\bar{1}}\}$  where each  $\pi_a$  is the line between 0 and  $\varepsilon_a$  (with the convention  $\varepsilon_{\bar{2}} = -\varepsilon_2$  and  $\varepsilon_{\bar{1}} = -\varepsilon_1$ ). The underlying crystal graph is

$$\pi_1 \xrightarrow{1} \pi_2 \xrightarrow{2} \pi_{\bar{2}} \xrightarrow{1} \pi_{\bar{1}}.$$

For  $(\tau_1, \tau_2) \in ]0, +\infty[^2$ , we obtain the probability distribution on  $B(\pi_\kappa)$

$$p_{\pi_1} = \frac{1}{1 + \tau_1 + \tau_1\tau_2 + \tau_1^2\tau_2}, p_{\pi_2} = \frac{\tau_1}{1 + \tau_1 + \tau_1\tau_2 + \tau_1^2\tau_2},$$

$$p_{\pi_{\bar{2}}} = \frac{\tau_1\tau_2}{1 + \tau_1 + \tau_1\tau_2 + \tau_1^2\tau_2} \text{ and } p_{\pi_{\bar{1}}} = \frac{\tau_1^2\tau_2}{1 + \tau_1 + \tau_1\tau_2 + \tau_1^2\tau_2}.$$

So we have

$$\bar{m} = \frac{1}{1 + \tau_1 + \tau_1\tau_2 + \tau_1^2\tau_2} ((1 - \tau_1^2\tau_2)\varepsilon_1 + (\tau_1 - \tau_1\tau_2)\varepsilon_2).$$

When the pair  $(\tau_1, \tau_2)$  runs over  $]0, 1[^2$ , one verifies by a direct computation that  $\mathcal{D}_\kappa$  coincide with the interior of the triangle with vertices  $0, \varepsilon_1, \varepsilon_2$ .

**Remark 4.8.** In the previous example, it is easy to show by a direct calculation that the adherence  $\overline{\mathcal{M}_\kappa}$  of  $\mathcal{M}_\kappa$  is the convex hull of the weight  $\{\pm\varepsilon_1, \pm\varepsilon_2\}$  of the representation  $V(\omega_1)$  considered (i.e. the interior of the square with vertices  $\{\pm\varepsilon_1, \pm\varepsilon_2\}$ ). In general, one can show that  $\overline{\mathcal{M}_\kappa}$  is contained in the convex hull of the weights of  $V(\kappa)$ . The problem of determining, for any dominant weight  $\kappa$ , whether or not both sets coincide seems to us interesting and not immediate.

**4.3. Random paths of arbitrary length.** With the previous convention, the product probability measure  $p^{\otimes \ell}$  on  $B(\pi_\kappa)^{\otimes \ell}$  satisfies

$$(13) \quad p^{\otimes \ell}(\pi_1 \otimes \dots \otimes \pi_\ell) = p(\pi_1) \dots p(\pi_\ell) = \frac{\tau^{\ell\kappa - (\pi_1(1) + \dots + \pi_\ell(1))}}{S_\kappa(\tau)^\ell} = \frac{\tau^{\ell\kappa - \text{wt}(b)}}{S_\kappa(\tau)^\ell}.$$

Let  $(X_\ell)_{\ell \geq 1}$  be a sequence of i.i.d. random variables with values in  $B(\pi_\kappa)$  and law  $p = (p_\pi)_{\pi \in B(\pi_\kappa)}$ ; for any  $\ell \geq 1$  we thus gets

$$(14) \quad \mathbb{P}(X_\ell = \pi) = p_\pi \text{ for any } \pi \in B(\pi_\kappa).$$

Consider  $\mu \in P$ . The random path  $\mathcal{W}$  starting at  $\mu$  is defined from the probability space  $\Omega$  with values in  $P_{\mathbb{R}}$  by

$$\mathcal{W}(t) := \Pi_{\ell}(\mathcal{W})(t) = \mu + (X_1 \otimes \cdots \otimes X_{\ell-1} \otimes X_{\ell})(t) \text{ for } t \in [\ell - 1, \ell].$$

For any integer  $\ell \geq 1$ , we set  $W_{\ell} = \mathcal{W}(\ell)$ . The sequence  $W = (W_{\ell})_{\ell \geq 1}$  defines a random walk starting at  $W_0 = \mu$  whose increments are the weights of the representation  $V(\kappa)$ . The following proposition was established in [9] (see Proposition 4.6).

**Proposition 4.9.**

(i) For any  $\beta, \eta \in P$ , one gets

$$\mathbb{P}(W_{\ell+1} = \beta \mid W_{\ell} = \eta) = K_{\kappa, \beta - \eta} \frac{\tau^{\kappa + \eta - \beta}}{S_{\kappa}(\tau)}.$$

(ii) Consider  $\lambda, \mu \in P^+$  we have

$$\mathbb{P}(W_{\ell} = \lambda, W_0 = \mu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [0, \ell]) = f_{\lambda/\mu}^{\ell} \frac{\tau^{\ell\kappa + \mu - \lambda}}{S_{\kappa}(\tau)^{\ell}}.$$

In particular

$$\mathbb{P}(W_{\ell+1} = \lambda, W_{\ell} = \mu, \mathcal{W}(t) \in \mathcal{C} \text{ for any } t \in [\ell, \ell + 1]) = m_{\mu, \kappa}^{\lambda} \frac{\tau^{\kappa + \mu - \lambda}}{S_{\kappa}(\tau)}.$$

**4.4. The generalized Pitman transform.** By assertion (viii) of Theorem 3.3, we know that  $B(\pi_{\kappa})^{\otimes \ell}$  is contained in  $\mathcal{L}_{\min \mathbb{Z}}$ . Therefore, if we consider a path  $\eta \in B(\pi_{\kappa})^{\otimes \ell}$ , its connected component  $B(\eta)$  is contained in  $\mathcal{L}_{\min \mathbb{Z}}$ . Now, if  $\eta^h \in B(b)$  is such that  $\tilde{e}_i(\eta^h) = 0$  for any  $i = 1, \dots, n$ , we should have  $\text{Im} \eta^h \subset \mathcal{C}$  by assertion (iii) of Proposition 3.2. Assertion (iii) of Theorem 3.3 thus implies that  $\eta^h$  is the unique path in  $B(\eta) = B(\eta^h)$  such that  $\tilde{e}_i(\eta^h) = 0$  for any  $i = 1, \dots, n$ . This permits to define the *generalized Pitman transform* on  $B(\pi_{\kappa})^{\otimes \ell}$  as the map  $\mathcal{P}$  which associates to any  $\eta \in B(\pi_{\kappa})^{\otimes \ell}$  the unique path  $\mathcal{P}(\eta) \in B(\eta)$  such that  $\tilde{e}_i(\mathcal{P}(\eta)) = 0$  for any  $i = 1, \dots, n$ . By definition, we have  $\text{Im} \mathcal{P}(\eta) \subset \mathcal{C}$  and  $\mathcal{P}(\eta)(\ell) \in P_+$ .

As observed in [1] the path transformation  $\mathcal{P}$  can be made more explicit (recall we have assumed that  $\mathfrak{g}$  is finite-dimensional). Consider a simple reflection  $\alpha$ . The Pitman transformation  $\mathcal{P}_{\alpha} : B(\pi_{\kappa})^{\otimes \ell} \rightarrow B(\pi_{\kappa})^{\otimes \ell}$  associated to  $\alpha$  is defined by

$$\mathcal{P}_{\alpha}(\eta)(t) = \eta(t) - 2 \inf_{s \in [0, t]} \langle \eta(s), \frac{\alpha}{\|\alpha\|^2} \rangle \alpha = \eta(t) - \inf_{s \in [0, t]} \langle \eta(s), \alpha^{\vee} \rangle \alpha$$

for any  $\eta \in B(\pi_{\kappa})^{\otimes \ell}$  and any  $t \in [0, \ell]$ . Let  $w_0$  be the maximal length element of  $W$  and fix a decomposition  $w_0 = s_{i_1} \cdots s_{i_r}$  of  $w_0$  as a product a reflections associated to simple roots.

**Proposition 4.10** ([1]). For any path  $\eta \in B(\pi_{\kappa})^{\otimes \ell}$ , we have

$$(15) \quad \mathcal{P}(\eta) = \mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta).$$

**Remarks 4.11.** (i) Since  $\mathcal{P}(\eta)$  corresponds to the highest weight vertex of the crystal  $B(\eta)$ , we have  $\mathcal{P}^2(\eta) = \mathcal{P}(\eta)$ .

(ii) One easily verifies that each transformation  $\mathcal{P}_{\alpha}$  is continuous for the topology of uniform convergence on the space of continuous maps from  $[0, \ell]$  to  $\mathbb{R}$ . Hence  $\mathcal{P}$  is also continuous for this topology.

(iii) It is also proved in [1] that  $\mathcal{P}$  does not depend on the reduced decomposition of  $w_0$  used in (15).

- (iv) Assume  $\eta \in B(\eta_\lambda) \subset B(\pi_\kappa)^{\otimes \ell}$  where  $\eta_\lambda$  is the highest weight path of  $B(\eta_\lambda)$ . Consider  $\eta^\lambda = w_0(\eta_\lambda)$  the lowest weight path in  $B(\eta_\lambda)$ . In this particular case, one can show that  $\mathcal{P}_{i_{a+1}} \cdots \mathcal{P}_{i_r}(\eta^\lambda) = s_{i_{a+1}} \cdots s_{i_r}(\eta^\lambda)$  for any  $a = 1, \dots, r-1$ . Moreover each path  $s_{i_{a+1}} \cdots s_{i_r}(\eta^\lambda)$  is extremal (i.e.  $s_{i_{a+1}} \cdots s_{i_r}(\eta^\lambda)$  is located at one end of any  $i$ -chain which contains it). So the successive applications of the transforms  $\mathcal{P}_\alpha$  used to define  $\mathcal{P}$  on lowest weight paths essentially reduce to the action of the generators  $s_\alpha$  of  $\mathbb{W}$ .

Let  $\mathcal{W}$  be the random path of § 4.3. We define the random process  $\mathcal{H}$  setting

$$(16) \quad \mathcal{H}(t) = \mathcal{P}(\Pi_\ell(\mathcal{W}))(t) \text{ for any } t \in [\ell - 1, \ell].$$

We will write for short  $\mathcal{H} = \mathcal{P}(\mathcal{W})$  in the sequel. For any  $\ell \geq 1$ , we set  $H_\ell := \mathcal{H}(\ell)$ . The following Theorem was established in [9].

**Theorem 4.12.** (i) *The random sequence  $H := (H_\ell)_{\ell \geq 1}$  is a Markov chain with transition matrix*

$$(17) \quad \Pi(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\kappa(\tau)S_\mu(\tau)} \tau^{\kappa+\mu-\lambda} m_{\mu, \kappa}^\lambda$$

where  $\lambda, \mu \in P_+$ .

- (ii) *Assume  $\eta \in B(\pi_\kappa)^{\otimes \ell}$  is a highest weight path of weight  $\lambda$ . Then*

$$\mathbb{P}(\mathcal{W}_\ell = \eta) = \frac{\tau^{\ell\kappa-\lambda} S_\lambda(\tau)}{S_\kappa(\tau)^\ell}$$

We shall also need the asymptotic behavior of the tensor product multiplicities established in [9].

**Theorem 4.13.** *Assume  $\bar{m} \in \mathcal{D}_\kappa$  (see (12)). For any  $\mu \in P$  and any sequence of dominant weights of the form  $\lambda^{(\ell)} = \ell\bar{m} + o(\ell)$ , we have*

- (i)  $\lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}-\gamma}^\ell}{f_{\lambda^{(\ell)}}^\ell} = \tau^{-\gamma}$  for any  $\gamma \in P$ .
- (ii)  $\lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^\ell}{f_{\lambda^{(\ell)}}^\ell} = \tau^{-\mu} S_\mu(\tau)$ .

**Corollary 4.14.** *Under the assumptions of the previous theorem, we also have*

$$\lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}}^{\ell-\ell_0}}{f_{\lambda^{(\ell)}}^\ell} = \frac{1}{\tau^{-\ell_0\kappa} S_\kappa^{\ell_0}(\tau)}$$

for any nonnegative integer  $\ell_0$ .

*Proof.* We first consider the case where  $\ell_0 = 1$ . By definition of the tensor product multiplicities in (3) we have  $s_\kappa^\ell = \sum_{\lambda \in P_+} f_\lambda^\ell s_\lambda$  but also  $s_\kappa^\ell = s_\kappa \times s_\kappa^{\ell-1} = \sum_{\lambda \in P_+} f_{\lambda/\kappa}^{\ell-1} s_\lambda$ . Therefore  $f_\lambda^\ell = f_{\lambda/\kappa}^{\ell-1}$  for any  $\ell \geq 1$  and any  $\lambda \in P_+$ . We get

$$(18) \quad \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}}^{\ell-1}}{f_{\lambda^{(\ell)}}^\ell} = \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}}^{\ell-1}}{f_{\lambda^{(\ell)}/\kappa}^{\ell-1}} = \frac{1}{\tau^{-\kappa} S_\kappa(\tau)}$$

by assertion (ii) of Theorem. Now observe that for any  $\ell_0 \geq 1$  we have

$$\frac{f_{\lambda^{(\ell)}}^{\ell-\ell_0}}{f_{\lambda^{(\ell)}}^\ell} = \frac{f_{\lambda^{(\ell)}}^{\ell-\ell_0}}{f_{\lambda^{(\ell)}-\ell_0+1}^{\ell-\ell_0+1}} \times \cdots \times \frac{f_{\lambda^{(\ell)}}^{\ell-1}}{f_{\lambda^{(\ell)}}^\ell}.$$

By using (18) each component of the previous product tends to  $\frac{1}{\tau^{-\kappa} S_\kappa(\tau)}$  when  $\ell$  tends to infinity which gives the desired limit.  $\square$

The previous theorem also implies that the drift  $\bar{m}$  determines the probability distribution on  $B(\pi_\kappa)$ . More precisely, consider  $p$  and  $p'$  two probability distributions defined on  $B(\pi_\kappa)$  from  $\tau \in ]0, 1[^n$  and  $\tau' \in ]0, 1[^n$ , respectively. Set  $m = \sum_{\pi \in B(\pi_\kappa)} p_\pi \pi$  and  $m' = \sum_{\pi \in B(\pi_\kappa)} p'_\pi \pi$ .

**Proposition 4.15.** *We have  $\bar{m} = \bar{m}'$  if and only if  $\tau = \tau'$ . Therefore, the map which associates to any  $\tau \in ]0, 1[^n$  the drift  $\bar{m} \in \mathcal{D}_\kappa$  is a one-to-one correspondence.*

*Proof.* Assume  $\bar{m} = \bar{m}'$ . By applying assertion (i) of Theorem 4.13, we get  $\tau^\gamma = (\tau')^\gamma$  for any  $\gamma \in P$ . Consider  $i \in \{1, \dots, n\}$ . For  $\gamma = \alpha_i$ , we obtain  $\tau_i = \tau'_i$ . Therefore  $\tau = \tau'$ .  $\square$

## 5. SOME LIMIT THEOREMS FOR THE PITMAN PROCESS

**5.1. The law of large numbers and the central limit theorem for  $\mathcal{W}$ .** We start by establishing two classical limit theorems for  $\mathcal{W}$ , deduced from the law of large numbers and the central limit theorem for the random walk  $W = (W_\ell)_{\ell \geq 1} = (X_1 + \dots + X_\ell)_{\ell \geq 1}$ . Recall that  $m = \sum_{\pi \in B(\pi_\kappa)} p_\pi \pi$  and  $\bar{m} = m(1)$ . Write  $m^{\otimes \infty}$  for the random path such that

$$m^{\otimes \infty}(t) = \ell \bar{m} + m(t - \ell) \text{ for any } t > 0$$

where  $\ell = \lfloor t \rfloor$ .

Let  $\Gamma = (\Gamma_{i,j})_{1 \leq i,j \leq n} = {}^t X_\ell \cdot X_\ell$  be the common covariance matrix of each random variable  $X_\ell$ .

**Theorem 5.1.** *Let  $\mathcal{W}$  be a random path defined on  $(B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}}, p^{\otimes \mathbb{Z}_{\geq 0}})$  with drift path  $m$ . Then, we have*

$$\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{t \in [0, \ell]} \|\mathcal{W}(t) - m^{\otimes \infty}(t)\| = 0 \text{ almost surely.}$$

Furthermore, the family of random variables  $\left( \frac{\mathcal{W}(t) - m^{\otimes \infty}(t)}{\sqrt{t}} \right)_{t > 0}$  converges in law as  $t \rightarrow +\infty$

towards a centered Gaussian law  $\mathcal{N}(0, \Gamma)$ .

More precisely, setting  $\mathcal{W}^{(\ell)}(t) := \frac{\mathcal{W}(\ell t) - m^{\otimes \infty}(\ell t)}{\sqrt{\ell}}$  for any  $0 \leq t \leq 1$  and  $\ell \geq 1$ , the sequence of random processes  $(\mathcal{W}^{(\ell)}(t))_{\ell \geq 1}$  converges to a  $n$ -dimensional Brownian motion  $(B_\Gamma(t))_{0 \leq t \leq 1}$  with covariance matrix  $\Gamma$ .

*Proof.* Fix  $\ell \geq 1$  and observe that

$$\sup_{t \in [0, \ell]} \|\mathcal{W}(t) - m^{\otimes \infty}(t)\| = \sup_{0 \leq k \leq \ell-1} \sup_{t \in [k, k+1]} \|\mathcal{W}(t) - k\bar{m} - m(t - k)\|.$$

For any  $0 \leq k \leq \ell$  and  $t \in [k, k+1]$ , we have  $\mathcal{W}(t) = W_k + X_{k+1}(t - k)$  so that

$$(19) \quad \mathcal{W}(t) - m^{\otimes \infty}(t) = W_k - k\bar{m} + (X_{k+1}(t - k) - m(t - k))$$

with  $\sup_{t \in [k, k+1]} \|X_{k+1}(t - k) - m(t - k)\| = \sup_{t \in [0, 1]} \|X_{k+1}(t) - m(t)\| \leq +2L$ , since both paths in  $B(\kappa)$  and  $m$  have length  $L$ , by Proposition 4.6. It readily follows that

$$(20) \quad \sup_{t \in [0, \ell]} \|\mathcal{W}(t) - m^{\otimes \infty}(t)\| \leq \sup_{0 \leq k \leq \ell} \|W_k - k\bar{m}\| + 2L.$$



By the law of large number for the random walk  $W = (W_k)_{k \geq 1}$ , one gets  $\lim_{k \rightarrow +\infty} \frac{1}{k} \|W_k - k\bar{m}\| = 0$  almost surely; this readily implies  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{1 \leq k \leq \ell} \|W_k - k\bar{m}\| = 0$  almost surely, so that, by (20),

$$\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{t \in [0, \ell]} \|\mathcal{W}(t) - m^{\otimes \infty}(t)\| = 0 \text{ almost surely.}$$

Let us now prove the central limit theorem; for any  $t > 0$ , set  $k_t := \lfloor t \rfloor$  and notice that decomposition (19) yields to

$$(21) \quad \frac{\mathcal{W}(t) - m^{\otimes \infty}(t)}{\sqrt{t}} = \sqrt{\frac{k_t}{t}} \times \frac{W_{k_t} - k_t \bar{m}}{\sqrt{k_t}} + \frac{X_{k_t+1}(t - k_t) - m(t - k_t)}{\sqrt{t}}$$

By the central limit theorem in  $\mathbb{R}^n$ , one knows that the sequence of random variables  $\left( \frac{W_k - k\bar{m}}{\sqrt{k}} \right)_{k \geq 1}$  converges in law as  $k \rightarrow +\infty$  towards a centered Gaussian law  $\mathcal{N}(0, \Gamma)$ ; on the other hand, one gets  $\lim_{t \rightarrow +\infty} \sqrt{\frac{k_t}{t}} = 1$  and  $\limsup_{t \rightarrow +\infty} \left\| \frac{X_{k_t+1}(t - k_t) - m(t - k_t)}{\sqrt{t}} \right\| \leq \limsup_{t \rightarrow +\infty} \frac{2L}{\sqrt{t}} = 0$ , so one may conclude using Slutsky theorem.

The convergence of the sequence  $(W_\ell(t))_{\ell \geq 1}$  towards a Brownian motion goes along the same line. Consider first for any  $\ell \geq 1$  the piecewise affine and continuous process  $(W^{(\ell)}(t))_{1 \leq t \leq 1}$  with value  $W_k/\sqrt{\ell}$  at point  $t = k/\ell$  for any  $1 \leq k \leq \ell$  and defined by:

$$\text{for all } \ell \geq 1 \text{ and } 0 \leq t \leq 1 \quad W^{(\ell)}(t) := \frac{W_{\lfloor \ell t \rfloor} + (\ell t - \lfloor \ell t \rfloor) X_{\lfloor \ell t \rfloor + 1}(1) - \ell t \bar{m}}{\sqrt{\ell}}.$$

By the invariance principle, one knows that the sequence  $(W^{(\ell)})_{\ell \geq 1}$  converges in distribution towards the Brownian motion  $B_\Gamma$ . One concludes noticing that

$$\begin{aligned} \left\| \mathcal{W}^{(\ell)}(t) - W^{(\ell)}(t) \right\| &= \left\| \frac{W_{\lfloor \ell t \rfloor} - \lfloor \ell t \rfloor \bar{m}}{\sqrt{\ell}} + \frac{X_{\lfloor \ell t \rfloor + 1}(\ell t - \lfloor \ell t \rfloor) - m(\ell t - \lfloor \ell t \rfloor)}{\sqrt{\ell}} \right. \\ &\quad \left. - \frac{W_{\lfloor \ell t \rfloor} - \lfloor \ell t \rfloor \bar{m} + (\ell t - \lfloor \ell t \rfloor) (X_{\lfloor \ell t \rfloor + 1}(1) - \bar{m})}{\sqrt{\ell}} \right\| \\ &\leq \frac{2}{\sqrt{\ell}} (\|m\|_\infty + \|X_{\lfloor \ell t \rfloor + 1}\|_\infty) \end{aligned}$$

and using again Slutsky lemma.  $\square$

**5.2. The law of large numbers and the central limit theorem for  $\mathcal{H}$ .** To prove the law of large numbers and the central limit theorem for  $\mathcal{H}$ , we need the two following preparatory lemmas. Consider a simple root  $\alpha$  and a trajectory  $\eta \in \Omega$  such that  $\frac{1}{\ell} \langle \eta(\ell), \alpha^\vee \rangle$  converges to a positive limit when  $\ell$  tends to infinity.

**Lemma 5.2.** *There exists a nonnegative integer  $\ell_0$  such that for any  $\ell \geq \ell_0$*

$$\inf_{t \in [0, \ell]} \langle \eta(t), \alpha^\vee \rangle = \inf_{t \in [0, \ell_0]} \langle \eta(t), \alpha^\vee \rangle.$$

*Proof.* Since  $\frac{1}{\ell} \langle \eta(\ell), \alpha^\vee \rangle$  converges to a positive limit, we have in particular that  $\lim_{\ell \rightarrow +\infty} \langle \eta(\ell), \alpha^\vee \rangle = +\infty$ . Consider  $t > 0$  and set  $\ell = \lfloor t \rfloor$ . We can write by definition of  $\eta \in \Omega$ ,  $\eta(t) = \eta(\ell) + \pi(t - \ell)$  where  $\pi$  is a path of  $B(\pi_\kappa)$ . So  $\langle \eta(t), \alpha^\vee \rangle = \langle \eta(\ell), \alpha^\vee \rangle + \langle \pi(t - \ell), \alpha^\vee \rangle$ . Since  $\pi \in B(\pi_\kappa)$ , we have

$$\|\pi(t - \ell)\| \leq L$$

where  $L$  is the common length of the paths in  $B(\pi_\kappa)$ . So the possible values of  $\langle \pi(t - \ell), \alpha^\vee \rangle$  are bounded. Since  $\lim_{\ell \rightarrow +\infty} \langle \eta(\ell), \alpha^\vee \rangle = +\infty$ , we also get  $\lim_{t \rightarrow +\infty} \langle \eta(t), \alpha^\vee \rangle = +\infty$ . Recall that  $\eta(0) = 0$ . Therefore  $\inf_{t \in [0, \ell]} \langle \eta(t), \alpha^\vee \rangle \leq 0$ . Since  $\lim_{t \rightarrow +\infty} \langle \eta(t), \alpha^\vee \rangle = +\infty$  and the path  $\eta$  is continuous, there should exist an integer  $\ell_0$  such that  $\inf_{t \in [0, \ell_0]} \langle \eta(t), \alpha^\vee \rangle = \inf_{t \in [0, \ell_0]} \langle \eta(t), \alpha^\vee \rangle$  for any  $\ell \geq \ell_0$ .  $\square$

**Lemma 5.3.**

- (i) Consider a simple root  $\alpha$  and a trajectory  $\eta \in \Omega$  such that  $\frac{1}{\ell} \langle \eta(\ell), \alpha^\vee \rangle$  converges to a positive limit when  $\ell$  tends to infinity. We have for any simple root  $\alpha$

$$\sup_{t \in [0, +\infty[} \|\mathcal{P}_\alpha(\eta)(t) - \eta(t)\| < +\infty$$

in particular,  $\frac{1}{\ell} \langle \mathcal{P}_\alpha(\eta)(\ell), \alpha^\vee \rangle$  also converges to a positive limit.

- (ii) More generally, let  $\alpha_{i_1}, \dots, \alpha_{i_r}, r \geq 1$ , be simple roots of  $\mathfrak{g}$  and  $\eta$  a path in  $\Omega$  satisfying  $\lim_{t \rightarrow +\infty} \langle \eta(t), \alpha_{i_j}^\vee \rangle = +\infty$  for  $1 \leq j \leq r$ . One gets

$$\sup_{t \in [0, +\infty[} \|\mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - \eta(t)\| < +\infty.$$

*Proof.* (i) By definition of the transform  $\mathcal{P}_\alpha$ , we have  $\|\mathcal{P}_\alpha(\eta)(t) - \eta(t)\| = |\inf_{s \in [0, t]} \langle \eta(s), \alpha^\vee \rangle| \|\alpha^\vee\|$  for any  $t \geq 0$ . By the previous lemma, there exists an integer  $\ell_0$  such that for any  $t \geq \ell_0$ ,  $\|\mathcal{P}_\alpha(\eta)(t) - \eta(t)\| = |\inf_{s \in [0, t]} \langle \eta(s), \alpha^\vee \rangle| \|\alpha^\vee\| = |\inf_{s \in [0, \ell_0]} \langle \eta(s), \alpha^\vee \rangle| \|\alpha^\vee\|$ . Since the infimum  $\inf_{s \in [0, \ell_0]} \langle \eta(s), \alpha^\vee \rangle$  does not depend on  $\ell$ , we are done. Now  $\frac{1}{\ell} \langle \mathcal{P}_\alpha(\eta(\ell)), \alpha^\vee \rangle$  and  $\frac{1}{\ell} \langle \eta(\ell), \alpha^\vee \rangle$  admit the same limit.

- (ii) Consider  $a \in \{2, \dots, r-1\}$  and assume by induction that we have

$$(22) \quad \sup_{t \in [0, +\infty[} \|\mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - m^{\otimes \infty}(t)\| < +\infty.$$

We then deduce

$$(23) \quad \lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(\ell), \alpha_{i_{a-1}}^\vee \rangle = \langle \bar{m}, \alpha_{i_{a-1}}^\vee \rangle > 0.$$

This permits to apply Lemma 5.3 with  $\eta' = \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)$  and  $\alpha = \alpha_{i_{a-1}}$ . We get

$$\sup_{t \in [0, +\infty[} \left\| \mathcal{P}_{\alpha_{i_{a-1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - \mathcal{P}_{\alpha_{i_a}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) \right\| < +\infty.$$

By using (22), this gives

$$(24) \quad \sup_{t \in [0, +\infty[} \left\| \mathcal{P}_{\alpha_{i_{a-1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta)(t) - m^{\otimes \infty}(t) \right\| < +\infty.$$

We thus have proved by induction that (24) holds for any  $a = 2, \dots, r-1$ .  $\square$

**Theorem 5.4.** Let  $\mathcal{W}$  be a random path defined on  $\Omega = (B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}}, p^{\otimes \mathbb{Z}_{\geq 0}})$  with drift path  $m$  and let  $\mathcal{H} = \mathcal{P}(\mathcal{W})$  be its Pitman transform. Assume  $\bar{m} \in \mathcal{D}_\kappa$ . Then, we have

$$\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{t \in [0, \ell]} \|\mathcal{H}(t) - m^{\otimes \infty}(t)\| = 0 \text{ almost surely.}$$

Furthermore, the family of random variables  $\left( \frac{\mathcal{H}(t) - m^{\otimes \infty}(t)}{\sqrt{t}} \right)_{t > 0}$  converges in law as  $t \rightarrow +\infty$  towards a centered Gaussian law  $\mathcal{N}(0, \Gamma)$ .

*Proof.* Recall we have  $\mathcal{P} = \mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}$  by Proposition ([1]). Consequently, by Theorem 5.1 and Lemma 5.3, the random variable  $\mathcal{H} - \mathcal{W} = \mathcal{P}(\mathcal{W}) - \mathcal{W}$  is finite almost surely. It follows that

$$\limsup_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{t \in [0, \ell]} \left\| \mathcal{H}(t) - m^{\otimes \ell}(t) \right\| \leq \limsup_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{t \in [0, \ell]} \left\| \mathcal{W}(t) - m^{\otimes \ell}(t) \right\| + \limsup_{\ell \rightarrow +\infty} \frac{1}{\ell} \sup_{t \geq 0} \|\mathcal{H}(t) - \mathcal{W}(t)\| = 0$$

almost surely. To get the central limit theorem for the process  $\mathcal{H}(t)$ , we write similarly

$$\frac{\mathcal{H}(t) - m^{\otimes \ell}(t)}{\sqrt{t}} = \frac{\mathcal{W}(t) - m^{\otimes \ell}(t)}{\sqrt{t}} + \frac{\mathcal{H}(t) - \mathcal{W}(t)}{\sqrt{t}}.$$

By Theorem 5.1, the first term in this decomposition satisfies the central limit theorem; on the other hand the second one tends to 0 almost surely and one concludes using Slutsky theorem  $\square$

The following Lemma will be useful in Section 7. Consider  $\pi \in B(\pi_\kappa)$  and  $\eta \in \Omega$  such that  $\frac{1}{\ell} \langle \eta(\ell), \alpha_i^\vee \rangle$  converges to a positive limit for any positive root  $\alpha_i, i = 1, \dots, n$ . For any  $\ell$ , set  $\Pi_\ell(\eta) = \eta_\ell$  so that we have  $\eta_\ell \in B(\pi_\kappa)^{\otimes \ell}$ . Since  $\pi \in B(\pi_\kappa)$ , the path  $\eta_\ell \otimes \pi$  is defined on  $[0, \ell + 1]$ . More precisely, we have  $\eta_\ell \otimes \pi(t) = \eta_\ell(t)$  for  $t \in [0, \ell]$  and  $\eta_\ell \otimes \pi(t) = \eta_\ell(\ell) + \pi(t - \ell)$  for  $t \in [\ell, \ell + 1]$ .

**Lemma 5.5.** *With the previous notation, we have*

$$\mathcal{P}(\eta_\ell \otimes \pi) = \mathcal{P}(\eta_\ell) \otimes \pi$$

for  $\ell$  sufficiently large.

*Proof.* Recall that  $\mathcal{P} = \mathcal{P}_{\alpha_{i_1}} \cdots \mathcal{P}_{\alpha_{i_r}}$ . We prove by induction that for any  $s = 1, \dots, r$ , there exists a nonnegative integer  $\ell_s$  such that

$$\mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell \otimes \pi) = \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell) \otimes \pi$$

for any  $\ell > \ell_s$  and  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell), \alpha^\vee \rangle = +\infty$  for any simple root  $\alpha$ .

Since  $\lim_{\ell \rightarrow +\infty} \langle \eta_\ell \otimes \pi, \alpha_{i_1}^\vee \rangle = +\infty$ , there exists by Lemma 5.2 a nonnegative integer  $\ell_r$  such that for any  $\ell > \ell_r$   $\inf_{s \in [0, \ell]} \langle \eta_\ell \otimes \pi(s), \alpha^\vee \rangle = \inf_{s \in [0, \ell_r]} \langle \eta_\ell \otimes \pi(s), \alpha^\vee \rangle = \inf_{s \in [0, \ell_r]} \langle \eta_\ell(s), \alpha^\vee \rangle$ . Here the last equality follows from the equality  $\eta_\ell \otimes \pi(s) = \eta_\ell(s)$  for  $s \in [0, \ell_r]$ . Then for any  $t \in [0, \ell + 1]$ , we have

$$(25) \quad \mathcal{P}_{\alpha_{i_r}}(\eta_\ell \otimes \pi)(t) = \begin{cases} \eta_\ell(t) - \inf_{s \in [0, t]} \langle \eta_\ell(s), \alpha^\vee \rangle \alpha^\vee & \text{for } t \in [0, \ell_r[, \\ \eta_\ell(t) - \inf_{s \in [0, \ell_r]} \langle \eta_\ell(s), \alpha^\vee \rangle \alpha^\vee & \text{for } t \in [\ell_r, \ell[, \\ \eta_\ell(\ell) + \pi(\ell - t) - \inf_{s \in [0, \ell_r]} \langle \eta_\ell(s) \otimes \pi, \alpha^\vee \rangle \alpha^\vee & \text{for } t \in [\ell, \ell + 1]. \end{cases}$$

Since  $\inf_{s \in [0, \ell]} \langle \eta_\ell(s), \alpha^\vee \rangle = \inf_{s \in [0, \ell_r]} \langle \eta_\ell(s), \alpha^\vee \rangle$ , we can write for any  $t \in [0, \ell]$

$$(26) \quad \mathcal{P}_{\alpha_{i_r}}(\eta_\ell)(t) = \begin{cases} \eta_\ell(t) - \inf_{s \in [0, t]} \langle \eta_\ell(s), \alpha^\vee \rangle \alpha^\vee & \text{for } t \in [0, \ell_r[, \\ \eta_\ell(t) - \inf_{s \in [0, \ell_r]} \langle \eta_\ell(s), \alpha^\vee \rangle \alpha^\vee & \text{for } t \in [\ell_r, \ell]. \end{cases}$$

We then deduce from (25) that  $\mathcal{P}_{\alpha_{i_r}}(\eta_\ell \otimes \pi) = \mathcal{P}_{\alpha_{i_r}}(\eta_\ell) \otimes \pi$ . The equalities (26) also show that  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \mathcal{P}_{\alpha_{i_r}}(\eta_\ell), \alpha_i^\vee \rangle = +\infty$  for any simple root  $\alpha_i$ .

Now assume we have a nonnegative integer  $\ell_{s+1}$  such that

$$\mathcal{P}_{\alpha_{i_{s+1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell \otimes \pi) = \mathcal{P}_{\alpha_{i_{s+1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell) \otimes \pi$$

for any  $\ell > \ell_{s+1}$  and  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \mathcal{P}_{\alpha_{i_{s+1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell), \alpha_i^\vee \rangle = +\infty$  for any simple root  $\alpha_i$ . Set  $\eta'_\ell = \mathcal{P}_{\alpha_{i_{s+1}}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell)$ . We can then apply the previous arguments to  $\eta'_\ell$  (instead of  $\eta_\ell$ ) and  $\alpha_{i_s}$  (instead of  $\alpha_{i_r}$ ). We obtain a nonnegative integer  $\ell_s \geq \ell_{s+1}$  such that  $\mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell \otimes \pi) = \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell) \otimes \pi$

$\pi) = \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell) \otimes \pi$  for any  $\ell > \ell_s$  and  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \mathcal{P}_{\alpha_{i_s}} \cdots \mathcal{P}_{\alpha_{i_r}}(\eta_\ell), \alpha_i^\vee \rangle = +\infty$  for any simple root  $\alpha_i$ . This proves the desired property by induction, thus also the lemma by considering  $s = 1$ .  $\square$

## 6. HARMONIC FUNCTIONS ON MULTIPLICATIVE GRAPHS ASSOCIATED TO A CENTRAL MEASURE

Harmonic functions on the Young lattice are the key ingredients in the study of the asymptotic representation theory of the symmetric group. In fact, it was shown by Kerov and Vershik that these harmonic functions completely determine the asymptotic characters of the symmetric groups. We refer the reader to [6] for a detailed review. The Young lattice is an oriented graph with set of vertices the set of all partitions (each partition is conveniently identified its Young diagram). We have an arrow  $\lambda \rightarrow \Lambda$  between the partitions  $\lambda$  and  $\Lambda$  when  $\Lambda$  can be obtained by adding a box to  $\lambda$ . The Young lattice is an example of branching graph in the sense that its structure reflects the branching rules between the representations theory of the groups  $S_\ell$  and  $S_{\ell+1}$  with  $\ell > 0$ . One can also consider harmonic functions on other interesting graphs.

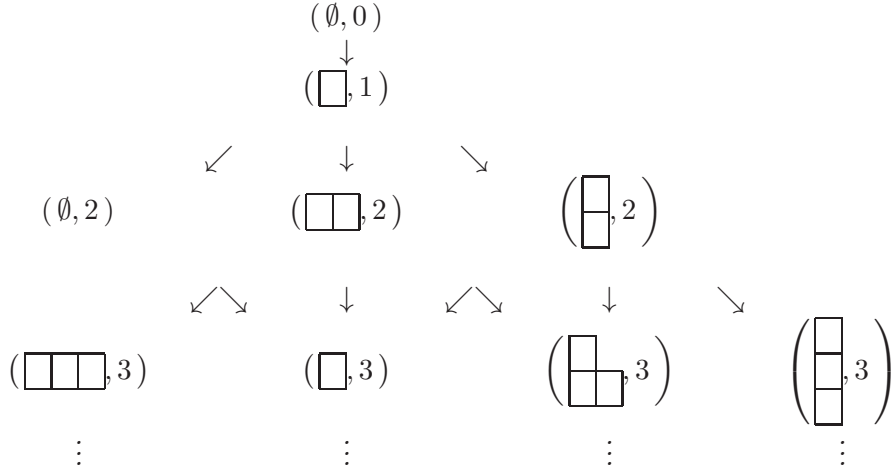
Here we focus on graphs defined from the weight lattice of  $\mathfrak{g}$ . These graphs depend on a fixed  $\kappa \in P_+$  and are multiplicative in the sense that a positive integer, equal to a tensor product multiplicity, is associated to each arrow. We call them the multiplicative tensor graphs. We are going to associate a Markov chain to each multiplicative tensor graph  $\mathcal{G}$ . The aim of this section is to determine the harmonic functions on  $\mathcal{G}$  when this associated Markov chain is assumed to have a drift. We will show this is equivalent to determine the central probability measure on the subset  $\Omega_{\mathcal{C}}$  containing all the trajectories which remains in  $\mathcal{C}$ . When  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  and  $\kappa = \omega_1$  (that is  $V(\kappa) = \mathbb{C}^{n+1}$  is the defining representation of  $\mathfrak{sl}_{n+1}$ ),  $\mathcal{G}$  is the subgraph of the Young lattice obtained by considering only the partitions with at most  $n+1$  parts and we recover the harmonic functions obtained by Kerov and Vershik from specializations of Schur polynomials.

**6.1. Multiplicative tensor graphs, harmonic functions and central measures.** So assume  $\kappa \in P_+$  is fixed. We denote by  $\mathcal{G}$  the oriented graph with set of vertices the pairs  $(\lambda, \ell) \in P_+ \times \mathbb{Z}_{\geq 0}$  and arrow

$$(\lambda, \ell) \xrightarrow{m_{\lambda, \kappa}^\Lambda} (\Lambda, \ell + 1)$$

with multiplicity  $m_{\lambda, \kappa}^\Lambda$  when  $m_{\lambda, \kappa}^\Lambda > 0$ . In particular there is no arrows between  $(\lambda, \ell)$  and  $(\Lambda, \ell + 1)$  when  $m_{\kappa, \kappa}^\Lambda = 0$ .

**Example 6.1.** Consider  $\mathfrak{g} = \mathfrak{sp}_{2n}$ . Then  $P = \mathbb{Z}^n$  and  $P_+$  can be identified with the set of partitions with at most  $n$  parts. For  $\kappa = \omega_1$  the graph  $\mathcal{G}$  is such that  $(\lambda, \ell) \rightarrow (\Lambda, \ell + 1)$  with  $m_{\lambda, \kappa}^\Lambda = 1$  if and only if the Young diagram of  $\Lambda$  is obtained from that of  $\lambda$  by adding or deleting one box. We have drawn below the connected component of  $(\emptyset, 0)$  up to  $\ell \leq 3$ .



Observe that in the case  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  and  $\kappa = \omega_1$ , we have  $m_{\lambda, \kappa}^\Lambda = 1$  if and only if of the Young diagram of  $\Lambda$  is obtained by adding one box to that of  $\lambda$  and  $m_{\lambda, \kappa}^\Lambda = 0$  otherwise. So in this very particular case, it is not useful to keep the second component  $\ell$  since it is equal to the rank of the partition  $\lambda$ . The vertices of  $\mathcal{G}$  are simply the partitions with at most  $n$  parts (i.e. whose Young diagram has at most  $n$  rows).

Now return to the general case. Our aim is now to relate the harmonic functions on  $\mathcal{G}$  and the central probability distributions on the set  $\Omega_{\mathcal{C}}$  of infinite trajectories with steps in  $B(\pi_\kappa)$  which remain in  $\mathcal{C}$ . We will identify the elements of  $P_+ \times \mathbb{Z}_{\geq 0}$  as elements of the  $\mathbb{R}$ -vector space  $P_{\mathbb{R}} \times \mathbb{R}$  (recall  $P_{\mathbb{R}} = \mathbb{R}^n$ ). For any  $\ell \geq 0$ , set  $H^\ell = \{\pi \in B(\pi_\kappa)^{\otimes \ell} \mid \text{Im } \pi \subset \mathcal{C}\}$ . Also if  $\lambda \in P_+$ , set  $H_\lambda^\ell = \{\pi \in H^\ell \mid \text{wt}(\pi) = \lambda\}$ . Given  $\pi \in H^\ell$ , we denote by

$$C_\pi = \{\omega \in \Omega_{\mathcal{C}} \mid \Pi_\ell(\omega) = \pi\}$$

the cylinder defined by  $\pi$ . We have  $C_\emptyset = \Omega_{\mathcal{C}}$ . Each probability distribution  $\mathbb{Q}$  on  $\Omega_{\mathcal{C}}$  is determined by its values on the cylinders and we must have

$$\sum_{\pi \in H^\ell} \mathbb{Q}(C_\pi) = 1$$

for any  $\ell \geq 0$ .

**Definition 6.2.** *A central probability distribution on  $\Omega_{\mathcal{C}}$  is a probability distribution  $\mathbb{Q}$  on  $\Omega_{\mathcal{C}}$  such that  $\mathbb{Q}(C_\pi) = \mathbb{Q}(C_{\pi'})$  provided that  $\text{wt}(\pi) = \text{wt}(\pi')$  and  $\pi, \pi'$  have the same length.*

Consider a central probability distribution  $\mathbb{Q}$  on  $\Omega_{\mathcal{C}}$ . We have  $\sum_{\pi \in H^\ell} \mathbb{Q}(C_\pi) = 1$ , so it is possible to define a probability distribution  $q$  on  $H^\ell$  by setting  $q_\pi = \mathbb{Q}(C_\pi)$  for any  $\pi \in H^\ell$ . Since  $\mathbb{Q}$  is central, we can also define the function

$$(27) \quad \varphi : \begin{cases} \mathcal{G} \rightarrow [0, 1] \\ (\lambda, \ell) \mapsto \mathbb{Q}(C_\pi) \end{cases}$$

where  $\pi$  is any path of  $H^\ell$ . Now observe that we have

$$C_\pi = \bigsqcup_{\eta \in B(\pi_\kappa) \mid \text{Im}(\pi \otimes \eta) \subset \mathcal{C}} C_{\pi \otimes \eta}$$

by considering the possible elementary paths  $\eta$  appearing as  $(\ell + 1)$ -steps of paths in  $C_\pi$ . This gives

$$(28) \quad \mathbb{Q}(C_\pi) = \sum_{\eta \in B(\pi_\kappa) \mid \text{Im}(\pi \otimes \eta) \subset \mathcal{C}} \mathbb{Q}(C_{\pi \otimes \eta}).$$

Assume  $\pi \in H_\lambda^\ell$ . The cardinality of the set  $\{\eta \in B(\pi_\kappa) \mid \text{Im}(\pi \otimes \eta) \subset \mathcal{C} \text{ and } \text{wt}(\pi \otimes \eta) = \Lambda\}$  is then equal to  $m_{\lambda, \kappa}^\Lambda$  by Theorem 3.3. Therefore, we get

$$(29) \quad \varphi(\lambda, \ell) = \sum_{\Lambda} m_{\lambda, \kappa}^\Lambda \varphi(\Lambda, \ell + 1).$$

This means that the function  $\varphi$  is *harmonic* on the multiplicative graph  $\mathcal{G}$ .

Conversely, if  $\varphi'$  is harmonic on the multiplicative graph  $\mathcal{G}$ , for any cylinder  $C_\pi$  in  $\Omega_{\mathcal{C}}$  with  $\pi \in H_\lambda^\ell$ , we set  $\mathbb{Q}'(C_\pi) = \varphi'(\lambda, \ell)$ . Then  $\mathbb{Q}'$  is a probability distribution on  $\Omega_{\mathcal{C}}$  since it verifies (28) and is clearly central. Therefore, *a central probability distribution on  $\Omega_{\mathcal{C}}$  is characterized by its associated harmonic function* defined by (27).

**6.2. Harmonic function on a multiplicative tensor graph.** Let  $\mathbb{Q}$  a central probability distribution on  $\Omega_{\mathcal{C}}$ . Consider  $\pi = \pi_1 \otimes \cdots \otimes \pi_\ell \in H_\lambda^\ell$  and  $\pi^\# = \pi_1 \otimes \cdots \otimes \pi_\ell \otimes \pi_{\ell+1} \in H_\Lambda^{\ell+1}$ . Since we have the inclusion of events  $C_{\pi^\#} \subset C_\pi$ , we get

$$\mathbb{Q}(C_{\pi^\#} \mid C_\pi) = \frac{\mathbb{Q}(C_{\pi^\#}, C_\pi)}{\mathbb{Q}(C_\pi)} = \frac{\mathbb{Q}(C_{\pi^\#})}{\mathbb{Q}(C_\pi)} = \frac{\varphi(\Lambda, \ell + 1)}{\varphi(\lambda, \ell)}.$$

We then define a Markov chain  $Z = (Z_\ell)_{\ell \geq 0}$  from  $(\Omega_{\mathcal{C}}, \mathbb{Q})$  with values in  $\mathcal{G}$  and starting from  $Z_0 = (\mu, \ell_0) \in \mathcal{G}$  by

$$Z_\ell(\pi) = (\text{wt}(\Pi_\ell(\omega)), \ell).$$

Its transition probabilities are given by

$$\Pi_Z((\lambda, \ell), (\Lambda, \ell + 1)) = \sum_{\pi^\#} \mathbb{Q}(C_{\pi^\#} \mid C_\pi)$$

where  $\pi \in H_\lambda^\ell$  is fixed and the sum runs over all the paths  $\pi^\# \in H_\Lambda^{\ell+1}$  such that  $\pi^\# = \pi \otimes \pi_{\ell+1}$ . Since there are  $m_{\lambda, \kappa}^\Lambda$  such pairs, we get

$$(30) \quad \Pi_Z((\lambda, \ell), (\Lambda, \ell + 1)) = \frac{m_{\lambda, \kappa}^\Lambda \varphi(\Lambda, \ell + 1)}{\varphi(\lambda, \ell)}$$

and by (29)  $Z = (Z_\ell)_{\ell \geq 0}$  is indeed a Markov chain. We then write  $\mathbb{Q}_{(\mu, \ell_0)}(Z_\ell = (\lambda, \ell))$  for the probability that  $Z_\ell = (\lambda, \ell)$  when the initial value is  $Z_0 = (\mu, \ell_0)$ . When  $Z_0 = (0, 0)$ , we simply write  $\mathbb{Q}(Z_\ell = (\lambda, \ell)) = \mathbb{Q}_{(0, 0)}(Z_\ell = (\lambda, \ell))$ .

**Lemma 6.3.** *For any  $\mu, \lambda \in P_+$  and any integer  $\ell_0 \geq 1$ , we have*

$$\mathbb{Q}_{(\mu, \ell_0)}(Z_{\ell - \ell_0} = (\lambda, \ell)) = f_{\lambda/\mu}^{(\ell - \ell_0)} \frac{\varphi(\lambda, \ell)}{\varphi(\mu, \ell_0)} \text{ for any } \ell \geq \ell_0.$$

*Proof.* By (30), the probability  $\mathbb{Q}_{(\mu, \ell_0)}(Z_{\ell - \ell_0} = (\lambda, \ell))$  is equal to the quotient  $\frac{\varphi(\lambda, \ell)}{\varphi(\mu, \ell_0)}$  times the number of paths in  $\mathcal{C}$  of length  $\ell - \ell_0$  starting at  $\mu$  and ending at  $\lambda$ . The lemma thus follows from the fact that this number is equal to  $f_{\lambda/\mu}^{(\ell - \ell_0)}$  by Theorem 3.3.  $\square$



We will say that the family of Markov chains  $Z$  with transition probabilities given by (30) and initial distributions of the form  $Z_0 = (\mu, \ell_0) \in \mathcal{G}$  admits a drift  $\bar{m} \in P_{\mathbb{R}}$  when

$$\lim_{\ell \rightarrow +\infty} \frac{Z_\ell}{\ell} = (\bar{m}, 1) \text{ } \mathbb{Q}\text{-almost surely}$$

for any initial distributions  $Z_0 = (\mu, \ell_0) \in \mathcal{G}$ .

**Theorem 6.4.** *Let  $\mathbb{Q}$  be a central probability distribution on  $\Omega_{\mathcal{C}}$  such that  $Z$  admits the drift  $\bar{m} \in \mathcal{D}_\kappa$  (see (12)).*

- (i) *The associated harmonic function  $\varphi$  on  $\mathcal{G}$  verifies  $\varphi(\mu, \ell_0) = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0 \kappa} S_\kappa^{\ell_0}(\tau)}$  for any  $\mu \in P_+$  and any  $\ell_0 \geq 0$  where  $\tau$  is determined by  $\bar{m}$  as prescribed by Proposition 4.15.*
- (ii) *The probability transitions (30) do not depend on  $\ell$ .*
- (iii) *For any  $\pi \in H_\mu^{\ell_0}$ , we have  $\mathbb{Q}(C_\pi) = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0 \kappa} S_\kappa^{\ell_0}(\tau)}$ . In particular,  $\mathbb{Q}$  is the unique central probability distribution on  $\Omega_{\mathcal{C}}$  such that  $Z$  admits the drift  $\bar{m}$ . We will denote it by  $\mathbb{Q}_{\bar{m}}$ .*

*Proof.* (i). Consider a sequence of random dominant weights of the form  $\lambda^{(\ell)} = \ell \bar{m} + o(\ell)$ . We have by using Lemma 6.3

$$\begin{aligned} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} \times \frac{1}{\varphi(\mu, \ell_0)} &= f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)} \times \frac{\varphi(\lambda^{(\ell)}, \ell)}{\varphi(\mu, \ell_0)} \times [f_{\lambda^{(\ell)}}^{(\ell)} \times \varphi(\lambda^{(\ell)}, \ell)]^{-1} \\ &= \frac{\mathbb{Q}_{(\mu, \ell_0)}(Z_{\ell-\ell_0} = (\lambda^{(\ell)}, \ell))}{\mathbb{Q}(Z_\ell = (\lambda^{(\ell)}, \ell))} = \frac{\mathbb{Q}_{(\mu, \ell_0)}(\frac{Z_{\ell-\ell_0}}{\ell-\ell_0} = (\frac{\lambda^{(\ell)}}{\ell-\ell_0}, \frac{\ell}{\ell-\ell_0}))}{\mathbb{Q}(\frac{Z_\ell}{\ell} = (\frac{\lambda^{(\ell)}}{\ell}, 1))}. \end{aligned}$$

Since  $Z$  admits the drift  $\bar{m}$ , we have

$$\lim_{\ell \rightarrow +\infty} \frac{\mathbb{Q}_{(\mu, \ell_0)}(\frac{Z_{\ell-\ell_0}}{\ell-\ell_0} = (\frac{\lambda^{(\ell)}}{\ell-\ell_0}, \frac{\ell}{\ell-\ell_0}))}{\mathbb{Q}(\frac{Z_\ell}{\ell} = (\frac{\lambda^{(\ell)}}{\ell}, 1))} = \frac{1}{1} = 1 \text{ and } \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} \times \frac{1}{\varphi(\mu, \ell_0)} = 1.$$

This means that

$$\varphi(\mu, \ell_0) = \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}}.$$

Now by Theorem 4.13 and since  $\bar{m} \in \mathcal{D}_\kappa$  we can write

$$\lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} = \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}/\mu}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell-\ell_0)}} \times \lim_{\ell \rightarrow +\infty} \frac{f_{\lambda^{(\ell)}}^{(\ell-\ell_0)}}{f_{\lambda^{(\ell)}}^{(\ell)}} = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0 \kappa} S_\kappa^{\ell_0}(\tau)}$$

where  $\tau \in ]0, 1[^n$  is determined by the drift  $\bar{m}$  as prescribed by Proposition 4.15. We thus obtain  $\varphi(\mu, \ell_0) = \frac{\tau^{-\mu} S_\mu(\tau)}{\tau^{-\ell_0 \kappa} S_\kappa^{\ell_0}(\tau)}$ .

(ii). We have  $\Pi_Z((\lambda, \ell), (\Lambda, \ell + 1)) = \frac{m_{\lambda, \kappa}^\Lambda \varphi(\Lambda, \ell + 1)}{\varphi(\lambda, \ell)} = \frac{S_\Lambda(\tau)}{S_\kappa(\tau) S_\lambda(\tau)} \tau^{\kappa + \lambda - \Lambda} m_{\lambda, \kappa}^\Lambda$  which does not depend on  $\ell$ .

(iii). This follows from the fact that  $\mathbb{Q}(C_\pi) = \varphi(\lambda, \ell)$  for any  $\pi \in H_\lambda^\ell$ .  $\square$

Consider  $\bar{m} \in \mathcal{D}_\kappa$  and write  $\tau$  for the corresponding  $n$ -tuple in  $]0, 1[^n$ . Let  $W$  be the random walk starting at 0 defined on  $P$  from  $\kappa$  and  $\tau$  as in § 4.3.

**Corollary 6.5.** *Let  $\mathbb{Q}$  be a central probability distribution on  $\Omega_{\mathcal{C}}$  such that  $Z$  admits the drift  $\bar{m} \in \mathcal{D}_\kappa$ . Then, the processes  $(Z_\ell)_\ell$  and  $((\mathcal{P}(W_\ell), \ell))_\ell$  have the same law.*

*Proof.* By the previous theorem, the transitions of the Markov chain  $Z$  on  $\mathcal{G}$  are given by  $\Pi_Z((\lambda, \ell), (\lambda, \ell + 1)) = \frac{m_{\lambda, \kappa}^{\lambda} \varphi(\lambda, \ell + 1)}{\varphi(\lambda, \ell)}$ . By Theorem 4.12, the transition matrix  $\Pi_Z$  thus coincides with the transition matrix of  $\mathcal{P}(W)$  as desired.  $\square$

Let  $\mathbb{P}_{\overline{m}}$  and  $\mathbb{Q}_{\overline{m}}$  be the probability distributions associated to  $\overline{m}$  (recall  $\overline{m}$  determines  $\tau \in ]0, 1[^n$ ) defined on the spaces  $\Omega$  and  $\Omega_{\mathcal{C}}$ , respectively.

**Corollary 6.6.** *The Pitman transform  $\mathcal{P}$  is a homomorphism of probability spaces between  $(\Omega, \mathbb{P}_{\overline{m}})$  and  $(\Omega_{\mathcal{C}}, \mathbb{Q}_{\overline{m}})$ , that is we have*

$$\mathbb{Q}_{\overline{m}}(C_{\pi}) = \mathbb{P}_{\overline{m}}(\mathcal{P}^{-1}(C_{\pi}))$$

for any  $\ell \geq 1$  and any  $\pi \in H^{\ell}$ .

*Proof.* Assume  $\pi \in H_{\lambda}^{\ell}$ . We have  $\mathbb{Q}_{\overline{m}}(C_{\pi}) = \varphi(\lambda, \ell) = \frac{\tau^{-\lambda} S_{\lambda}(\tau)}{\tau^{-\ell \kappa} S_{\kappa}^{\ell}(\tau)}$ . By definition of the generalized Pitman transform  $\mathcal{P}$ ,  $\mathcal{P}^{-1}(C_{\pi}) = \{\omega \in \Omega \mid \mathcal{P}(\Pi_{\ell}(\omega)) = \pi\}$ , that is  $\mathcal{P}^{-1}(C_{\pi})$  is the set of all trajectories in  $\Omega$  which remains in the connected component  $B(\pi) \subset B(\pi_{\kappa})^{\otimes \ell}$  for any  $t \in [0, \ell]$ . We thus have  $\mathbb{P}_{\overline{m}}(\mathcal{P}^{-1}(C_{\pi})) = p^{\otimes \ell}(B(\pi)) = \frac{\tau^{-\lambda} S_{\lambda}(\tau)}{\tau^{-\ell \kappa} S_{\kappa}^{\ell}(\tau)}$  by assertion (ii) of Theorem 4.12. Therefore we get  $\mathbb{P}_{\overline{m}}(\mathcal{P}^{-1}(C_{\pi})) = \mathbb{Q}_{\overline{m}}(C_{\pi})$  as desired.  $\square$

## 7. SOME CONSEQUENCES

In this section, we first explain how the trajectories in  $\Omega$  and  $\Omega_{\mathcal{C}}$  can be equipped with natural shifts  $S$  and  $J$ , respectively. We then prove that the generalized Pitman transform  $\mathcal{P}$  intertwines  $S$  and  $J$ . When  $\mathfrak{g} = \mathfrak{sl}_{n+1}$  and  $\kappa = \omega_1$ , we recover in particular some analogue results of [17]. Next, we show that the  $\ell$ -th elementary paths in  $\mathcal{W}$  and  $\mathcal{H} = \mathcal{P}(\mathcal{W})$  almost surely coincide when  $\ell$  tends to infinity provided  $\overline{m} \in \mathcal{D}_{\kappa}$  (i.e. the drift of  $\mathcal{W}$  belongs to  $\mathring{\mathcal{C}}$ ).

**7.1. Isomorphism of dynamical systems.** Let  $S : \Omega \rightarrow \Omega$  be the shift operator on  $\Omega$  defined by

$$S(\pi) = S(\pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \dots) := (\pi_2 \otimes \pi_3 \otimes \dots)$$

for any trajectory  $\pi = \pi_1 \otimes \pi_2 \otimes \pi_3 \otimes \dots \in \Omega$ . Observe that  $S$  is measure preserving for the probability distribution  $\mathbb{P}_{\overline{m}}$ . We now introduce the map  $J : \Omega_{\mathcal{C}} \rightarrow \Omega_{\mathcal{C}}$  defined by

$$J(\pi) = \mathcal{P} \circ S(\pi)$$

for any trajectory  $\pi \in \Omega_{\mathcal{C}}$ . Observe that  $S(\pi)$  does not belong to  $\Omega_{\mathcal{C}}$  in general so we need to apply the Pitman transform  $\mathcal{P}$  to ensure that  $J$  takes values in  $\Omega_{\mathcal{C}}$ .

**Theorem 7.1.**

- (i) *The Pitman transform is a factor map of dynamical systems, i.e. the following diagram commutes :*

$$\begin{array}{ccc} \Omega & \xrightarrow{S} & \Omega \\ \mathcal{P} \downarrow & & \downarrow \mathcal{P} \\ \Omega_{\mathcal{C}} & \xrightarrow{J} & \Omega_{\mathcal{C}} \end{array}$$

- (ii) *For any  $\overline{m} \in \mathcal{D}_{\kappa}$ , the transformation  $J : \Omega_{\mathcal{C}} \rightarrow \Omega_{\mathcal{C}}$  is measure preserving with respect to the (unique) central probability distribution  $\mathbb{Q}_{\overline{m}}$  with drift  $\overline{m}$ .*

*Proof.* (i). To prove assertion (i), it suffices to establish that the above diagram commutes on trajectories of any finite length  $\ell > 0$ . So consider  $\pi = \pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell \in B(\pi_\kappa)^{\otimes \ell}$  and set  $\mathcal{P}(\pi) = \pi_1^+ \otimes \pi_2^+ \otimes \cdots \otimes \pi_\ell^+$ . We have to prove that

$$\mathcal{P}(\pi_2 \otimes \cdots \otimes \pi_\ell) = \mathcal{P}(\pi_2^+ \otimes \cdots \otimes \pi_\ell^+)$$

which means that both vertices  $\pi_2 \otimes \cdots \otimes \pi_\ell$  and  $\pi_2^+ \otimes \cdots \otimes \pi_\ell^+$  belong to the same connected component of  $B(\pi_\kappa)^{\otimes \ell-1}$ . We know that  $\mathcal{P}(\pi)$  is the highest weight vertex of  $B(\pi)$ . This implies that there exists a sequence of root operators  $\tilde{e}_{i_1}, \dots, \tilde{e}_{i_r}$  such that

$$(31) \quad \pi_1^+ \otimes \pi_2^+ \otimes \cdots \otimes \pi_\ell^+ = \tilde{e}_{i_1} \cdots \tilde{e}_{i_r}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell).$$

By (6), we can define a subset  $X := \{k \in \{1, \dots, r\} \text{ such that } \tilde{e}_{i_k} \text{ acts on the first component of the tensor product } \tilde{e}_{i_{k+1}} \cdots \tilde{e}_{i_r}(\pi_1 \otimes \pi_2 \otimes \cdots \otimes \pi_\ell)\}$ . We thus obtain

$$\pi_2^+ \otimes \cdots \otimes \pi_\ell^+ = \prod_{k \in \{1, \dots, r\} \setminus X} \tilde{e}_{i_k}(\pi_2 \otimes \cdots \otimes \pi_\ell)$$

which shows that  $\pi_2 \otimes \cdots \otimes \pi_\ell$  and  $\pi_2^+ \otimes \cdots \otimes \pi_\ell^+$  belong to the same connected component of  $B(\pi_\kappa)^{\otimes \ell-1}$ . They thus have the same highest weight path as desired.

(ii). Let  $A \subset \Omega_{\mathcal{C}}$  be a  $\mathbb{Q}$ -measurable set. We have  $\mathbb{Q}(J^{-1}(A)) = \mathbb{P}(\mathcal{P}^{-1}(J^{-1}(A)))$  since  $\mathcal{P}$  is an homomorphism. Using the fact that previous diagram commutes and  $S$  preserves  $\mathbb{P}$ , we get  $\mathbb{Q}(J^{-1}(A)) = \mathbb{P}(S^{-1}(\mathcal{P}^{-1}(A))) = \mathbb{P}(\mathcal{P})$ , so that so  $\mathbb{Q}(J^{-1}(A)) = \mathbb{Q}(A)$  since  $\mathcal{P}$  is an homomorphism.  $\square$

**7.2. Asymptotic behavior in a fixed component.** Consider  $\bar{m} \in \mathcal{D}_\kappa$  and the associated distributions  $\mathbb{P}_{\bar{m}}$  and  $\mathbb{Q}_{\bar{m}}$  defined on  $\Omega$  and  $\Omega_{\mathcal{C}}$ , respectively. We introduce the subsets of typical trajectories in  $\Omega$  and  $\Omega_{\mathcal{C}}$  as

$$\begin{aligned} \Omega^{\text{typ}} &= \{\omega \in \Omega \mid \lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \pi(\ell), \alpha_i^\vee \rangle = \langle \bar{m}, \alpha_i^\vee \rangle \in \mathbb{R}_{>0} \quad \forall i = 1, \dots, n\}. \\ \Omega_{\mathcal{C}}^{\text{typ}} &= \{\omega \in \Omega_{\mathcal{C}} \mid \lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \pi(\ell), \alpha_i^\vee \rangle = \langle \bar{m}, \alpha_i^\vee \rangle \in \mathbb{R}_{>0} \quad \forall i = 1, \dots, n\}. \end{aligned}$$

By Theorems 5.1 and 5.4, we have

$$\mathbb{P}_{\bar{m}}(\Omega^{\text{typ}}) = 1 \text{ and } \mathbb{Q}_{\bar{m}}(\Omega_{\mathcal{C}}^{\text{typ}}) = 1.$$

Let  $\mathcal{H} = (\mathcal{H}_\ell)_{\ell \geq 1}$  be a random process in  $\Omega_{\mathcal{C}} \subset \Omega$  with distribution  $\mathbb{Q}_{\bar{m}}$ . Since  $\mathcal{H}$  takes value in  $\Omega$ , we can write  $\mathcal{H}_\ell = T_1 \otimes \cdots \otimes T_\ell$  for any  $\ell \geq 1$ , where the random variable  $T_i$  takes values in  $B(\pi_\kappa)$  for any  $i \geq 1$ . By Corollary 6.6, there exists a random process  $\mathcal{W}$  with values in  $\Omega$  and distribution  $\mathbb{P}_{\bar{m}}$  such that  $\mathcal{H}$  and  $\mathcal{P}(\mathcal{W})$  coincide  $\mathbb{P}_{\bar{m}}$ -almost surely. Notice that we also have  $\mathcal{W}_\ell = X_1 \otimes \cdots \otimes X_\ell$  for any  $\ell \geq 1$ , where  $X_\ell$  is a random variable with values in  $B(\pi_\kappa)$  with the law defined in (14).

**Proposition 7.2.**  *$\mathbb{P}_{\bar{m}}$ -almost surely, the random variables  $T_\ell$  and  $X_\ell$  coincide for any large enough  $\ell$ .*

*Proof.* Consider a trajectory  $\omega \in \Omega^{\text{typ}}$ . For any  $\ell \geq 1$  and set  $\Pi_\ell(\omega) = \pi_1 \otimes \cdots \otimes \pi_\ell$ . We can apply Lemma 5.5 to  $\pi_1 \otimes \cdots \otimes \pi_{\ell-1} \otimes \pi_\ell$  since we have  $\omega \in \Omega^{\text{typ}}$ . Hence, for  $\ell$  sufficiently large, we have

$$\mathcal{P}(\pi_1 \otimes \cdots \otimes \pi_{\ell-1} \otimes \pi_\ell) = \mathcal{P}(\pi_1 \otimes \cdots \otimes \pi_{\ell-1}) \otimes \pi_\ell.$$

We thus have  $\lim_{\ell \rightarrow +\infty} (T_\ell - X_\ell) = 0$  on  $\Omega^{\text{typ}}$ . We are done since  $\mathbb{P}_{\bar{m}}(\Omega^{\text{typ}}) = 1$ .  $\square$

## 8. DUAL RANDOM PATH AND THE INVERSE PITMAN TRANSFORM

It is well known (see [16]) that the Pitman transform on the line is reversible. Assume that  $\omega : [0, +\infty[ \rightarrow \mathbb{R}$  is an infinite trajectory on the line such that  $\omega(0) = 0$ . Its Pitman transform  $\mathcal{P}(\omega)$  is then defined by

$$\mathcal{P}(\omega)(t) = \omega(t) - 2 \inf_{s \in [0, t]} \{\omega(s)\}$$

for any  $t \geq 0$ . Now starting from an infinite trajectory  $\omega_+$  on  $\mathbb{R}_{\geq 0}$  such that  $\lim_{t \rightarrow +\infty} \omega_+(t) = +\infty$ , one can associate an infinite trajectory  $\mathcal{P}^{-1}(\omega_+)$  on  $\mathbb{R}$  by setting

$$\mathcal{P}^{-1}(\omega_+)(t) = \omega_+ - 2 \inf_{s \in [t, +\infty[} \{\omega_+(s)\}.$$

We have  $\mathcal{P} \circ \mathcal{P}^{-1}(\omega_+) = \omega_+$  and  $\mathcal{P}^{-1} \circ \mathcal{P}(\omega) = \omega$  so that  $\mathcal{P}^{-1}$  can be regarded as the inverse transform of  $\mathcal{P}$ .

The aim of this paragraph is to define an inverse Pitman transform for any Lie algebra  $\mathfrak{g}$  and any dominant weight  $\kappa$  of  $\mathfrak{g}$ , that is to define an inverse  $\mathcal{P}^{-1}$  for the generalized Pitman transform. We assume in the remaining of the paper that  $\bar{m} \in D_\kappa$ . This permits to define a random walk  $\mathcal{W}$  and a Markov chain  $\mathcal{H} = \mathcal{P}(\mathcal{W})$  as in Section 4. Since  $\bar{m}$  is fixed, we will denote for short by  $\mathbb{P}$  and  $\mathbb{Q}$  the probability distributions  $\mathbb{P}_{\bar{m}}$  and  $\mathbb{Q}_{\bar{m}}$ , respectively. The transform  $\mathcal{P}^{-1}$  will be defined on  $\Omega_{\mathcal{C}}^{\text{typ}}$  and we will show that  $\mathcal{P}^{-1}(\mathcal{H})$  is a random trajectory with drift  $w_0(\bar{m})$  where  $w_0$  is the longest elements of the Weyl group  $W$ . Observe that  $\mathcal{P}^{-1}$  will only be defined  $\mathbb{Q}$ -almost surely since  $\mathbb{Q}(\Omega_{\mathcal{C}}^{\text{typ}}) = 1$ . The case  $\mathfrak{g} = \mathfrak{sl}_2$  and  $\kappa = \omega_1$  corresponds to the Markov chain  $\mathcal{H} = (\mathcal{H}_{\ell \geq 1})$  on  $\mathbb{Z}_{\geq 0}$  with transitions  $\pm 1$ . We obtain a random walk  $\mathcal{P}^{-1}(\mathcal{H})$  on  $\mathbb{Z}$  with transitions  $\pm 1$ . Moreover, if  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \mathcal{H}_\ell = \bar{m}$ , the random walk  $\mathcal{P}^{-1}(\mathcal{H})$  has drift  $-\bar{m}$ , that is  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \mathcal{P}^{-1}(\mathcal{H}) = -\bar{m}$ . Indeed the Weyl group  $W$  acts then on  $P = \mathbb{Z}$  as  $W = \{\pm id\}$  and we have  $w_0 = -id$ .

**8.1. The bar involution.** Let us now review the Lusztig involution defined on any connected crystal  $B(\pi_\lambda)$  of highest path  $\pi_\lambda$ . We refer the reader to [10] for a complete exposition. The longest element  $w_0$  of the Weyl group  $W$  (which is an involution) induces an involution  $*$  on the set of simple roots defined by  $\alpha_{i^*} = -w_0(\alpha_i)$  for any  $i = 1, \dots, n$ . Write  $\pi_\lambda^{\text{low}}$  for the lowest weight vertex of  $B(\pi_\lambda)$ , that is  $\pi_\lambda^{\text{low}}$  is the unique vertex of  $B(\pi_\lambda)$  such that  $\tilde{f}_i(\pi_\lambda^{\text{low}}) = 0$  for any  $i = 1, \dots, n$ . The Schützenberger involution  $\iota$  is defined on the crystal  $B(\pi_\lambda)$  by

$$\iota(\pi_\lambda) = \pi_\lambda^{\text{low}} \text{ and } \iota(\tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \pi_\lambda) = \tilde{e}_{i_1^*} \cdots \tilde{e}_{i_r^*}(\pi_k^{\text{low}})$$

for any sequence of crystal operators  $\tilde{f}_{i_1}, \dots, \tilde{f}_{i_r}$  with  $r > 0$ . This means that  $\iota$  flips the orientation of the arrows of  $B(\pi_\lambda)$  and each label  $i$  is changed in  $i^*$ . In particular, we have  $\text{wt}(\iota(\pi)) = w_0(\text{wt}(\pi))$  for any  $\pi \in B(\pi_\lambda)$ . We extend  $\iota$  by linearity on the linear combinations of paths in  $B(\pi_\lambda)$ . In fact the involution  $\iota$  can also be understood from the Weyl group  $W$  action (see Theorem 3.3) on the vertices of  $B(\pi_\lambda)$ . One can indeed show that the involution  $\iota$  corresponds to the action of  $w_0$  as described in assertion (vii) of Theorem 3.3.

Now consider  $\kappa \in P_+$ . The involution  $\iota$  can only be defined (connected) component-wise (here we regard  $B(\pi_\kappa)^{\otimes \ell}$  as the disjoint union of its connected components). In the following we need in fact an involution on each  $B(\pi_\kappa)^{\otimes \ell}$ ,  $\ell \geq 1$  which moreover *reverses the order the elementary paths in the tensor product*. We define the bar involution on  $B(\pi_\kappa)^{\otimes \ell}$  by setting

$$\overline{\pi_1 \otimes \cdots \otimes \pi_\ell} = \iota(\pi_\ell) \otimes \cdots \otimes \iota(\pi_1)$$

for any  $\pi_1 \otimes \cdots \otimes \pi_\ell \in B(\pi_\kappa)^{\otimes \ell}$ . It then follows from (6) that for any any  $i = 1, \dots, n$  we have

$$\overline{\tilde{f}_i(\pi_1 \otimes \cdots \otimes \pi_\ell)} = \tilde{e}_{i^*}(\overline{\pi_1 \otimes \cdots \otimes \pi_\ell}).$$

Thus the bar involution flips the lowest and highest weight paths and reverse the arrows. It follows that, for any connected component  $B(\eta)$  of  $B(\pi_\kappa)^{\otimes \ell}$ , the set  $\overline{B(\eta)}$  is also a connected component of  $B(\pi_\kappa)^{\otimes \ell}$ . In addition, we have

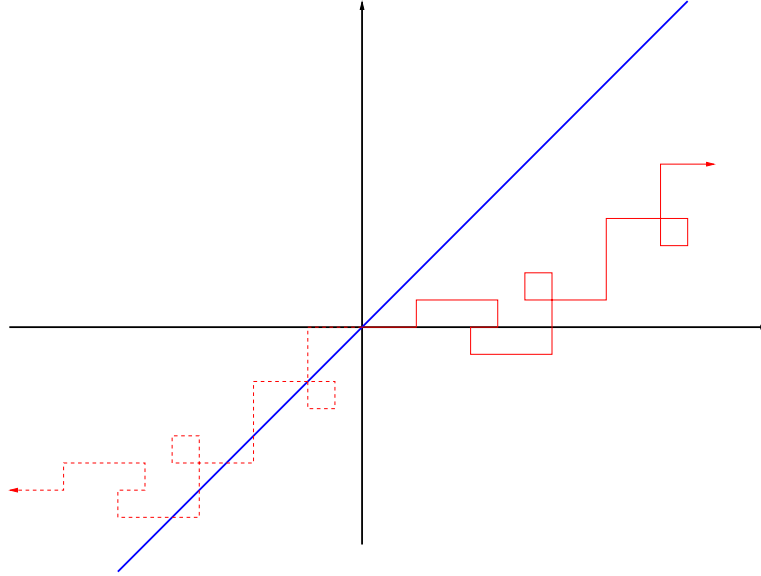
$$\text{wt}(\overline{\pi_1 \otimes \cdots \otimes \pi_\ell}) = w_0(\text{wt}(\pi_1 \otimes \cdots \otimes \pi_\ell)).$$

**Remark 8.1.** Consider  $B(\pi_\lambda) \subset B(\pi_\kappa)^{\otimes \ell}$  a connected component with highest weight path  $\pi_\lambda$ . The connected components  $B(\pi_\lambda)$  and  $\overline{B(\pi_\lambda)}$  do not coincide in general. So the restriction of the bar involution on the connected component is not in general the Schützenberger involution as soon as  $\ell \geq 2$ . Nevertheless,  $B(\pi_\lambda)$  and  $\overline{B(\pi_\lambda)}$  are isomorphic since the highest weight path of  $\overline{B(\pi_\lambda)}$  is  $\iota(\overline{\pi_\lambda})$  which has weight  $\lambda = w_0^2(\lambda)$  (recall  $w_0$  is an involution).

**Example 8.2.** We resume Example 4.7 and consider  $\mathfrak{g} = \mathfrak{sp}_4$  and  $\kappa = \omega_1$ . In this case we get  $w_0 = -id$ . We then have  $\iota(\pi_1) = \pi_{\overline{1}}$  and  $\iota(\pi_2) = \pi_{\overline{2}}$ . In the picture below we have drawn the path  $\eta$  and  $\overline{\eta}$  where

$$\begin{aligned} \eta &= 112111\overline{2}\overline{1}\overline{2}\overline{1}111222\overline{1}\overline{2}\overline{1}1112222111\overline{2}\overline{1}\overline{2}22211, \\ \overline{\eta} &= \overline{1}\overline{1}\overline{2}\overline{2}\overline{2}12\overline{1}\overline{1}\overline{1}\overline{2}\overline{2}\overline{2}\overline{1}\overline{1}\overline{1}\overline{2}1\overline{2}\overline{2}\overline{2}\overline{1}\overline{1}\overline{1}\overline{2}12\overline{1}\overline{1}\overline{1}\overline{2}\overline{1}\overline{1}. \end{aligned}$$

Here we simply write  $a \in \{\overline{2}, \overline{1}, 1, 2\}$  instead of  $\pi_a$  and omitted for short the symbols  $\otimes$ .



The paths  $\eta$  (in red) and  $\overline{\eta}$  (in dashed red)

**8.2. Dual random path.** Let us define the probability distribution  $p^\iota$  on  $B(\pi_\kappa)$  by setting

$$(32) \quad p_\pi^\iota = p_{\iota(\pi)} = \frac{\tau^{\kappa - w_0 \text{wt}(\pi)}}{S_\kappa(\tau)} \text{ for any } \pi \in B(\pi_\kappa)$$

and consider a random variable  $Y$  defined on some probability space  $(\Omega, \mathcal{T}, \mathbb{P})$  with values in  $B(\pi_\kappa)$  and probability distribution  $p^\iota$ . Set  $m^\iota = E(Y)$ ,  $\overline{m}^\iota = m^\iota(1)$  and  $\mathcal{D}_\kappa^\iota = w_0(\mathcal{D}_\kappa)$ .

**Lemma 8.3.** We have

- (i)  $m^\iota = \iota(m)$
- (ii)  $\overline{m}^\iota = w_0(\overline{m})$ . In particular,  $\overline{m} \in \mathcal{D}_\kappa$  if and only if  $\overline{m}^\iota \in \mathcal{D}_\kappa$ .

*Proof.* By using that  $\iota$  is an involution on  $B(\pi_\kappa)$ , we get

$$m^\iota = \sum_{\pi \in B(\pi_\kappa)} p_\pi^\iota \pi = \sum_{\pi \in B(\pi_\kappa)} p_{\iota(\pi)} \pi = \iota \left( \sum_{\pi \in B(\pi_\kappa)} p_{\iota(\pi)} \iota(\pi) \right) = \iota(m)$$

which proves assertion (i). In particular, if we set  $\overline{m}^\iota = m^\iota(1)$ , we have  $\overline{m}^\iota = w_0(\overline{m})$  and assertion 2 follows.  $\square$

Similarly, we may consider the probability measure  $(p^\iota)^{\otimes \ell}$  on  $B(\pi_\kappa)^{\otimes \ell}$  defined by

$$(p^\iota)^{\otimes \ell}(\pi_1 \otimes \cdots \otimes \pi_\ell) = p^\iota(\pi_1) \cdots p^\iota(\pi_\ell) = \frac{\tau^{\ell\kappa - w_0(\pi_1(1) + \cdots + \pi_\ell(1))}}{S_\kappa(\tau)^\ell} = \frac{\tau^{\ell\kappa - w_0(\text{wt}(b))}}{S_\kappa(\tau)^\ell}. \quad (2)$$

By the Kolmogorov extension theorem, the family of probability measure  $((p^\iota)^{\otimes \ell})_\ell$  admits a unique extension  $\mathbb{P}^\iota := (p^\iota)^{\otimes \mathbb{Z}_{\geq 0}}$  to the space  $B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}}$ . For any  $\ell \geq 1$ , let  $Y_\ell : B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}} \rightarrow B(\pi_\kappa)$  be the canonical projection on the  $\ell^{\text{th}}$  coordinate; by construction, the variables  $Y_1, Y_2, \dots$  are independent and identically distributed with the same law as  $Y$ . We denote by  $\mathcal{W}^\iota$  the random path defined by

$$\mathcal{W}^\iota(t) := Y_1(1) + Y_2(1) + \cdots + Y_{\ell-1}(1) + Y_\ell(t - \ell + 1) \text{ for } t \in [\ell - 1, \ell].$$

Then  $\mathcal{W}^\iota$  is defined on the probability space  $\Omega^\iota = (B(\pi_\kappa)^{\otimes \mathbb{Z}_{\geq 0}}, \mathbb{P}^\iota)$ ; notice that the set of trajectories of  $\Omega^\iota$  is the same as the one of  $\Omega$  but that the probability  $\mathbb{P}^\iota$  is defined from  $p^\iota$ . We also define the random walk  $W^\iota = (W_\ell^\iota)_{\ell \geq 1}$  such that  $W_\ell^\iota = \mathcal{W}^\iota(\ell)$  for any  $\ell \geq 1$ . Let  $\mathcal{H}^\iota$  be the random process  $\mathcal{H}^\iota = \mathcal{P}(\mathcal{W}^\iota)$  and define  $H^\iota = (H_\ell^\iota)_{\ell \geq 1}$  such that  $H_\ell^\iota = \mathcal{H}^\iota(\ell)$  for any  $\ell \geq 1$ . We then have (see Proposition 4.6 in [9])

**Theorem 8.4.**

- (i) For any  $\beta, \eta \in P$ , one gets

$$\mathbb{P}^\iota(W_{\ell+1}^\iota = \beta \mid W_\ell^\iota = \eta) = K_{\kappa, \beta - \eta} \frac{\tau^{\kappa - w_0(\beta - \eta)}}{S_\kappa(\tau)}.$$

- (ii) The random sequence  $H^\iota$  is a Markov chain with the same law as  $H$ , that is with transition matrix

$$\Pi(\mu, \lambda) = \frac{S_\lambda(\tau)}{S_\kappa(\tau)S_\mu(\tau)} \tau^{\kappa + \mu - \lambda} m_{\mu, \kappa}^\lambda$$

where  $\lambda, \mu \in P_+$ .

- (iii) For any path  $\pi \in H_\lambda^\ell$ , we have

$$\mathbb{P}^\iota(\mathcal{H}^\iota = \pi) = \mathbb{P}(\mathcal{H} = \pi) = \frac{\tau^{\ell\kappa - \lambda} S_\lambda(\tau)}{S_\kappa(\tau)^\ell}.$$

---

<sup>2</sup>We now have two probability measures  $p^{\otimes \ell}$  and  $(p^\iota)^{\otimes \ell}$  on  $B(\pi_\kappa)^{\otimes \ell}$ . Observe that for any event  $E \subset B(\pi_\kappa)^{\otimes \ell}$ , we get

$$(33) \quad (p^\iota)^{\otimes \ell}(E) = p^{\otimes \ell}(\iota(E)).$$



**8.3. The stabilization phenomenon.** Recall the reduced decomposition  $w_0 = s_{i_1} \cdots s_{i_r}$  of the longest element of  $W$ . It is then easy to verify that  $w_0 = s_{i_r^*} \cdots s_{i_1^*}$  is also a reduced decomposition. Consider a trajectory  $\omega \in \Omega_{\mathcal{C}}^{\text{typ}}$  ( $\overline{m} \in \mathcal{D}_\kappa$  by hypothesis), that is  $\frac{1}{\ell} \langle \pi(\ell), \alpha_i^\vee \rangle$  converges to a *positive* limit for any simple root  $\alpha_i$ . For any  $\ell \geq 1$ , the path  $\Pi_\ell(\omega)$  is then a highest weight path of length  $\ell$ . Also, for  $\ell$  sufficiently large, the weight  $\Pi_\ell(\omega)(\ell)$  of  $\Pi_\ell(\omega)$  belongs to  $\mathring{\mathcal{C}}$ . In particular, its orbit under the action of the Weyl group  $W$  has a trivial stabilizer. This is therefore also true for the orbit of the path  $\Pi_\ell(\omega)$  under the action of  $W$  ( $W$  acts on the paths as prescribed by assertion (vii) of Theorem 3.3).

We are going to prove that the last elementary path in  $\mathcal{P}(\overline{\Pi_\ell(\omega)})$  stabilizes for  $\ell$  sufficiently large. Since  $\overline{\Pi_\ell(\omega)}$  is a lowest weight path, we know that we have in fact  $\mathcal{P}(\overline{\Pi_\ell(\omega)}) = s_{i_1} \cdots s_{i_r}(\overline{\Pi_\ell(\omega)})$  (see assertion (iii) of the remark following Proposition 4.10). Now, by definition of the bar involution, we get

$$\mathcal{P}(\overline{\Pi_\ell(\omega)}) = s_{i_1} \cdots s_{i_r}(\overline{\Pi_\ell(\omega)}) = \overline{s_{i_1^*} \cdots s_{i_r^*}(\Pi_\ell(\omega))}.$$

This leads us to define the paths

$$(34) \quad \omega^{(r)} = \Pi_\ell(\omega), \quad \omega^{(a)} = s_{i_{a+1}^*} \cdots s_{i_r^*}(\Pi_\ell(\omega)) \text{ for } a = 0, \dots, r-1.$$

and

$$\eta^{(r)} = \overline{\omega^{(r)}}, \quad \eta^{(a)} = \overline{\omega^{(a)}} = s_{i_{a+1}} \cdots s_{i_r}(\overline{\Pi_\ell(\omega)}) \text{ for } a = 0, \dots, r-1.$$

This indeed defines two families of  $r+1$  distinct paths of length  $\ell$  for  $\ell$  sufficiently large because the stabilizer of  $\Pi_\ell(\omega)$  under the action of  $W$  is then trivial. Observe also that we have  $\eta^{(0)} = \mathcal{P}(\overline{\Pi_\ell(\omega)})$ .

**Lemma 8.5.** *For any  $a = 1, \dots, r$ ,  $\frac{1}{\ell} \langle \omega^{(a)}(\ell), \alpha_{i_a^*}^\vee \rangle$  tends to a positive limit.*

*Proof.* Since  $\omega \in \Omega_{\mathcal{C}}^{\text{typ}}$ , we have that  $\langle \omega^{(r)}(\ell), \alpha_{i_r^*}^\vee \rangle$  tends to a positive limit. Now assume that  $a = 1, \dots, r-1$ , we have

$$\langle \omega^{(a)}(\ell), \alpha_{i_a^*}^\vee \rangle = \frac{2}{\langle \alpha_{i_a^*}, \alpha_{i_a^*} \rangle} \langle s_{i_{a+1}^*} \cdots s_{i_r^*} \omega^{(a)}(\ell), \alpha_{i_a^*}^\vee \rangle = \frac{2}{\langle \alpha_{i_a^*}, \alpha_{i_a^*} \rangle} \langle \omega^{(a)}(\ell), s_{i_r^*} \cdots s_{i_{a+1}^*}(\alpha_{i_a^*}^\vee) \rangle$$

because the elements of  $W$  preserve the scalar product  $\langle \cdot, \cdot \rangle$ . Now  $s_{i_r^*} \cdots s_{i_{a+1}^*}$  is also a reduced decomposition of  $w_0$ . Hence we have as in (2)

$$R_+ = \{ \alpha_{i_r^*}, s_{i_r^*} \cdots s_{i_{a+1}^*}(\alpha_{i_a^*}) \text{ with } a = 1, \dots, r-1 \}.$$

Then  $s_{i_r^*} \cdots s_{i_{a+1}^*}(\alpha_{i_a^*}) = \alpha$  with  $\alpha \in R_+$ . We can decompose it on the basis of simple root as  $\alpha = m_1 \alpha_1 + \cdots + m_n \alpha_n$  where each  $m_i$  is a nonnegative integer. We thus get

$$\frac{1}{\ell} \langle \omega^{(a)}(\ell), \alpha_{i_a^*}^\vee \rangle = \sum_{i=1}^n \frac{m_i \langle \alpha_i, \alpha_{i_a^*} \rangle}{\langle \alpha_{i_a^*}, \alpha_{i_a^*} \rangle} \times \frac{1}{\ell} \langle \omega^{(a)}(\ell), \alpha_i^\vee \rangle.$$

Now since at least one of the coordinates  $m_i$  is nonzero and each  $\frac{1}{\ell} \langle \omega^{(a)}(\ell), \alpha_i^\vee \rangle$  tends to a positive limit, we obtain that  $\frac{1}{\ell} \langle \omega^{(a)}(\ell), \alpha_{i_a^*}^\vee \rangle$  also tends to a positive limit.  $\square$

Assume that for any  $\ell \geq 1$ ,  ${}^{(\ell)}\widehat{\eta}$  is a path of length  $\ell$ . To prove the crucial Proposition 8.8, we need the two following lemmas.

**Lemma 8.6.** *Let  $\alpha_i$  be a simple root such that  $\frac{1}{\ell} \langle {}^{(\ell)}\widehat{\eta}(\ell), \alpha_i^\vee \rangle$  converges to a positive limit. For any integer  $p \geq 1$ , there exists an integer  $q \geq p$  such that for any  $t \in [0, p]$  and any  $\ell \geq q$ , we have  $\min_{s \in [t, \ell]} \langle {}^{(\ell)}\widehat{\eta}(s), \alpha_i^\vee \rangle = \min_{s \in [t, q]} \langle {}^{(\ell)}\widehat{\eta}(s), \alpha_i^\vee \rangle$ .*

*Proof.* Define the map  $f$  on  $[0, +\infty[$  by setting  $f(t) = \langle {}^{(\ell)}\hat{\eta}(t), \alpha_i^\vee \rangle$ . Since the length of the elementary paths in  $B(\pi_\kappa)$  is fixed and  $\lim_{\ell \rightarrow +\infty} f(\ell) = +\infty$ , we also have  $\lim_{s \rightarrow +\infty} f(s) = +\infty$ . Let  $M(a) = \sup_{t \in [0, p]} f(t)$ . There exists an integer  $q \geq p$  such that  $f(s) \geq M(a)$  for any  $s \in [q, +\infty[$ . For any  $t \in [0, p]$  and any  $\ell \geq q$ , we have then  $\min_{s \in [t, \ell]} f(s) = \min_{s \in [t, q]} f(s)$  since  $f(s) \geq M(a) \geq f(t) \geq \min_{s \in [t, \ell]} f(s)$  for any  $s \geq q$ .  $\square$

**Lemma 8.7.** *Let  $\omega \in \Omega_{\mathcal{C}}$  and  $\alpha_i$  be a simple root such that  $\frac{1}{\ell} \langle \omega(\ell), \alpha_{i^*}^\vee \rangle$  converges to a positive limit. For any  $\ell \geq 1$ , set  ${}^{(\ell)}\eta = \overline{\Pi_\ell(\omega)}$  and  ${}^{(\ell)}\eta = \eta_1 \otimes \eta_2 \otimes \cdots \otimes \eta_\ell$  its decomposition in elementary paths<sup>3</sup>. For any integer  $p \geq 1$ , there exists an integer  $q \geq p$  such that for any  $\ell \geq q$  the  $p$  last steps in the paths  $\mathcal{P}_{\alpha_i}({}^{(\ell)}\eta)$  and  $\mathcal{P}_{\alpha_i}(\eta_{\ell-q} \otimes \cdots \otimes \eta_\ell)$  coincide.*

*Proof.* Define the path  ${}^{(\ell)}\hat{\eta}$  on  $[0, \ell]$  by

$$(35) \quad \hat{\eta}(t) = {}^{(\ell)}\eta(\ell - t) - {}^{(\ell)}\eta(\ell).$$

Hence  ${}^{(\ell)}\hat{\eta}$  is the reverse path of  ${}^{(\ell)}\eta$  translated by  $-{}^{(\ell)}\eta(\ell)$ . We obtain  ${}^{(\ell)}\hat{\eta}(0) = 0$  and  ${}^{(\ell)}\hat{\eta}(\ell) = -{}^{(\ell)}\eta(\ell) = -w_0(\omega(\ell))$  since  ${}^{(\ell)}\hat{\eta} = \overline{\Pi_\ell(\omega)}$ . We have

$$\frac{1}{\ell} \langle {}^{(\ell)}\hat{\eta}(\ell), \alpha_i^\vee \rangle = -\frac{1}{\ell} \langle w_0(\omega(\ell)), \alpha_i^\vee \rangle = \frac{1}{\ell} \langle \omega(\ell), \alpha_{i^*}^\vee \rangle$$

since  $w_0$  is an involution in  $W$  (thus preserves the scalar product) and  $w_0(\alpha_{i^*}^\vee) = -\alpha_i^\vee$ . Since  $\frac{1}{\ell} \langle \omega(\ell), \alpha_{i^*}^\vee \rangle$  converges to a positive limit, we also have that  $\frac{1}{\ell} \langle {}^{(\ell)}\hat{\eta}(\ell), \alpha_i^\vee \rangle$  tends to the same positive limit. Consider  $p \geq 1$ . By applying Lemma 8.6, we obtain an integer  $q \geq p$  such that for any  $t \in [0, p]$  and any  $\ell \geq q$  we have  $\min_{s \in [t, \ell]} \langle {}^{(\ell)}\hat{\eta}(s), \alpha_i^\vee \rangle = \min_{s \in [t, q]} \langle {}^{(\ell)}\hat{\eta}(s), \alpha_i^\vee \rangle$ . Now by definition (35) of the paths  ${}^{(\ell)}\hat{\eta}$ , we get for any  $\ell \geq q$  and any  $t \in [\ell - p, \ell]$

$$\min_{s \in [0, t]} \langle {}^{(\ell)}\eta(s), \alpha_i^\vee \rangle = \min_{s \in [\ell - q, t]} \langle {}^{(\ell)}\eta(s), \alpha_i^\vee \rangle$$

because  ${}^{(\ell)}\eta(\ell)$  is a constant (i.e. does not depend on  $t$ ). For any  $\ell \geq q$  and any  $t \in [\ell - p, \ell]$  we finally obtain

$$\mathcal{P}_{\alpha_i}({}^{(\ell)}\eta)(t) = {}^{(\ell)}\eta(t) - \min_{s \in [0, t]} \langle {}^{(\ell)}\eta(s), \alpha_i^\vee \rangle = {}^{(\ell)}\eta(t) - \min_{s \in [\ell - q, t]} \langle {}^{(\ell)}\eta(s), \alpha_i^\vee \rangle.$$

Since  $[\ell - p, \ell] \subset [\ell - q, \ell]$ , this implies that the  $p$  last elementary path in the paths  $\mathcal{P}_{\alpha_i}({}^{(\ell)}\eta)$  and  $\mathcal{P}_{\alpha_i}(\eta_{\ell-q} \otimes \cdots \otimes \eta_\ell)$  (obtained by restriction to  $[\ell - p, \ell]$ ) coincide as desired.  $\square$

Recall that we have defined  $\eta^{(a)} = \overline{\omega^{(a)}}$  for any  $a = 0, \dots, r$ . For any  $a = 1, \dots, r$ , decompose the path  $\eta^{(a)} = \eta_1^{(a)} \otimes \cdots \otimes \eta_\ell^{(a)}$  as the concatenation of  $\ell$  elementary paths. The following proposition shows that there exists an integer  $\ell_r \geq 1$  such that  $\eta_\ell^{(0)}$ , the final elementary path of  $\eta^{(0)} = \mathcal{P}(\overline{\Pi_\ell(\omega)})$ , is determined by the  $\ell_r$ -last elementary paths in  $\eta^{(r)}$ .

**Proposition 8.8.**

- (i) *For any  $a = 0, \dots, r - 1$ , we have  $\mathcal{P}_{i_{a+1}} \cdots \mathcal{P}_{i_r}(\eta^{(r)}) = \eta^{(a)}$ . Hence  $\eta^{(a-1)} = \mathcal{P}_{\alpha_{i_a}}(\eta^{(a)})$  for  $a = 1, \dots, r$  and for  $a = 0$  we get  $\mathcal{P}(\eta^{(r)}) = \eta^{(0)}$ .*
- (ii) *There exists an integer  $\ell_r \geq 1$  such that for any  $\ell \geq \ell_r$ , the last elementary path in  $\eta^{(0)}$  only depends on the  $\ell_r$  last elementary paths of  $\eta^{(r)}$ .*

<sup>3</sup>Observe that the first step of  ${}^{(\ell)}\eta$  depends on  $\ell$  so one cannot define an infinite path in  $\Omega$  from the sequence  $({}^{(\ell)}\eta)_{\ell \geq 1}$ .

*Proof.* Observe first that  $\eta^{(r)}$  is a lowest weight path since  $\omega^{(r)}$  is a highest weight path and  $\eta^{(r)} = \overline{\omega^{(r)}}$ . So we have  $\mathcal{P}_{i_{a+1}} \cdots \mathcal{P}_{i_r}(\eta^{(r)}) = \overline{s_{i_{a+1}} \cdots s_{i_r}(\eta^{(r)})}$  (see assertion (iv) of the remark following Proposition 4.10). But  $\eta^{(a)} = \overline{\omega^{(a)}} = \overline{s_{i_{a+1}}^* \cdots s_{i_r}^*(\omega^{(r)})} = s_{i_{a+1}} \cdots s_{i_r}(\overline{\omega^{(r)}}) = s_{i_{a+1}} \cdots s_{i_r}(\eta^{(r)})$ . This gives assertion (i).

To prove assertion (ii), set  $\ell_0 = 1$ . We have  $\omega^{(0)} = \mathcal{P}_{\alpha_{i_1}^*}(\omega^{(1)})$  and by Lemma 8.5,  $\frac{1}{\ell} \langle \omega^{(1)}(\ell), \alpha_{i_1}^{\vee} \rangle$  tends to a positive limit. So we can apply Lemma 8.7 with  $p = 1$  and obtain an integer  $\ell_1$  such that for any  $\ell \geq \ell_1$  the last elementary path in  $\eta^{(0)}$  and  $\mathcal{P}_{\alpha_1}(\eta_{\ell-\ell_1}^{(1)} \otimes \cdots \otimes \eta_{\ell}^{(1)})$  coincide. This means that one can compute the last elementary path in  $\eta^{(0)}$  by applying  $\mathcal{P}_{\alpha_1}$  to the  $\ell_1$  last elementary paths in  $\eta^{(1)}$  and restricting to  $[\ell-1, \ell]$ . Assume by induction that we have defined a sequence  $1 = \ell_0 \leq \ell_1 \leq \cdots \leq \ell_{a-1}$  such that for any  $k = 1, \dots, a$  the  $\ell_{k-1}$  last elementary paths in  $\eta^{(k-1)}$  and  $\mathcal{P}_{\alpha_k}(\eta_{\ell-\ell_k}^{(k)} \otimes \cdots \otimes \eta_{\ell}^{(k)})$  coincide. We have  $\omega^{(a)} = \mathcal{P}_{\alpha_{i_{a+1}}^*}(\omega^{(a+1)})$  and by Lemma 8.5,  $\frac{1}{\ell} \langle \omega^{(a+1)}(\ell), \alpha_{i_{a+1}}^{\vee} \rangle$  tends to a positive limit. So we can apply Lemma 8.7 with  $p = \ell_a$  and obtain an integer  $\ell_{a+1} \geq \ell_a$  such that for any  $\ell \geq \ell_{a+1}$  the  $\ell_a$  last elementary paths in  $\eta^{(a)}$  and  $\mathcal{P}_{\alpha_{a+1}}(\eta_{\ell-\ell_{a+1}}^{(a+1)} \otimes \cdots \otimes \eta_{\ell}^{(a+1)})$  coincide. By induction we thus have obtained a sequence  $1 = \ell_0 \leq \ell_1 \leq \cdots \leq \ell_r$  such that for any  $a = 1, \dots, r$  and any  $\ell \geq \ell_r$ , the  $\ell_{a-1}$  last elementary paths in  $\eta^{(a-1)}$  are obtained by applying  $\mathcal{P}_{\alpha_a}$  to the last  $\ell_a$  elementary paths in  $\eta^{(a)}$  and restricting to  $[\ell - \ell_{a-1}, \ell]$ . This can be written

$$\eta_{\ell-\ell_{a-1}}^{(a-1)} \otimes \cdots \otimes \eta_{\ell}^{(a-1)} = \mathcal{P}_{\alpha_a}(\eta_{\ell-\ell_a}^{(a)} \otimes \cdots \otimes \eta_{\ell}^{(a)}) \downarrow_{\ell_{a-1}}$$

where  $\downarrow_{\ell_{a-1}}$  means we restrict to the last  $\ell_{a-1}$  elementary paths. This proves that the last elementary path in  $\eta^{(0)}$  can be obtained from  $\eta_{\ell-\ell_r}^{(r)} \otimes \cdots \otimes \eta_{\ell}^{(r)}$  by successive restrictions and applications of the transforms  $\mathcal{P}_{\alpha_r}, \dots, \mathcal{P}_{\alpha_1}$ . Hence the last elementary path in  $\eta^{(0)}$  only depends on the  $\ell_r$  last elementary paths in  $\eta^{(r)}$ .  $\square$

Recall that  $\omega \in \Omega_C^{\text{typ}}$  and  $\overline{m} \in \mathcal{D}_{\kappa}$ . For any  $\ell \geq 1$ , we have set  $\Pi_{\ell}(\omega) = \omega^{(r)}$  and obtained  $\eta^{(0)} = \mathcal{P}(\eta^{(r)}) = \mathcal{P}(\overline{\omega}^{(r)})$ . In particular  $\eta^{(0)}$  is the concatenation of  $\ell$  elementary paths and we can write it on the form

$$\eta^{(0)} = \eta_1^{(0)} \otimes \cdots \otimes \eta_{\ell}^{(0)}$$

where  $\eta_k^{(0)} \in B(\pi_{\kappa})$  for any  $k = 1, \dots, \ell$ . The following corollary shows that the final path  $\eta_{\ell}^{(0)}$  stabilizes when  $\ell$  tends to infinity.

**Corollary 8.9** (stabilization phenomenon). *There exists an elementary path  $\eta \in B(\pi_{\kappa})$  and an integer  $\ell_r \geq 1$  such that  $\eta_{\ell}^{(0)} = \eta$  for any  $\ell \geq \ell_r$ .*

*Proof.* Recall the decomposition of the path  $\eta^{(r)} = \eta_1^{(r)} \otimes \cdots \otimes \eta_{\ell}^{(r)}$  as the concatenation of  $\ell$  elementary paths and the similar decomposition  $\omega^{(r)} = \pi_1 \otimes \cdots \otimes \pi_{\ell}$  of  $\Pi_{\ell}(\omega) = \omega^{(r)}$ . By assertion (ii) of the previous proposition, for any  $\ell \geq \ell_r$ , the elementary path  $\eta_{\ell}^{(0)}$  only depends on  $\eta_{\ell-\ell_r}^{(r)} \otimes \cdots \otimes \eta_{\ell}^{(r)}$ . Since  $\eta^{(r)} = \overline{\omega}^{(r)}$ , we have

$$\eta_1^{(r)} \otimes \cdots \otimes \eta_{\ell}^{(r)} = \iota(\pi_{\ell}) \otimes \cdots \otimes \iota(\pi_1).$$

Hence  $\eta_{\ell}^{(0)}$  is completely determined by  $\iota(\pi_{\ell_r}) \otimes \cdots \otimes \iota(\pi_1)$  which is independent of  $\ell$ .  $\square$

**Definition 8.10.** We define the map  $\psi$  from  $\Omega_C^{\text{typ}}$  to  $B(\pi_\kappa)$  by setting  $\psi(\omega) = \eta$  where  $\eta$  is obtained from  $\omega \in \Omega_C^{\text{typ}}$  as in the previous corollary. Namely, we have

$$\psi(\omega) = \lim_{\ell \rightarrow +\infty} \mathcal{P}(\overline{\Pi_\ell(\omega)})_\ell$$

where  $\mathcal{P}(\overline{\Pi_\ell(\omega)})_\ell$  is the last elementary path in  $\mathcal{P}(\overline{\Pi_\ell(\omega)})$ .

**Remarks 8.11.** (i) By the proof of the previous corollary, one can compute  $\psi(\omega)$  from  $\omega = \pi_1 \otimes \pi_2 \otimes \cdots$  by considering the final elementary paths in the sequences of paths  $\mathcal{P}(\iota(\pi_k) \otimes \cdots \otimes \iota(\pi_1))$ ,  $k \geq 1$ . This elementary path should stabilize to  $\psi(\omega)$  as soon as  $\omega \in \Omega_C^{\text{typ}}$ .

(ii) The arguments in the proof of Proposition 8.8 also imply a stronger statement: for any fixed integer  $p \geq 1$ , the  $p$  last elementary paths in  $\eta^{(0)}$  stabilizes for  $\ell$  sufficiently large. This is illustrated by the example below.

**Example 8.12.** Assume  $\mathfrak{g} = \mathfrak{sp}_4$  and take  $\kappa = \omega_1$ . Consider  $\omega$  such that  $\Pi_{23}(\omega) = u$  with

$$u = 12\bar{2}\bar{1}112\bar{2}\bar{2}121112\bar{1}2121\bar{2}1\bar{2}\bar{2}.$$

For any  $k \in \{1 \dots 23\}$ , set  $v^{(k)} = \mathcal{P}(\overline{u_1 \cdots u_k})$  (here, we also omit for short the symbol  $\otimes$  in the paths we consider). We obtain successively for the paths  $v^{(k)}$ ,  $k = 1, \dots, 23$

$$\begin{aligned} & 1 \\ & 12 \\ & 112 \\ & 1112 \\ & 1211\bar{2} \\ & 11221\bar{2} \\ & 111221\bar{2} \\ & 1211221\bar{2} \\ & 11211221\bar{2} \\ & 111211221\bar{2} \\ & 1211211221\bar{2} \\ & 12\bar{2}\bar{2}1211221\bar{2} \\ & 112\bar{2}\bar{2}1211222\bar{2} \\ & 1112\bar{2}\bar{2}1211222\bar{2} \\ & 12112\bar{2}\bar{2}1211222\bar{2} \\ & 112112\bar{2}\bar{2}1211222\bar{2} \\ & 121\bar{2}\bar{1}12\bar{2}\bar{2}1211222\bar{2} \\ & 1121\bar{2}\bar{1}12\bar{2}\bar{2}1211222\bar{2} \\ & 12121\bar{2}\bar{1}12\bar{2}\bar{2}\bar{2}1211222\bar{2} \\ & 112122\bar{2}\bar{1}12\bar{2}\bar{2}\bar{2}1211222\bar{2}. \end{aligned}$$

**8.4. Inversion of the generalized Pitman transform.** Write  $\mathcal{W}^\iota = Y_1 \otimes Y_2 \cdots$  the dual random path with drift  $\iota(\bar{m})$ . The following proposition shows that the action of  $\psi$  on  $\mathcal{P}(\mathcal{W}^\iota)$  is easy to compute.

**Proposition 8.13.** We have  $\overline{\psi \circ \mathcal{P}(\mathcal{W}^\iota)} = Y_1$   $\mathbb{P}^\iota$ -almost surely.

*Proof.* Consider  $\Pi_\ell(\mathcal{W}^\iota) = Y_1 \otimes Y_2 \cdots \otimes Y_\ell$ . To compute  $\psi \circ \mathcal{P}(\mathcal{W}^\iota)$ , we need to obtain the limit of the last elementary path in  $\mathcal{P}(\overline{\Pi_\ell(\mathcal{P}(\mathcal{W}^\iota))})$ . Since  $\mathcal{W}^\iota$  has drift  $w_0(\bar{m})$ , we have for any simple root  $\alpha_i$ ,  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \Pi_\ell(\mathcal{W}^\iota)(\ell), \alpha_i^\vee \rangle = \langle w_0(\bar{m}), \alpha_i^\vee \rangle$   $\mathbb{P}^\iota$ -almost surely. It follows by (33)

that  $\overline{\Pi_\ell(\mathcal{W}^\iota)} = \iota(Y_\ell) \otimes \cdots \otimes \iota(Y_2) \otimes \iota(Y_1)$  verifies  $\lim_{\ell \rightarrow +\infty} \frac{1}{\ell} \langle \overline{\Pi_\ell(\mathcal{W}^\iota)}(\ell), \alpha_i^\vee \rangle = \langle \overline{m}, \alpha_i^\vee \rangle$   $\mathbb{P}$ -almost surely for any simple root  $\alpha_i$ . Also the last elementary random path in  $\overline{\Pi_\ell(\mathcal{W}^\iota)}$ , is always equal to  $\iota(Y_1)$  for any  $\ell \geq 1$ . Set

$$\mathcal{P}(\overline{\Pi_\ell(\mathcal{W}^\iota)}) = \mathcal{P}(\iota(Y_\ell) \otimes \cdots \otimes \iota(Y_2) \otimes \iota(Y_1)) = T_1 \otimes \cdots \otimes T_\ell.$$

By applying Proposition 7.2, we know that  $\lim_{\ell \rightarrow +\infty} T_\ell = \iota(Y_1)$   $\mathbb{P}$ -almost surely. Now recall that the bar involution is an anti-isomorphism of crystal graphs<sup>4</sup>. Since  $\mathcal{P}(\Pi_\ell(\mathcal{W}^\iota))$  and  $\Pi_\ell(\mathcal{W}^\iota)$  belong to the same connected component, this is also true for  $\overline{\mathcal{P}(\Pi_\ell(\mathcal{W}^\iota))}$  and  $\overline{\Pi_\ell(\mathcal{W}^\iota)}$ . Therefore

$$\mathcal{P}(\overline{\Pi_\ell(\mathcal{W}^\iota)}) = \mathcal{P}(\iota(Y_\ell) \otimes \cdots \otimes \iota(Y_2) \otimes \iota(Y_1)) = T_1 \otimes \cdots \otimes T_\ell$$

is also the highest weight path of  $\overline{\mathcal{P}(\Pi_\ell(\mathcal{W}^\iota))}$ . But we have  $\overline{\mathcal{P}(\Pi_\ell(\mathcal{W}^\iota))} = \Pi_\ell(\mathcal{P}(\mathcal{W}^\iota))$  because the  $\ell$ -first elementary paths in  $\mathcal{P}(\mathcal{W}^\iota)$  are obtained by applying the Pitman transform to the  $\ell$ -first elementary paths in  $\mathcal{W}^\iota$ . So we also have  $\overline{\mathcal{P}(\Pi_\ell(\mathcal{W}^\iota))} = \overline{\Pi_\ell(\mathcal{P}(\mathcal{W}^\iota))}$ . We therefore get

$$\mathcal{P}(\overline{\Pi_\ell(\mathcal{P}(\mathcal{W}^\iota))}) = T_1 \otimes \cdots \otimes T_\ell.$$

Since  $\lim_{\ell \rightarrow +\infty} T_\ell = \iota(Y_1)$   $\mathbb{P}$ -almost surely, this implies that  $\psi \circ \mathcal{P}(\mathcal{W}^\iota) = \iota(Y_1)$   $\mathbb{P}$ -almost surely, that is  $\psi \circ \mathcal{P}(\mathcal{W}^\iota) = Y_1$   $\mathbb{P}^\iota$  almost surely.  $\square$

**Definition 8.14.** We define the transformation  $\mathcal{P}^{-1}$  from  $\Omega_{\mathcal{C}}^{\text{stab}}$  to  $\Omega$  by setting  $\mathcal{P}^{-1}(\omega) = \mathcal{P}^{-1}(\omega)_1 \otimes \mathcal{P}^{-1}(\omega)_2 \otimes \cdots$  where

$$\mathcal{P}^{-1}(\omega)_\ell = \overline{\psi \circ J^{\ell-1}(\omega)}.$$

The following Theorem shows that the transformation  $\mathcal{P}^{-1}$  can be regarded as the inverse of the generalized Pitman transform  $\mathcal{P}$ . Observe that, contrary to  $\mathcal{P}$  which is defined on all the trajectories of  $\Omega$ , the transformation  $\mathcal{P}^{-1}$  is only defined on  $\Omega_{\mathcal{C}}^{\text{stab}}$ , that is  $\mathbb{Q}$ -almost surely since  $\mathbb{Q}(\Omega_{\mathcal{C}}^{\text{stab}}) = 1$ . Recall that for both random trajectories  $\mathcal{W}^\iota$  and  $\mathcal{W}$ , we have  $\mathcal{H} = \mathcal{P}(\mathcal{W}) = \mathcal{P}(\mathcal{W}^\iota)$ .

**Theorem 8.15.** Assume  $\overline{m} \in \mathcal{D}_\kappa$ . Then we have

- (i)  $\mathcal{P}^{-1} \circ \mathcal{P}(\mathcal{W}^\iota) = \mathcal{W}^\iota$   $\mathbb{P}^\iota$ -almost surely,
- (ii) We have  $\mathcal{P}^{-1}(\mathcal{H}) = Y_1 \otimes Y_2 \otimes \cdots$  where the sequence of random variable  $(Y_\ell)_{\ell \geq 1}$  is i.i.d. and each variable  $Y_\ell, \ell \geq 1$  has law  $Y$  as defined in (32).
- (iii)  $\mathcal{P} \circ \mathcal{P}^{-1}(\mathcal{H}) = \mathcal{H}$   $\mathbb{Q}$ -almost surely.

*Proof.* (i) Write  $\mathcal{W}^\iota = Y_1 \otimes Y_2 \otimes \cdots$ . We can set  $\mathcal{P}^{-1} \circ \mathcal{P}(\mathcal{W}^\iota) = U_1 \otimes U_2 \otimes \cdots$  where for any  $\ell \geq 1$ ,  $U_\ell = \overline{\psi \circ J^{\ell-1}(\mathcal{P}(Y_1 \otimes Y_2 \otimes \cdots))}$ . Now by Theorem 7.1, we have  $U_\ell = \overline{\psi \circ \mathcal{P}(Y_\ell \otimes Y_{\ell+1} \otimes \cdots)}$ . By Proposition 8.13, we obtain  $U_\ell = Y_\ell$   $\mathbb{P}^\iota$ -almost surely for any  $\ell \geq 1$ . This proves that  $\mathcal{P}^{-1} \circ \mathcal{P}(\mathcal{W}^\iota) = \mathcal{W}^\iota$   $\mathbb{P}^\iota$ -almost surely.

Since  $\mathcal{P}(\mathcal{W}^\iota) = \mathcal{H}$ , we have  $\mathcal{P}^{-1}(\mathcal{H}) = \mathcal{P}^{-1} \circ \mathcal{P}(\mathcal{W}^\iota)$ . By assertion (i), this means that  $\mathcal{P}^{-1}(\mathcal{H}) = \mathcal{W}^\iota$  which proves assertion (ii).

To prove assertion (iii), set  $\mathcal{H} = \mathcal{P}(\mathcal{W}^\iota)$  so that  $\mathcal{P} \circ \mathcal{P}^{-1}(\mathcal{H}) = \mathcal{P} \circ \mathcal{P}^{-1} \circ \mathcal{P}(\mathcal{W}^\iota)$ . By using assertion (i), we get  $\mathcal{P} \circ (\mathcal{P}^{-1} \circ \mathcal{P})(\mathcal{W}^\iota) = \mathcal{P}(\mathcal{W}^\iota)$   $\mathbb{P}^\iota$ -almost surely which is equivalent to  $\mathcal{P} \circ \mathcal{P}^{-1}(\mathcal{H}) = \mathcal{H}$   $\mathbb{Q}$ -almost surely since  $\mathcal{P}(\mathcal{W}^\iota) = \mathcal{H}$ .  $\square$

<sup>4</sup>Here we need that the bar involution matches the connected components

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