Typing weak MSOL properties

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Abstract

We consider non-interpreted functional programs: the result of the execution of a program is its normal form, that can be seen as the tree of calls to built-in operations. Weak monadic second-order logic (wMSO) is well suited to express properties of such trees. This is an extension of first order logic with quantification over finite sets. Many behavioral properties of programs can be expressed in wMSO. We use the simply typed lambda calculus with the fixpoint operator, λY-calculus, as an abstraction of functional programs that faithfully represents the higher-order control flow.

We give a type system for ensuring that the result of the execution of a λY-program satisfies a given wMSO property. The type system is an extension of a standard intersection type system with both: the least-fixpoint rule, and a restricted version of the greatest-fixpoint rule. In order to prove soundness and completeness of the system we construct a denotational semantics of λY-calculus that is capable of computing properties expressed in wMSO. The model presents many symmetries reflecting dualities in the logic and has also other applications on its own. The type system is obtained from the model following the domain in logical form approach.

Categories and Subject Descriptors D [*2*]: 4; F [*3*]: 1; F [*4*]: 2

1. Introduction

We are interested in dynamics of program behaviors rather than in the final result. The analysis presented here aims particularly at programs that interact with their environment without ever terminating. The objective is to ensure, or verify, properties of the behavior of a program without actually running it. We present a kind of type and effect discipline [26] where a well typed program will satisfy behavioral properties expressed in weak monadic second-order logic (wMSO).

We consider the class of programs written in the simply-typed calculus with recursion and finite base types: λY-calculus. This calculus offers an abstraction of higher-order programs that faithfully represents higher-order control. The dynamics of an interaction of a program with its environment is represented by the Böhm tree of a λY-term that is a tree reflecting the control flow of the program. For example, the Böhm tree of the term $Yx.ax$ is the infinite sequence of a's, representing that the program does an infinite sequence of a actions without ever terminating. Another example is presented in Figure 1. A functional program for the factorial function is written as a λY-term Fct and the value of Fct applied to a constant c is calculated. Observe that all constants in Fct are non-interpreted. The Böhm tree semantics means call-by-name evaluation strategy. Nevertheless, call-by-value evaluation can be encoded, so can be finite data domains, and conditionals over them [13, 20]. The approach is then to translate a functional program to a λY-term and to examine the Böhm tree it generates.

$$Factorial(x) \;\equiv\; \mathbf{if}\ x = 0\ \mathbf{then}\ 1\ \mathbf{else}\ x \cdot Factorial(x-1)$$

$$Fct \;\equiv\; YF.\,\lambda x.\,\mathbf{if}\ (z\ x)\ 1\ (m\ x\ (Fct(-\ x\ 1))))$$

$BT(Fct\ c)$ is

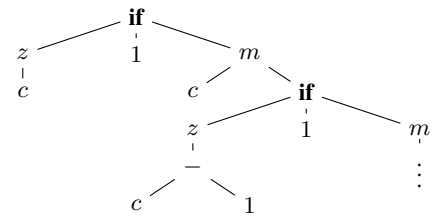
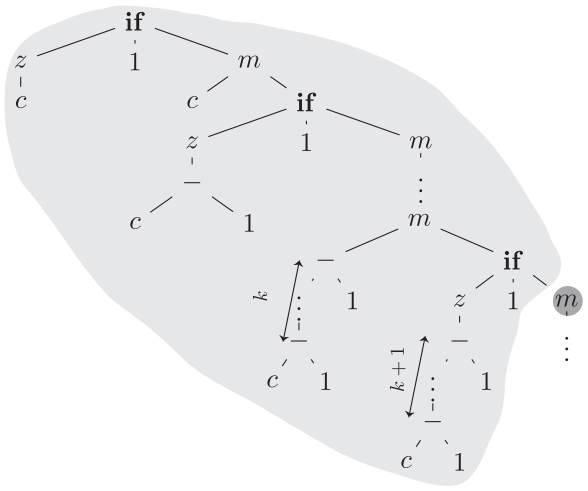


Figure 1. Böhm tree of the factorial function

Since the dynamics of the program is represented by a potentially infinite tree, monadic second-order logic (MSOL) is a natural candidate for the language to formulate properties in. This logic is an extension of first-order logic with quantification over sets. MSOL captures precisely regular properties of trees [29], and it is decidable if the Böhm tree generated by a given λY-term satisfies a given property [27]. In this paper we will restrict to weak monadic second-order logic (wMSO). The difference is that in wMSO quantification is restricted to range over finite sets. While wMSO is a proper fragment of MSO, it is sufficiently strong to express safety, reachability, and many liveness properties. Over sequences, that is degenerated trees where every node has one successor, wMSO is equivalent to full MSO.

The basic judgments we are interested in are of the form $BT(M) \models \alpha$ meaning that the result of the evaluation of M, i.e. the Böhm tree of M, has the property α formulated in wMSO. Going back to the example of the factorial function from Figure 1, we can consider formula $\alpha \equiv \forall X.\mathit{finite}(X)$ saying that the whole tree is finite. Of course this formula is not true in $BT(Fct\ c)$. Observe that in α we have used quantification $\forall X$ over sets of nodes in the tree, and a predicate finite over sets of nodes. This predicate is definable in the logic. A more complicated property would be "all computations that eventually take the "if" branch of the conditional are finite". This property holds in $BT(Fct\ c)$. The formula describing the property is presented in Figure 2. It says that for all nodes x that are labeled by m, the set of nodes that is not below x is finite. The nodes not below x are those that contribute to the computation that takes "if" branch just before x. Observe by the way that $BT(Fct\ c)$ is not regular – it has infinitely many non-isomorphic subtrees as the number of subtractions is growing. In general the interest of judgments of the form $BT(M) \models \alpha$ is to

be able to express liveness and fairness properties of executions, like: "every *open* action is eventually followed by a *close* action", or that "there are infinitely many read actions". Various other verification problems for functional programs can be reduced to this problem [12, 20, 22, 28, 37].



$$\forall x.\ m(x) \Rightarrow (\forall Z.\ notbelow(Z, x) \Rightarrow finite(Z))$$

Figure 2. A wMSO property of a Böhm tree. A possible valuation of quantified variables x and Z is highlighted.

Technically, the judgment $BT(M) \vDash \alpha$ is equivalent to determining whether a Böhm tree of a given λY-term is accepted by a given weak alternating automaton. This problem is known to be decidable thanks to the result of Ong [27], but here we present a denotational approach. Our two main contributions are:

- A construction of a finitary model for a given weak alternating automaton. The value of a term in this model determines if the Böhm tree of the term is accepted by the automaton. So verification is reduced to evaluation.
- Two typing systems. A typing system deriving statements of the form "the value of a term M is bigger than an element d of the model"; and a typing system for dual properties. These typing systems use standard fixpoint rules and follow the methodology coined as *Domains in Logical Form* [1]. Thanks to the first item, these typing systems can directly talk about acceptance/rejection of the Böhm tree of a term by an automaton. These type systems are decidable, and every term has a "best" type that is simply its value in the model.

Having a model and a typing system has several advantages over having just a decision procedure. First, it makes verification compositional: the result for a term is calculated from the results for its subterms. In particular, it opens possibilities for a modular approach to the verification of large programs. Next, it allows semantic based program transformations as for example reflection of a given property in a given term [8, 13, 32]. It also implies the transfer theorem for wMSO [33] with a number of consequences offered by this theorem. Finally, models open a way to novel verification algorithms be it through evaluation, type system, or through hybrid algorithms using typing and evaluation at the same time [36].

Historically, Muller and Schupp [24] have proved that the MSO theory of trees generated by pushdown automata is decidable. Knapik, Niwinski, and Urzyczyn [18] have established the relation between higher-order pushdown automata and λY-calculus by showing that Böhm trees of *safe* λY-terms are precisely trees generated by such automata. They have also established the decidability of the MSO theory of such trees. Ong [27] has shown the decidability of the MSO theory of Böhm trees for all λY-terms. This last result has been revisited in several different ways. Some approaches take a term of the base type, and unroll it to some infinite object: tree with pointers [27], computation of a higher-order pushdown automaton with collapse [14], a collection of typing judgments that are used to define a game [21], a computation of a Krivine machine [31]. Very recently Tsukada and Ong [38] have presented a compositional approach: a typing system is used to reduce the verification problem to a series of game solving problems. Another recent advance is given by Hofmann and Chen who provide a type system for verifying path properties of trees generated by first-order λY-terms [11]. In other words, this last result gives a typing system for verifying path properties of trees generated by deterministic pushdown automata. Compared to this last work, we consider the whole λY-calculus and an incomparable set of properties.

Already some time ago, Aeligh [2] has discovered an easy way to prove Ong's theorem restricted to properties expressed by tree automata with trivial acceptance conditions (TAC automata). The core of his approach can be formulated by saying that the verification problem for such properties can be reduced to evaluation in a specially constructed and simple model. Building on this result Kobayashi proposed a type system for such properties and constructed a tool based on it [20]. This in turn opened a way to an active ongoing research resulting in the steady improvement of the capacities of the verification tools [9, 10, 19, 30]. TAC automata can express only safety properties. Our model and typing systems set the stage for practical verification of wMSO properties.

The model approach to verification of λY-calculus is quite recent. In [32] it is shown that simple models with greatest fixpoints capture exactly properties expressed with TAC automata. An extension is then proposed to allow to detect divergence. The simplicity offered by models is exemplified by Haddad's recent work [13] giving simple semantic based transformations of λY-terms.

We would also like to mention two other quite different approaches to integrate properties of infinite behaviors into typing. Naik and Palsberg [25] make a connection between model-checking and typing. They consider only safety properties, and since their setting is much more general than ours, their type system is more complex too. Jeffrey [16, 17] have shown how to incorporate Linear Temporal Logic into to types using a much richer dependent types paradigm. The calculus is intended to talk about control and data in functional reactive programming framework, and aims at using SMT solvers.

In the next section we introduce the main objects of our study: λY-calculus, weak alternating automata, and models. The central definition of a model recognizing a set of terms is also formulated in this section. Section 3 presents the two type systems and states their soundness and completeness properties. The soundness and completeness can be straightforwardly formulated only for closed terms of atomic type. For the proof though we need a statement about all terms. This is where the model based approach helps. Section 4 presents a model construction and proves some of its basic properties. We then show in Section 5 that our models indeed capture weak alternating automata (Theorem 17). In Section 6 we come back to our type systems. The model construction allows us to formulate a more general soundness and completeness property: we show that types can denote every element of the model, and that the type systems can derive precisely the judgments that hold in the model (Theorem 25). In the conclusion section we mention other applications of our model. Most of the proofs are included.

2. Preliminaries

We quickly fix notations related to the simply typed λY-calculus and to Böhm trees. We then recall the definition of weak alternating automata on ranked trees. These will be used to specify properties of Böhm trees. Finally, we introduce the notion of the greatest fixpoint models for the λY-calculus. This notion allows us to adapt the definition of recognizability from language theory, so models can be used to define sets of terms. These sets of terms are closed under reductions of the λY-calculus, moreover the meaning of a term and its Böhm tree are the same in a model. We recall the characterization, in terms of automata, of the sets of terms recognizable by the greatest fixpoint models.

λY-calculus

The *set of types* $\mathcal{T}$ is constructed from a unique *basic type* o using a binary operation $\to$ that associates to the right. Thus o is a type and if A, B are types, so is $(A \to B)$. The order of a type is defined by: $order(o) = 0$, and $order(A \to B) = max(1 + order(A), order(B))$. We work with *tree signatures* that are finite sets of *typed constants of order at most* 1. Types of order 1 are of the form $o \to \cdots \to o \to o$ that we abbreviate $o^i \to o$ when they contain $i+1$ occurrences of o. For convenience we assume that $o^0 \to o$ is just o. If Σ is a signature, we write Σ^i for the set of constants of type $o^i \to o$. In examples we will often use constants of type $o \to o$ as this makes the examples more succinct. In proofs we will sometimes restrict to type $o^2 \to o$ that is representative for the general case.

The set of *simply typed λY-terms* is built from the constants in the signature, and constants Y^A, Ω^A for every type A. These stand for the *fixpoint combinator* and *undefined term*, respectively. Apart from constants, for each type A there is a countable set of variables $x^A, y^A, \ldots$. Terms are built from these constants and variables using typed application and λ-abstraction. We shall write sequences of λ-abstractions $\lambda x_1 \ldots \lambda x_n.\ M$ with only one λ: either $\lambda x_1 \ldots x_n.\ M$, or even shorter $\lambda \vec{x}.\ M$. We will often write $Yx.M$ instead of $Y(\lambda x.M)$. Every λY-term can be written in this notation since YN has the same Böhm tree as $Y(\lambda x.Nx)$, and the later term is $Yx.(Nx)$. We take for granted the operational semantics of the calculus given by β and δ reductions.

Definition 1 A *Böhm tree* of a term M is obtained in the following way.

- If $M \to^*_{\beta\delta} \lambda\vec{x}.N_0 N_1 \ldots N_k$ with N_0 a variable or a constant then $BT(M)$ is a tree having its root labeled by $\lambda\vec{x}.N_0$ and having $BT(N_1), \ldots, BT(N_k)$ as subtrees.
- Otherwise $BT(M) = \Omega^A$, where A is the type of M.

Böhm trees are infinite normal forms of λY-terms. It is immediate from the definition that a Böhm tree of a closed term of type o over a tree signature is a potentially infinite ranked tree: a node labeled by a constant a of type $o^i \to o$ has i successors.

Example : As a simple example, let us take $M = \lambda x.ax$. We have a reduction sequence

$$YM \to_\delta (\lambda x.ax)(YM) \to_\beta a(YM) \to_{\beta\delta} a(a(YM)) \to \ldots$$

So $BT(YM)$ is the infinite sequence $aa\ldots$

For a more complicated example take $(YF.\ N)a$ where $N = \lambda g.g(b(F(\lambda x.g(g\,x))))$. Both a and b have the type $o \to o$; while F has type $(o \to o) \to o$, and so does N. Observe that we are using a more convenient notation YF here. The Böhm tree of $(YF.N)a$ is

$$BT((YF.N)a) = aba^2ba^4b \ldots a^{2^n}b \ldots$$

after every consecutive occurrence of b the number of occurrences of a doubles because of the double application of g inside N.

wMSO and weak alternating automata

We will be interested in properties of trees expressed in weak monadic second-order logic. This is an extension of first-order logic with quantification over finite sets of elements. The interplay of negation and quantification allows the logic to express many infinitary properties. The logic is closed for example under constructs: "for infinitely many vertices a given property holds", "every path consisting of vertices having a given property is finite". From the automata point of view, the expressive power of the logic is captured by weak alternating automata.

A *weak alternating automaton* accepts trees over a fixed tree signature Σ.

Definition 2 A *weak alternating tree automaton* over the signature Σ is:

$$\mathcal{A} = \langle Q, \Sigma, q^0 \in Q, \{\delta_i\}_{i\in\mathbb{N}}, \rho : Q \to \mathbb{N}\rangle$$

where Q is a finite set of states, $q^0 \in Q$ is the initial state, ρ is the *rank function*, and $\delta_i : Q \times \Sigma^i \to \mathcal{P}(\mathcal{P}(Q)^i)$ is the transition function. For q in Q, we call $\rho(q)$ *its rank*. The automaton is *weak* in the sense that when $(S_1, \ldots, S_i)$ is in $\delta_i(q, a)$, then the rank of every q' in $\bigcup_{1\le j\le i} S_j$ is not bigger than the rank of q, $\rho(q') \le \rho(q)$.

Observe that since Σ is finite, only finitely many δ_i are nontrivial. From the definition it follows that $\delta_2 : Q \times \Sigma^2 \to \mathcal{P}(\mathcal{P}(Q) \times \mathcal{P}(Q))$ and $\delta_0 : Q \times \Sigma^0 \to \{0, 1\}$. We will simply write δ without a subscript when this causes no ambiguity. The weakness requirement means that in the run the ranks cannot increase.

Automata will work on Σ-labeled binary trees that are partial functions $t : \mathbb{N}^* \dot{\to} \Sigma \cup \{\Omega\}$ such that the number successors of a node is determined by the label of the node. In particular, if $t(u) \in \Sigma^0 \cup \{\Omega\}$ then u is a leaf.

The acceptance of a tree is defined in terms of *games* between two players that we call Eve and Adam. A *play* between Eve and Adam from some node v of a tree t and some state $q \in Q$ proceeds as follows. If v is a leaf and is labeled by some $c \in \Sigma_o$ then Eve wins iff $\delta_o(q, c)$ holds. If the node is labeled by Ω then Eve wins iff the rank of q is even. Otherwise, v is an internal node; Eve chooses a tuple of sets of states $(S_1, \ldots, S_i) \in \delta(q, t(v))$. Then Adam chooses S_j (for $j = 1, \ldots, i$) and a state $q' \in S_j$. The play continues from the j-th son of v and state q'. If the play is infinite then the winner is decided by looking at ranks of states appearing on the play. Due to the weakness of $\mathcal{A}$ the rank of states in a play can never increase, so it eventually stabilizes at some value. Eve wins if this value is even. A tree t is *accepted* by $\mathcal{A}$ from a state $q \in Q$ if Eve has a winning strategy in the game started from the root of t and from q.

Observe that without a loss of generality we can assume that δ is monotone, i.e. if $(S_1, \ldots, S_i) \in \delta(q, a)$ then for every $(S'_1, \ldots, S'_i)$ such that $S_j \subseteq S'_j \subseteq \{q' : \rho(q') \le \rho(q)\}$ we have $(S'_1, \ldots, S'_i) \in \delta(q, a)$. Indeed, adding the transitions needed to satisfy the monotonicity condition does not give Eve more winning possibilities.

An automaton defines a language of closed terms of type o whose Böhm trees it accepts from its initial state q^0:

$$L(\mathcal{A}) = \{M : M \text{ is closed term of type } o, \text{ and } BT(M) \text{ is accepted by } \mathcal{A} \text{ from } q^0\}$$

Observe that $L(\mathcal{A})$ is closed under $\beta\delta$-conversion since the Church-Roser property of the calculus implies that two $\beta\delta$-convertible terms have the same Böhm tree.

Example Consider a weak alternating automaton $\mathcal{A}$ defining the property "action b appears infinitely often". The automaton has states $Q = \{q_1, q_2\}$, and the signature $\Sigma = \{a, b\}$ consisting of two constants of type $o \to o$. Over this signature, the Böhm trees are just sequences. The transitions of $\mathcal{A}$ are:

$$\delta(q_1, a) = \{q_1\} \qquad \delta(q_2, a) = \{q_1, q_2\}$$
$$\delta(q_1, b) = \emptyset \qquad \delta(q_2, b) = q_2$$

The ranks of states are indicated by their subscripts. When started in q_2 the automaton spawns a run from q_1 each time it sees letter a. The spawned runs must stop in order to accept, and they stop when they see letter b (cf. Figure 3). So every a must be eventually followed by b.

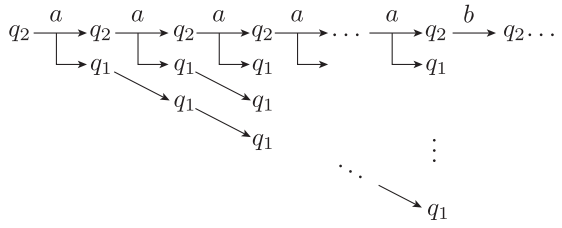


Figure 3. A run of an alternating automaton on $a \dots ab \dots$

Models

We use standard notions and notations for models for λY-calculus, in particular for *valuation/variable assignment* and of *interpretation of a term* (see [15]). We shall write $[\![M]\!]^{\mathcal{S}}_{\nu}$ for the interpretation of a term M in a model $\mathcal{S}$ with the valuation ν. As usual, we will omit subscripts or superscripts in the notation of the semantic function if they are clear from the context.

The simplest models of λY-calculus are based on monotone functions. A *GFP-model* of a signature Σ is a tuple $\mathcal{S} = \langle \{\mathcal{S}_A\}_{A \in \mathcal{T}}, \rho \rangle$ where $\mathcal{S}_o$ is a finite lattice, called the *base set* of the model, and for every type $A \to B \in \mathcal{T}$, $\mathcal{S}_{A \to B}$ is the lattice $\mathrm{mon}[\mathcal{S}_A \mapsto \mathcal{S}_B]$ of monotone functions from $\mathcal{S}_A$ to $\mathcal{S}_B$ ordered coordinate-wise. The valuation function ρ is required to interpret Ω^A as the greatest element of $\mathcal{S}_A$, and Y^A as the greatest fixpoint of functions in $\mathcal{S}_{A \to A}$. Observe that every $\mathcal{S}_A$ is finite, hence all the greatest fixpoints exist without any additional assumptions on the lattice.

We can now adapt the definition of recognizability by semigorups taken from language theory to our richer models.

Definition 3 A GFP model $\mathcal{S}$ over the base set $\mathcal{S}_o$ *recognizes a language* L of closed λY-terms of type o if there is a subset $F \subseteq \mathcal{S}_o$ such that $L = \{M \mid [\![M]\!]^{\mathcal{S}} \in F\}$.

An direct consequence of Statman's finite completeness theorem [35] is that such models can characterize a term up to equality: $BT(M) = BT(N)$ iff the values of M and N are the same in every monotone models. This property is sufficient for our purposes. The celebrated result of Loader [23] implies that we cannot hope for a much stronger completeness property, and have good algorithmic qualities at the same time.

The following theorem characterizes the recognizing power of GFP models.

Theorem 4 ([32]) *A language L of λY-terms is recognized by a GFP-model iff it is a boolean combination of languages recognized by weak automata whose all states have rank* 0.

3. Type systems for wMSOL

In this section we describe the main result of the paper. We present two type systems to reason about wMSO properties of Böhm trees of terms. We will rely on the equivalence of wMSO and weak alternating automata, and construct a type system for an automaton. For a fixed weak alternating automaton $\mathcal{A}$ we want to characterize the terms whose Böhm trees are accepted by $\mathcal{A}$, i.e. the set $L(\mathcal{A})$. The characterization will be purely type theoretic (cf. Theorem 5).

Fix an automaton $\mathcal{A} = \langle Q, \Sigma, q^0, \{\delta_i\}_{i \in \mathbb{N}}, \rho \rangle$. Let m be the maximal rank, i.e., the maximal value ρ takes on Q. For every $0 \leq k \leq m$ we write

$$Q_k = \{q \in Q : \rho(q) = k\} \quad \text{and} \quad Q_{\leq k} = \{q \in Q : \rho(q) \leq k\} .$$

The type system we propose is obtained by allowing to use intersections inside simple types. This idea has been used by Kobayashi [20] to give a typing characterization for languages of automata with trivial acceptance conditions. We work with, more general, weak acceptance conditions, and this will be reflected in stratification of types, and two fixpoint rules: greatest fixpoint rule for even strata, and the least fixpoint rule for odd strata.

First, we define the sets of intersection types. They are indexed by a rank of the automaton and by a simple type. In particular, every intersection type will have a corresponding simple type. This is a crucial difference with intersection types characterizing strongly normalizing terms [4].

$$\mathrm{types}^k_o = \{q \in Q : \rho(q) = k\} ,$$
$$\mathrm{types}^k_{A \to B} = \{T \to s : T \subseteq \mathrm{Types}^k_A \text{ and } s \in \mathrm{types}^k_B\} ,$$
$$\mathrm{Types}^k_A = \bigcup_{0 \leq l \leq k} \mathrm{types}^l_A .$$

The difference with simple types is that now we have a set constructor that will be interpreted as the intersection of its elements.

Example Suppose that in Q we have states q_0, r_0, q_1 with ranks given by their subscripts. A type $\{q_0, r_0\} \subseteq \mathrm{Types}^0_o$ will type terms whose Böhm tree is accepted both from q_0 and from r_0. A type $\{\{q_0, r_0\} \to \{q_1\}\} \subseteq \mathrm{types}^1_{o \to o}$ will type terms that when given a term of type $\{q_0, r_0\}$ produce a Böhm tree accepted from q_1. Observe that, for example, $\{\{q_0, r_0\} \to \{q_1\}, \{q_1\} \to \{q_1\}\} \subseteq \mathrm{Types}^1_{o \to o}$ while $\{\{q_1\}, \{q_1\} \to \{q_1\}\}$ is not an intersection type in our sense since the two types in the set have different underlying simple types.

When we write types_A or Types_A we mean types^m_A and Types^m_A respectively; where m is the maximal rank used by the automaton $\mathcal{A}$.

For $S \subseteq \mathrm{Types}^k_A$ and $T \subseteq \mathrm{types}^k_B$ we write $S \to T$ for $\{S \to t : t \in T\}$. Notice that $S \to T$ is included in $\mathrm{types}^k_{A \to B}$. Given $S \subseteq \mathrm{Types}_{A \to B}$ and $T \subseteq \mathrm{Types}_A$ we write $S(T)$ for the set $\{t : (U \to t) \in S \wedge U \sqsubseteq T\}$.

We now give subsumption rules that express the intuitive dependence between types. So as to make the connection with the model construction later, we have adopted an ordering of intersection types that is dual to the usual one.

$$\frac{S \subseteq T \subseteq Q}{S \sqsubseteq_0 T} \qquad \frac{\forall s \in S, \exists t \in T, s \sqsubseteq_A t}{S \sqsubseteq_A T}$$
$$\frac{s = t}{s \sqsubseteq_0 t} \qquad \frac{T \sqsubseteq_A S \quad s \sqsubseteq_B t}{S \to s \sqsubseteq_{A \to B} T \to t}$$

The typing system presented in Figure 4 derives judgments of the form $\Gamma \vdash M \geq S$ where Γ is an environment containing all free variables of the term M, and $S \subseteq \mathrm{Types}_A$ with A the type of M.

As usual, an environment Γ is a finite list $x_1 \geq S_1, \ldots, x_n \geq S_n$ where $x_1 \ldots, x_n$ are pairwise distinct variables of type A_i, and $S_i \subseteq \mathrm{Types}_{A_i}$. We will use a functional notation and write $\Gamma(x_i)$ for S_i. We shall also write $\Gamma, x \geq S$ with its usual meaning.

The rules in the first row of Figure 4 express standard in intersection types dependencies between typing and the subsumption on types. The rules in the second line are specific to our fixed automaton: they express the meaning of constraints. For clarity we have separated the case of constants of type o. The third line contains the usual rules for application and abstraction. The least fixpoint rule in the next line is standard. Observe that we use Yx notation as explained on page 3. The greatest fixpoint rule in the last line is more intricate. It is allowed only on even strata. If taken for $k = 0$ the rule becomes the standard rule for the greatest fixpoint. For $k > 0$ the rule permits to incorporate T that is the result of the fixpoint computation on the lower level. Intuitively, since we consider judgments of the form $\Gamma \vdash M \geq q$, the least fixpoint rule can be safely applied in any context, while the greatest fixpoint rule needs to be restricted as otherwise we would be back in the case of automata with trivial acceptance conditions from Theorem 4.

The main result of the paper says that the typing in this system is equivalent to accepting with our fixed weak alternating automaton.

Theorem 5 *For every closed term M of type o and every state q of $\mathcal{A}$: judgment $\vdash M \geq q$ is derivable iff $\mathcal{A}$ accepts $BT(M)$ from q.*

Observe that typing is decidable. For a fixed automaton, there are finitely many different types, and moreover the size of the derivation tree is bounded by the size of the term being typed. From an interpretation of judgments in a model it will immediately follow that every term has a "best type", that is the type denoting its value in the model.

In order to prove Theorem 5 we will need to formulate and prove an extension of it to terms of all types (Theorem 25). To describe the properties of the type system in higher types, we will construct a model from our fixed automaton $\mathcal{A}$, and show (Theorem 17) that the model recognizes $L(\mathcal{A})$ in the sense of Definition 3. Then Theorem 25 will say that the type system reflects the values of the terms in the model.

Due to the symmetries in weak alternating automata, and in the model we are going to construct, we will obtain also a dual type system. This system can be used to show that the Böhm tree of a term is not accepted by the automaton.

The dual type system is presented in Figure 7 on page 12. The notation is as before but we now define $S(T)$ to be $\{s : U \to s \in S \wedge U \sqsupseteq T\}$. The rules for application, abstraction and variable do not change and are not presented in Figure 7. By duality we obtain

Corollary 6 For every closed term M of type o and every state q of $\mathcal{A}$: judgment $\vdash M \not\geq q$ is derivable iff $\mathcal{A}$ does not accept $BT(M)$ from q.

So the two type systems together allow us to derive both positive and negative information about a program.

We finish this section with two moderate size examples of typing derivations.

Example 1 Take a signature consisting of two constants $c : o$ and $a : o \to o$. We consider an extremely simple weak alternating automaton with just one state q of rank 1 and transitions:

$$\delta(q, a) = \{q\} \qquad \delta(q, c) = \emptyset\ .$$

This automaton accepts the finite sequences of a's ending in c. Observe that these transition rules give us typing axioms

$$\frac{}{\vdash a \geq \{q\} \to q} \qquad \frac{}{\vdash c \geq q}$$

Notice that we omit some set parenthesis over singletons; so for example we write $c \geq q$ instead of $c \geq \{q\}$. In this example we will still keep the parenthesis to the left of the arrow to underline that we are in our type system, and not in simple types. In Example 2 we will omit them too.

First, let us look at a term $G_1 \equiv \lambda f^{o \to o} \lambda x^o.\ f(fx)$. It has a simple type $\tau_1 \equiv (o \to o) \to o \to o$. The judgment $\vdash G_1 \geq \gamma_1$ is derivable in our system; where $\gamma_1 \equiv \{\{q\} \to q\} \to \{q\} \to q$.

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash f \geq \{q\} \to q} \quad \dfrac{\Gamma \vdash f \geq \{q\} \to q \quad \Gamma \vdash x \geq q}{\Gamma \vdash fx \geq q}}{\Gamma \vdash f(fx) \geq q}}{\vdash \lambda f \lambda x.\ f(fx) \geq \gamma_1}$$

here $\Gamma \equiv f \geq \{q\} \to q,\ x \geq q$.

Consider now $G_2 \equiv \lambda f^{\tau_1} \lambda x^{o \to o}.\ f(fx)$. We have that G_2 is of type $\tau_2 \equiv \tau_1 \to \tau_1$. A derivation very similar to the above will show $\vdash G_2 \geq \gamma_2$ where $\gamma_2 \equiv \{\gamma_1\} \to \gamma_1$.

Then by induction we can define $G_i \equiv \lambda f^{\tau_{i-1}} \lambda x^{\tau_{i-2}}.\ f(fx)$ that is of a type $\tau_i \equiv \tau_{i-1} \to \tau_{i-1}$. Still a similar derivation as above will show $\vdash G_i \geq \gamma_i$ where $\gamma_i \equiv \{\gamma_{i-1}\} \to \gamma_{i-1}$.

One use of application rule then shows that $G_i G_{i-1} \geq \gamma_{i-1}$:

$$\frac{\vdash G_i \geq \{\gamma_{i-1}\} \to \gamma_{i-1} \quad \vdash G_{i-1} \geq \gamma_{i-1}}{\vdash G_i G_{i-1} \geq \gamma_{i-1}}$$

In consequence, we can construct by induction a derivation of

$$\vdash G_i G_{i-1} \ldots G_1 ac \geq q$$

This derivation proves that the Böhm tree of $G_i G_{i-1} \ldots G_1 ac$ is a sequence of a's ending in c. While the length of this sequence is a tower of exponentials in the height i, the typing derivation we have constructed is linear in i (if types are represented succinctly). This simple example, already analyzed in [20], shows the power of modular reasoning provided by the typing approach. We should note though that if the initial automaton had two states, the number of potential types would also be roughly the tower of exponentials in i. Due to the complexity bounds [36], there are terms and automata for which there is no small derivation. Yet one can hope that in many cases a small derivation exists. For example, if we wanted to show that the length of the sequence is even then automaton would have two states but the derivation would be essentially the same.

Example 2 In the first example we have not used the fixpoint rules. To see them at work consider once again the term $(YF.N)a$ where $N = \lambda g.g(b(F(\lambda x.g(g\,x))))$. As we have seen in the example on page 2, the Böhm tree of $(YF.N)a$ is the sequence $aba^2ba^4b \ldots a^{2^n}b \ldots$. We will construct a typing derivation showing that there are infinitely many occurrences of b in the Böhm tree of $(YF.N)a$. To this end we take the automaton from the example on page 4 expressing exactly this property. So our goal is to derive

$$\vdash (YF.N)a \geq q_2$$

First observe that from the definition of the transitions of the automaton we get axioms:

$$\frac{}{\vdash a \geq q_1 \to q_1} \quad \frac{}{\vdash a \geq \{q_1, q_2\} \to q_2} \quad \frac{}{\vdash b \geq \emptyset \to q_1} \quad \frac{}{\vdash b \geq q_2 \to q_2}$$

Looking at the typings of a, we can see that we will get our desired judgment from the application rule if we prove

$$\vdash YF.N \geq S \qquad \text{where } S \text{ is } \{q_1 \to q_1, \{q_1, q_2\} \to q_2\} \to q_2.$$

To this end, we apply weakening rule and the greatest fixpoint rule:

$$\frac{}{\Gamma, x \geq S \vdash x \geq S} \qquad \frac{\Gamma \vdash M \geq S \quad \Gamma \vdash M \geq T}{\Gamma \vdash M \geq S \cup T} \qquad \frac{\Gamma \vdash M \geq S \quad T \sqsubseteq S}{\Gamma \vdash M \geq T}$$

$$\frac{}{\Gamma \vdash c \geq \{q : \delta_o(q, c) \text{ holds}\}} \qquad \frac{(S_1, \ldots, S_i) \in \delta(a, q)}{\Gamma \vdash a \geq \{S_1 \to \cdots \to S_i \to q\}}$$

$$\frac{\Gamma \vdash M \geq S \quad \Gamma \vdash N \geq T}{\Gamma \vdash MN \geq S(T)} \qquad \frac{S \subseteq \text{Types}^k,\, T \subseteq \text{types}^k \quad \Gamma, x \geq S \vdash M \geq T}{\Gamma \vdash \lambda x.M \geq S \to T}$$

$$\frac{\Gamma \vdash (\lambda x.M) \geq S \quad \Gamma \vdash (Yx.M) \geq T}{\Gamma \vdash Yx.M \geq S(T)}\, Y \textit{ odd}$$

$$\frac{S \subseteq \text{types}_A^{2k}, \quad T \subseteq \text{Types}_A^{2k-1}, \quad \Gamma \vdash \lambda x.M \geq (S \cup T) \to S \quad \Gamma \vdash Yx.M \geq T}{\Gamma \vdash Yx.M \geq S \cup T}\, Y \textit{ even}$$

Figure 4. Type system

$$\dfrac{\dfrac{\dfrac{}{\vdash \lambda F.N \geq (S \cup T) \to S} \quad \vdash YF.N \geq T}{\vdash YF.N \geq S \cup T}\, Y \textit{ even}}{\vdash YF.N \geq S} \quad \text{where } T = \{(q_1 \to q_1) \to q_1\}$$

The derivation of the top right judgment uses the least fixpoint rule:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{g \geq q_1 \to q_1 \vdash g \geq q_1 \to q_1} \quad g \geq q_1 \to q_1 \vdash b(F(\lambda x.g(g\,x))) \geq q_1}{g \geq q_1 \to q_1 \vdash g(b(F(\lambda x.g(g\,x)))) \geq q_1}}{\vdash \lambda F \lambda g.g(b(F(\lambda x.g(g\,x)))) \geq \emptyset \to (q_1 \to q_1) \to q_1}\, Y \textit{ odd}}{\vdash FY.N \geq (q_1 \to q_1) \to q_1}$$

We have displayed only one of the two premises of the $Y odd$ rule since the other is of the form $\geq \emptyset$ so it is vacuously true. The top right judgment is derivable directly from the axiom on b. The derivation of the remaining judgment $\vdash \lambda F.N \geq (S \cup T) \to S$ is as follows.

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash g \geq \{q_1, q_2\} \to q_2} \qquad \dfrac{\vdots}{\Gamma \vdash b(F(\lambda x.g(g\,x))) \geq q_1, q_2}}{\Gamma \vdash g(b(F(\lambda x.g(g\,x)))) \geq q_2}}{\vdash \lambda F \lambda g.g(b(F(\lambda x.g(g\,x)))) \geq (S \cup T) \to S}$$

where Γ is $F \geq S \cup T, g \geq \{q_1 \to q_1, \{q_1, q_2\} \to q_2\}$. So the upper left judgment is an axiom. The other judgment on the top is an abbreviation of two judgments: one to show $\geq q_1$ and the other one to show $\geq q_2$. These two judgments are proven directly using application and intersection rules.

4. Models for weak automata

The first step towards proving soundness and completeness of the typing systems is to capture weak alternating automata with models similarly to Theorem 4. The model we construct will depend only on the states of the automaton and the ranks of those states. The transitions of the automaton will be encoded in the interpretation of constants. In this section we will define a model for λY-terms constructed from the states of an automaton. In the next section we will show that such a model can indeed recognize the set of terms accepted by the automaton.

Theorem 4 points out the challenge for such a construction. The fixpoints in a model capturing wMSO must be something different than the least or the greatest fixpoint. The structure of a weak automaton will provide a solution here. The automaton is stratified into states of different rank: the transitions for states of rank i depend only on states of rank $\leq i$. We will find this stratification in our model too. At each stratum we will use the least or the greatest fixpoint depending on the parity of the stratum. In the whole model, the fixpoint computation will perform a sort of zig-zag as represented in Figure 6.

We fix a finite set of states Q and a ranking function $\rho : Q \to \mathbb{N}$. Let m be the maximal rank, i.e., the maximal value ρ takes on Q. Recall that for every $0 \leq k \leq m$ we let $Q_k = \{q \in Q : \rho(q) = k\}$ and $Q_{\leq k} = \{q \in Q : \rho(q) \leq k\}$.

We define by induction on $k \leq m$ an applicative structure $\mathcal{D}^k = (\mathcal{D}^k_A)_{A \in \text{types}}$ and a logical relation $\mathcal{L}^k$ (for $0 < k$) between $\mathcal{D}^{k-1}$ and $\mathcal{D}^k$.

For $k = 0$, the model $\mathcal{D}^0$ is just the model of monotone functions over the powerset of Q_0:

$$\mathcal{D}^0_0 = \mathcal{P}(Q_0) \quad \text{and} \quad \mathcal{D}^0_{A \to B} = \text{mon}[\mathcal{D}^0_A \mapsto \mathcal{D}^0_B],$$

For $k > 0$ we have:

$$\begin{aligned}
\mathcal{D}^k_0 &= \mathcal{P}(Q_{\leq k}) \\
\mathcal{L}^k_0 &= \{(R, P) \in \mathcal{D}^{k-1}_0 \times \mathcal{D}^k_0 \;:\; R = P \cap Q_{\leq (k-1)}\}, \\
\mathcal{L}^k_{A \to B} &= \{(f_1, f_2) \in \mathcal{D}^{k-1}_{A \to B} \times \text{mon}[\mathcal{D}^k_A \mapsto \mathcal{D}^k_B] \;: \\
&\qquad\qquad \forall (g_1, g_2) \in \mathcal{L}^k_A.\, (f_1(g_1), f_2(g_2)) \in \mathcal{L}^k_B\} \\
\mathcal{D}^k_{A \to B} &= \{f_2 \;:\; \exists f_1 \in \mathcal{D}^{k-1}_{A \to B}.\, (f_1, f_2) \in \mathcal{L}^k_{A \to B})\}
\end{aligned}$$

Observe that the structures $\mathcal{D}^k$ are defined by a double induction: the outermost on k and the auxiliary induction on the size of the type. Since $\mathcal{L}^k$ is a logical relation between $\mathcal{D}^{k-1}$ and $\mathcal{D}^k$, each $\mathcal{D}^k$ is an applicative structure. Each $\mathcal{D}^k_A$ is ordered. Indeed, each $\mathcal{D}^k$ can be seen as a part of the model of monotone functions over $\mathcal{P}(Q_{\leq k})$. The order in the later is the inclusion in type 0, and the coordinate-wise ordering for other types.

The next lemma is the main technical lemma summarizing the essential properties of the model. These properties are expressed in terms of the logical relation $\mathcal{L}^k_A \subseteq \mathcal{D}^{k-1}_A \times \mathcal{D}^k_A$. For notational convenience we write $\mathcal{L}^k_A(d_1)$ for the set of elements d_2 in the relation with d_1: $\mathcal{L}^k_A(d_1) = \{d_2 : \mathcal{L}^k_A(d_1, d_2)\}$. Figure 5 represents schematically some of the properties stated in the lemma.

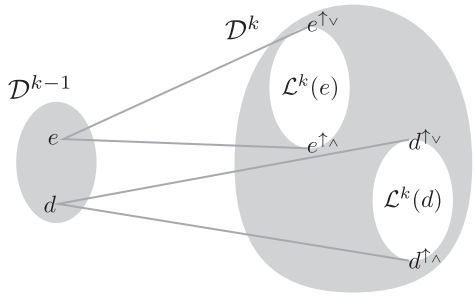


Figure 5. Relation between models $\mathcal{D}^{k-1}$ and $\mathcal{D}^k$. Every element in $\mathcal{D}^{k-1}$ is related to a sub-lattice of elements in $\mathcal{D}^k$.

Lemma 7 For every $0 < k \leq m$, and every type A, we have:

0. $\mathcal{L}^k_A$ is a lattice: if (d_1, d_2), (e_1, e_2) are in $\mathcal{L}^k_A$, then so are $(d_1 \vee e_1, d_2 \vee e_2)$ and $(d_1 \wedge e_1, d_2 \wedge e_2)$.
1. Given (d_1, d_2) and (e_1, e_2) in $\mathcal{L}_{k,A}$, if $d_2 \leq e_2$ then $d_1 \leq e_1$.
2. Given d_2 in $\mathcal{D}^k_A$, there is a unique d_1 in $\mathcal{D}^{k-1}_A$, so that (d_1, d_2) in $\mathcal{L}^k_A$. Let us denote this unique element $d_2^{\downarrow}$.
3. If $d_1 \leq e_1$ in $\mathcal{D}^{k-1}_A$, then:
   (a) there is $e_1^{\uparrow\vee}$ in $\mathcal{L}^k_A(e_1)$ so that for every d_2 in $\mathcal{L}^k_A(d_1)$ we have $d_2 \leq e_1^{\uparrow\vee}$,
   (b) there is $d_1^{\uparrow\wedge}$ in $\mathcal{L}^k_A(d_1)$ so that for every e_2 in $\mathcal{L}^k_A(e_1)$ we have $d_2^{\uparrow\wedge} \leq e_2$.

Proof
Item 0 can be proved by a straightforward induction on the size of types.

The proof of items $1, 2, 3$, is by simultaneous induction on the size of A. Notice that item 2 is an immediate consequence of item 1. We shall therefore not prove it, but we feel free to use it as an induction hypothesis.

We start with the case when A is the base type 0:
Ad 1. In that case $\mathcal{D}^k_A = Q_{\leq k}$, $e_1 = e_2 \cap Q_{\leq k-1}$ and $d_1 = d_2 \cap Q_{\leq k-1}$. Thus, we indeed have that $d_2 \leq e_2$ implies that $d_1 \leq e_1$.
Ad 3. Here, we have $\mathcal{D}^{k-1,A} = Q_{\leq k-1}$ and then letting $e_1^{\uparrow\vee} = e_1 \cup Q_k$ and $d_2^{\uparrow\wedge} = d_2$ is enough to conclude.

Let us now suppose that $A = B \to C$:
Ad 1. Given f_1 in $\mathcal{D}^{k-1}_B$, by induction hypothesis, using item 3, we know that there exist f_2 in $\mathcal{D}^k_B$ so that (f_1, f_2) is in $\mathcal{L}^k_B$. Thus, we have $(d_1(f_1), d_2(f_2))$ and $(e_1(f_1), e_2(f_2))$ in $\mathcal{L}_{k,C}$. With the assumption that $d_2 \leq e_2$, we obtain $d_2(f_2) \leq e_2(f_2)$. By induction hypothesis we get $e_1(f_1) \leq d_1(f_1)$. As f_1 is arbitrary, we can conclude that $e_1 \leq d_1$.
Ad 3. By induction hypothesis, using item 2, for f_2 in $\mathcal{D}^k_B$, there is a unique element $f_2^{\downarrow}$ of $\mathcal{D}^{k-1}_B$ so that $(f_2^{\downarrow}, f_2)$ is in $\mathcal{L}^k_B$. Given h_1 in $\mathcal{D}^{k-1}_A$ we define for every element f_2 in $\mathcal{D}^k_B$:

$$h_1^{\uparrow\vee}(f_2) = (h_1(f_2^{\downarrow}))^{\uparrow\vee} \qquad h_1^{\uparrow\wedge}(f_2) = (h_1(f_2^{\downarrow}))^{\uparrow\wedge} .$$

We will verify only item 3(a), the case of $h_1^{\uparrow\wedge}$ being analogous.

We need to check that $h_1^{\uparrow\vee}$ is in $\mathcal{D}^k_A$. First of all we need to check that it is in $\mathrm{mon}[\mathcal{D}^k_B \mapsto \mathcal{D}^k_C]$. Take g_2 and f_2 in $\mathcal{D}^k_B$ so that $g_2 \leq f_2$. By induction hypothesis, using item 1, we have that $g_2^{\downarrow} \leq f_2^{\downarrow}$. Then $h_1(g_2^{\downarrow}) \leq h_1(f_2^{\downarrow})$ by monotonicity of h_1. From item 3 of induction hypothesis we obtain $(h_1(g_2^{\downarrow}))^{\uparrow\vee} \leq (h_1(f_2^{\downarrow}))^{\uparrow\vee}$; proving that $h_1^{\uparrow\vee}$ is monotone.

Next, we show that $(h_1, h_1^{\uparrow\vee})$ is in $\mathcal{L}^k_A$. If we take $(f_2^{\downarrow}, f_2)$ in $\mathcal{L}_{k,B}$, we obtain by the induction hypothesis, item 3, that $(h_1(f_2^{\downarrow}), (h_1(f_2^{\downarrow}))^{\uparrow\vee})$ is in $\mathcal{L}^k_C$. But, by our definition, this implies that $(h_1(f_2^{\downarrow}), h_1^{\uparrow\vee}(f_2))$ is in $\mathcal{L}^k_C$. As f_2 is arbitrary we obtain that $(h_1, h_1^{\uparrow\vee})$ is indeed in $\mathcal{L}^k_A$ proving then that $h_1^{\uparrow\vee}$ is in $\mathcal{D}^k_A$.

It remains to prove that, given $d_1 \leq e_1$ in $\mathcal{D}^k_A$, for all d_2 in $\mathcal{L}^k_A(d_1)$ we have $d_2 \leq e_1^{\uparrow\vee}$. Given $(f_2^{\downarrow}, f_2)$ in $\mathcal{L}^k_B$, we have $(d_1(f_2^{\downarrow}), d_2(f_2))$ in $\mathcal{L}^k_C$. Using induction hypothesis, item 3, we get $d_2(f_2) \leq (d_1(f_2^{\downarrow}))^{\uparrow\vee}$. Since $d_1 \leq e_1$ we obtain $d_2(f_2) \leq (e_1(f_2^{\downarrow}))^{\uparrow\vee}$ that is the desired $d_2(f_2) \leq e_1^{\uparrow\vee}(f_2)$. As f_2 is arbitrary, this shows that $d_2 \leq e_1^{\uparrow\vee}$. □

Some consequence of this lemma are spelled out in the following corollaries.

Corollary 8 For each $k \leq m$, and each type A, $\mathcal{D}^k_A$ is a non-empty lattice.

The mappings $(\cdot)^{\downarrow}$ and $(\cdot)^{\uparrow\vee}$ form a Galois connection between $\mathcal{D}^k_A$ and $\mathcal{D}^{k-1}_A$. For every $d_2 \in \mathcal{D}^k_A$ and $e_1 \in \mathcal{D}^{k-1}_A$ we have: $d_2^{\downarrow} \leq e_1$ iff $d_2 \leq e_1^{\uparrow\vee}$.

Similarly $(\cdot)^{\downarrow}$ and $(\cdot)^{\uparrow\wedge}$ form a Galois connection between $\mathcal{D}^{k-1}_A$ and $\mathcal{D}^k_A$. For every $d_1 \in \mathcal{D}^{k-1}_A$ and $e_2 \in \mathcal{D}^k_A$ we have: $d_1^{\uparrow\wedge} \leq e_2$ iff $d_1 \leq e_2^{\downarrow}$.

From now on, $\bot^k_A$ and $\top^k_A$ will denote the least and the greatest element of $\mathcal{D}^k_A$.

Corollary 9 Given d_2 in $\mathcal{D}^k_{B\to C}$ and e_2 in $\mathcal{D}^k_B$, we have $(d_2(e_2))^{\downarrow} = d_2^{\downarrow}(e_2^{\downarrow})$. Given d_1 in $\mathcal{D}^{k-1}_{B\to C}$ and e_2 in $\mathcal{D}^k_B$, we have: $d_1^{\uparrow\wedge}(e_2) = (d_1(e_2^{\downarrow}))^{\uparrow\wedge}$ and $d_1^{\uparrow\vee}(e_2) = (d_1(e_2^{\downarrow}))^{\uparrow\vee}$.

Finally, we show one more decomposition property of our models that will allow to show a correspondence between types and elements of the model (Lemma 24).

Definition 10 We define an operation $\overline{d}$ on the elements d of $\mathcal{D}^k_A$ by induction on A:

- if $A = 0$, then $\overline{d} = d \cap Q_k$,
- if $A = B \to C$, then for e in $\mathcal{D}^{k,B}$, $\overline{d}(e) = \overline{d(e)}$.

Lemma 11 For every $0 < k \leq m$, and every d in $\mathcal{D}^k_A$; let $\overline{d}$ be as defined above, and let $\widetilde{d} = (\top^{k-1}_A)^{\uparrow\wedge} \vee \overline{d}$. We have $d = ((d^{\downarrow})^{\uparrow\wedge}) \vee \overline{d}$ and $d = ((d^{\downarrow})^{\uparrow\vee}) \wedge \widetilde{d}$.

Proof
We prove only the first identity. The proof is by induction on the size of A.

In case $A = o$, from definitions we get $\overline{d} = d \cap Q_k$ while $(d^{\downarrow})^{\uparrow\wedge} = d \cap Q_{\leq k-1}$. Thus we indeed have $d = ((d^{\downarrow})^{\uparrow\wedge}) \vee \overline{d}$.

In case $A = B \to C$. Given e in $\mathcal{D}^k_B$, we have that $(d^{\downarrow})^{\uparrow\wedge}(e) = (d^{\downarrow}(e^{\downarrow}))^{\uparrow\wedge} = ((d(e))^{\downarrow})^{\uparrow\wedge}$. Moreover, $\overline{d}(e) = \overline{d(e)}$. Therefore, $(((d^{\downarrow})^{\uparrow\wedge}) \vee \overline{d})(e) = (((d(e))^{\downarrow})^{\uparrow\wedge}) \vee \overline{d(e)}$. But, by induction, we have $d(e) = (((d^{\downarrow})^{\uparrow\wedge}) \vee \overline{d})(e) = (((d(e))^{\downarrow})^{\uparrow\wedge}) \vee \overline{d(e)}$. As e is arbitrary, we get the identity. □

This lemma shows that every element d_2 in $\mathcal{L}^k_A(d_1)$ is of the form $d_1^{\uparrow\wedge} \vee e_2$ with e_2 in $\mathcal{L}^k_A(\bot^{k-1}_A)$ and of the form $d_1^{\uparrow\vee} \wedge e_2$ with e_2 in $\mathcal{L}^k_A(\top^{k-1}_A)$. Thus not only $\mathcal{L}^k_A$ puts every element d_1 in relation with a lattice $\mathcal{L}^k(d_1)$, but also this lattice is isomorphic to $\mathcal{L}^k_A(\bot^{k-1}_A)$ and to $\mathcal{L}^k_A(\top^{k-1}_A)$. Therefore, we can see $\mathcal{D}^k_A$ as isomorphic to the lattice $\mathcal{D}^{k-1}_A \times \mathcal{L}_{k,A}(\bot^{k-1}_A)$ or to the lattice $\mathcal{D}^{k-1}_A \times \mathcal{L}_{k,A}(\top^{k-1}_A)$.

We can now define fixpoint operators in every applicative structure $\mathcal{D}^k$.

Definition 12 For $f \in \mathcal{D}^0_{A\to A}$ we define

$$\mathrm{fix}^0_A(f) = \bigwedge\{f^n(\top^0) : n \geq 0\}$$

For $0 < 2k \leq m$ and $f \in \mathcal{D}^{2k}_{A\to A}$ we define

$$\mathrm{fix}^{2k}_A(f) = \bigwedge\{f^n(e) : n \geq 0\} \text{ where } e = (\mathrm{fix}^{2k-1}_A(f^\downarrow))^{\uparrow\vee}$$

For $0 < 2k+1 \leq m$ and $f \in \mathcal{D}^{2k+1}_{A\to A}$ we define

$$\mathrm{fix}^{2k+1}_A(f) = \bigvee\{f^n(d) : n \geq 0\} \text{ where } d = (\mathrm{fix}^{2k}_A(f^\downarrow))^{\uparrow\wedge}$$

Observe that, for even k, e is obtained with $(\cdot)^{\uparrow\vee}$; while for odd k, $(\cdot)^{\uparrow\wedge}$ is used.

The intuitive idea behind the definition of the fixpoint is presented in Figure 6. On stratum 0 it is just the greatest fixpoint. Then this greatest fixpoint is lifted to stratum 1, and the least fixpoint computation is started from it. The result is then lifted to stratum 2, and once again the greatest fixpoint computation is started, and so on. The Galois connections between strata guarantee that this process makes sense.

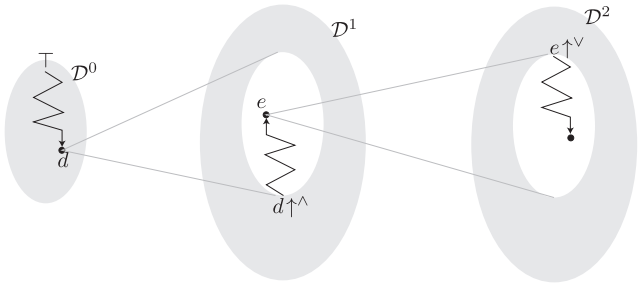


Figure 6. A computation of a fixpoint: it starts in $\mathcal{D}^0$, and then the least and the greatest fixpoints alternate.

It remains to show that equipped with the interpretation of fixpoints given by Definition 12 the applicative structure $\mathcal{D}^k$ is a model of the λY-calculus. First, we check that fix^k_A is indeed an element of the model and that it is a fixpoint.

Lemma 13 For every $0 \leq k \leq m$ and every type A we have that fix^k_A is monotone, and if $k > 0$ then $(\mathrm{fix}^{k-1}_A, \mathrm{fix}^k_A) \in \mathcal{L}^k_{(A\to A)\to A}$. Moreover for every f in $\mathcal{D}^k_{A\to A}$, $f(\mathrm{fix}^k_A(f)) = \mathrm{fix}^k_A(f)$.

Proof
For 0 the statement is obvious. We will only consider the case where k is even, the other being dual.

Consider the case $2k > 0$. First we show monotonicity. Suppose $g \leq h$ are two elements of $\mathcal{D}^{2k}_{A\to A}$. By Lemma 7 we get $g^\downarrow \leq h^\downarrow$. Consider $g^1 = \mathrm{fix}^{2k-1}_A(g^\downarrow)$ and $h^1 = \mathrm{fix}^{2k-1}_A(h^\downarrow)$. By induction hypothesis fix^{2k-1}_A is monotone, so $g^1 \leq h^1$. Then, once again using the Lemma 7, we have $(g^1)^{\uparrow\vee} \leq (h^1)^{\uparrow\vee}$. This implies $\mathrm{fix}^{2k}_A(g) \leq \mathrm{fix}^{2k}_A(h)$.

Now we show $(\mathrm{fix}^{2k-1}_A, \mathrm{fix}^{2k}_A) \in \mathcal{L}^{2k}_{(A\to A)\to A}$. We take an arbitrary pair $(f_1, f_2) \in \mathcal{L}^{2k}_{A\to A}$ and we need to show that $(\mathrm{fix}^{2k-1}_A(f_1), \mathrm{fix}^{2k}_A(f_2)) \in \mathcal{L}^{2k}_A$. This follows from the following calculation

$$\begin{aligned}
(\mathrm{fix}^{2k-1}(f_1), (\mathrm{fix}^{2k-1}(f_1))^{\uparrow\vee}) &\in \mathcal{L}^{2k}_A && \text{by Lemma 7}\\
(f_1(\mathrm{fix}^{2k-1}(f_1)), f_2((\mathrm{fix}^{2k-1}(f_1)^{\uparrow\vee})) &\in \mathcal{L}^{2k}_A && \text{by logical relation}\\
(\mathrm{fix}^{2k-1}(f_1), f_2((\mathrm{fix}^{2k-1}(f_1)^{\uparrow\vee})) &\in \mathcal{L}^{2k}_A && \text{since } \mathrm{fix}^{2k-1}(f_1)\\
& && \text{is a fixpoint of } f_1\\
(\mathrm{fix}^{2k-1}(f_1), f_2^i((\mathrm{fix}^{2k-1}(f_1)^{\uparrow\vee})) &\in \mathcal{L}^{2k}_A && \text{for every } i \geq 0
\end{aligned}$$

Moreover, from Lemma 7, we have that $f_2((\mathrm{fix}^{2k-1}(f_1)^{\uparrow\vee}) \leq (\mathrm{fix}^{2k-1}(f_1))^{\uparrow\vee}$. This implies

$$f_2^{i+1}((\mathrm{fix}^{2k-1}(f_1)^{\uparrow\vee}) \leq f_2^i((\mathrm{fix}^{2k-1}(f_1))^{\uparrow\vee})$$

for every $i \in \mathbb{N}$. Therefore, $(f_2^i((\mathrm{fix}^{2k-1}(f_1))^{\uparrow\vee})_{i\in\mathbb{N}}$ is a decreasing sequence of $\mathcal{D}^{2k}_A$. Since the model is finite this sequence reaches the fixpoint, namely $\mathrm{fix}^{2k}_A(f_2) = f_2^i((\mathrm{fix}^{2k-1}_A(f_1)^{\uparrow\vee}))$ for some i. Thus, at the same time, this shows that $(\mathrm{fix}^{2k-1}_A(f_1), \mathrm{fix}^{2k}_A(f_2)) \in \mathcal{L}^{2k}_A$ and that $\mathrm{fix}^{2k}(f_2)$ is a fixpoint of f_2.

□

This lemma has the following interesting corollary that will prove useful in the study of type systems.

Corollary 14 For $k > 0$, A a type, and $f \in \mathcal{D}^k_{A\to A}$ we have

$$\mathrm{fix}^{2k}_A(f) = \bigvee\{d \mid f(d) \geq d \text{ and } d^\downarrow = \mathrm{fix}^{2k-1}_A(f)\}$$
$$\mathrm{fix}^{2k+1}_A(f) = \bigwedge\{d \mid f(d) \leq d \text{ and } d^\downarrow = \mathrm{fix}^{2k}_A(f)\}$$

Moreover the fundamental lemma on logical relation has the following consequence.

Lemma 15 For every $k > 0$, every term M and valuation ν into $\mathcal{D}^{k-1}$ we have $([\![M]\!]^{k-1}_\nu, [\![M]\!]^k_{\nu^{\uparrow\wedge}}) \in \mathcal{L}^k$ and $([\![M]\!]^{k-1}_\nu, [\![M]\!]^k_{\nu^{\uparrow\vee}}) \in \mathcal{L}^k$; where $\nu^{\uparrow\wedge}$ and $\nu^{\uparrow\vee}$ are as expected.

Standard techniques show that equipped with this interpretation of the fixpoint, each $\mathcal{D}^k$ forms a model of the λY-calculus

Theorem 16 *For every finite set Q, and function $\rho : Q \to \mathbb{N}$. For every $k \leq 0$ the applicative structure $\mathcal{D}^k$ is a model of the λY-calculus.*

5. Correctness and completeness of the model

We show that the models introduced in the previous section are expressive enough to recognize all properties definable by weak alternating automata. For a given automaton we will take a model as defined above, and show that with the right interpretation of constants the model can recognize the set of terms whose Böhm trees are accepted by the automaton (Theorem 17).

For the whole section we fix a weak alternating automaton $\mathcal{A} = \langle Q, \Sigma, q^0, \delta_o, \delta_{o^2\to o}, \rho\rangle$ where Q is a set of states, Σ is the alphabet, $\delta_o \subseteq Q \times \Sigma$ and $\delta_{o^2\to o} : Q \times \Sigma \to \mathcal{P}(\mathcal{P}(Q) \times \mathcal{P}(Q))$ are transition functions, and $\rho : Q \to \mathbb{N}$ is a ranking function. For simplicity of the notation in this section we assume that the only constants in the signature are either of type o or $o^2 \to o$.

Recall that weak means that the states in a transition for a state q have ranks at most $\rho(q)$, in other words, for every $(S_0, S_1) \in \delta(q, a)$, $S_0, S_1 \subseteq Q_{\leq\rho(q)}$. As noted before, without a loss of generality, we assume that δ is monotone, i.e. if $(S_0, S_1) \in \delta(q, a)$ and $S_0 \subseteq S_0' \subseteq Q_{\leq\rho(q)}$ and $S_1 \subseteq S_1' \subseteq Q_{\leq\rho(q)}$ then $(S_0', S_q') \in \delta(q, a)$. Let

$$\mathcal{A}(M) = \{q \in Q : \mathcal{A} \text{ accepts } BT(M) \text{ from } q\}$$

be the set of states from which $\mathcal{A}$ accepts the tree $BT(M)$.

We want to show that our model $\mathcal{D}^m$ can calculate $\mathcal{A}(M)$; here m is the maximal value of the rank function of $\mathcal{A}$. The following theorem states a slightly more general fact. Before proceeding we need to fix the meaning of constants:

$$[\![c]\!]^k = \{q \in Q_{\leq k} : (c, q) \in \delta_o\}$$
$$[\![a]\!]^k(S_0, S_1) = \{q \in Q_{\leq k} : (S_0, S_1) \in \delta_{o^2\to o}(q, a)\}$$

Notice that, by our assumption about monotonicity of δ, these functions are monotone.

Theorem 17 *For every closed term M of type 0, and for every $0 \leq k \leq m$ we have: $[\![M]\!]^k = \mathcal{A}(M) \cap Q_{\leq k}$.*

The rest of this section is devoted to the proof of the theorem.

For $k = 0$ the model $\mathcal{D}^0$ is just the GFP model over Q_0. Moreover $\mathcal{A}$ restricted to the states in Q_0 is an automaton with trivial acceptance conditions. The theorem follows from Theorem 4.

For the induction step consider even $k > 0$. The case of odd k is similar and we will not present it here. The two directions of Theorem 17 are proved using different techniques. The next lemma shows left to right inclusion and is based on a rather simple unrolling. The other inclusion is proved using logical relations (Lemma 22)

Lemma 18 $[\![M]\!]^k \subseteq \mathcal{A}(M)$.

Proof
We take $q \in [\![M]\!]^k$ and describe a winning strategy for Eve in the acceptance game of $\mathcal{A}$ on $BT(M)$ from q (cf. page 3). If the rank of $q < k$ then such a strategy exist by the induction assumption. So we suppose that $\rho(q) = k$.

If M does not have a head normal form then $BT(M)$ consists just of the root ε labeled with Ω. Then Eve wins by the definition of the game since k is even.

If the head normal form of M is a constant $c : o$ then since $q \in [\![c]\!]^k$ we have $(q, c) \in \delta_o$. Eve wins by the definition of the game.

Suppose then that M has a head normal form aM_0M_1. As $[\![M]\!]^k = [\![aM_0M_1]\!]^k$ we have $q \in [\![aM_0M_1]\!]^k$. By the semantics of a we know that $([\![M_0]\!]^k, [\![M_1]\!]^k) \in \delta(q, a)$. The strategy of Eve is to choose $([\![M_0]\!]^k, [\![M_1]\!]^k)$. Suppose Adam then selects $i \in \{0, 1\}$ and $q_i \in [\![M_i]\!]^k$. If $\rho(q_i) < k$ then Eve has a winning strategy by induction hypothesis. Otherwise, if $\rho(q_i) = k$ we repeat the reasoning.

This strategy is winning for Eve since a play either stays in states of even rank k or switches to a play following a winning strategy for smaller ranks. □

It remains to show that $\mathcal{A}(M) \cap Q_{\leq k} \subseteq [\![M]\!]^k$. For this we will define one logical relation between $\mathcal{D}^k$ and the syntactic model of λY and show a couple of lemmas.

Definition 19 We define a logical relation between the model $\mathcal{D}^k$ and closed terms

$$R^k_0 = \{(P, M) : \mathcal{A}(M) \cap Q_{\leq k} \subseteq P\}$$
$$R^k_{A\to B} = \{(f, M) : \forall_{(g,N)\in R_A} . (f(g), MN) \in R_B\} .$$

Since R^k is a logical relation we have:

Lemma 20 If $M =_{\beta\delta} N$ and $(f, M) \in R^k_A$ then $(f, N) \in R^k_A$.

The next lemma shows a relation between R^k and R^{k-1}.

Lemma 21 For every type A, $f \in \mathcal{D}^k_A$, $g \in \mathcal{D}^{k-1}_A$:

- if $(f, M) \in R^k_A$ then $(f^\downarrow, M) \in R^{k-1}_A$;
- if $(g, M) \in R^{k-1}_A$ then $(g^{\uparrow\vee}, M) \in R^k$;

Proof
The proof is an induction on the size of the type. The base case is when $A = o$.

For the first item suppose $(f, M) \in R^k_A$. By definition, this means $\mathcal{A}(M) \cap Q_{\leq k} \subseteq f$. Then $\mathcal{A}(M) \cap Q_{\leq k-1} \subseteq f \cap Q_{\leq k-1} = f^\downarrow$. So $(f^\downarrow, M) \in R^{k-1}_A$.

For the second item suppose $(g, M) \in R^{k-1}_A$. So $\mathcal{A}(M) \cap Q_{\leq k-1} \subseteq g$. We have $\mathcal{A}(M) \cap Q_{\leq k} \subseteq (\mathcal{A}(M) \cap Q_{\leq k-1}) \cup Q_k \subseteq g \cup Q_k = g^{\uparrow\vee}$.

For the induction step let A be $B \to C$. Let us consider the first item. Suppose $(f, M) \in R^k_{B\to C}$. Take $(h, N) \in R^{k-1}_B$, we need to show that $(f^\downarrow(h), MN) \in R^{k-1}_C$. By the second item of the induction hypothesis we get $(h^{\uparrow\vee}, N) \in R^k_B$. Then $(f(h^{\uparrow\vee}), N) \in R^k_C$, by the definition of $R^k_{B\to C}$. Using the first item of the induction hypothesis we get $((f(h^{\uparrow\vee}))^\downarrow, N) \in R^{k-1}_C$. Then using Corollaries 8 and 9 we obtain $(f(h^{\uparrow\vee}))^\downarrow = f^\downarrow((h^{\uparrow\vee})^\downarrow) = f^\downarrow(h)$.

For the proof of the second item consider $(g, M) \in R^{k-1}_{B\to C}$ and $(h, N) \in R^k_B$. We need to show that $(g^{\uparrow\vee}(h), MN) \in R^k_C$. From the first item of the induction hypothesis we obtain $(h^\downarrow, N) \in R^{k-1}_B$, so $(g(h^\downarrow), MN) \in R^{k-1}_C$. The second item of the induction hypothesis gives us $((g(h^\downarrow))^{\uparrow\vee}, MN) \in R^k_C$. We are done since $g^{\uparrow\vee}(h) = (g(h^\downarrow))^{\uparrow\vee}$ by Corollary 9. □

Lemma 22 Let v be a valuation, and σ a substitution of closed terms such that $(v(x^A), \sigma(x^A)) \in R^k_A$ for every variable x^A in the domain of σ. For every term M of a type A we have $([\![M]\!]^k_v, M.\sigma) \in R^k_A$.

Proof
The proof is by induction on the structure of M.

If M is a variable then the proof is immediate.

If M is a constant a then we show that $([\![a]\!], a) \in R^k_{0\to 0\to 0}$. For this we take arbitrary $(S_0, N_0), (S_1, N_1) \in R^k_0$, and we show that $([\![a]\!]^k(S_0, S_1), aN_0N_1) \in R^k_0$. Take $q \in \mathcal{A}(aN_0N_1) \cap Q_{\leq k}$. Let us look at Eve's winning strategy in the acceptance game from q on $BT(aN_0N_1)$. In the first round of this game she chooses some $(T_0, T_1) \in \delta(a, q)$. So $q \in [\![a]\!](T_0, T_1)$. Since her strategy is winning we have $T_0 \subseteq \mathcal{A}(N_0)$ and $T_1 \subseteq \mathcal{A}(N_1)$, and by weakness of the automaton $T_0, T_1 \subseteq Q_{\leq k}$. From the definition of R^k_0 we get $\mathcal{A}(N_0) \cap Q_{\leq k} \subseteq S_0$ and $\mathcal{A}(N_1) \cap Q_{\leq k} \subseteq S_1$. By monotonicity we get the desired $q \in [\![a]\!](S_0, S_1)$.

If M is an application NP then the conclusion is immediate from the definition of R^k_A.

If M is an abstraction $\lambda x.\, N : B \to C$, then we take $(g, P) \in R^k_B$. By induction hypothesis $([\![N]\!]^k_{v[g/x]}, N.\sigma[P/x]) \in R^k_C$. So $([\![M]\!]^k_v(g), MP) \in R^k_C$ by Lemma 20.

If $M = Y^{(A\to A)\to A}$. Take $(f, P) \in R^k_{A\to A}$. By Lemma 21 we have $(f^\downarrow, P) \in R^{k-1}_{A\to A}$. As by the outermost induction hypothesis $(\mathrm{fix}^{k-1}_A, Y^{(A\to A)\to A}) \in R^{k-1}_{(A\to A)\to A}$, and we obtain $(\mathrm{fix}^{k-1}_A(f^\downarrow), YP) \in R^{k-1}_A$. Once again using Lemma 21 we get $((\mathrm{fix}^{k-1}_A(f^\downarrow))^{\uparrow\vee}, YP) \in R^k_A$. By the choice of (f, P) we obtain $(f((\mathrm{fix}^{k-1}_A(f^\downarrow))^{\uparrow\vee}), P(YP)) \in R^k_A$. Since $YP =_{\beta\delta} P(YP)$, we have $(f^i((\mathrm{fix}^{k-1}_A(f^\downarrow))^{\uparrow\vee}), YP) \in R^k_A$ for all $i \geq 0$. Since the sequence of $f^i((\mathrm{fix}^{k-1}_A(f^\downarrow))^{\uparrow\vee})$ is decreasing, it reaches the fixpoint $\mathrm{fix}^k_A(f)$ in a finite number of steps and $(\mathrm{fix}^k_A(f), YP)$ is in R_A. As (f, P) is an arbitrary element of $R^k_{A\to A}$, this shows that (fix^k_A, Y) is in $R^k_{(A\to A)\to A}$. □

6. From models to type systems

We are now in a position to show that our type system from Figure 4 can reason about the values of λY-terms in a stratified model (Theorem 25). Thanks to Theorem 17 this means that the type system can talk about acceptance of the Böhm tree of a term by the

automaton. This implies soundness and completeness of our type system, Theorem 5.

Throughout this section we work with a fixed signature Σ and a fixed weak alternating automaton $\mathcal{A} = \langle Q, \Sigma, q^0, \delta_o, \delta_{o^2 \to o}, \rho \rangle$. As in the previous section, for simplicity of notation we will assume that the constants in the signature are of type o or $o \to o \to o$. We will also prefer notation $Yx.M$ to $Y(\lambda x.M)$.

The type system we introduced in Section 3 can be understood as means for reasoning about the values of a term in a model. Indeed, we will show that types can denote all elements in the model (Lemma 24). This follows Abramsky's idea of domains in logical form [1] further adapted to the typed setting in [34].

The arrow constructor in types will be interpreted as a step function in the model. For the dual system we will also need co-step functions, so we recall the two notions here. Step and co-step functions are particular monotone functions from a lattice $\mathcal{L}_1$ to a lattice $\mathcal{L}_2$. For d in $\mathcal{L}_1$ and e in $\mathcal{L}_2$, the *step function* $d \rightharpoonup e$ and the *co-step function* $d \rightharpoondown g$ are defined by:

$$(d \rightharpoonup e)(h) = \begin{cases} e \text{ when } d \leq h \\ \bot \text{ otherwise} \end{cases}$$

$$(d \rightharpoondown e)(h) = \begin{cases} e \text{ when } h \leq d \\ \top \text{ otherwise .} \end{cases}$$

To emphasize that we work in $\mathcal{D}^l$ we will write $d \rightharpoonup^l e$ and $d \rightharpoondown^l e$.

Types defined on page 4 can be meaningfully interpreted at every level of the model. So $[\![t]\!]^l$ will denote the interpretation of t in $\mathcal{D}^l$ defined as follows.

$$[\![q]\!]^l = \begin{cases} \{q\} & \text{if the rank of } q \text{ is at most } l \\ \emptyset & \text{otherwise} \end{cases}$$

$$[\![S]\!]^l = \bigvee \{ [\![t]\!]^l : t \in S \} \qquad \text{for } S \subseteq \mathrm{Types}_A$$

$$[\![T \to s]\!]^l = [\![T]\!]^l \rightharpoonup^l [\![s]\!]^l \qquad \text{for } (T \to s) \in \mathrm{Types}_A$$

Directly from the definition we have $[\![S_1 \cup S_2]\!]^l = [\![S_1]\!]^l \vee [\![S_2]\!]^l$, and $[\![S \to T]\!]^l = [\![S]\!]^l \rightharpoonup^l [\![T]\!]^l$.

The next lemma summarizes basic facts about the interpretation of types. Recall that the application operation $S(T)$ on types (cf. page 4) means $\{t : (U \to t) \in S \wedge U \sqsubseteq T\}$. The proof of the lemma uses Corollaries 9 and 8.

Lemma 23 For every type A, if $S \subseteq \mathrm{Types}_A$ and $k \leq m$ we have: $[\![S]\!]^k = [\![S \cap \mathrm{Types}_A^k]\!]^k$, $[\![S \cap \mathrm{Types}_A^k]\!]^{k+1} = ([\![S]\!]_A^k)^{\uparrow\wedge}$ and $[\![S]\!]^k = ([\![S]\!]^{k+1})^{\downarrow}$. For every $S \subseteq \mathrm{Types}_{A \to B}^k$ and $T \subseteq \mathrm{Types}_A^k$ we have: $[\![S(T)]\!]^k = [\![S]\!]^k([\![T]\!]^k)$.

We can now show that every element of $\mathcal{D}_A^k$ is representable by a type. For this we use $\overline{(\cdot)}$ operation (cf. Definition 10).

Lemma 24 For every $k \leq m$ and every type A. For every d in $\mathcal{D}_A^k$ there is $S \subseteq \mathrm{Types}_A^k$ so that $[\![S]\!]^k = d$, and there is $S' \subseteq \mathrm{types}_A^k$ so that $[\![S']\!]^k = \overline{d}$.

Proof
We proceed by induction on k.

The case where $k = 0$ has been proved in [32].

For the case $k > 0$, as we have seen with Lemma 11, that $f = (f^{\downarrow})^{\uparrow\wedge} \vee \overline{f}$. From the induction hypothesis there is $S_1 \subseteq \mathrm{Types}_A^{k-1}$ such that $[\![S_1]\!]^{k-1} = f^{\downarrow}$. By Lemma 23 we get $[\![S_1]\!]^k = (f^{\downarrow})^{\uparrow\wedge}$.

It remains to describe $\overline{f}$ with types from $\mathrm{types}_{B \to C}^k$. Take $d \in \mathcal{D}_B^k$ and recall that $\overline{f}(d) = \overline{f(d)}$. By induction hypothesis we have $S_d \subseteq \mathrm{Types}_B^k$ and $S_{\overline{f(d)}} \subseteq \mathrm{types}_C^k$ such that $[\![S_d]\!]^k = d$ and $[\![S_{\overline{f(d)}}]\!]^k = f(d)$. So the set of types $S_d \to S_{\overline{f(d)}}$ is included in $\mathrm{types}_{B \to C}^k$ and $[\![S_d \to S_{\overline{f(d)}}]\!] = d \rightharpoonup^k \overline{f}(d)$. It remains to take $S_2 = \bigcup \{ S_d | d \in \mathcal{D}_B^k \}$. We can conclude that $S_2 \subseteq \mathrm{types}_A^k$ and $[\![S_2]\!]^k = \overline{f}$. Therefore $[\![S_1 \cup S_2]\!]^k = f$. □

The next theorem is the main technical result of the paper. It says that the type system can derive all lower-approximations of the meanings of terms in the model. For an environment Γ, we write $[\![\Gamma]\!]^k$ for the valuation such that $[\![\Gamma]\!]^k(x) = [\![\Gamma(x)]\!]^k$.

Theorem 25 *For $k = 0, \ldots, m$ and $S \subseteq \mathrm{Types}^k$: $[\![M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![S]\!]^k$ iff $\Gamma \vdash M \geq S$ is derivable.*

The above theorem implies Theorem 5 stating soundness and completeness of the type system. Indeed, let us take a closed term M of type o, and a state q of our fixed automaton $\mathcal{A}$. Theorem 17 tells us that $[\![M]\!] = \mathcal{A}(M)$; where $\mathcal{A}(M)$ is the set of states from which $\mathcal{A}$ accepts $BT(M)$. So $\vdash M \geq q$ is derivable iff $[\![M]\!] \supseteq \{q\}$ iff $q \in \mathcal{A}(M)$.

The theorem is proved by the following two lemmas.

Lemma 26 If $\Gamma \vdash M \geq S$ is derivable, then for every $k \leq m$: $[\![M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![S]\!]^k$.

Proof
This proof is done by a simple induction on the structure of the derivation of $\Gamma \vdash M \geq S$. For most of the rules, the conclusion follows immediately from the induction hypothesis (using Lemma 23). We shall only treat here the case of the rules Y *odd* and Y *even*.

In the case of Y *odd*, when we derive $\Gamma \vdash Yx.M \geq S(T)$ from $\Gamma \vdash \lambda x.M \geq S$ and $\Gamma \vdash Yx.M \geq T$ with $S, T \in \mathrm{Types}_A^{2l+1}$, the induction hypothesis gives that for every k, $[\![\lambda x.M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![S]\!]^k$ and $[\![Yx.M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![T]\!]^k$. Therefore $[\![Yx.M]\!]^k_{[\![\Gamma]\!]^k} = [\![(\lambda x.M)(Yx.M)]\!]^k_{[\![\Gamma]\!]^k} \geq [\![S]\!]^k([\![T]\!]^k) = [\![S(T)]\!]^k$, using Lemma 23.

In the case of Y *even* we consider the case $k = 2l$. Let ν_{k-1} stand for $[\![\Gamma]\!]^{k-1}$ and ν_k for $[\![\Gamma]\!]^k$.

By induction hypothesis we have $[\![Yx.M]\!]^{k-1}_{\nu_{k-1}} \geq [\![T]\!]^{k-1}$. Since Lemma 15 implies $([\![Yx.M]\!]^{k-1}_{\nu_{k-1}}, [\![Yx.M]\!]^k_{\nu_k}) \in \mathcal{L}^k$, we have $[\![Yx.M]\!]^k_{\nu_k} \geq ([\![T]\!]^{k-1})^{\uparrow\wedge}$ by Lemma 7. By Lemma 23 we know $([\![T]\!]^{k-1})^{\uparrow\wedge} = [\![T]\!]^k$. In consequence we have

$$[\![\lambda x.M]\!]^k_{\nu_k}([\![T]\!]^k) \geq [\![T]\!]^k.$$

Also by induction hypothesis we have $[\![\lambda x.M]\!]^k_{\nu_k} \geq [\![(S \cup T) \to S]\!]^k$. This means $[\![\lambda x.M]\!]^k_{\nu_k}([\![S \cup T]\!]^k) \geq [\![S]\!]^k$. Put together with what we have concluded about $[\![T]\!]^k$ we get

$$[\![\lambda x.M]\!]^k_{\nu_k}([\![S \cup T]\!]^k) \geq [\![S \cup T]\!]^k.$$

Now we use Lemma 14 telling us that

$$[\![Yx.M]\!]^k_{\nu_k} = \bigvee \{ d \mid [\![\lambda x.M]\!]^k(d) \geq d \text{ and } d^{\downarrow} = [\![Yx.M]\!]^{k-1}_{\nu_{k-1}} \}.$$

This gives us immediately the desired $[\![Yx.M]\!]^k_{\nu_k} \geq [\![S \cup T]\!]^k$. □

Lemma 27 Given a type $S \subseteq \mathrm{Types}^k$, if $[\![M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![S]\!]^k$ then $\Gamma \vdash M \geq S$.

Proof
This theorem is proved by induction on the pairs (M, k) ordered component-wise. Suppose that the statement is true for M, we are going to show that it is true for $Yx.M$, the other cases are straightforward.

The first observation is that, if $T \to S$ is such that $[\![\lambda x.M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![T \to S]\!]^k$, then $\Gamma \vdash \lambda x.M \geq T \to S$ is derivable. Indeed, since, letting $\nu = [\![\Gamma, x \geq T]\!]^k$, if $[\![M]\!]^k_\nu \geq [\![S]\!]^k$ holds then, $\Gamma, x \geq T \vdash M \geq S$ is derivable by induction hypothesis. So $\Gamma \vdash \lambda x.M \geq T \to S$ is derivable.

There are now two cases depending on the parity of k. First let us assume that k is even. Suppose $[\![Yx.M]\!]^k_{[\![\Gamma]\!]^k} = [\![S \cup T]\!]^k$ where $S \subseteq \text{types}^k$ and $T \subseteq \text{Types}^{k-1}$. Lemma 24 guaranties the existence of such S and T as every element of $\mathcal{D}^k$ is expressible by a set of types. We have $[\![\lambda x.M]\!]^k_{[\![\Gamma]\!]^k} \geq [\![S \cup T]\!]^k \rightarrow^k [\![S \cup T]\!]^k$. By the above we get that $\Gamma \vdash \lambda x.M \geq (S \cup T) \to (S \cup T)$ is derivable and thus $\Gamma \vdash \lambda x.M \geq (S \cup T) \to S$ is also derivable. We also have $[\![Yx.M]\!]^{k-1} \geq [\![T]\!]^{k-1}$ which gives that $\Gamma \vdash Yx.M \geq T$ is derivable. This allows us to derive $\Gamma \vdash Yx.M \geq S \cup T$. Using the subsumption rule and the fact that the subsumption reflects the order on types, every other valid judgment $\Gamma \vdash Yx.M \geq U$ is derivable.

Now consider the case where k is odd. Suppose $[\![Yx.M]\!]^k_{[\![\Gamma]\!]^k} = [\![S \cup T]\!]^k$ with $T \subseteq \text{Types}^{k-1}$ and $S \subseteq \text{types}^k$. By induction hypothesis on k, we have that $\Gamma \vdash M \geq T$ is derivable. Take $d = [\![\lambda x.M]\!]^k_{[\![\Gamma]\!]_k}$. Lemma 24 guarantees us a set of types $U \subseteq \text{Types}^k$ such that $[\![U]\!]^k = d$. By the observation we have made above, there is a derivation of $\Gamma \vdash \lambda x.M \geq U$. Then iteratively using the rule Y *odd* we compute the least fixpoint by letting $U^0(T) = T$ and $U^{n+1}(T) = U(U^n(T))$. □

As we have seen, the applicative structure $\mathcal{D}^k_A$ is a lattice, therefore each construction can be dualized: in Abramsky's methodology, this consists in considering $\wedge$-prime elements of the models, meets and co-step functions instead of $\vee$-primes, joins and step functions. It is worth noticing that dualizing at the level of the model amounts to dualizing the automaton. So, in particular, we can define a system so that $BT(M)$ is not accepted by $\mathcal{A}$ from state q iff $\Gamma \vdash M \not\geq q$ is derivable. While the first typing system establishes positive facts about the semantics, the second one refutes them. For this, we use the same syntax to denote types, but we give types a different semantics that is dual to the first semantics we have used.

$$\langle\!\langle q \rangle\!\rangle^k = Q_{\leq k} - \{q\}\ ,$$
$$\langle\!\langle S \to f \rangle\!\rangle^k = \Big(\bigwedge \{\langle\!\langle g \rangle\!\rangle^k : g \in S\}\Big) \rightharpoonup^k \langle\!\langle f \rangle\!\rangle^k\ .$$

The dual type system is presented in Figure 7. The notation is as before but we use $\not\geq$ instead of $\geq$. Similarly to the definition of $[\![\cdot]\!]^k$, we write $\langle\!\langle S \rangle\!\rangle^k$ for $\bigwedge\{\langle\!\langle s \rangle\!\rangle^k : s \in S\}$ and we have that $\langle\!\langle T \to S \rangle\!\rangle^k = \langle\!\langle T \rangle\!\rangle^k \rightharpoonup^k \langle\!\langle S \rangle\!\rangle^k$. We also need to redefine $S(T)$ to be $\{s : U \to s \in S \wedge U \sqsupseteq T\}$. With those notations, the rules for application, abstraction, and variable do not change and are not presented in the Figure 4. By duality, from Theorem 25 we obtain:

Theorem 28 *For $S \subseteq \text{Types}^k$: $\Gamma \vdash M \not\geq S$ is derivable iff $[\![M]\!]^k_{[\![\Gamma]\!]^k} \leq \langle\!\langle S \rangle\!\rangle^k$.*

Together Theorems 25 and 28 give a characterization by typing of $[\![M]\!] = L(\mathcal{A})$, that is the set of states from which our fixed automaton $\mathcal{A}$ accepts $BT(M)$.

Corollary 29 For a closed term M of type o:

$$[\![M]\!] = [\![S]\!] \quad \text{iff} \quad \text{both} \quad \vdash M \geq S \text{ and } \quad \vdash M \not\geq (Q - S).$$

7. Conclusions

We have shown how to construct a model for a given weak alternating tree automaton so that the value of a term in the model determines if the Böhm tree of the term is accepted by the automaton. Our construction builds on ideas from [32] but requires to bring out the modular structure of the model. This structure is very rich, as testified by Galois connections (Corollary 8) and decomposition principles (Lemma 11). This structure allows us to derive type systems for wMSO properties following the "domains in logical form" approach.

The type systems are relatively streamlined: the novelty is the stratification of types used to restrict applicability of the greatest fixpoint rule.

Typing is decidable, actually the height of the derivation is bounded by the size of the term. Yet the width can be large, that is unavoidable given that the typability is n-EXPTIME hard for terms of order n [36]. Due to the correspondence of the typing with semantics, every term has a "best" type.

This paper focuses on typing, but our model construction can be also used in other contexts. It allows us to immediately deduce reflection [8] and transfer [33] theorems for wMSO. Our techniques used to construct models and prove their correctness rely on usual techniques of domain theory [3], offering an alternative, and arguably simpler, point of view to techniques based on unrolling.

The idea behind the reflection construction is to transform a given term so that at every moment of its evaluation every subterm "knows" its meaning in the model. In [8] this property is formulated slightly differently and is proved using a detour to higher-order pushdown automata. Recently Haddad [13] has given a direct proof for all MSO properties. The proof is based on some notion of applicative structure that is less constrained than a model of the λY-calculus. One could apply his construction, or take the one from [32].

The transfer theorem says that for a fixed finite vocabulary of terms, an MSOL formula φ can be effectively transformed into an MSOL formula $\widehat{\varphi}$ such that for every term M of type 0 over the fixed vocabulary: M satisfies $\widehat{\varphi}$ iff the Böhm tree of M satisfies φ. Since the MSO theory of a term, that is a finite graph, is decidable, the transfer theorem implies decidability of MSO theory of Böhm trees of λY-terms. As shown in [33] it gives also a number of other results.

A transfer theorem for wMSO can be deduced from our model construction. For every wMSO formula φ we need to find a formula $\widehat{\varphi}$ as above. For this we transform φ into a weak alternating automaton $\mathcal{A}$, and construct a model $\mathcal{D}_\varphi$ based on $\mathcal{A}$. Thanks to the restriction on the vocabulary, it is quite easy to write for every element d of the model $\mathcal{D}_\varphi$ a wMSO formula α_d such that for every term M of type 0 in the restricted vocabulary: $M \vDash \alpha_d$ iff $[\![M]\!]^{\mathcal{D}_\varphi} = d$. The formula $\widehat{\varphi}$ is then just a disjunction $\bigvee_{d \in F} \alpha_d$, where F is the set elements of $\mathcal{D}_\varphi$ characterizing terms whose Böhm tree satisfies φ.

The fixpoints in our models are non-extremal: they are neither the least nor the greatest fixpoints. From Theorem 4 we know that this is unavoidable. We are aware of very few works considering such cases. Our models are an instance of cartesian closed categories with internal fixpoint operation as studied by Bloom and Esik [6]. Our model satisfies not only Conway identities but also a generalization of the *commutative axioms* of iteration theories [5]. Thus it is possible to give semantics to the infinitary λ-calculus in our models. It is an essential step towards obtaining an algebraic framework for weak regular languages [7].

$$\frac{S \subseteq T \subseteq Q}{S \sqsupseteq_0 T} \qquad \frac{\forall s \in S, \exists t \in T, s \sqsupseteq_A t}{S \sqsupseteq_A T} \qquad \frac{T \sqsupseteq_A S \quad s \sqsupseteq_B t}{S \to s \sqsupseteq_{A\to B} T \to t}$$

$$\frac{}{\Gamma \vdash c \geq \{q : \delta_o(q,c) \text{ does not hold}\}} \qquad \frac{\forall (S_1,S_2) \in \delta(a,q), (T_1 \cap S_1) \cup (T_2 \cap S_2) \neq \emptyset}{\Gamma \vdash a \not\geq T_1 \to T_2 \to q}$$

$$\frac{S \subseteq \text{types}_A^{2k+1}, \quad T \in \text{Types}_A^{2k}, \quad \Gamma \vdash \lambda x.M \not\geq (S \cup T) \to S \quad \Gamma \vdash Yx.M \not\geq T}{\Gamma \vdash Yx.M \not\geq S \cup T}\ Y \text{ odd}$$

$$\frac{\Gamma \vdash (\lambda x.M) \not\geq S \quad \Gamma \vdash (Yx.M) \not\geq T}{\Gamma \vdash Yx.M \not\geq S(T)}\ Y \text{ even}$$

Figure 7. Dual type system

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